Powers of large random unitary matrices and Toeplitz determinants

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Abstract

We study the limiting behavior of $\operatorname{Tr} U^{k(n)}$, where U is a $n \times n$ random unitary matrix and $k(n)$ is a natural number that may vary with n in an arbitrary way. Our analysis is based on the connection with Toeplitz determinants. The central observation of this paper is a strong Szegö limit theorem for Toeplitz determinants associated to symbols depending on n in a particular way. As a consequence to this result, we find that for each fixed $m \in \mathbb{N}$, the random variables $\operatorname{Tr} U^{k_j(n)}/\sqrt{\min(k_j(n), n)}$, $j = 1, \ldots, m$, converge to independent standard complex normals.

1 Introduction and statement of results

Random matrix theory

Let U be a random unitary matrix with respect to the Haar measure on $U(n)$, where $U(n)$ is the group of unitary matrices of size $n \times n$. Denote the eigenvalues of U by $e^{i\theta_\mu}$, for $\mu = 1, \ldots, n$ with $\theta_\mu \in [-\pi, \pi)$. Throughout this paper we will consider the random variable X_n defined by

$$X_n(U) = \sum_{\mu=1}^{n} f_n(e^{i\theta_\mu}), \tag{1.1}$$

where f_n is a square integrable function on $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ with Fourier-series

$$f_n(z) = \sum_{|j|>0} \frac{\alpha_j z^{k_j(n)}}{\sqrt{\min(|k_j(n)|, n)}}. \tag{1.2}$$

Here we assume that $\{\alpha_j\}_{j\in\mathbb{Z}}$ is a square summable sequence satisfying $\alpha_{-j} = \overline{\alpha_j}$, for each $n \in \mathbb{N}$ the sequence $\{k_j(n)\}_{j\in\mathbb{N}}$ consists of mutually distinct positive integers

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and $k_{-j}(n) = -k_j(n)$. Under these conditions f_n is real-valued. Alternatively, we may write X_n as

$$X_n(U) = \sum_{|j|>0} \frac{\alpha_j}{\sqrt{\min(|k_j(n)|, n)}} \operatorname{Tr} U^{k_j(n)}. \tag{1.3}$$

The main result we obtain is the following theorem.

Theorem 1.1. *We have that*

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{\mathrm{i}X_n}] = \mathrm{e}^{-\sum_{j=1}^{\infty} |\alpha_j|^2}. \tag{1.4}$$

Hence, for each fixed $m \in \mathbb{N}$, the random variables $\frac{1}{\sqrt{min(k_j(n),n)}} \operatorname{Tr} U^{k_j(n)}$, for $j = 1, \dots, m$, converge to independent standard complex normals.

The latter result was obtained before in several special cases. When $k_j(n)$, $1 \leq j \leq m$, do not depend on n, this result is proved by Diaconis and Shahshahani [7] using moment identities. In this case it is in fact a direct consequence of the strong Szegö limit theorem for Toeplitz determinants and the Weyl integration formula. If we consider a single $k_j(n) \geq n$ this result is due to Rains, see [11]. More details and an extensive list of references can be found in the survey article by Diaconis [5]. See also Diaconis and Evans [6].

There is a remarkable difference in normalization between the two cases $k_j(n) \leq n$ and $k_j(n) > n$. For the single case $k_j(n) > n$, Rains proved that the eigenvalues of $U^{k_j(n)}$ behave like n independently and uniformly distributed points on the unit circle. Therefore (1.1) follows from the classical central limit theorem. In particular, the sum of the eigenvalues is of order $\sqrt{n}$.

For $k_j(n) \leq n$, the term $\operatorname{Tr} U^{k_j(n)}$ is normalized by $\sqrt{k_j(n)}$. This normalization follows from the correlation between the eigenvalues of $U^{k_j(n)}$. Due to repulsion, the typical picture one finds for the eigenvalues is that of a small perturbation of n equidistant points on the unit circle and we have a very effective cancellation. Note that the sum of n equidistant points on the unit circle is zero.

Our result generalizes previous results by allowing arbitrary powers depending on n and thus combines the result from Szegö's theorem with that of Rains.

An interesting generalization of the problem we consider would be to allow the coefficients α_j to depend on n. In this case it seems difficult to formulate a general theorem. See section 5 for a remark.

Strong Szegö limit for n-dependent symbols

The starting point of our analysis is the connection with Toeplitz determinants. If $a \in \mathbb{L}_1(\mathbb{T})$, let $T_n(a)$ be the $n \times n$ matrix given by $\big(T_n(a)\big)_{jk} = a_{j-k}$, where the a_k are the Fourier-coefficients of a. The Heine-Szegö identity states that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}X_n}] = \det T_n(\mathrm{e}^{\mathrm{i}f_n}), \tag{1.5}$$

see [5]. Using this identity we see that in case $k_j(n)$, $1 \leq j \leq m$, do not depend on n, Theorem 1.1 is nothing else then the strong Szegö limit for Toeplitz determinants.

In order to prove Theorem 1.1 in the general case, we will prove a strong Szegö limit for n-dependent symbols of the type (1.2).

Note that f_n as defined in (1.2) is a real-valued function. The strong Szegö limit that we prove holds for complex-valued functions as well, but with a stronger condition on the coefficients α_j. For the sake of completeness we will prove the general complex-valued case.

Let $\{\alpha_j\}_{j\in\mathbb{Z}}$ be any sequence of complex numbers satisfying $\sum_j |\alpha_j| < \infty$. For each $n \in \mathbb{N}$ let $\{k_j(n)\}_{j\in\mathbb{N}}$ again be a sequence of mutually distinct positive integers and set $k_{-j}(n) = -k_j(n)$. Define $g_n : \mathbb{T} \to \mathbb{C}$ by

$$g_n(z) = \sum_{|j|>0} \frac{\alpha_j z^{k_j(n)}}{\sqrt{\min(|k_j(n)|, n)}}, \tag{1.6}$$

for all $z \in \mathbb{T}$ and $n \in \mathbb{N}$. Our main result is the following

Theorem 1.2. *If* $\sum_j |\alpha_j| < \infty$, *then*

$$\lim_{n\to\infty} \det T_n(\mathrm{e}^{g_n}) = \exp \sum_{j=1}^{\infty} \alpha_j \alpha_{-j}. \tag{1.7}$$

This is the analogue of the strong Szegö theorem for Toeplitz determinants, but now for symbols that vary with n in a particular way.

