brane has only eight supercharges)[6], the instanton partition function $Z(a, \hbar, \mathbf{t})$, $\mathbf{t} = (t_1, t_2, \ldots)$, can be shown to be given by the sum over the Young diagrams, i.e. over the partitions

$$Z(a, \mathbf{t}, \hbar) = \sum_{\mathbf{k}} \frac{\mathbf{m}_{\mathbf{k}}^2}{(-\hbar^2)^{|\mathbf{k}|}} \exp \frac{1}{\hbar^2} \sum_{k>0} t_k \frac{\mathrm{ch}_{k+1}(a, \mathbf{k}, \hbar)}{k+1} \tag{78}$$ zuone

where $\mathbf{m}_{\mathbf{k}}$ is the Plancherel measure (95), and the Chern polynomials $\mathrm{ch}_{k+1}(a, \mathbf{k}, \hbar)$ can be introduced, e.g. via

$$\left(e^{\frac{\hbar u}{2}} - e^{-\frac{\hbar u}{2}}\right) \sum_{i=1}^{\infty} e^{u(a+\hbar(\frac{1}{2}-i+k_i))} = \sum_{l=0}^{\infty} \frac{u^l}{l!} \mathrm{ch}_l(a, \mathbf{k}, \hbar) \tag{79}$$ chern

If the theory has the gauge group $U(N)$, e.g. it is realized on the stack of N fractional D3 branes, the corresponding partition function is given by the generalization of (78):

$$Z(\vec{a}, \mathbf{t}, \hbar) = Z^{\mathrm{pert}}(\vec{a}, \mathbf{t}, \hbar) \sum_{\vec{\mathbf{k}}} \left(\mathbf{m}(\vec{a}, \vec{\mathbf{k}}, \hbar)\right)^2 (-1)^{|\vec{\mathbf{k}}|} \exp \frac{1}{\hbar^2} \sum_{k>0} t_k \frac{\mathrm{ch}_{k+1}(\vec{a}, \vec{\mathbf{k}}, \hbar)}{k+1} \tag{80}$$ zunntre

where $\mathbf{m}(\vec{a}, \vec{\mathbf{k}}, \hbar)$ is the $U(N)$ generalization of Plancherel measure [6] and $Z^{\mathrm{pert}}(\vec{a}, \mathbf{t}, \hbar)$ is the perturbative partition function.

2.5 Partitions and free fermions

Let us introduce, first, the main definitions and notations for the two-dimensional theory of a single free complex fermion the action $\int \tilde{\psi} \bar{\partial} \psi$ on a cylinder. One can expand the solutions to Dirac equation in holomorphic co-ordinate $w \in \mathbb{C}^*$:

$$\begin{aligned} \psi(w) &= \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r \, w^{-r} \left(\frac{dw}{w}\right)^{\frac{1}{2}}, \\ \widetilde{\psi}(w) &= \sum_{r \in \mathbb{Z}+\frac{1}{2}} \widetilde{\psi}_r \, w^{r} \left(\frac{dw}{w}\right)^{\frac{1}{2}}, \end{aligned} \tag{81}$$ frem

so that the modes after quantization satisfy the (anti)commutational relations

$$\{\psi_r, \widetilde{\psi}_s\} = \delta_{rs} \tag{82}$$

[6] This theory can also be realized at a special point on the moduli space of $U(N)$ gauge theory with $2N-2$ fundamental hypermultiplets.

The fermionic Fock space is constructed with the help of the charge M vacuum state (a Dirac sea)

$$|M\rangle = \psi_{-M+\frac{1}{2}}\psi_{-M+\frac{3}{2}}\psi_{-M+\frac{5}{2}}\ldots = \bigwedge_{r>-M} \psi_r \tag{83}$$ vcms

with

$$\psi_r|M\rangle = 0,\ r > -M, \qquad \widetilde{\psi}_r|M\rangle = 0,\ r < -M \tag{84}$$

and these definitions correspond to the two-point function

$$\langle 0|\tilde{\psi}(z)\psi(w)|0\rangle = \frac{\sqrt{dzdw}}{z-w} \tag{85}$$ vaccor

More conventional "Japanese" conventions (with the integer-valued fermionic operators ψ_i, ψ_i^*, $i \in \mathbb{Z}$, see e.g. [?]) can be got from these by

$$\tilde{\psi}_r \to \psi_{r+\frac{1}{2}}, \qquad \psi_r \to \psi^*_{r+\frac{1}{2}}, \qquad M = -n \tag{86}$$ japfer

It is also convenient to use the basis of the so-called partition states: for each partition $\mathbf{k} = (k_1 \geq k_2 \geq \ldots \geq k_{\ell_{\mathbf{k}}} = 0 \geq 0 \ldots)$ one introduces the state:

$$|M;\mathbf{k}\rangle = \psi_{-M+\frac{1}{2}-k_1}\psi_{-M+\frac{3}{2}-k_2}\ldots = \bigwedge_{r>-M} \psi_{r-k_i} \tag{87}$$ prtst

and defines the $U(1)$ current as:

$$J =: \widetilde{\psi}\psi := \sum_{n\in\mathbb{Z}} J_n w^{-n}\frac{dw}{w}, \qquad J_n = \sum_{r\in\mathbb{Z}+\frac{1}{2}} : \widetilde{\psi}_r\psi_{r+n} : \tag{88}$$ crnt

Obviously

$$\begin{aligned} [J_n, \psi_r] = -\psi_{r+n}, &\qquad \left[J_n, \tilde{\psi}_r\right] = \tilde{\psi}_{r-n} \\ [J_n, \psi(w)] = -w^n\psi(w), &\qquad \left[J_n, \tilde{\psi}(w)\right] = w^n\tilde{\psi}(w) \end{aligned} \tag{89}$$ Jpsi

