Integrable systems in gauge & string theory

- Integrable systems & integrability

$\{q_i, p_j\} = \delta_{ij}$ $\quad$ 2N-dimensional phase space:

$i,j = 1, \dots, N$

$$\Omega = \sum dp_i \wedge dq_i$$

$$H = H_2 = \frac{1}{2}\sum_{j=1}^{N} p_j^{\,2} + V(q) = \frac{1}{2}\sum_{j=1}^{N} p_j^{\,2} + \sum e^{q_{i+1}-q_i}$$

Toda chain

$$H_1 = \sum_{j=1}^{N} p_j = \underline{P}$$

$H_1, \dots, H_N$

$$H_3 = \sum p_i p_j p_k + \dots$$

$\{H_1, H_2\} = 0$

$\vdots$

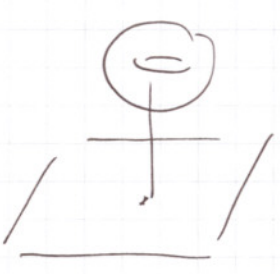
$$\Omega = \sum dp_i \wedge dq_i = \sum_{i=1}^{N} da_i \wedge dz_i$$

$a_i = a_i(\vec{H})$ $\qquad$ $\{a_i, a_j\} = 0$

canonical "actions" – integrals of motion

$\{z_i\} \subset T^N$ co-ordinates on the Liouville torus

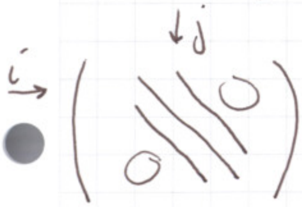
Toric fibration of the 2N-dimensional phase space



Two aspects: – how to solve an integrable system
– what do we need for gauge & string theory (complex geometrical aspects)

$\exists$ Lax representation:

$$\mathcal{L}_{ij} = p_i \cdot \delta_{ij} + e^{\frac{1}{2}(q_{i+1}-q_i)}\delta_{i+1,j} + e^{\frac{1}{2}(q_i-q_{i-1})}\delta_{i,j+1}$$



$$\frac{\partial \mathcal{L}}{\partial t} = [B, \mathcal{L}]$$

$$B_{ij} = \frac{1}{2}(R \circ \mathcal{L})_{ij} = \frac{1}{2}\left\{ e^{\frac{1}{2}(q_{i+1}-q_i)}\delta_{i+1,j} - e^{\frac{1}{2}(q_i-q_{i-1})}\delta_{i,j+1} \right\}$$

$$\frac{\partial p_k}{\partial t} = \frac{\partial}{\partial t}\mathcal{L}_{kk} = \sum_i \left(B_{ki}\mathcal{L}_{ik} - \mathcal{L}_{ki} B_{ik}\right) = e^{q_{k+1}-q_k} - e^{q_k-q_{k-1}}$$

$$B_{k\,k+1}\mathcal{L}_{k+1\,k} + B_{k\,k-1}\mathcal{L}_{k-1\,k} =$$

$$= e^{2\frac{1}{2}(q_{k+1}-q_k)} - e^{q_k-q_{k-1}}$$

Higher flows, generated by H_{k+1}; $k = 2, \dots, N-1$

$$\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}] \qquad B \sim R \circ \mathcal{L}^k$$

Nontrivial over-determined system of ~~[illegible]~~ differential equations

Auxiliary linear problem $\left(\frac{\partial}{\partial t_k} - B_k\right)\Psi = 0$

Where is here complex geometry etc ?

$\Omega = \sum_{i=1}^{N} da_i \wedge dz_i$ can be considered as $(2,0)$ holomorphic 2-form

$\{a_i\} \subset \mathcal{M}$ (moduli space)

$\{z_i\} \subset$ Jacobian of a complex curve

How to understand the meaning of this curve:

$\Omega^\infty = \sum dp_i \wedge dq_i$ $\quad H = \frac{1}{2}\sum p_i^2 + \sum e^{q_{i+1}-q_i}$

$i \in \mathbb{Z}$ "1-dimensional" system (like KdV)

$\mathcal{L} = \diagdown\!\diagdown\!\diagdown$

N-reduction

$T_N : \begin{array}{l} q_i \to q_{i+N} \\ p_i \to p_{i+N} \end{array}$ $\qquad T_N \Psi = w \Psi$

quasiperiodicity of Ψ

$[\mathcal{L}, T_N] = 0 \quad : \quad P(\mathcal{L}, T_N) = 0$

$$\begin{pmatrix} \diagdown\!\diagdown\!\diagdown & w e^{\frac{1}{2}(q_1 - q_N)} \\ \frac{1}{w} e^{\frac{1}{2}(q_N - q_1)} & \diagdown\!\diagdown\!\diagdown \end{pmatrix}$$