Now Theorem 1.1 follows from (1.5) and Theorem 1.2 with $g_n = \mathrm{i} f_n$, but under the extra condition $\sum_j |\alpha_j| < \infty$. This condition can however be eliminated by a standard approximation argument which is described in Section 4. However, we want to emphasize that this argument depends on the fact that f_n is real-valued.

Overview of the proof

We will omit the dependence on n in the notation and simply write g and k_j. Split g in

$$g(z) = g^{(1)}(z) + g^{(2)}(z) = \sum_{0<|k_j|\leq n} \frac{\alpha_j z^{k_j}}{\sqrt{|k_j|}} + \sum_{|k_j|>n} \frac{\alpha_j z^{k_j}}{\sqrt{n}} \tag{1.8}$$

Let a and b be defined by

$$a = \mathrm{e}^{g^{(1)}} \qquad \text{and} \qquad b = \mathrm{e}^{g^{(2)}}. \tag{1.9}$$

Define

$$C^{(1)} = \sum_{0<k_j\leq n} \alpha_j \alpha_{-j}, \qquad C^{(2)} = \sum_{k_j>n} \alpha_j \alpha_{-j}, \qquad C = \sum_{j=1}^{\infty} \alpha_j \alpha_{-j}, \tag{1.10}$$

Note that $C^{(1)}$ and $C^{(2)}$ depend on n, whereas C does not.

The terms a and b are very different in behavior. As a consequence, we analyze them separately. We therefore divide the proof of Theorem 1.2 into two parts. The first part consists of proving that

$$\lim_{n\to\infty} \mathrm{e}^{-C^{(1)}} \det T_n(a) = 1. \tag{1.11}$$

To this end we need the Fredholm determinant identity for Toeplitz determinants, which was found by Case and Geronimo [8] and independently by Borodin and Okounkov [3].

The second part consists of proving that

$$\lim_{n\to\infty} \frac{\mathrm{e}^{-C^{(2)}} \det T_n(ab)}{\det T_n(a)} = 1. \tag{1.12}$$

Indeed if we can prove that (1.11) and (1.12) hold, then a simple multiplication of the two gives

$$\lim_{n\to\infty} \mathrm{e}^{-C^{(1)}-C^{(2)}} \det T_n(ab) = \lim_{n\to\infty} \mathrm{e}^{-C} \det T_n(ab) = 1. \tag{1.13}$$

Now, since C does not depend on n we can multiply both sides with e^{C} which proves Theorem 1.2.

For reasons of clarity we will prepare the proof of (1.12) and first prove

$$\lim_{n\to\infty} \mathrm{e}^{-C^{(2)}} \det T_n(b) = 1. \tag{1.14}$$

The proof of this result follows by a fairly direct computation. The results of this computation can be used for proving (1.12). Hence, in the remaining proof of (1.12) we can restrict ourselves to only those parts that come in by interaction of $g^{(1)}$ and $g^{(2)}$. In our opinion, it helps to get a better understanding of the problem. Moreover, combining (1.11), (1.12) and (1.14) we immediately find the following result.

Proposition 1.3. *We have that*

$$\lim_{n\to\infty} \frac{\det T_n(ab)}{\det T_n(a) \det T_n(b)} = 1. \tag{1.15}$$

This is a so-called separation theorem. Such results have been often investigated before, see for example [2, 14]. However, all the results known thus far use the fact that $H(a)H(\tilde{b})$ is of trace class. This is not necessarily true in our case, which makes Theorem 1.3 an interesting result in its own right.

2 Preliminaries

To fix notation, we recall some definitions of certain operators and Banach algebras we need later. For a more detailed discussion we refer to [4].

For $c \in \mathbb{L}_\infty(\mathbb{T})$, define infinite matrices $T(c)$ and $H(c)$ by

$$T(c) = (c_{j-l})_{j,l=1}^{\infty} \qquad \text{and} \qquad H(c) = (c_{j+l-1})_{j,l=1}^{\infty}, \tag{2.1}$$

where c_k are the Fourier coefficients of c. These matrices induce bounded operators on $\ell_2(\mathbb{N})$. Moreover, $\|T(c)\|_\infty = \|c\|_{\mathbb{L}_\infty}$ and $\|H(c)\|_\infty \le \|c\|_\infty$.

Denote with P_n the projection operator on ℓ_2 that projects on the subspace of all $x \in \ell_2(\mathbb{N})$ for which $x_k = 0$ for all $k > n$. Define $Q_n = I - P_n$. Let $W_n : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the operator defined by

$$(W_n x)_k = \begin{cases} x_{n-k+1}, & 1 \le k \le n, \\ 0, & k > n \end{cases}, \tag{2.2}$$

for all $x \in \ell_2(\mathbb{N})$. If $c \in \mathbb{L}_\infty$, then

$$W_n T_n(c) W_n = T_n(\tilde{c}), \tag{2.3}$$

where $\tilde{c}(z) = c(1/z)$.

Next we recall the definition of certain Banach algebras which will appear frequently in the sequel.

The space $B_2^{1/2}$ consists of all $f \in \mathbb{L}_2(\mathbb{T})$ for which $\sum_k |k||f_k|^2 < \infty$, equipped with norm defined by

$$\|f\|^2_{B_2^{1/2}} = \sum_k (1 + |k|)|f_k|^2. \tag{2.4}$$

Again, the f_k denote the Fourier coefficients of f. The space $B_2^{1/2}$ is a Sobolev space and a Banach algebra.