Recall the bosonization rules:

$$\tilde{\psi} =: e^{i\phi} :, \quad \psi =: e^{-i\phi} :, \quad J = i\partial\phi \tag{90}$$ bsnz

where

$$\phi(z)\phi(0) \sim -\log z + \ldots \tag{91}$$ fiope

and a useful fact from $U(\hat{N})$ and permutation's group theory: the Schur-Weyl correspondence, which states that

$$(\mathbf{C}^{\hat{N}})^{\otimes k} = \bigoplus_{\mathbf{k},|\mathbf{k}|=k} R_{\mathbf{k}} \otimes \mathcal{R}_{\mathbf{k}} \tag{92}$$ wlcrsp

as $\mathcal{S}_k \times U(\hat{N})$ representation. Now let $U = \mathrm{diag}\,(u_1, \ldots, u_{\hat{N}})$ be a $U(\hat{N})$ matrix. Then one easily gets using the Weyl character formula, and the bosonization rules (90), that:

$$\mathrm{Tr}_{\mathcal{R}_{\mathbf{k}}} U = \langle \hat{N}; \mathbf{k} | : e^{i \sum_{n=1}^{\hat{N}} \phi(u_n)} : |0\rangle \;=\; s_{\mathbf{k}}(u_1, \ldots, u_{\hat{N}}) = \frac{\det u_j^{k_i + \hat{N} - i}}{\det u_j^{\hat{N}-i}} \tag{93}$$ chrt

gives the (ratio of the) standard Schur functions for any partition $\mathbf{k}$, a very nice review of their properties can be found in [4]. In particular, from this formula one derives:

$$e^{\frac{J_{-1}}{\hbar}} \, |M\rangle = \sum_{\mathbf{k}} \frac{\mathbf{m}_{\mathbf{k}}}{\hbar^k} \, |M; \mathbf{k}\rangle = \sum_{\mathbf{k}} \frac{\mathrm{dim} R_{\mathbf{k}}}{\hbar^k \, k!} \, |M; \mathbf{k}\rangle \tag{94}$$ nashevse

with

$$\begin{aligned} \mathbf{m}_{\mathbf{k}} &= \frac{\mathrm{dim} R_{\mathbf{k}}}{k!} = \prod_{i<j} \frac{k_i - k_j + j - i}{j - i} = \\ &= \prod_{i=1}^{\ell_{\mathbf{k}}} \frac{(\ell_{\mathbf{k}} - i)!}{(\ell_{\mathbf{k}} + k_i - i)!} \prod_{1 \leq i < j \leq \ell_{\mathbf{k}}} \frac{k_i - k_j + j - i}{j - i} = \frac{\prod_{1 \leq i < j \leq \ell_{\mathbf{k}}} (k_i - k_j + j - i)}{\prod_{i=1}^{\ell_{\mathbf{k}}} (\ell_{\mathbf{k}} + k_i - i)!} \end{aligned} \tag{95}$$ planch

being the Plancherel measure. It follows from the fact that for particular values $u_1 = \ldots = u_{\hat{N}} = \frac{1}{\hbar \hat{N}}$

$$s_{\mathbf{k}} \left(\frac{1}{\hbar \hat{N}}, \ldots, \frac{1}{\hbar \hat{N}} \right) \underset{\hat{N} \to \infty}{=} \frac{\mathbf{m}_{\mathbf{k}}}{\hbar^k} \tag{96}$$ schupla

The Virasoro generators can be defined by

$$T(w) = \tfrac{1}{2} : J(w)^2 := \left(\frac{dw}{w} \right)^2 \sum_{n \in \mathbb{Z}} \frac{L_n}{w^n} \tag{97}$$ Tvir

or

$$T = -\tfrac{1}{2} \left(\partial \phi \right)^2 = \tfrac{1}{2} \left(\tilde{\psi} \partial \psi - \partial \tilde{\psi} \psi \right) \tag{98}$$ Tff

or

$$L_n = \tfrac{1}{2} \sum_{k \in \mathbb{Z}} : J_k J_{n-k} := - \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{n}{2} \right) : \tilde{\psi}_n \psi_{r+n} : \tag{99}$$ LJpsi

and satisfy $[L_n, L_m] = -(n-m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n+m,0}$.

Okounkov-Pandharipande operators

Following [3], one can introduce

$$\tilde{\psi}\left(e^{\zeta/2}w\right)\psi\left(e^{-\zeta/2}w\right) = \frac{dw}{w}\sum_{k\in\mathbb{Z}}\frac{\mathcal{E}_k(\zeta)}{w^k}$$
$$\mathcal{E}_k(\zeta) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{\zeta\left(r+\frac{k}{2}\right)} : \tilde{\psi}_r\psi_{r+k} : + \frac{\delta_{k,0}}{e^{\zeta/2}-e^{-\zeta/2}} \qquad (100) \quad \texttt{opop}$$

and, the "Fourier transform"

$$A(\zeta) = \oint_{\frac{dw}{w}} \tilde{\psi}\left(e^{\zeta/2}w\right)\psi\left(e^{-\zeta/2}w\right)\exp\left(\frac{e^{\zeta/2}-e^{-\zeta/2}}{w}\right) = \sum_{n\geq 0}\frac{\left(e^{\zeta/2}-e^{-\zeta/2}\right)^n}{n!}\mathcal{E}_{-n}(\zeta) \qquad (101)$$

From $gl(\infty)$ relations $[E_{ij},E_{rs}] = \delta_{jr}E_{is} - \delta_{is}E_{rj}$ for $E_{rs} =: \tilde{\psi}_r\psi_s :$ one easily gets

$$[\mathcal{E}_k(z),\mathcal{E}_l(w)] = \left(e^{(kw-lz)/2} - e^{-(kw-lz)/2}\right)\mathcal{E}_{k+l}(z+w) \qquad (102)$$

and from (89) one concludes that

$$e^{J_{-1}}A(\zeta)e^{-J_{-1}} = \oint_{\frac{dw}{w}} \tilde{\psi}\left(e^{\zeta/2}w\right)\psi\left(e^{-\zeta/2}w\right) = \mathcal{E}_0(\zeta) \qquad (103)$$

