where $v = v(\mathbf{t})$ and $\Lambda = \Lambda(\mathbf{t})$ are given by (167), or obey

$$v - a = 2t_2\Lambda^2, \qquad \log\Lambda^2 = t_1 + 2t_2 v \tag{173}$$ rel2

Formula (172) is a direct consequence of (143) and directly following from (152) upon the relations (173) expression

$$\begin{aligned}\Phi(w; t_1, t_2, a) &= 2t_2 y - 2\log w + (t_1 + 2t_2 v - \log\Lambda^2)\frac{z-v}{y} + 2\frac{a - v + 2t_2\Lambda^2}{y} = \\ &\underset{(173)}{=} 2t_2\Lambda\left(w - \frac{1}{w}\right) - 2\log w\end{aligned} \tag{174}$$ fit2

Note, that (172) stays that the Vershik-Kerov "arcsin law" [10] for the limiting shape is deformed by the Wigner semicircle distribution.

If the first three Toda times t_1, t_2, t_3 are nonvanishing, instead of (172) one gets

$$\begin{gathered}f'(x) = \frac{2}{\pi}\left(\arcsin\left(\frac{x-v}{2\Lambda}\right) + (t_2 + 3t_3 v)\sqrt{4\Lambda^2 - (x-v)^2} + \frac{3}{2}t_3(x-v)\sqrt{4\Lambda^2 - (x-v)^2}\right) \\ v - 2\Lambda \le x \le v + 2\Lambda\end{gathered} \tag{175}$$ shape3

where v and Λ are now subjected to

$$\begin{aligned}v - a &= 2(t_2 + 3t_3 v)\Lambda^2 \\ \log\Lambda^2 &= t_1 + 2t_2 v + 3t_3(v^2 + 2\Lambda^2)\end{aligned} \tag{176}$$ rel3

Generally we obtain for the limit shape

$$\begin{gathered}f'(x) = \frac{2}{\pi}\left(\arcsin\left(\frac{x-v}{2\Lambda}\right) + \sum_{k>1} t_k Q_k(x)\sqrt{4\Lambda^2 - (x-v)^2}\right) \\ v - 2\Lambda \le x \le v + 2\Lambda\end{gathered} \tag{177}$$ shapegen

where v and Λ obey some sort of hodograph equations $v - a = P_v(v, \Lambda; \mathbf{t})$, $\log\Lambda^2 = P_\Lambda(v, \Lambda; \mathbf{t})$ for some polynomials P_v and P_Λ, whose expansion in Toda times $\mathbf{t}$ can be easily reconstructed from the presented above formulas of general solution.

Hurwitz numbers

For $t_2 \neq 0$ one gets the (asymptotic) generation function for the Hurwitz numbers

$$H_{g,d} = \langle \sigma_1(\varpi)^{2g+2d-2} \rangle_{g,d}$$

the number of genus g, d-sheeted covers of $\mathbb{P}^1$, with a fixed general branch divisor of degree

$$d \cdot \chi(\mathbb{P}^1) - \chi(\Sigma_g) = 2d + 2g - 2$$

(the Riemann-Hurwitz formula).

Indeed,

$$\mathcal{F}(a=0,t_1,t_2=\tfrac{1}{2},0,\dots)=\sum_{d>0}\frac{H_{d,0}}{(2d-2)!}e^{dt_1}$$

From our solution

$$\left.\frac{\partial\mathcal{F}}{\partial t_1^2}\right|_{a=0,t_2=1/2}=\Lambda^2=-\mathbf{L}\left(-e^{t_1}\right) \tag{178}$$

therefore $H_{d,0}=\frac{(2d-2)!}{d!}d^{d-3}$ since

$$\mathbf{L}(t)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}t^n}{n!} \tag{179}$$

3.3 Extended Toda hierarchy

Formula (152) can be naturally generalized to

$$S=\sum_{k>0}t_k\Omega_k(w)+2a\log w-2\sum_{k>0}T_kH_k(z,w) \tag{180}$$ SwE

so that (152) is a particular case of (180), corresponding to

$$T_k=\delta_{k,1} \tag{181}$$ nopun

which is implied by the following definition of extra Hamiltonians

$$H_k(z,w)=z^k\log w+\sum_{j=1}^{k}C_j^{(k)}\Omega_j(w) \tag{182}$$ HEk

which are odd under involution $w\leftrightarrow\frac{1}{w}$ and fixed by the asymptotics

$$\begin{gathered}H_k(z,w)\underset{z\to\infty}{=}\pm H_k^{(+)}(z)+O(1)\\ H_k^{(+)}(z)=z^k(\log z-c_k),\quad c_k=\sum_{i=1}^{k}\frac{1}{i}\\ dH_k^{(+)}=kH_{k-1}^{(+)}dz\end{gathered} \tag{183}$$ HEas

i.e.