The Krein algebra $K_2^{1/2}$ is defined as $B_2^{1/2} \cap \mathbb{L}_\infty(\mathbb{T})$. This is a (non-closed) subalgebra of $\mathbb{L}_\infty(\mathbb{T})$. However, the norm defined by

$$\|f\|_{K_2^{1/2}} = \|f\|_{\mathbb{L}_\infty} + \|f\|_{B_2^{1/2}}, \tag{2.5}$$

for all $f \in K_2^{1/2}$, turns $K_2^{1/2}$ into a Banach algebra.

The Wiener algebra consists of all $f \in \mathbb{L}_\infty$, for which $\sum_k |f_k| < \infty$ and has norm

$$\|f\|_W = \sum_k |f_k|, \tag{2.6}$$

for all $f \in W$. It is well-known that this is again a Banach algebra.

Note that due to the assumption $\sum |\alpha_j| < \infty$ we have that $g^{(1)} \in K_2^{1/2}$ and $g^{(2)} \in W$. In particular this shows that a and b in (1.9) are well-defined. Moreover, $a \in K_2^{1/2}$, $b \in W$ and we have the following inequalities

$$\|a\|_{B_2^{1/2}} \le \mathrm{e}^{\|g^{(1)}\|_{B_2^{1/2}}} < \mathrm{e}^{\left(2\sum|\alpha_j|^2\right)^{1/2}}, \tag{2.7}$$

$$\|b\|_W \le \mathrm{e}^{\|g^{(2)}\|_W} \le \mathrm{e}^{\sum|\alpha_j|/\sqrt{n}}. \tag{2.8}$$

Hence, $\|a\|_{B_2^{1/2}}$ and $\|b\|_W$ are uniformly bounded in n. For convenience we define

$$A_1 = \sum |\alpha_j| \qquad \text{and} \qquad A_2 = \left(\sum |\alpha_j|^2\right)^{1/2}. \tag{2.9}$$

These constants will appear frequently in upcoming inequalities.

Besides the operator norm $\|\cdot\|_\infty$ we will also use the trace norm, denoted by $\|\cdot\|_1$, and the Hilbert-Schmidt norm, denoted by $\|\cdot\|_2$. Note that if $c \in K_2^{1/2}$, then $H(c)$ is a Hilbert-Schmidt operator and

$$\|H(c)\|_2^2 = \sum_{j,l=1}^{\infty} |c_{j+l-1}|^2 = \sum_{j=1}^{\infty} j|c_j|^2 \leq \|c\|^2_{B_2^{1/2}}. \tag{2.10}$$

This will be used frequently in the sequel.

3 Proof of Theorem 1.2

3.1 Proof of (1.11)

First, we will prove (1.11). To this end we will use a celebrated Fredholm identity for Toeplitz determinants. Let $g_+^{(1)}$ be the projection of $g^{(1)}$ onto the subspace of all $f \in K_2^{1/2}$ for which $f_k = 0$ for all $k < 0$. Moreover, define $g_-^{(1)} = g^{(1)} - g_+^{(1)}$, $a_+ = \mathrm{e}^{g_+^{(1)}}$ and $a_- = \mathrm{e}^{g_-^{(1)}}$. Finally, define $\phi = a_+^{-1} a_-$ and $\psi = \widetilde{a_+ a_-^{-1}}$.

The Borodin-Okounkov-Geronimo-Case identity now states that

$$\det T_n(a) = \mathrm{e}^{C^{(1)}} \det(I - Q_n H(\phi) H(\psi) Q_n), \tag{3.1}$$

for all $n \in \mathbb{N}$. Note that since $K_2^{1/2}$ is a Banach algebra, we find that $\phi, \psi \in K_2^{1/2}$ and hence $Q_n H(\phi) H(\psi) Q_n$ is a trace class operator. The determinant on the right-hand side is a Fredholm-determinant. Note that we use the formulation by Basor and Widom, see [1], which is slightly different from the one by Borodin and Okounkov in [3].

So we need to prove that the Fredholm-determinant converges to 1 to obtain (1.11).

Lemma 3.1. *We have that*

$$|\det(I - Q_n H(\phi) H(\psi) Q_n) - 1| \leq \exp\left(\left(\sum_{k=1}^{\infty} k|\phi_{k+n}|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} k|\psi_{k+n}|^2\right)^{1/2}\right) - 1, \tag{3.2}$$

for all $n \in \mathbb{N}$.

Proof. A standard inequality for Fredholm-determinants gives

$$|\det(I - Q_n H(\phi) H(\psi) Q_n) - 1| \leq \mathrm{e}^{\|Q_n H(\phi) H(\psi) Q_n\|_1} - 1.$$

The trace norm can be estimated by

$$\|Q_n H(\phi) H(\psi) Q_n\|_1 \leq \|Q_n H(\phi)\|_2 \|H(\psi) Q_n\|_2.$$

A straightforward calculation shows that

$$\|Q_n H(\phi)\|_2^2 = \sum_{k=1}^{\infty} k|\phi_{k+n}|^2, \qquad \text{and} \qquad \|H(\psi)Q_n\|_2^2 = \sum_{k=1}^{\infty} k|\psi_{k+n}|^2,$$

which proves the statement. □

Hence we need to show that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} k|\phi_{k+n}|^2 = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} k|\psi_{k+n}|^2 = 0. \tag{3.3}$$

Note that if ϕ and ψ did not depend on n (as in the classical case), then this trivially holds. But since they depend on n there is still some work to be done.

Lemma 3.2. *Let $N \in \mathbb{N}$ and t be defined by the Fourier series $t(z) = \sum_{0<j\leq N} \frac{t_j z^j}{\sqrt{|j|}}$. Define F_t associated to t by $F_t(z) = \sum_{0<j\leq N} |t_j| z^j$. Then*

$$|(\mathrm{e}^t)_{k+N}| < \frac{1}{\sqrt{k(N+k)}} \big(F_t(\mathrm{e}^{F_t} - 1)\big)_{N+k} \tag{3.4}$$

for all $k \in \mathbb{N}$.