Toda chain and tau-functions

Consider, first, the well-known formula for the tau-function of Toda molecule (or the open N-Toda chain with co-ordinates $q_n(\mathbf{t},\mathsf{a}) = \log\frac{Z(\mathbf{t};n|\mathsf{a})}{Z(\mathbf{t};n-1|\mathsf{a})}$), given by all principal n-minors

$$Z(\mathbf{t};n|\mathsf{a}) = \sum_{K:i_1<\ldots<i_n}\mu_K(\mathsf{a})^2\exp\sum_{l,i_k}t_lH_l(\mathsf{a}_{i_k}) = \Delta_{n\times n}\left(A\cdot D\cdot A^T\right) \qquad (104) \quad \texttt{tauto}$$

of the $N\times N$ matrix, expressed as a matrix product with

$$A_{ij}\sim \mathsf{a}_i^{j-1},\; D_{ij} = \delta_{ij}\exp(z_i)\prod_{k=1,k\neq i}^N|\mathsf{a}_i-\mathsf{a}_k|^{-1} \qquad (105)$$

where

$$z_i = \sum_l t_lH_l(\mathsf{a}_i) = \sum_l(t_l\mathsf{a}_i^l+\ldots)+z_i^{(0)} \qquad (106) \quad \texttt{angles}$$

with some appropriately chosen "initial phases" $z_i^{(0)}$. Rewriting (104) in the form

$$Z(\mathbf{t};n|\mathsf{a}) = \sum\nolimits_{|K|=n}\prod_{i\in K}\frac{e^{z_i}}{\prod_{k=1,k\neq i}^N|\mathsf{a}_i-\mathsf{a}_k|}\prod_{i,j\in K,i\neq j}(\mathsf{a}_i-\mathsf{a}_j)^2 \qquad (107) \quad \texttt{brama}$$

Following [3], one can introduce

$$\tilde{\psi}\left(e^{\zeta/2}w\right)\psi\left(e^{-\zeta/2}w\right) = \frac{dw}{w}\sum_{k\in\mathbb{Z}}\frac{\mathcal{E}_k(\zeta)}{w^k}$$
$$\mathcal{E}_k(\zeta) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{\zeta\left(r+\frac{k}{2}\right)} : \tilde{\psi}_r\psi_{r+k} : + \frac{\delta_{k,0}}{e^{\zeta/2}-e^{-\zeta/2}} \qquad (100) \quad \text{opop}$$

and, the "Fourier transform"

$$A(\zeta) = \oint_{\frac{dw}{w}} \tilde{\psi}\left(e^{\zeta/2}w\right)\psi\left(e^{-\zeta/2}w\right)\exp\left(\frac{e^{\zeta/2}-e^{-\zeta/2}}{w}\right) = \sum_{n\geq 0}\frac{\left(e^{\zeta/2}-e^{-\zeta/2}\right)^n}{n!}\mathcal{E}_{-n}(\zeta) \qquad (101)$$

From $gl(\infty)$ relations $[E_{ij}, E_{rs}] = \delta_{jr}E_{is} - \delta_{is}E_{rj}$ for $E_{rs} =: \tilde{\psi}_r\psi_s :$ one easily gets

$$[\mathcal{E}_k(z), \mathcal{E}_l(w)] = \left(e^{(kw-lz)/2} - e^{-(kw-lz)/2}\right)\mathcal{E}_{k+l}(z+w) \qquad (102)$$

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$$e^{J_{-1}}A(\zeta)e^{-J_{-1}} = \oint_{\frac{dw}{w}} \tilde{\psi}\left(e^{\zeta/2}w\right)\psi\left(e^{-\zeta/2}w\right) = \mathcal{E}_0(\zeta) \qquad (103)$$

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$$Z(\mathbf{t};n|\mathsf{a}) = \sum_{K:i_1<\ldots<i_n}\mu_K(\mathsf{a})^2\exp\sum_{l,i_k}t_lH_l(\mathsf{a}_{i_k}) = \Delta_{n\times n}\left(A\cdot D\cdot A^T\right) \qquad (104) \quad \text{tauto}$$

of the $N\times N$ matrix, expressed as a matrix product with

$$A_{ij}\sim\mathsf{a}_i^{j-1},\ D_{ij} = \delta_{ij}\exp(z_i)\prod_{k=1,k\neq i}^N|\mathsf{a}_i-\mathsf{a}_k|^{-1} \qquad (105)$$

where

$$z_i = \sum_l t_lH_l(\mathsf{a}_i) = \sum_l(t_l\mathsf{a}_i^l+\ldots) + z_i^{(0)} \qquad (106) \quad \text{angles}$$

with some appropriately chosen "initial phases" $z_i^{(0)}$. Rewriting (104) in the form

$$Z(\mathbf{t};n|\mathsf{a}) = \sum_{|K|=n}\prod_{i\in K}\frac{e^{z_i}}{\prod_{k=1,k\neq i}^N|\mathsf{a}_i-\mathsf{a}_k|}\prod_{i,j\in K,i\neq j}(\mathsf{a}_i-\mathsf{a}_j)^2 \qquad (107) \quad \text{brama}$$

we see that the sum in (104) is in fact taken over the partitions $(k_1 \geq k_2 \geq \ldots \geq k_n)$ with the fixed length and

$$k_j = i_{n-j+1} + j - n - 1, \quad j = 1, \ldots, n \tag{108}$$

jk

For the particular solution of the Toda chain with $\mathsf{a}_i \simeq i$, one gets for (107)

$$Z(\mathbf{t}; n) = \sum\nolimits_{|K|=n} \prod_{i \in K} e^{z_i} \frac{\prod_{i,j \in K, i \neq j}(i-j)^2}{\prod_{i \in K}(i-1)!(N-i)!} \tag{109}$$

toi

This is a singular or "stringy" solution, presenting a collection of particles, moving each with a constant speed, proportional to its number. In KP/KdV-theory the analog is $u \propto x/t$, a linear growing potential of Kontsevich model, which never topples.

Comparing (109) to (104), one finds that

$$\mu_K(\mathsf{a})^2\big|_{\mathsf{a}_i=i} = \left(\frac{\prod_{i,j \in K}(i-j)}{\prod_{i \in K}(i-1)!}\right)^2 \prod_{i \in K} \frac{(i-1)!}{(N-i)!} = \mathbf{m}_{\mathbf{k}}^2\big|_{\ell_{\mathbf{k}}=n} \prod_{i \in K} e^{z_i^{(0)}} \tag{110}$$

mum

In the limit $N \to \infty$, after particular choice of the Hamiltonians (79)

$$H_l(\mathsf{a}_i)\big|_{\mathsf{a}_i=i} \to \frac{\mathrm{ch}_{l+1}(a; i, \hbar)}{l+1} \tag{111}$$