$$\begin{gathered}C_k^{(k)}=\log\Lambda-c_k=H_k^{(+)}(\Lambda)\Lambda^{-k}\\ C_j^{(k)}=\omega_{k-j},\quad j=1,\dots,k-1\\ \log w\underset{z\to\infty}{=}\log z-\log\Lambda-\sum_{k>0}\frac{\omega_k}{z^k}\end{gathered} \tag{184}$$ condHE

which upon substitution into (198), and using (189) give rise to

$$\mathcal{F}(t_1, a, T_1) = \frac{a^2 t_1}{2T_1} + T_1^2 \exp\frac{t_1}{T_1} \tag{200}$$ Ft1T1

It is interesting to point out that at $T_1 \to \infty$, (200) gives

$$\begin{aligned}\mathcal{F}(t_1, a, T_1) \underset{T_1\to\infty}{\sim} \left(T_1^2 + T_1 t_1 + \frac{t_1^2}{2}\right) + \frac{1}{T_1}\left(\frac{a^2 t_1}{2} + \frac{t_1^3}{6T_1}\right) + \ldots = \\ = \ldots + \frac{1}{6T_1}\left((t_1+a)^3 + (t_1-a)^3\right) + \ldots\end{aligned} \tag{201}$$ Ft1T1inf

modulo quadratic terms.

3.4 Landau-Ginzburg calculations

4 Non-abelian theory

4.1 Quasiclassical tau-functions and Seiberg-Witten prepotentials

A generic finite-gap integrable system can be defined by a generating differential dS (the Jacobi differential "pdq") on spectral curve Σ whose derivatives over g independent directions in moduli space, give rise to (some) linearly independent holomorphic differentials

$$\delta_{\mathrm{moduli}} dS = \text{holomorphic} \tag{202}$$ hol

It is easy to see that the holomorphic differentials in the r.h.s. of (202) become canonically normalized (30) if one takes as co-ordinates on moduli space the canonically normalized integrals of motion – the canonical action variables or **A**-periods of the differential dS

$$a_j = \oint_{A_j} dS \tag{203}$$ aper

Indeed,

$$\delta a_j = \oint_{A_j} \frac{\partial dS}{\partial a_i}\delta a_i \tag{204}$$ vara

and comparing (204) with (30) one gets

$$\frac{\partial dS}{\partial a_i} \cong d\omega_i \tag{205}$$ varsa

where $\cong$ means equality modulo total derivatives, disappearing from the contour integral. By accepted convention the corresponding "dual" **B**-periods will be called $\mathbf{a}_D$

$$a_j^D = \oint_{B_j} dS \tag{206}$$ adper

Taking derivatives of both sides w.r.t. a_i and using (205), (31) one gets that

$$\frac{\partial a_j^D}{\partial a_i} = \oint_{B_j} \frac{\partial dS}{\partial a_i} = \oint_{B_j} d\omega_i = T_{ij} \tag{207}$$ ada

The existence of the relation (202) can be checked for all known g-parametric families of curves.

4.2 Elliptic U(1) example

Consider elliptic curve $\mathcal{E}_\tau = \mathbb{C}/(1,\tau)$ with a marked point P. Choose $z(P) = 0$ where $dz \in H^1(\mathcal{E}_\tau)$ is the canonical holomorphic differential

$$\oint_A dz = 1, \qquad \oint_B dz = \tau \tag{208}$$ dzper

Consider also the second kind Abelian differential

$$d\lambda = -m dz\,(\wp(z) + 2\eta) = -m\frac{dz}{z^2} + \ldots \tag{209}$$ dlam

where $\eta = \zeta(1/2)$ is chosen to ensure

$$\begin{gathered}\oint_A d\lambda = 0\\ \oint_B d\lambda = -m(2\eta\tau - 2\eta') = -2\pi i m = \text{const}\end{gathered} \tag{210}$$ dlamper