Proof. First consider powers t^l for $l \geq 2$. Then

$$\big(t^l\big)_{k+N} = \sum_{j_1+j_2+\cdots+j_l=k+N} \frac{t_{j_1}\cdots t_{j_l}}{\sqrt{|j_1\cdots j_l|}}$$

Since $j_1 + j_2 + \cdots + j_l = k + N$, there should be at least one j_s, with $j_s \geq (N+k)/l$. But $j_s \leq N$ and hence $j_1 + j_2 + \cdots + j_l - j_s \geq k$. Hence there exists a $j_r \neq j_s$ such that $j_r \geq k/(l-1) > k/l$.

Therefore

$$\left|\big(t^l\big)_{k+N}\right| < \frac{l}{\sqrt{k(k+N)}} \sum_{j_1+j_2+\cdots+j_l=k+N} |t_{j_1}\cdots t_{j_l}| = \frac{l}{\sqrt{k(k+N)}} \big(F_t^l\big)_{k+N}.$$

Hence,

$$\left|\big(\mathrm{e}^t\big)_{k+N}\right| \leq \sum_{l=2}^{\infty} \frac{\left|\big(t^l\big)_{k+N}\right|}{l!} < \sum_{l=2}^{\infty} \frac{\big(F_t^l\big)_{k+N}}{(l-1)!\sqrt{k(k+N)}} = \frac{1}{\sqrt{k(k+N)}} \big(F_t(\mathrm{e}^{F_t} - 1)\big)_{k+N}$$

This proves the statement. □

Now we immediately find the following corollary.

Corollary 3.3. *With A_1 as in (2.9) we have that*

$$\sum_{k=1}^{\infty} k|\phi_{k+n}|^2 < \frac{A_1(\mathrm{e}^{A_1} - 1)}{n} \tag{3.5}$$

for all n. The same estimate holds for ψ.

Proof. Applying Lemma 3.2 with $t = \phi$ and $N = n$, we find

$$\sum_{k=1}^{\infty} k|\phi_{k+n}|^2 < \frac{\|F_\phi(\mathrm{e}^{F_\phi} - 1)\|_{\mathbb{L}_2}^2}{n}.$$

The statement now follows from the fact that $\|\cdot\|_{\mathbb{L}_2} \leq \|\cdot\|_W$, the fact that W is a Banach algebra and $\|F_\phi\|_W \leq A_1$. ☐

Now (1.11) follows by combining Corollary 3.3, Lemma 3.1 and (3.1).

3.2 Proof of (1.14)

Next we analyze $\det T_n(b)$. In this case the identity (3.1) breaks down at two places. First, the factor in front of the Fredholm-determinant is infinite, since b is not necessarily contained in $K_2^{1/2}$. Second, the operator in the Fredholm-determinant is no longer of trace class and the determinant is therefore not well-defined. However, there is no need for such a strong result as (3.1), since a direct analysis on $\det T_n(b)$ will suffice.

We will use the notion of regularized determinants. For a trace class operator A the regularized determinant is defined by

$$\det\nolimits_2\,(I + A) = \mathrm{e}^{-\operatorname{Tr} A} \det(I + A). \tag{3.6}$$

One can prove that $A \mapsto \det_2\,(I + A)$ is a continuous function defined on a dense subspace (namely the space of all trace class operators) of the space of Hilbert-Schmidt operators. Therefore it can be extended and defined for all Hilbert-Schmidt operators. Moreover, we have that

$$|\det\nolimits_2\,(I + A) - 1| \leq \|A\|_2 \exp\left(\tfrac{1}{2}(\|A\|_2 + 1)^2\right), \tag{3.7}$$

for all Hilbert-Schmidt operators.

We will use the regularized determinant only for matrices, but (3.7) plays a crucial role. Write

$$\det T_n(b) = \det(I + T_n(b-1)) = \mathrm{e}^{\operatorname{Tr} T_n(b-1)} \det\nolimits_2\,(I + T_n(b-1)). \tag{3.8}$$

The proof of (1.14) falls into two parts. First we will show that the Hilbert-Schmidt norm of $T_n(b-1)$ tends to 0 as $n \to \infty$, hence the regularized determinant tends to 1. And second, we show that $\operatorname{Tr} T_n(b-1) - C^{(2)}$ tends to 0 as $n \to \infty$. Then (1.14) follows by (3.7) and (3.8).

We start with the trace of $T_n(b-1)$. We define $g_+^{(2)}$ as $\sum_{|k_j|>n} \frac{\alpha_j z^{k_j}}{\sqrt{n}}$ and $g_-^{(2)} = g^{(2)} - g_+^{(2)}$. Moreover, we let $b_\pm = \mathrm{e}^{g_\pm^{(2)}}$.

Lemma 3.4. *With A_1 as in* (2.9) *we have that*

$$\left|\operatorname{Tr} T_n(b-1) - C^{(2)}\right| \leq n(\mathrm{e}^{A_1/\sqrt{n}} - 1)^2 - A_1^2, \tag{3.9}$$

for all $n \in \mathbb{N}$.

Proof. First note that $\operatorname{Tr} T_n(b-1) = n(b_0 - 1)$. Now

$$b_0 - 1 = \sum_{j\ge 0} \left(e^{g_+^{(2)}}\right)_j \left(e^{g_-^{(2)}}\right)_{-j} - 1 = \sum_{j\ge 0} \left(e^{g_+^{(2)}} - 1\right)_j \left(e^{g_-^{(2)}} - 1\right)_{-j}$$
$$= \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j\ge 0} \frac{\left({g_+^{(2)}}^l\right)_j \left({g_-^{(2)}}^m\right)_{-j}}{l!m!}$$

Since $\sum_{j>n} \alpha_j \alpha_{-j} = n \sum_{j\ge 0} \left(g_+^{(2)}\right)_j \left(g_-^{(2)}\right)_{-j}$, we find

$$\frac{1}{n}\left| \operatorname{Tr} T_n(b-1) - \sum_{j>n} \alpha_j \alpha_{-j} \right| = \left| b_0 - 1 - \sum_{j\ge 0} \left(g_+^{(2)}\right)_j \left(g_-^{(2)}\right)_{-j} \right|$$
$$= \left| \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j\ge 0} \frac{\left({g_+^{(2)}}^l\right)_j \left({g_-^{(2)}}^m\right)_{-j}}{l!m!} - \sum_{j\ge 0} \left(g_+^{(2)}\right)_j \left(g_-^{(2)}\right)_{-j} \right|$$