hamchern

and renormalization of the initial phase $z_i^{(0)}$, passing from summation over partitions with a fixed length $\ell_{\mathbf{k}} = n$ to a "grand-canonical" ensemble by a sort of Fourier transform, one gets $Z(\mathbf{t}; n) \to Z(a, \mathbf{t}, \hbar)$ the (78) partition function. By (94), it becomes equivalent to the following fermionic correlator

$$Z(a, \mathbf{t}, \hbar) = \sum_{\mathbf{k}} \frac{\mathbf{m}_{\mathbf{k}}^2}{(-\hbar^2)^{|\mathbf{k}|}} \, e^{\frac{1}{\hbar^2}\sum_{k>0} t_k \frac{\mathrm{ch}_{k+1}(a,\mathbf{k})}{k+1}} = \langle M | e^{-\frac{J_1}{\hbar}} e^{\frac{1}{\hbar}\sum_{k>0} t_k W_{k+1}} e^{\frac{J_{-1}}{\hbar}} | M \rangle \tag{112}$$

zumat

where the mutually commuting modes of the W-infinity generators can be defined as

$$\begin{aligned} W_{k+1} &= -\frac{\hbar^k}{k+1} \oint : \tilde{\psi} \left(\left(w\frac{d}{dw} + \frac{1}{2} \right)^{k+1} - \left(w\frac{d}{dw} - \frac{1}{2} \right)^{k+1} \right) \psi : = \\ &= \frac{\hbar^k}{k+1} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \left[(-r+\tfrac{1}{2})^{k+1} - (-r-\tfrac{1}{2})^{k+1} \right] \, : \psi_r \tilde{\psi}_r : \end{aligned} \tag{113}$$

wgen

The matrix element (112) is a particular non-standard fermionic representation of the tau-function, where the Toda times are coupled to the W-generators (113) instead of the modes of the $U(1)$ current (88), and it has been discussed in [3].

If only $t_1 \neq 0$ the correlator in (112) gives

$$Z(a=\hbar M, t_1, 0, 0, \ldots) = \langle M| e^{-\frac{J_1}{\hbar}} e^{\frac{1}{\hbar}t_1 L_0} e^{\frac{J_{-1}}{\hbar}} |M\rangle = \exp\left[-\frac{1}{\hbar^2}\left(\tfrac{1}{2}t_1 a^2 + e^{t_1}\right)\right] = \exp\left(-\frac{\mathcal{F}_{\mathbb{P}^1}}{\hbar^2}\right) \quad (114) \quad \texttt{taucp1}$$

the partition function of topological string on $\mathbb{P}^1$. This is the only case when summing over partitions can be performed straightforwardly, using the Burnside theorem

$$\sum_{\mathbf{k}, |\mathbf{k}|=k} \mathbf{m}_{\mathbf{k}}^2 = \frac{1}{k!^2} \sum_{\mathbf{k}, |\mathbf{k}|=k} \dim R_{\mathbf{k}}^2 = \frac{1}{k!} \quad (115) \quad \texttt{burn}$$

Baker-Akhiezer functions

In addition to (94), one can consider

$$\tilde{\psi}_{-r}\, e^{\frac{J_{-1}}{\hbar}} |M+1; \emptyset\rangle = \sum_{\mathbf{k}} \tilde{C}_{\mathbf{k}} |M; \mathbf{k}\rangle \quad (116) \quad \texttt{tpsiact}$$

with (computed by the Wick theorem and using the properties of the Schur functions (93))

$$\tilde{C}_{\mathbf{k}} = \langle M, \mathbf{k}| \tilde{\psi}_{-r}\, e^{\frac{J_{-1}}{\hbar}} |M+1; \emptyset\rangle = \hbar^{M-|\mathbf{k}|-r-\frac{1}{2}} \prod_{i=1}^{\infty} \frac{i - k_i + r - \frac{1}{2} - M}{i}\, \mathbf{m}_{\mathbf{k}} \quad (117) \quad \texttt{nashekha}$$

where the infinite product is actually finite

$$\prod_{i=1}^{\infty} \frac{i - k_i + r - \frac{1}{2} - M}{i} = \frac{1}{\Gamma(r + \frac{1}{2} - M)} \prod_{i=1}^{\ell_{\mathbf{k}}} \frac{i - k_i + r - \frac{1}{2} - M}{i + r - \frac{1}{2} - M} \quad (118) \quad \texttt{infprd}$$

Therefore, one gets for the Baker-Akhiezer functions

$$\begin{aligned}
\tilde{\Psi}(r) &= \frac{\langle M| e^{-\frac{J_1}{\hbar}} \tilde{\psi}_{-r} e^{\frac{1}{\hbar}\sum_{k>0} t_k W_{k+1}} e^{\frac{J_{-1}}{\hbar}} |M+1\rangle}{\langle M| e^{-\frac{J_1}{\hbar}} e^{\frac{1}{\hbar}\sum_{k>0} t_k W_{k+1}} e^{\frac{J_{-1}}{\hbar}} |M\rangle} = \\
&= \frac{\hbar^{M-r-\frac{1}{2}}}{Z(a, \mathbf{t}, \hbar)} e^{\frac{1}{\hbar^2}\sum_{k>0} \frac{t_k \hbar^k}{k+1}\left((r+\frac{1}{2})^{k+1} - (r-\frac{1}{2})^{k+1}\right)} \cdot \\
&\cdot \sum_{\mathbf{k}} \frac{\mathbf{m}_{\mathbf{k}}^2}{(-\hbar^2)^{|\mathbf{k}|}}\, e^{\frac{1}{\hbar^2}\sum_{k>0} t_k \frac{\mathrm{ch}_{k+1}(a,\mathbf{k},\hbar)}{k+1}} \prod_{i=1}^{\infty} \frac{i - k_i + r - \frac{1}{2} - M}{i} = \\
&= \hbar^{M-r-\frac{1}{2}} \exp \frac{1}{\hbar^2} \sum_{k>0} \frac{t_k \hbar^k}{k+1} \left((r+\tfrac{1}{2})^{k+1} - (r-\tfrac{1}{2})^{k+1}\right) \cdot \frac{e^{M\Gamma'(r+\frac{1}{2})/\Gamma(r+\frac{1}{2})}}{\Gamma(r+\frac{1}{2})} \cdot \frac{Z(a, \mathbf{t} - \boldsymbol{\delta}(r), \hbar)}{Z(a, \mathbf{t}, \hbar)}
\end{aligned} \quad (119) \quad \texttt{batpsir}$$

with $a = M\hbar$ and the shift $\boldsymbol{\delta}(r) = (\delta_1(r), \delta_2(r), \ldots)$ generated by

$$\frac{1}{\hbar^2}\sum_{k>0}\delta_k(r)\frac{x^{k+1}}{k+1} = \log\frac{\Gamma\left(r+\frac{1}{2}-\frac{x}{\hbar}\right)}{\Gamma\left(r+\frac{1}{2}\right)} + \frac{x}{\hbar}\frac{\Gamma'(r+\frac{1}{2})}{\Gamma(r+\frac{1}{2})} \qquad (120) \quad \texttt{dellog}$$