$\mathcal{L}\Psi = z\Psi$

$\det(\mathcal{L} - z) = 0$

$P_N(z) = \left(w + \frac{1}{w}\right)\Lambda^N$

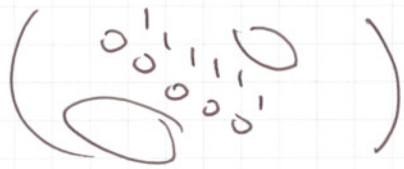
coefficients of P_N: integrals of motion

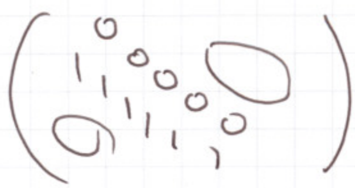
$q_{N+i} = q_i + \log \Lambda$ "quasiperiodicity"

N=1 example:

$$z - v = \Lambda\left(w + \frac{1}{w}\right)$$ can be considered as a Lax operator itself!

trilinear structure:

 $\rightarrow e^{\partial} \rightarrow w$

 $\rightarrow e^{-\partial} \rightarrow w^{-1}$

$\partial \equiv \frac{\partial}{\partial t_0}$ (derivative in zero-th time q_i, p_i)

$$\mathcal{L} \rightarrow \quad z = v + \Lambda\left(w + \frac{1}{w}\right)$$

$$B \rightarrow \quad \Omega = \frac{1}{2}\Lambda\left(w - \frac{1}{w}\right)$$

$$[B, \mathcal{L}] \rightarrow \{\Omega, z\} = -\frac{\partial \Omega}{\partial t_0}\frac{\partial z}{\partial \log w} + \frac{\partial \Omega}{\partial \log w}\frac{\partial z}{\partial t_0}$$

$$\dot\Lambda\left(w - \frac{1}{w}\right)\Lambda\left(w - \frac{1}{w}\right) - \Lambda\left(w + \frac{1}{w}\right)\left(\dot v + \dot\Lambda\left(w + \frac{1}{w}\right)\right) =$$

$$= \Lambda\dot\Lambda\left(w^2 - 2 + \frac{1}{w^2}\right) - \Lambda\dot v\left(w + \frac{1}{w}\right) - \Lambda\dot\Lambda\left(w^2 + 2 + \frac{1}{w^2}\right)$$

$$= -4\Lambda\dot\Lambda - \Lambda\dot v\left(w + \frac{1}{w}\right)$$

$$\dot v = 2\Lambda\dot\Lambda = \frac{d}{dt}(\Lambda^2) \qquad \dot\Lambda = \frac{1}{2}\Lambda\dot v$$

Dispersionless Toda equation $\qquad \Lambda \Rightarrow e^{q/2} \qquad v \Rightarrow p$

τ- functions

$$\exp q_n = \frac{\tau_n}{\tau_{n+1}}$$

For example:

$$\tau_0 = 1$$
$$\tau_1 = \frac{1}{a}\left(e^{z} + e^{-z}\right)$$
$$\tau_2 = 1$$

$$\boxed{H = \tfrac{1}{2} p^2 + e^{q}}$$

$$e^{q/2} = \frac{\sqrt{H}}{\cosh z}$$

$$\frac{1}{2} e^{q/2}\, \dot q = - \frac{\sqrt{H}}{(\mathrm{ch} z)^2}\, \mathrm{sh} z\, \dot z$$

$$\dot q = -2\, \mathrm{th} z\, \dot z = -2\omega\, \mathrm{th}\, \omega t$$

$$\ddot q = -2\omega^2 \frac{1}{\mathrm{ch}^2 \omega t} = -e^{q}$$

$$\omega = \sqrt{H} = a$$

$$\tau(\vec t, n \,|\, \vec a) = \sum_{|K| = n} \prod_{i \in K} \frac{e^{z_i}}{\prod_{k \neq i} |a_i - a_k|} \prod_{\substack{i,j \in K \\ i \neq j}} (a_i - a_j)^2$$

degenerate
Θ- function

or determinant representation

$$z_i = \sum t_k a_i^{k}$$

solves open N-Toda chain or Toda molecule

$0 \leq n \leq N-1$

Matrix-model
$N \to \infty$

$$\tau_n = \frac{1}{n!} \int d\phi_1 \dots d\phi_n \prod_{i<j} (\phi_i - \phi_j)^2 e^{W(\phi)}$$

$$W(\phi) = \sum t_k \phi^k$$

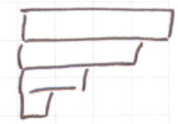
Strange Toda τ-function

$$\tau(a,\underbrace{\vec{t}}_{t_0};\hbar) = \sum_{\vec{k}} \frac{m_k^2}{(\hbar^2)^{|k|}} \exp \frac{1}{\hbar^2} \sum_{k>0} t_k \frac{ch_{k+1}(a,\vec{t},\hbar)}{k+1}$$

?

zuqum?