where $\eta' = \zeta(\tau/2)$ and the last equation is due to the Legender identity. Integrating (209) one gets

$$\lambda = m(\zeta(z) - 2\eta z + i\pi) + a = m\frac{\theta'(z)}{\theta(z)} + a + i\pi m \tag{211}$$ lam

where $\theta(z) \equiv \theta_1(z|\tau)$ is the only odd Jacobi theta-function, while the integration constant is taken to get

$$\oint_A dS = \oint_A \lambda dz = a \tag{212}$$ aper

For the dual period one gets

$$\begin{aligned}\oint_B dS = \oint_B \lambda dz = (a + i\pi m)\tau + m \log\frac{\theta(z_0+\tau)}{\theta(z_0)} =\\ = a\tau + i\pi m(\pm 1 - 2z_0)\end{aligned} \tag{213}$$ bper

For the derivatives of generating one-form one gets

$$\frac{\partial}{\partial a}dS = \left.\frac{\partial\lambda}{\partial a}\right|_{z,m,\tau} dz = dz$$
$$\frac{\partial}{\partial\tau}dS = \left.\frac{\partial\lambda}{\partial\tau}\right|_{z,m,a} dz = m\left(\frac{\partial}{\partial\tau}\log\theta(z)\right)' dz = 4\pi i m\left(\frac{\theta''(z)}{\theta(z)}\right)' dz \tag{214}$$

Sder

i.e. the derivative over τ gives rise to a non-single valued differential, since

$$\Omega = \frac{\theta''(z)}{\theta(z)} = (\log\theta(z))'' + \left((\log\theta(z))'\right)^2 \equiv J'(z) + J^2(z)$$
$$\Omega(z_0+1) - \Omega(z_0) = 0$$
$$\Omega(z_0+\tau) - \Omega(z_0) = ((J(z_0+\tau) - J(z_0))((J(z_0+\tau) + J(z_0)) =$$
$$= -4\pi i(J(z_0) - i\pi) = -4\pi i\left(\frac{\theta(z_0)'}{\theta(z_0)} - i\pi\right) \tag{215}$$

omega

so that

$$\Delta_A d\Omega = d\Omega(z_0+\tau) - d\Omega(z_0) = -4\pi i d\left(\frac{\theta(z_0)'}{\theta(z_0)}\right) = -\frac{4\pi i}{m}d\lambda \tag{216}$$

jumpdom

In order to get the nontrivial part of prepotential one needs now to compute

$$\frac{\partial\mathcal{F}}{\partial\tau} = \tfrac{1}{2}\oint_A \lambda^2 dz = \frac{1}{2}\left(a^2 + \pi^2 m^2 + m^2\oint_A\left(\frac{\theta(z)'}{\theta(z)}\right)^2 dz\right) \tag{217}$$

Ftau

For the last integral one gets

$$\oint_A\left(\frac{\theta(z)'}{\theta(z)}\right)^2 dz = \pi^2\int_{z_0}^{z_0+1}\cot^2(\pi z)dz + 8\pi^2\sum_{n>0}\frac{q^{2n}}{1-q^{2n}}\int_{z_0}^{z_0+1}\cot(\pi z)\sin(2\pi n z)dz +$$
$$+16\pi^2\sum_{n,k>0}\frac{q^{2n}}{1-q^{2n}}\frac{q^{2k}}{1-q^{2k}}\int_{z_0}^{z_0+1}\sin(2\pi n z)\sin(2\pi k z)dz \tag{218}$$

serint

where $q = e^{i\pi\tau}$. Computing the integrals one gets

$$\int_{z_0}^{z_0+1}\cot^2(\pi z)dz = -\pi^2$$
$$\int_{z_0}^{z_0+1}\cot(\pi z)\sin(2\pi n z)dz = 1, \quad n>0$$
$$\int_{z_0}^{z_0+1}\sin(2\pi n z)\sin(2\pi k z)dz = \tfrac{1}{2}\delta_{n,k}, \quad n,k>0 \tag{219}$$

intcomp

Therefore

$$\frac{\partial \mathcal{F}}{\partial \tau} = \oint_A \lambda^2 dz = a^2 + 8\pi^2 m^2 \sum_{n>0} \frac{q^{2n}}{(1-q^{2n})^2} \qquad (220) \quad \texttt{Ftaures}$$