Now apply the Cauchy-Schwarz inequality to obtain

$$\frac{1}{n}\left| \operatorname{Tr} T_n(b-1) - \sum_{j>n} \alpha_j \alpha_{-j} \right| \le \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{\|{g_+^{(2)}}^l\|_{\mathbb{L}_2} \|{g_-^{(2)}}^m\|_{\mathbb{L}_2}}{l!m!} - \|g_+^{(2)}\|_{\mathbb{L}_2} \|g_-^{(2)}\|_{\mathbb{L}_2}$$
$$\le \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{\|{g_+^{(2)}}^l\|_W \|{g_-^{(2)}}^m\|_W}{l!m!} - \|g_+^{(2)}\|_W \|g_-^{(2)}\|_W$$
$$\le \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{\|g_+^{(2)}\|_W^l \|g_-^{(2)}\|_W^m}{l!m!} - \|g_+^{(2)}\|_W \|g_-^{(2)}\|_W$$

Now $\|g_\pm^{(2)}\|_W \le A_1/\sqrt{n}$ proves the statement. ☐

Next we proceed with the Hilbert-Schmidt norm of $T_n(b-1)$.

Lemma 3.5. *With A_1 as in (2.9), we have that*

$$\|T_n(b-1)\|_2 \le \sqrt{n}(e^{A_1/\sqrt{n}} - 1)^2, \tag{3.10}$$

for all $n \in \mathbb{N}$.

Proof. Since $(b_\pm - 1)_j = 0$ for $j = -n+1, \ldots, n-1$ we find

$$\|T_n(b-1)\|_2^2 \le n \sum_{j=-n+1}^{n-1} |(b-1)_j|^2 = n \sum_{j=-n+1}^{n-1} |((b_+ - 1)(b_- - 1))_j|^2$$
$$\le n\|(b_+ - 1)(b_- - 1)\|_{\mathbb{L}_2}^2 \le n\|(b_+ - 1)(b_- - 1)\|_W^2$$
$$\le n\|b_+ - 1\|_W^2 \|b_- - 1\|_W^2 \le n(e^{\|g_+^{(2)}\|_W} - 1)^2 (e^{\|g_-^{(2)}\|_W} - 1)^2.$$

By $\|g_\pm^2\|_W \le A_1/\sqrt{n}$ we obtain the statement. ☐

By Lemma 3.5 and (3.7) we obtain

$$\lim_{n\to\infty} \det\nolimits_2 \left(I + T_n(b-1)\right) = 1. \tag{3.11}$$

By substituting this in (3.8) and using Lemma 3.4 we obtain (1.14).

3.3 Proof of (1.12)

Since we proved the result for the cases (1.11) and (1.14) in a completely different way, a natural way to deal with the general case is to split the two cases. To this end we use a factorization theorem due to Widom

$$T_n(ab) = T_n(a)T_n(b) + P_nH(a)H(\widetilde{b})P_n + W_nH(\tilde{a})H(b)W_n, \tag{3.12}$$

and the operator B_n defined by

$$B_n = T_n(a^{-1}) - P_nH(a_+^{-1})H(\widetilde{a_-^{-1}})P_n - W_nH(\widetilde{a_-^{-1}})H(a_+^{-1})W_n. \tag{3.13}$$

The operator B_n is a good approximation of the inverse of $T_n(a)$. In the case that a does not depend on n, this observation is due to Widom. Moreover, the operator can be used to prove the strong Szegö limit, see [4, 13]. We will prove that it is also a good approximation in our case. One can show, see [4, 13], that

$$B_nT_n(a) = I + P_nH(a_+^{-1})H(\widetilde{a_-^{-1}})Q_nT(a)P_n + W_nH(\widetilde{a_-^{-1}})H(a_+^{-1})Q_nT(\tilde{a})W_n \tag{3.14}$$

for all $n \in \mathbb{N}$. Even in our case where a depends on n, the operators on the right-hand side are small in trace norm.

Lemma 3.6. *We have*

$$\|B_nT_n(a) - I\|_1 = \mathcal{O}(n^{-\frac{1}{2}}), \tag{3.15}$$

for $n \to \infty$.

Proof. First note that

$$\|P_nH(a_+^{-1})H(\widetilde{a_-^{-1}})Q_nT(a)P_n\|_1 \le \|P_nH(a_+^{-1})\|_2\|H(\widetilde{a_-^{-1}})Q_n\|_2\|T(a)P_n\|_\infty.$$

Now

$$\|P_nH(a_+^{-1})\|_2 \le \|H(a_+^{-1})\|_2 \le \|a_+^{-1}\|_{B_2^{1/2}} < \exp(\sqrt{2}A_2),$$

and

$$\|T(a)P_n\|_\infty \le \|a\|_\infty \le \|a\|_W \le \exp(A_1),$$

and finally

$$\|H(\widetilde{a_-^{-1}})Q_n\|_2^2 = \sum_{k=1}^{\infty} k|(\widetilde{a_-^{-1}})_{k+n}|^2.$$

By Lemma 3.2 and the same arguments as in Corollary 3.3, the latter is $\mathcal{O}(n^{-1})$, as $n \to \infty$. This proves the statement. □

Therefore the following corollary is immediate.

Corollary 3.7. *We have that*

$$\lim_{n\to\infty} \det B_n T_n(a) = 1. \tag{3.16}$$

In view of this corollary, it is enough to show that

$$\lim_{n\to\infty} \exp(-C^{(2)}) \det B_n T_n(ab) = 1, \tag{3.17}$$

to prove (1.12). This will cover the rest of this section.