In the quasiclassical asymptotic $\hbar \to 0$, with $\hbar r = z$, (119) gives

$$\tilde{\Psi}(z, a, \mathbf{t}, \hbar) \sim \exp\frac{1}{\hbar}\left(\sum_{k>0} t_k z^k - z(\log z - 1) + a\log z + \ldots\right) \qquad (121) \quad \texttt{quasf}$$

In the same way one can define the two-point function

$$\mathcal{E}(r) = \frac{\langle M+1|e^{-\frac{J_1}{\hbar}}\psi_{-r}\tilde{\psi}_{-r}e^{\frac{1}{\hbar}\sum_{k>0}t_k W_{k+1}}e^{\frac{J_{-1}}{\hbar}}|M+1\rangle}{\langle M|e^{-\frac{J_1}{\hbar}}e^{\frac{1}{\hbar}\sum_{k>0}t_k W_{k+1}}e^{\frac{J_{-1}}{\hbar}}|M\rangle} \sim$$

$$\sim \exp\frac{1}{\hbar^2}\sum_{k>0}\frac{t_k\hbar^k}{k+1}\left((r+\tfrac{1}{2})^{k+1} - (r-\tfrac{1}{2})^{k+1}\right)\cdot\frac{e^{2M\Gamma'(r+\frac{1}{2})/\Gamma(r+\frac{1}{2})}}{\Gamma(r+\frac{1}{2})^2}\cdot\frac{Z(a,\mathbf{t}-2\cdot\boldsymbol{\delta}(r),\hbar)}{Z(a,\mathbf{t},\hbar)} \sim$$

$$\underset{\hbar\to 0}{\sim} \exp\frac{S(z,a,\mathbf{t})}{\hbar} \qquad (122) \quad \texttt{nashepp}$$

with

$$S(z,a,\mathbf{t}) = \sum_{k>0} t_k z^k - 2z(\log z - 1) + 2a\log z + \ldots \qquad (123) \quad \texttt{Sas}$$

The asymptotics (123) plays an essential role in the study of the quasiclassical solution. Formula (122) can be also interpreted as average of the r-th Fourier mode of the "symmetrically splitted" bi-fermionic operator $\mathcal{E}(\zeta) = \oint_{\frac{dw}{w}}\psi\left(we^{-\zeta/2}\right)\tilde{\psi}\left(we^{\zeta/2}\right)$, introduced in [3]. The "doubling" of the fermions and their symmetric splitting along the w-cylinder turn into the double covering of z-plane by the quasiclassical spectral curve.

Non-Abelian matrix element

The dual $N = 2$ partition function is

$$Z^D = \langle\lambda'|e^{\beta\left(J_0^+ + J_1^-\right)}e^{xL_0}e^{\xi H_0}e^{\alpha\left(J_0^- + J_{-1}^+\right)}|\lambda\rangle \qquad (124) \quad \texttt{ZDsl2}$$

where we have used the modes of the $\widehat{sl(2)}_{k=1}$ currents, and $|\lambda\rangle$, $\langle\lambda'|$ are the highest vector in the basic representation and its dual. The commutation relations

$$\begin{aligned} \left[J_n^+, J_m^-\right] &= 2H_{n+m} + k\delta_{n+m,0} \\ \left[H_n, J_m^\pm\right] &= \pm J_{n+m}^\pm \\ \left[H_n, H_m\right] &= k\delta_{n+m,0} \\ \left[L_n, J_m^a\right] &= -mJ_{n+m}^a \end{aligned} \qquad (125)$$

give

$$Z^D = \langle\lambda'|e^{\beta\left(J_0^+ + J_1^-\right)}e^{\alpha\left(e^{-\xi}J_0^- + e^{\xi+x}J_{-1}^+\right)}|\lambda\rangle e^{x\Delta_0}e^{\xi h_0} \tag{126}$$ ZDcom

3 Quasiclassical limit and dispersionless hierarchies

3.1 Functional representation

The sum over partitions can be effectively rewritten as a functional problem

$$Z(a,\mathbf{t},\hbar) = \sum_{\mathbf{k}} \exp\frac{1}{\hbar^2}\mathcal{E}_{\mathbf{k}}(a,\mathbf{t},\hbar) \underset{\hbar\to 0}{\sim} \exp\frac{1}{\hbar^2}\mathcal{F}(a,\mathbf{t}) \tag{127}$$ ZEF

so that the main contribution in the quasiclassical limit $\hbar \to 0$ is given by a contribution of extremal partition $\mathbf{k}_*$, by the value of the effective potential $\mathcal{E}_{\mathbf{k}_*}(a,\mathbf{t},\hbar) = \mathcal{F}(a,\mathbf{t}) + o(\hbar)$ calculated on this partition. The effective potential $\mathcal{E}_{\mathbf{k}}$ can be read off from (78) and elegantly written in terms of the profile function, defined for each partition $\mathbf{k} = k_1 \geq k_2 \geq \ldots \geq k_{\ell_{\mathbf{k}}} \geq k_{\ell_{\mathbf{k}}+1} = 0, \ldots$ by

$$\begin{gathered} f_{\mathbf{k}}(x) = |x-a| + \\ + \sum_{i=1}^{\ell_{\mathbf{k}}} \left(|x-a-\hbar(k_i-i+1)| - |x-a-\hbar(k_i-i)| - |x-a-\hbar(1-i)| + |x-a+\hbar i|\right) \sim \\ \sim \sum_{i=1}^{\infty} \left(|x-a-\hbar(k_i-i+1)| - |x-a-\hbar(k_i-i)|\right) \end{gathered} \tag{128}$$ fpart

where the last expression should be understood as regularized in the above sense. In particular, one can write for (79), using (128)

$$\mathrm{ch}_l(a,\mathbf{k}) = \frac{1}{2}\int dx\ f''_{\mathbf{k}}(x)x^l \sim \sum_{i=1}^{\infty}\left((a+\hbar(k_i-i+1))^l - (a+\hbar(k_i-i))^l\right) \tag{129}$$ chernf