$\vec{k} = k_1 \geq k_2 \geq k_3 \geq \dots k_{\ell_{\vec{k}}} = 0 \dots 0$



$|k| = k_1 + \dots + k_\ell$

$$m_{\vec{k}} = \prod_{i<j} \frac{k_i - k_j + j - i}{j-i} = \frac{\prod_{1\leq i<j\leq \ell_k} (k_i - k_j + j - i)}{\prod_{i=1}^{\ell_k} (\ell_k + k_i - i)!}$$

Young diagram

Plancherel measure

$$\left(e^{\frac{\hbar u}{2}} - e^{-\frac{\hbar u}{2}}\right) \sum_{i=1}^{\infty} e^{u(a+\hbar(\frac{1}{2} - i + k_i))} = \sum_{\ell\geq 0} \frac{u^\ell}{\ell!} ch_\ell(a,\vec{k},\hbar)$$

$ch_0 = 1, \quad ch_1 = a, \quad ch_2 = a^2 + \hbar^2|k|, \dots$

1. Global aim: to study this function at all arbitrary $t_k \neq 0$

   $\hbar \to 0$ the problem is "solvable"

2. Let $t_2 = t_3 = \dots = 0$.

$$\tau(a,t_1,\hbar) = \sum_{\vec{k}} \frac{m_k^2}{(+\hbar^2)^{|k|}} \exp\left(\frac{1}{\hbar^2} t_1 \frac{a^2}{2} + t_1|k|\right)$$

$$= e^{\frac{1}{\hbar^2} t_1 \frac{a^2}{2}} \sum_{|k|} \frac{e^{t_1|k|}}{\hbar^{2|k|}} \sum_{\substack{\vec{k} \\ |\vec{k}|=k}} m_{\vec{k}}^2$$

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$$\sum_{|\vec{k}|=k} m_{\vec{k}}^2 = \frac{1}{k!}$$

$$\sum (\dim R_{\vec{k}})^2 = k!$$

$$\tau = \exp \frac{1}{\hbar^2}\left(a^2 t_1 + e^{t_1}\right)$$

$$\frac{\dim R_{\vec{k}}}{k!}$$

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$$\tau(a,\vec{t};\hbar) = \sum_{\vec{k}} \frac{m_{\vec{k}}^2}{\hbar^{2|k|}} \exp \frac{1}{\hbar^2} \sum_{k>0} t_k \frac{ch_{k+1}(a,\vec{t};\hbar)}{k+1}$$

$k \equiv |k| = \sum k_i$ $\qquad k_1 \geq k_2 \geq \ldots k_{\ell_k} \geq 0 \ldots 0$

$$\left(e^{\frac{\hbar u}{2}} - e^{-\frac{\hbar u}{2}}\right) \sum_{i=1}^{\infty} e^{u(a+\hbar(\frac{1}{2}-i+k_i))} = \sum \frac{u^\ell}{\ell!} ch_\ell(a,\vec{k};\hbar)$$

$ch_0 = 1 \qquad ch_1 = a \qquad ch_2 = a^2 + \hbar^2|k|$

$$ch_n \sim \sum_{i=1}^{\infty} (a+\hbar(k_i-i+1))^n - (a+\hbar(k_i-i))^n$$

$$ch_3 = a^3 + 6\hbar^2 a|k| + 3\hbar^3 \underbrace{\sum k_i(k_i+1-2i)}_{\text{"Hurwitz" expression}}$$

$$m_{\vec{k}} = \prod_{i<j} \frac{k_i-k_j+j-i}{j-i} = \frac{\prod_{1\leq i<j\leq \ell_k}(k_i-k_j+j-i)}{\prod_{i=1}^{\ell_k}(\ell_k+k_i-i)!} =$$

$$= \frac{\dim R_{\vec{k}}}{|k|!}$$

Instanton moduli spaces

N D3 - branes $\qquad N=1 \qquad U(1)$ theory

$|k|$ D(-1) branes attached to each D-brane

$k_1 \geq k_2 \geq \ldots \geq k_\ell$ partition

generally N-tuples of partitions

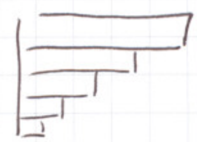
Burnside theorem computation:

$$\sum_{\substack{\vec{k} \\ |\vec{k}|=k}} \dim R_{\vec{k}} = k!$$

Combinatorics:

Hook formulas

$\vdots$

$Y_{\vec{k}}$ 

$$m_{\vec{k}} = \prod_{\square \in Y_{\vec{k}}} \frac{1}{h(\square)}$$

Fermions

Bosonization

$$S = \int \tilde{\psi}\, \bar{\partial}\, \psi$$

$$\psi = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r\, w^{-r} \sqrt{\frac{dw}{w}}$$

$$J = :\tilde{\psi}\psi:$$

$$\tilde{\psi} = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \tilde{\psi}_r\, w^{r} \sqrt{\frac{dw}{w}}$$

$$|M\rangle = \bigwedge_{r > -M} \psi_r$$

M-charged vacuum (Dirac sea)

$$|M, \vec{k}\rangle = \bigwedge_{r > -M} \psi_{r-k_i} = \psi_{-M+\frac{1}{2}-k_1}\, \psi_{-M+\frac{1}{2}-k_2} \cdots$$

$$\tilde{\psi} = e^{i\phi} \qquad \psi = e^{-i\phi}$$

$$\langle 0|\, \tilde{\psi}(z)\psi(w)\, |0\rangle = \frac{\sqrt{dz\,dw}}{z-w}$$

$$\phi(z)\phi(w) \sim -\log(z-w)$$

$$\langle \overset{m}{M}, \vec{k}|\, e^{i\phi(u_1)} \cdots e^{i\phi(u_M)}\, |0\rangle = S_{\vec{k}}(u_1 \ldots u_M) \cancel{\det u_j^{k_i+M}}$$

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$$S_{\vec{k}}(u_1 \ldots u_n) = \frac{\det u_j^{k_i+M-i}}{\det u_j^{M-i}} \xrightarrow[u \to \frac{1}{M}]{} \frac{m_{\vec{k}}}{\hbar^k}$$

$$u_i \to \frac{1}{M} \qquad J(w) = \sum \frac{J_n}{w^n} \frac{dw}{w}$$

$$e^{\frac{J_{-1}}{\hbar}} = \sum_{\vec{k}} \frac{m_{\vec{k}}}{k!} |M, \vec{k}\rangle$$

$$\tau(a, \vec{t}; \hbar) = \langle \tfrac{a}{\hbar} | e^{\frac{J_1}{\hbar}} e^{\frac{1}{\hbar}\sum t_k W_{k+1}} e^{\frac{J_{-1}}{\hbar}} | \tfrac{a}{\hbar} \rangle$$

$M = \frac{a}{\hbar}$ $\rightsquigarrow$ charge of the vacuum (discrete variable in Toda chain)

$$W_{k+1} = \frac{\hbar^k}{k+1} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \left[\left(-r+\tfrac{1}{2}\right)^{k+1} - \left(-r-\tfrac{1}{2}\right)^{k+1} \right] \psi_r \tilde{\psi}_r$$

$$W_\ell |M, \vec{k}\rangle = \frac{1}{\hbar \ell} ch_\ell(\hbar M, \vec{k}; \hbar) |M, \vec{k}\rangle$$

Fermionic matrix element: a tautology

More on bosonization: only $t_1 \neq 0$

$$\tau = \langle M | e^{J_1} e^{t_1 L_0} e^{J_{-1}} | M \rangle$$

$$J_{-1} \to a^+ \qquad J_1 \to a \qquad [a, a^+] = 1$$

$$Q = J_0 = \text{charge} \qquad L_0 = \frac{1}{2} Q^2 + a^+ a + \dots$$

$$J(w) = \sum \frac{J_n}{w^n} \frac{dw}{w} \qquad T(w) = :J^2(w): = \sum \frac{L_n}{w^n} \left(\frac{dw}{w}\right)^2$$

$$\tau = e^{\frac{1}{2} M^2 t_1} \underbrace{\langle 0 | e^{a} e^{t_1 a^+ a} e^{a^+} | 0 \rangle}$$

kernel of the evolution operator for harmonic oscillator in holomorphic representation

$$e^{a^+} |0\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} |n\rangle \qquad |n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$$

$$\langle \dots \rangle = \sum_{n=0}^{\infty} \frac{e^{t_1 n}}{n!} = e^{e^{t_1}}$$

restore $\hbar$, $a = M\hbar$ $\qquad \tau = e^{\frac{1}{\hbar^2}(a^2 t_1 + e^{t_1})}$

$$F = \frac{1}{2} a^2 t_1 + e^{t_1} \qquad \frac{\partial^2 F}{\partial t_1^2} = \exp \frac{\partial^2 F}{\partial a^2}$$

$$\exp q_n \sim \frac{\tau_n}{\tau_{n+1}} \qquad q \rightsquigarrow \frac{\partial}{\partial a} \log \tau \qquad a_D = \frac{\partial F}{\partial a}$$