We will again use the regularized determinant. Write

$$\det B_n T_n(ab) = \mathrm{e}^{\mathrm{Tr}(B_n T_n(ab) - I)} \det{}_2\, B_n T_n(ab). \tag{3.18}$$

In view of (3.8) and (3.7), to prove (3.17) it is enough to (1) prove that $B_n T_n(ab) - I$ converges to zero in Hilbert-Schmidt norm and (2) calculate its trace.

If we introduce the notations

$$E_n = -P_n H(a_+^{-1}) H(\widetilde{a_-^{-1}}) P_n - W_n H(\widetilde{a_-^{-1}}) H(a_+^{-1}) W_n, \tag{3.19}$$

and

$$F_n = P_n H(a) H(\widetilde{b}) P_n + W_n H(\widetilde{a}) H(b) W_n, \tag{3.20}$$

and multiply (3.12) from the left with B_n we find by (3.13)

$$B_n T_n(ab) = B_n T_n(a) T_n(b) + T_n(a^{-1}) F_n + E_n F_n. \tag{3.21}$$

We will analyze the three terms on the right-hand side separately. In the following lemma, we state results about the Hilbert-Schmidt norms and the trace of each of these three terms, except for the trace of $T_n(a^{-1})F_n$. All the statements follow from earlier results. However, $\mathrm{Tr}\, T_n(a^{-1})F_n$ is more subtle and needs some extra attention.

Lemma 3.8. *We have that*

1. $\|B_n T_n(a) T_n(b) - I\|_2 \to 0,$
2. $|\,\mathrm{Tr}(B_n T_n(a) T_n(b) - I) - C^{(2)}| \to 0,$
3. $\|E_n F_n\|_1 \to 0,$
4. $\|T_n(a^{-1}) F_n\|_2 \to 0,$

for $n \to \infty$.

Proof. 1. We estimate the Hilbert-Schmidt norm by

$$\begin{aligned} \|B_n T_n(a) T_n(b) - I\|_2 &\leq \|\Big(B_n T_n(a) - I\Big) T_n(b)\|_2 + \|T_n(b-1)\|_2 \\ &\leq \|B_n T_n(a) - I\|_2 \|T_n(b)\|_\infty + \|T_n(b-1)\|_2. \end{aligned}$$

Note that $\|T_n(b)\|_\infty \leq \|b\|_{\mathbb{L}_\infty} \leq \|b\|_W$. The statement now follows from Lemma 3.5, Lemma 3.6 and (2.8).

2. Note that

$$\begin{aligned}
&|\operatorname{Tr}(B_nT_n(a)T_n(b)-I)-C^{(2)}| \\
&\qquad\le |\operatorname{Tr}\left((B_nT_n(a)-I)\,T_n(b)\right)|+|\operatorname{Tr}T_n(b-1)-C^{(2)}| \\
&\qquad\le \|\,(B_nT_n(a)-I)\,\|_1\|T_n(b)\|_\infty+|\operatorname{Tr}T_n(b-1)-C^{(2)}|.
\end{aligned}$$

The statement now follows from Lemma 3.4 and Lemma 3.6.

3. First note that $\|E_nF_n\|_1\le\|E_n\|_2\|F_n\|_2$. Now

$$\begin{aligned}
\|F_n\|_2 &\le \|P_nH(a)\|_2\|P_nH(\tilde{b})\|_\infty+\|W_nH(\tilde{a})\|_2\|H(b)W_n\|_\infty \\
&\le \|a\|_{B_2^{1/2}}\|b-1\|_\infty\le\|a\|_{B_2^{1/2}}\|b-1\|_W \\
&\le \|a\|_{B_2^{1/2}}\Big(\exp(A_1/\sqrt{n})-1\Big),
\end{aligned}\tag{3.22}$$

with A_1 as in (2.9). By combining (3.22) with (2.7) we obtain $\|F_n\|_2\to 0$. By similar estimates one finds that $\|E_n\|_2$ is bounded in n.

4. This follows from (3.22) and the estimate $\|T_n(a^{-1})F_n\|_2\le\|T_n(a^{-1})\|_\infty\|F_n\|_2$. Note that $\|T_n(a^{-1})\|_\infty\le\|a^{-1}\|_\infty\le\|a^{-1}\|_W$ and the latter is uniformly bounded in n.

□

From this lemma, (3.18), (3.7) and (3.21) it follows that

$$\lim_{n\to\infty}\exp\Big(-C-\operatorname{Tr}(T_n(a^{-1})F_n)\Big)\det B_nT_n(ab)=1.\tag{3.23}$$

Hence it remains to prove that $\operatorname{Tr}T_n(a^{-1})F_n$ tends to 0 as $n\to\infty$, which is the most difficult part of the proof. We start with an estimate that follows from a subtle cancellation.

Lemma 3.9. *There exists a constant D such that*

$$\sum_{s=-\sqrt{n}}^{\sqrt{n}}\left(a^{-1}\right)_s\left(a\right)_{N-s}\le\frac{D}{n^{3/4}},\tag{3.24}$$

for all $n,N\in\mathbb{N}$ with $N>n$.

Proof. Let $n,N\in\mathbb{N}$ with $N>n$. Define $j^*=\sup\{j\mid k_j<n\}$. The proof follows by an induction-like argument with respect to j^*.

Suppose first that k_{j^*} is such that $N-k_{j^*}>\sqrt{n}/2$. In this case split the sum into two parts

$$\sum_{s=-\sqrt{n}}^{\sqrt{n}}\left(a^{-1}\right)_s\left(a\right)_{N-s}=\sum_{|s|<\sqrt{n}/3}\left(a^{-1}\right)_s\left(a\right)_{N-s}+\sum_{\sqrt{n}/3\le|s|\le\sqrt{n}}\left(a^{-1}\right)_s\left(a\right)_{N-s}\tag{3.25}$$