The effective potential in (127) can be written therefore

$$\mathcal{E}_{\mathbf{k}} = \frac{1}{2}\int dx f''_{\mathbf{k}}(x)\sum_{k>0} t_k\frac{x^{k+1}}{k+1} - \frac{1}{2}\int_{x_1>x_2} dx_1dx_2 f''_{\mathbf{k}}(x_1)f''_{\mathbf{k}}(x_2)\gamma(x_1-x_2;\hbar) \tag{130}$$ Ef

where the form of the linear term directly follows from (129), while the interaction is totally determined by the Plancherel measure (95)

$$\begin{gathered} \mathbf{m}_{\mathbf{k}}^2 \sim \prod_{i,j}(k_i-k_j+j-i) = \exp\sum_{i,j}\log(k_i-k_j+j-i) \sim \\ \sim \exp\left(-\frac{1}{2\hbar^2}\int_{x_1>x_2} dx_1dx_2 f''_{\mathbf{k}}(x_1)f''_{\mathbf{k}}(x_2)\gamma(x_1-x_2;\hbar)\right) \end{gathered} \tag{131}$$ plg

where $\frac{1}{2}f''_{\mathbf{k}}(x) \sim \sum_{i=1}^{\infty} (\delta(x - a - \hbar(k_i - i + 1)) - \delta(x - a - \hbar(k_i - i)))$ and the kernel $\gamma(x, \hbar)$ satisfies

$$\gamma(x + \hbar) + \gamma(x - \hbar) - 2\gamma(x) = \log x \tag{132}$$

the second-order difference equation. In the limit $\hbar \to 0$ one can replace it by differential equation, and the function $\gamma(x) = F(x) + o(\hbar)$ by

$$F(x) = \frac{x^2}{2}\left(\log x - \frac{3}{2}\right) \tag{133}$$ SWkern

coinciding with the perturbative prepotential of pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory.

The quasiclassical contribution to (78) is therefore given by the functional $\mathcal{E}_{\mathbf{k}}$ (130), extremized w.r.t. second derivative of the profile function $f''_{\mathbf{k}}(x) = \frac{d^2 f_{\mathbf{k}}}{dx^2}$, i.e. by

$$\mathcal{F} = \frac{1}{2}\int dx f''(x) \sum_{k>0} t_k \frac{x^{k+1}}{k+1} - \frac{1}{2}\int_{x_1 > x_2} dx_1 dx_2 f''(x_1) f''(x_2) F(x_1 - x_2) \tag{134}$$ functnl

Formula (134) means that the quasiclassical free energy for the partition functions (78) and (80) is saturated onto a single "large" partition $\mathbf{k}_*$ with the profile function $f_{\mathbf{k}_*}(x) = f(x)$. The variational problem for the functional (130) or (134) should be solved upon constraint

$$a = \tfrac{1}{2}\int dx\ x f''(x) \tag{135}$$

which can be in standard way taken into account by adding it with the Lagrange multiplier

$$\mathcal{F} \to \mathcal{F} + a^D \left(a - \tfrac{1}{2}\int dx\ x f''(x)\right) \tag{136}$$ Lagr

having a sense of the $k = 0$ term in the summation in formula (134). The whole setup of (134) is almost identical to the standard quasiclassics of the matrix models, where the Coulomb gas kernel is replaces by a (multivalued!) Seiberg-Witten function (133).

The extremal equation for the (134) gives

$$\sum_{k>0} t_k x^k - \int d\tilde{x} f''(\tilde{x})(x - \tilde{x})(\log|x - \tilde{x}| - 1) = a^D \tag{137}$$ exeq

on the support $\mathbf{I}$ where $f''(x) \neq 0$. Generally, for the microscopic non-abelian theory this support consists of a set of several (disjoint) segments along the real axis in the complex plane, where the filling fractions are fixed separately with the help of several Lagrange multipliers, see below. Equation (137) means that

$$S(z) = \sum_{k>0} t_k z^k - \int dx f''(x)(z - x)(\log(z - x) - 1) - a^D \tag{138}$$ Sfun

is an analytic multivalued function on the double-cover of the z-plane with the following properties:

- The real part of the multivalued function (138) vanishes

$$S(x) = \tfrac{1}{2}\left(S(x+i0) + S(x-i0)\right) = 0, \qquad x \in \mathbf{I} \qquad (139) \quad \mathtt{Scut}$$

  on the cut, due to (137).

- For its imaginary part one can write

$$\frac{1}{\pi}\mathrm{Im}\; S(z \pm i0) = \mp \int_z^\infty dx f''(x)(z-x) = \mp \left\{ \begin{array}{cc} 0\,, & z > x^+ \\ a - z + f(z)\,, & x^- < z < x^+ \\ 2(a-z)\,, & z < x^- \end{array} \right. \qquad (140) \quad \mathtt{ImS1}$$

- We see from (140) that even the differential dS is multivalued. Indeed, one can easily establish for

$$\Phi = \frac{dS}{dz} = \mathbf{t}''(z) - \int dx f''(x)\log(z-x) \qquad (141) \quad \mathtt{fi}$$

  that

$$\frac{1}{\pi}\mathrm{Im}\; \Phi(z \pm i0) = \pm \left\{ \begin{array}{cc} 0\,, & z > x^+ \\ 1 - f'(z)\,, & x^- < z < x^+ \\ 2\,, & z < x^- \end{array} \right. \qquad (142) \quad \mathtt{Imfi}$$

  However, the differential

$$d\Phi = \mathbf{t}'''(z)dz + dz\int \frac{dx f''(x)}{z-x} \qquad (143) \quad \mathtt{dfi}$$

  is already single-valued on the double cover of the cut z-plane with the periods $\oint d\Phi \sim 4\pi i\mathbb{Z}$, so dS is defined modulo $4\pi i dz$, and one can make sense of the periods $\oint dS$ due to $\oint dz = 0$. Therefore, the exponent $\exp(\Phi/2)$ is already single-valued on the double cover and equals to unity on the cut.