Integrating this relation one finally gets

$$\mathcal{F} = \tfrac{1}{2}\tau a^2 + \frac{4\pi m^2}{i} \sum_{n>0} \int \frac{q^{2n-1} dq}{(1-q^{2n})^2} = \tfrac{1}{2}\tau a^2 + \frac{2\pi m^2}{i} \sum_{n>0} \frac{1}{n} \left(\frac{1}{1-q^{2n}} - 1 \right) \qquad (221)$$

Ressumation of the last expression gives

$$\sum_{n>0} \frac{1}{n} \left(\frac{1}{1-q^{2n}} - 1 \right) = \sum_{n,k>0} \frac{q^{2nk}}{n} = - \sum_{k>0} \log\left(1 - q^{2k}\right) \qquad (222)$$

so that

$$\mathcal{F} = \tfrac{1}{2}\tau a^2 + 2\pi i m^2 \log \prod_{k>0} \left(1 - q^{2k}\right) \sim \tfrac{1}{2}\tau a^2 + 2\pi i m^2 \log \eta(\tau) \qquad (223) \quad \texttt{prep}$$

where $\eta(\tau) = q^{1/12} \prod_{k>0} \left(1-q^{2k}\right)$ is the Dedekind function.

4.3 Extended non-abelian theory

In the case of $U(N)$ gauge theory one has to consider solution with N cuts $\{\mathbf{I}_i\}$, $i = 1 \ldots, N$, which arises after adding to the functional (134) N constraints with the Lagrange multipliers

$$\mathcal{F} \to \mathcal{F} + \sum_{i=1}^{N} a_i^D \left(a_i - \frac{1}{2} \int_{\mathbf{I}_i} dx \; x f''(x) \right) \qquad (224)$$

i.e. solution to the integral equation

$$\sum_{k>0} t_k x^k - \int d\tilde{x} f''(\tilde{x})(x-\tilde{x}) \left(\log|x-\tilde{x}| - 1\right) = a_i^D, \qquad x \in \mathbf{I}_i, \quad i = 1, \ldots, N \qquad (225) \quad \texttt{exeqN}$$

Now it can be expressed in terms of the Abelian integrals on the double cover

$$y^2 = \prod_{i=1}^{N} (z - x_i^+)(z - x_i^-) \qquad (226) \quad \texttt{dcN}$$

which is a hyperelliptic curve of genus $g = N - 1$. Define, as before:

$$S(z) = \mathbf{t}'(z) - \int dx f''(x)(z-x)(\log(z-x) - 1) - a^D \qquad (227) \quad \texttt{sfunn}$$

where the integral is taken over the whole support $\mathbf{I} = \cup_{i=1}^N \mathbf{I}_i$, $a^D = \frac{1}{N}\sum_{j=1}^N a_j^D$, and consider its differential, or

$$\Phi(z) = \frac{dS}{dz} = \sum_{k>0} kt_k z^{k-1} - \int dx f''(x)\log(z-x) \tag{228}$$ dS

satisfying

$$\Phi(x+i0) + \Phi(x-i0) = 0, \quad x \in \mathbf{I}_i, \quad i = 1,\dots,N \tag{229}$$ ficut

on each cut, and normalized to

$$\begin{gathered} \Phi(x_N^+) = 0, \\ \Phi(x_j^- \pm i0) = \Phi(x_{j-1}^+ \pm i0) = \pm 2\pi i(N-j+1), \quad j = 2,\dots,N \\ \Phi(x_1^-) = \pm 2\pi i N \end{gathered} \tag{230}$$ ImfiN

Vanishing microscopic times

Consider, first, all $t_k = 0$ for $k \neq 1$, and define $\Lambda^{2N} = e^{t_1}$. Now $\Phi = \frac{dS}{dz}$ is an Abelian integral on the curve (226) with the asymptotic

$$\Phi \underset{P\to P_\pm}{=} \mp 2N\log z \pm 2N\log\Lambda + O(z^{-1}) \tag{231}$$ fiasysw

whose jumps are integer-valued due to (230), or $\oint d\Phi \sim 4\pi i\mathbb{Z}$. It means that the hyperelliptic curve (226) can be seen also as an algebraic Riemann surface for the function $w = \exp(-\Phi/2)$, satisfying quadratic equation

$$\Lambda^N\left(w + \frac{1}{w}\right) = P_N(z) = \prod_{i=1}^N (z - v_i) \tag{232}$$ Todacu

since for the two branches $w_+ = w$ and $w_- = \frac{1}{w}$ one immediately finds that their product $w_+ \cdot w_-$ and sum $w_+ + w_-$ are polynomials of z of given powers (zero and N correspondingly).