The second sum of the right-hand side of (3.25) is estimated by

$$\begin{aligned}\left|\sum_{\sqrt{n}/3<|s|<\sqrt{n}}\big(a^{-1}\big)_s\big(a\big)_{N-s}\right| &\le \left(\sum_{\sqrt{n}/3<|s|<\sqrt{n}}|\big(a^{-1}\big)_s|^2\right)^{1/2}\left(\sum_{\sqrt{n}/3<|s|<\sqrt{n}}|\big(a\big)_{N-s}|^2\right)^{1/2}\\ &= \left(\sum_{\sqrt{n}/3<|s|<\sqrt{n}}\frac{|s|\,|\big(a^{-1}\big)_s|^2}{|s|}\right)^{1/2}\left(\sum_{\sqrt{n}/3<|s|<\sqrt{n}}\frac{|N-s|\,|\big(a\big)_{N-s}|^2}{|N-s|}\right)^{1/2}\\ &\le \frac{\sqrt{3}\|a^{-1}\|_{B_2^{1/2}}\|a\|_{B_2^{1/2}}}{n^{1/4}\sqrt{N-\sqrt{n}}} \le \frac{\sqrt{6}\|a^{-1}\|_{B_2^{1/2}}\|a\|_{B_2^{1/2}}}{n^{3/4}}, \qquad (3.26)\end{aligned}$$

where we used that $N-\sqrt{n}\ge n+1-\sqrt{n}>n/2$. Note that $\|a^{-1}\|_{B_2^{1/2}}$ and $\|a\|_{B_2^{1/2}}$ are uniformly bounded in n by (2.7).

The first sum of the right-hand side of (3.25) is estimated in a similar way

$$\left|\sum_{|s|<\sqrt{n}/3}\big(a^{-1}\big)_s(a)_{N-s}\right| \le \|a^{-1}\|_{\mathbb{L}_2}\left(\sum_{|s|<\sqrt{n}/3}|(a)_{N-s}|^2\right)^{1/2}.$$

The term $\|a^{-1}\|_{\mathbb{L}_2}$ is uniformly bounded in n. Applying Lemma 3.2, with $t=a$, gives

$$\begin{aligned}\sum_{|s|<\sqrt{n}/3}|(a)_{N-s}|^2 &< \sum_{|s|<\sqrt{n}/3}\frac{|\big(F_a(\mathrm{e}^{F_a}-1)\big)_{N-s}|^2}{(N-s-k_{j^*})(N-s)}\\ &\le \frac{1}{(N-\sqrt{n}/3-k_{j^*})(N-\sqrt{n}/3)}\sum_{s<\sqrt{n}/3}|\big(F_a(\mathrm{e}^{F_a}-1)\big)_{N-s}|^2\\ &\le \frac{18}{n^{3/2}}\|\big(F_a(\mathrm{e}^{F_a}-1)\|_{\mathbb{L}_2} \le \frac{18}{n^{3/2}}\|\big(F_a(\mathrm{e}^{F_a}-1)\|_W\\ &\le \frac{18}{n^{3/2}}A_1(\mathrm{e}^{A_1}-1). \qquad (3.27)\end{aligned}$$

By combining (3.25), (3.26) and (3.27) we obtain the statement in the case $N-k_{j^*}>\sqrt{n}/2$.

Now suppose $N-k_{j^*}\le\sqrt{n}/2$. We will then show that the terms that come from j^* are negligible. To be precise, define

$$c_1 = \exp\big((\alpha_{j^*}z^{k_j^*}+\alpha_{-j^*}z^{-k_{j^*}})/\sqrt{k_{j^*}}\big), \qquad (3.28)$$

$$a_1 = ac_1^{-1}. \qquad (3.29)$$

We will show that

$$\left|\sum_{s=-\sqrt{n}}^{\sqrt{n}}\big(a^{-1}\big)_s\big(a\big)_{N-s} - \sum_{s=-\sqrt{n}}^{\sqrt{n}}\big(a_1^{-1}\big)_s\big(a_1\big)_{N-s}\right| \le (|\alpha_{j^*}|+|\alpha_{-j^*}|)D_1/n, \qquad (3.30)$$

where D_1 is a constant independent of j^*, n and N that can be expressed in terms of A_1 and A_2 only. Redefine j^*, now with respect to a_1. If $N - k_{j^*} > \sqrt{n}/2$, then the above arguments show that (3.24) holds for a_1. By combining this with (3.30) we see that (3.24) also holds for a. If however $N - k_{j^*} < \sqrt{n}/2$ then we define a_2 and c_2 as in (3.28) and (3.29) and redefine j^* with respect to a_1. We also have that the inequality (3.30) holds with a_1 replaced by a_2, a replaced by a_1 and j^* is with respect to a_1. If $N - k_{j^*} > \sqrt{n}/2$ then we are again done. Otherwise we continue by defining a_3 and c_3 and so on. After a finite number of steps, say $m \leq n + \sqrt{n}/2 - N$, we do find $N - k_{j^*} > \sqrt{n}/2$. At each step l we have the inequality (3.30) with a replaced by a_l and a_1 replaced by a_{l+1} and j^* is with respect to a_l. We can reduce all the inequalities together to the single inequality

$$\left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a^{-1}\right)_s \left(a\right)_{N-s} - \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_m^{-1}\right)_s \left(a_m\right)_{N-s} \right| \leq \frac{D_1 A_1}{n}.$$

Combining this inequality with the fact that the above arguments show that (3.24) holds for a_m leads to the statement.

Hence it remains to prove (3.30). First note that

$$\left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a^{-1}\right)_s \left(a\right)_{N-s} - \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}\right)_s \left(a_1\right)_{N-s} \right| \leq I_1 + I_2 + I_3, \tag{3.31}$$

where

$$\begin{aligned} I_1 &= \left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a^{-1}\right)_s \left(a\right)_{N-s} - \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}\right)_s \left(a\right)_{N-s} \right| = \left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}(c_1^{-1} - 1)\right)_s \left(a\right)_{N-s} \right|, \\ I_2 &= \left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}\right)_s \left(a\right)_{N-s} - \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}\right)_s \left(a_1(1 + \log c_1)\right)_{N-s} \right| \\ &= \left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}\right)_s \left(a_1(c_1 - 1 - \log c_1)\right)_{N-s} \right|, \end{aligned}$$

and

$$I_3 = \left| \sum_{s=-\sqrt{n}}^{\sqrt{n}} \left(a_1^{-1}\right)_s \left(a_1 \log c_1\right)_{N-s} \right|.$$