- In order to consider the asymptotic of (138) in what follows we shall always choose a branch, which is real along the real axis, i.e. take it at *real* $x \to +\infty$. In particular, all residues below could be understood in this sense, as coefficients of expansion of generally multivalued differential at $x \to +\infty$.

- Taking derivatives of (137) in x-variable, or integrating by parts, one can bring it literally to the form, arising in the context of matrix model. However, for the purposes of Seiberg-Witten theory one needs a solution with different analytic properties: in matrix models the resolvent $G \sim \frac{dS}{dz}$ does not have poles at the branching points where $dz = 0$, which is not true for (138).

Asymptotically from (138) one gets

$$S(z) \underset{z\to\infty}{=} -2z(\log z - 1) + \sum_{k>0} t_k z^k + \log z \int dx x f''(x) - \\ -a^D - 2\sum_{k=1}^{\infty} \frac{1}{kz^k} \int dx f''(x) \frac{x^{k+1}}{k+1} = \\ = -2z(\log z - 1) + \sum_{k>0} t_k z^k + 2a \log z - \frac{\partial \mathcal{F}}{\partial a} - 2\sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial \mathcal{F}}{\partial t_k} \tag{144}$$ sasympt

where, according to (134), (with convention that $\mathrm{res}_\infty \frac{dz}{z} = 1$)

$$\frac{\partial \mathcal{F}}{\partial t_k} = \frac{1}{2(k+1)} \int dx f''(x) x^{k+1}, \qquad k > 0 \tag{145}$$ derF

and, due to (136)

$$a^D = \frac{\partial \mathcal{F}}{\partial a} \tag{146}$$ aad

The coefficient at the $z(\log z - 1)$ term is fixed by

$$\int_{\mathbf{I}} dx f''(x) = f'(x^+) - f'(x^-) = 2 \tag{147}$$ normlog

where $x^\pm$ (in the one-cut case) can be defined as two solutions to the equation

$$f(x^\pm) = |x^\pm - a| \tag{148}$$ eqends

Using variational equation (137), one can also write for the functional (134) the double-integral representation (cf. with [8])

$$\mathcal{F} = a a^D + \frac{1}{2} \int_{x_1 > x_2} dx_1 dx_2 f''(x_1) f''(x_2) F(x_1 - x_2) \tag{149}$$ doubint

expressing it in terms of the perturbative kernel (133) and extremal shape $f(x)$, solving (137).

3.2 Dispersionless Toda chain

In the case of a single cut let us present the double cover of the z-plane $y^2 = (z - x^+)(z - x^-)$ in the form

$$z = v + \Lambda \left(w + \frac{1}{w} \right) \tag{150}$$ u1curve

with $x^\pm = v \pm 2\Lambda$ and

$$y^2 = (z - v)^2 - 4\Lambda^2 \tag{151}$$ yu1

On the double cover (150), which is in the case of single cut just $\mathbb{P}^1$ with two marked points $P_\pm$, with $z(P_\pm) = \infty$, $w^{\pm 1}(P_\pm) = \infty$, formula (138) defines a function with a logarithmic cut and asymptotic behavior (144), odd under the involution $w \leftrightarrow \frac{1}{w}$ of the curve (150). In terms of the uniformizing variable w one can globally write

$$S = -2\left(v + \Lambda\left(w + \frac{1}{w}\right)\right)\log w - 2\Lambda(\log\Lambda - 1)\left(w - \frac{1}{w}\right) + \sum_{k>0} t_k\Omega_k(w) + 2a\log w \tag{152}$$

Sw

where

$$\Omega_k(w) = z_+^k - z_-^k, \quad k > 0 \tag{153}$$

Ow

are the Laurent polynomials, odd under $w \leftrightarrow \frac{1}{w}$. The first term in (152) comes from the Legendre transform of the Seiberg-Witten differential $d\Sigma \sim z\frac{dw}{w}$.

The canonical Toda chain times are defined by the coefficients at the singular terms in (144)

$$t_0 = \mathrm{res}_{P_+} dS = -\mathrm{res}_{P_-} dS = 2a \tag{154}$$

t0res

and

$$t_k = \frac{1}{k}\,\mathrm{res}_{P_+} z^{-k} dS = -\frac{1}{k}\,\mathrm{res}_{P_-} z^{-k} dS, \quad k > 0 \tag{155}$$

tP

From the expansion (144) it also immediately follows, that

$$\frac{\partial\mathcal{F}}{\partial t_k} = \frac{1}{2}\mathrm{res}_{P_+} z^k dS = -\frac{1}{2}\mathrm{res}_{P_-} z^k dS, \quad k > 0 \tag{156}$$

tPd

Formulas (154), (155), (156) together with (146) identify the generating function (134) with the logarithm of quasiclassical tau-function, being here, in the case of a single cut, a tau-function of dispersionless Toda chain hierarchy.

The consistency condition for (156) is ensured by the symmetricity of second derivatives

$$\frac{\partial^2\mathcal{F}}{\partial t_n \partial t_k} = \frac{1}{2}\mathrm{res}_{P_+}(z^k d\Omega_n) \tag{157}$$

sysi

where the time derivatives of (144)

$$\begin{aligned}
\Omega_0 = \frac{\partial S}{\partial a} &\underset{z\to P_\pm}{=} \pm\left(2\log z - \frac{\partial^2\mathcal{F}}{\partial a^2} - 2\sum_{n>0}\frac{\partial^2\mathcal{F}}{\partial a\partial t_n}\frac{1}{nz^n}\right) \\
\Omega_k = \frac{\partial S}{\partial t_k} &\underset{z\to P_\pm}{=} \pm\left(z^k - \frac{\partial^2\mathcal{F}}{\partial a\partial t_k} - 2\sum_{n>0}\frac{\partial^2\mathcal{F}}{\partial t_k\partial t_n}\frac{1}{nz^n}\right), \quad k > 0
\end{aligned} \tag{158}$$

Oz

form a basis of meromorphic functions with poles at the points $P_\pm$, with $z(P_\pm) = \infty$. All time-derivatives here are taken at constant z.