Equivalently, the ends of the cuts in (226) are restricted by N constraints in such a way, that this equation can be rewritten as

$$y^2 = P_N(z)^2 - 4\Lambda^{2N} \tag{233}$$ Today

i.e. $\{x_i^\pm\}$ are roots of $P_N(z) \mp 2\Lambda^N = 0$, and

$$y = \Lambda^N\left(w - \frac{1}{w}\right) \tag{234}$$ yw

The generating differential (228) is now

$$dS = -2\log w dz = -d(2z\log w) + 2z\frac{dw}{w} \tag{235}$$ dSsw

just the Legendre transform of the Seiberg-Witten differential $d\Sigma \sim z\frac{dw}{w}$ on the curve (232), (233). It periods

$$a_i = \frac{1}{2\pi i}\oint_{A_i} z\frac{dw}{w} \tag{236}$$ SWper

coincide with the Seiberg-Witten integrals and the only nontrivial residues at infinity give

$$\begin{aligned}\mathrm{res}_{P_+}\left(z^{-1}dS\right) = -\mathrm{res}_{P_-}\left(z^{-1}dS\right) = \log\Lambda^{2N}\\ \mathrm{res}_{P_+}(dS) = -\mathrm{res}_{P_-}(dS) = 2\sum_{j=1}^{N} v_j\end{aligned} \tag{237}$$

The differential (235) satisfies the condition

$$\delta dS \sim \frac{\delta w}{w}dz = \frac{\delta P(z)}{y}dz \underset{\sum_{j=1}^N v_j=0}{=} \text{holomorphic} \tag{238}$$

where the variation is taken at constant co-ordinate z and constant scale factor Λ. Thus, the integrable system on "small phase space" is solved for the scale $\Lambda^{2N} = e^{t_1}$ and the moduli v_j, $j = 1,\ldots,N$ of vacua of the $U(N)$ gauge theory, satisfying the equation $\sum_{j=1}^N v_j = a$ and the transcendental equations for the Seiberg-Witten periods (249).

Nonvanishing microscopic times

When we switch on "adiabatically" the higher times (77) with $k > 1$, the number of cuts in (225) remains intact, and the differential (228) can be still defined on hyperelliptic curve (226). However, now the role of bipole differential $\frac{dw}{w}$ of the third kind is played by

$$\begin{aligned} d\Phi &= dz\left(\sum_{k>1} k(k-1)t_k z^{k-2} - \int\frac{dx f''(x)}{z-x}\right) = \\ &= \sum_{k>1} k(k-1)t_k d\Omega_{k-1} - 2N d\Omega_0 - 4\pi i\sum_{j=1}^{N-1} d\omega_j \end{aligned} \tag{239}$$ dpsi

where $d\omega_i$, $i = 1,\ldots,N-1$ are canonical holomorphic differentials normalized to the A-cycles, surrounding first $N-1$ cuts. The differentials $d\Omega_k$ in (239) are fixed by their asymptotic at $z\to\infty$

$$d\Omega_k \underset{z\to\infty}{=} \begin{cases} kz^{k-1}dz + O(z^{-2}), & k>0 \\ \dfrac{dz}{z} + O(z^{-2}), & k=0\end{cases} \tag{240}$$ Omas

and vanishing A-periods

$$\oint_{A_i} d\Omega_k = 0, \quad k \geq 0, \quad \forall\, i = 1, \dots, N-1 \tag{241}$$

The nonvanishing periods of $d\Phi$ are fixed by

$$\oint_{A_j} d\Phi = -2\pi i \int_{\mathbf{I}_j} f''(x)dx = -2\pi i \left(f'(x_j^+) - f'(x_j^-) \right) = -4\pi i \tag{242}$$ dpsiperA

which justifies that generating differential dS is still defined modulo $4\pi i dz$. The only important difference with the previous case is that integrality of the periods $\oint d\Phi \sim 4\pi i \mathbb{Z}$, which was reformulated in terms of an algebraic equation (232) for the theory on "small phase space", remains now a transcendental equation, which cannot be resolved explicitly.