Expansion (158) of the Hamiltonian functions (153) expresses the second derivatives of $\mathcal{F}$ in terms of the coefficients of the equation of the curve (150), e.g.

$$\begin{aligned}\Omega_0 &\underset{z\to\infty}{=} 2\log z - 2\log\Lambda - \frac{2v}{z} - \frac{2\Lambda^2+v^2}{z^2} + \ldots\\ \Omega_1 &\underset{z\to\infty}{=} z - v - \frac{2\Lambda^2}{z} - \frac{2v\Lambda^2}{z^2} + \ldots\\ \Omega_2 &\underset{z\to\infty}{=} z^2 - (v^2+2\Lambda^2) - \frac{4v\Lambda^2}{z} - \frac{2\Lambda^2(\Lambda^2+2v^2)}{z^2} + \ldots\end{aligned} \tag{159}$$ Omexp

Comparison of the coefficients in (159) gives, in particular,

$$\frac{\partial^2\mathcal{F}}{\partial a^2} = \log\Lambda^2, \qquad \frac{\partial^2\mathcal{F}}{\partial a\partial t_1} = v \tag{160}$$ F2der

and

$$\frac{\partial^2\mathcal{F}}{\partial t_1^2} = \Lambda^2 = \exp\frac{\partial^2\mathcal{F}}{\partial a^2} \tag{161}$$ todaeq

which becomes the long-wave limit of the Toda chain equations after an extra derivative with respect to a is taken

$$\frac{\partial^2 a^D}{\partial t_1^2} = \frac{\partial}{\partial a}\exp\frac{\partial a^D}{\partial a} \tag{162}$$ todaequ

for the Toda co-ordinate $a^D = \frac{\partial\mathcal{F}}{\partial a}$. Substituting expansions (159) into (157), one gets the expressions for the so called contact terms in the $U(1)$ case, which are the polynomials of a single variable v with Λ-dependent coefficients.

One can now find the dependence of the coefficients of the curve (150) on the deformation parameters $\mathbf{t}$ of the microscopic theory by requiring $dS = 0$ at the ramification points, where $dz = 0$. This condition avoids from arising of extra singularities at the branch points in the variation of dS w.r.t. moduli of the curve. Equation

$$\frac{dz}{d\log w} = \Lambda\left(w - \frac{1}{w}\right) = 0 \tag{163}$$ dz

gives $w = \pm 1$, where now

$$\left.\frac{dS}{d\log w}\right|_{w=\pm1} = \sum_{k>0} t_k \left.\frac{d\Omega_k}{d\log w}\right|_{w=\pm1} + 2a - 2v \mp 4\Lambda\log\Lambda = 0 \tag{164}$$ eqscu

If $t_k = 0$ for $k > 1$, solution to (164) immediately gives

$$v = a, \quad \Lambda^2 = e^{t_1} \tag{165}$$ t01

and the prepotential

$$\mathcal{F} = \frac{1}{2}aa^D + \frac{1}{2}\text{res}_{P_+}(zdS) - \frac{a^2}{2} = \frac{1}{2}a^2t_1 + e^{t_1} \tag{166}$$ Fsphs

Adding nonvanishing t_2, one finds

$$\begin{aligned} v &= a - \frac{1}{2t_2}\mathbf{L}\left(-4t_2^2e^{t_1+2t_2a}\right) \\ \log\Lambda^2 &= t_1 + 2t_2a - \mathbf{L}\left(-4t_2^2e^{t_1+2t_2a}\right) \end{aligned} \tag{167}$$ t012

where the Lambert function $\mathbf{L}(t)$ is defined by an expansion

$$\mathbf{L}(t) = \sum_{n=1}^{\infty}\frac{(-n)^{n-1}t^n}{n!} = t - t^2 + \frac{3}{2}t^3 - \frac{8}{3}t^4 + \ldots \tag{168}$$

and satisfies to the functional equation

$$\mathbf{L}(t)e^{\mathbf{L}(t)} = t \tag{169}$$ Lambert

Hence, for the prepotential with $t_1, t_2 \neq 0$ one gets

$$\begin{aligned} \mathcal{F} &= \frac{1}{2}\left(a\frac{\partial\mathcal{F}}{\partial a} + + \sum_{k>1}(1-k)t_k\frac{\partial\mathcal{F}}{\partial t_k}\right) + \frac{\partial\mathcal{F}}{\partial t_1} - \frac{a^2}{2} = \\ &= \frac{1}{2}aa^D + \frac{1}{4}\sum_{k>1}(1-k)t_k\text{res}_{P_+}\left(z^kdS\right) + \frac{1}{2}\text{res}_{P_+}(zdS) - \frac{a^2}{2} = \\ &\underset{t_k=0,k>2}{=} \frac{1}{2}aa^D + \frac{1}{2}\text{res}_{P_+}(zdS) - \frac{1}{4}t_2\text{res}_{P_+}\left(z^2dS\right) - \frac{a^2}{2} \end{aligned} \tag{170}$$ F

wherefrom the instanton expansion can be computed (which can be strictly got as an expansion in parameter q after $t_1 \to t_1 + \frac{1}{2}\log q$)

$$\begin{aligned} \mathcal{F} &= \mathbf{t}(a) + e^{\mathbf{t}''(a)} + t_2^2\,e^{2\mathbf{t}''(a)} + \frac{8}{3}t_2^4\,e^{3\mathbf{t}''(a)} + \frac{32}{3}t_2^6\,e^{4\mathbf{t}''(a)} + \\ &+ \frac{160}{3}t_2^8\,e^{5\mathbf{t}''(a)} + \frac{1536}{5}t_2^{10}\,e^{6\mathbf{t}''(a)} + \ldots = \mathbf{t}(a) + S(a) + \frac{1}{4}S(a)S''(a) + \ldots \end{aligned} \tag{171}$$ F1inst

with $S(a) = \exp\mathbf{t}''(a)$. Expansion (171) directly corresponds to summing over connected tree diagrams in bosonic model, see Appendix B.

It is also easy to compute the explicit form of the extremal shape for nonvanishing t_1, t_2, which reads

$$\begin{aligned} f'(x) = \frac{2}{\pi}\left(\arcsin\left(\frac{x-v}{2\Lambda}\right) + t_2\sqrt{4\Lambda^2 - (x-v)^2}\right) \\ v - 2\Lambda \le x \le v + 2\Lambda \end{aligned} \tag{172}$$ shape2