Nevertheless, on the curve (226) any odd under hyperelliptic involution differential can be always presented as

$$d\Phi = \frac{s(z)dz}{y} \tag{243}$$ dpsiy

where $s(z)$ is a polynomial of power $N+K-2$ in case of nonvanishing microscopic times $t_1, \dots, t_K$ up to the K-th order. Its higher K coefficients are fixed by leading asymptotic $(t_2, \dots, t_K)$ and the residue at infinity

$$\mathrm{res}_{P_\pm} d\Phi = \mp 2N \tag{244}$$

and the rest $N-1$ coefficients can be determined from (242). This fixes completely the differential $d\Phi$ on the curve (226) which still remain to be dependent upon $2N$ (yet arbitrary) branch points $\{x_j^\pm\}$.

The generating differential dS can be now defined in terms of the Abelian integral $\Phi(z)$

$$dS = \Phi dz = dz \int_{z_0}^{z} d\Phi \tag{245}$$ dSpsi

The dependence upon $2N+1$ parameters (the positions of the branch points in (226) together with z_0) is constrained by additional to (242) vanishing of the B-periods

$$\oint_{B_j} d\Phi = 0, \quad j = 1, \dots, N-1 \tag{246}$$ dpsiperB

Integral representation (228) suggests a natural normalization (230), i.e.

$$z_0 = x_N^+, \quad \Phi(z_0) = \Phi(x_N^+) = 0 \tag{247}$$ normfi

where x_N^+ is the largest among real ramification points $\{x_j^\pm\}$. These conditions lead to the following form of expansion of $\Phi(z)$ in the vicinity of ramification points

$$\begin{aligned} \Phi(z) &\underset{z\to x_j^\pm}{=} \Phi(x_j^\pm) + \phi_j^\pm\sqrt{z - x_j^\pm} + \ldots \\ \phi_j^\pm &= \frac{2s(x_j^\pm)}{\prod_k' \sqrt{(x_j^\pm - x_k^+)(x_j^\pm - x_k^-)}}, \qquad j = 1, \ldots, N \end{aligned} \tag{248}$$ psibranc

where the constants $\Phi(x_j^\pm)$ are given by (230).

The rest $N+1$ parameters are eaten by the periods

$$a_j = \frac{1}{2}\int_{\mathbf{I}_j} dx x f''(x) = -\frac{1}{4\pi i}\oint_{A_j} z d\Phi = \frac{1}{4\pi i}\oint_{A_j} dS, \quad j = 1, \ldots, N-1 \tag{249}$$ Aper

together with the residues

$$a = \tfrac{1}{2}\int_{\mathbf{I}} dx x f''(x) = -\tfrac{1}{2}\mathrm{res}_{P_+}(z d\Phi) \tag{250}$$ respdz

and the "free term" or scaling factor

$$t_1 = \mathrm{res}_{P_+}\left(z^{-1}\Phi dz\right) \tag{251}$$ t1pdz

Recall once more, that an essential difference with the case of vanishing times is that for $t_k \neq 0$, the exponent $\exp(\Phi)$ acquires an essential singularity at the points $P_\pm$, and the constraints (242), (246) cannot be resolved algebraically. The form of the expansion (248) ensures that variation of the generating differential at constant z w.r.t. moduli of the curve (226)

$$\begin{aligned} &\delta(dS) = \delta(\Phi dz) = \\ &\underset{z\to x_j^\pm}{=} \frac{-s(x_j^\pm)\delta x_j^\pm}{\prod_k' \sqrt{(x_j^\pm - x_k^+)(x_j^\pm - x_k^-)}}\frac{dz}{\sqrt{z - x_j^\pm}} + \ldots \simeq \text{holomorphic} \end{aligned} \tag{252}$$ delShol

is indeed holomorphic.

The Lagrange multipliers

$$a_i^D = \frac{\partial\mathcal{F}}{\partial a_i} \tag{253}$$ ad

can be computed by a standard trick. Consider equation (225) for $i \neq j$ and fix there x-variables to be at the ends of corresponding cuts. Then

$$a_i^D - a_j^D = \mathrm{Re}\int_{x_j^+}^{x_i^-} dS = \frac{1}{2}\oint_{B_{ij}} dS \tag{254}$$ adper