The terms I_1 and I_2 can be estimated by the Cauchy-Schwarz inequality,

$$I_1 \leq \|a_1^{-1}(c_1^{-1} - 1)\|_{\mathbb{L}_2} \left(\sum_{s=-\sqrt{n}}^{\sqrt{n}} \left| \left(a\right)_{N-s} \right|^2 \right)^{1/2} \leq \frac{\|a_1^{-1}\|_W \|c_1^{-1} - 1\|_W \|a\|_{B_2^{1/2}}}{\sqrt{N - \sqrt{n}}},$$

and

$$I_2 \le \|a_1^{-1}\|_2 \|a_1(c_1 - 1 - \log c_1)\|_2 \le \|a_1^{-1}\|_W \|a_1\|_W \|c_1 - 1 - \log c_1\|_W.$$

Note that $\|a_1\|_W, \|a_1^{-1}\|_W$ and $\|a\|_{B_2^{1/2}}$ are all uniformly bounded in n and N. Now

$$\|c_1^{-1} - 1\|_W \le \exp\left(\frac{|\alpha_{j^*}| + |\alpha_{-j^*}|}{\sqrt{k_{j^*}}}\right) - 1,$$
$$\|c_1 - 1 - \log c_1\|_W \le \exp\left(\frac{|\alpha_{j^*}| + |\alpha_{-j^*}|}{\sqrt{k_{j^*}}}\right) - 1 - \frac{|\alpha_{j^*}| + |\alpha_{-j^*}|}{\sqrt{k_{j^*}}}.$$

Since $k_{j^*} > n/2$ it follows that

$$I_{1,2} \le (|\alpha_{j^*}| + |\alpha_{-j^*}|) D_2 n^{-1}, \tag{3.32}$$

for some constant D_2.

This brings us to the most important part of the proof, namely estimating I_3. Note that $\log c_1 = (a_{j^*} z^{k_{j^*}} + \alpha_{-j^*} z^{-k_{j^*}})/\sqrt{k_{j^*}}$. Write

$$I_3 \le I_{31} + I_{32},$$

where

$$I_{31} = \left| \frac{\alpha_{k_{j^*}}}{\sqrt{k_{j^*}}} \sum_{s=-\sqrt{n}}^{\sqrt{n}} (a_1^{-1})_s (a_1)_{N-s-k_{j^*}} \right|, \qquad I_{32} = \left| \frac{\alpha_{-k_{j^*}}}{\sqrt{k_{j^*}}} \sum_{s=-\sqrt{n}}^{\sqrt{n}} (a_1^{-1})_s (a_1)_{N-s+k_{j^*}} \right|.$$

The term I_{32} can again be estimated by the Cauchy-Schwarz inequality. The result is that

$$I_{32} \le \frac{\sqrt{2}|\alpha_{-k_{j^*}}| \|a_1^{-1}\|_{\mathbb{L}_2} \|a_1\|_{B_2^{1/2}}}{\sqrt{k_{j^*} n}} \le \frac{2|\alpha_{-k_{j^*}}| \|a_1^{-1}\|_{\mathbb{L}_2} \|a_1\|_{B_2^{1/2}}}{n}, \tag{3.33}$$

where we used the fact that $N - s + k_{j^*} \ge n/2$ if $|s| \le \sqrt{n}$ and $k_{j^*} \ge n/2$.

The term I_{31} is more subtle. Since $N > k_{j^*}$ we find

$$\sum_{s=-\sqrt{n}}^{\sqrt{n}} (a_1^{-1})_s (a_1)_{N-s-k_{j^*}} + \sum_{|s|>\sqrt{n}} (a_1^{-1})_s (a_1)_{N-s-k_{j^*}} = \sum_s (a_1^{-1})_s (a_1)_{N-s-k_{j^*}}$$
$$= (a_1^{-1} a_1)_{N-k_{j^*}} = 0.$$

Therefore

$$I_{31} = \frac{|a_{j^*}|}{\sqrt{k_{j^*}}} \left| \sum_{|s|>\sqrt{n}} (a_1^{-1})_s (a_1)_{N-s-k_{j^*}} \right|.$$

Now we estimate the latter expression by the Cauchy-Schwarz inequality again. Note that $|N-s-k_{j^*}| > \sqrt{n}/2$ if $|s| > \sqrt{n}$. The result is that

$$I_{31} \leq \frac{2|\alpha_{j^*}| \|a_1^{-1}\|_{B_2^{1/2}} \|a_1\|_{B_2^{1/2}}}{n\sqrt{k_{j^*}}} \leq \frac{2^{3/2}|\alpha_{j^*}| \|a_1^{-1}\|_{B_2^{1/2}} \|a_1\|_{B_2^{1/2}}}{n^{3/2}}, \tag{3.34}$$

where we also used $k_{j^*} \geq n/2$. So from (3.32), (3.33) and (3.34) we find

$$I_{1,2,3} \leq (|a_{k_{j^*}}| + |\alpha_{-k_{j^*}}|)D_3/n \tag{3.35}$$

for some constant D_3. Now (3.30) follows by (3.31) and (3.35). This proves the statement. ☐

Now we can prove the following corollary by fairly direct estimates.

Corollary 3.10. *We have that*

$$\operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\widetilde{g^{(2)}}) P_n = \mathcal{O}(n^{-1/4}), \tag{3.36}$$

for $n \to \infty$.

Proof. A straightforward calculation leads to

$$\operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\widetilde{g^{(2)}}) P_n = \sum_{k_j > n} \frac{\alpha_{-j}}{\sqrt{n}} \sum_{s=-n}^{n} (a^{-1})_s (a)_{k_j - s}(n - |s|). \tag{3.37}$$

We estimate each term in the sum with respect to k_j separately. So let $k_j > n$. Write

$$\begin{aligned}\sum_{s=-n}^{n} (a^{-1})_s (a)_{k_j-s}(n-|s|) &= \sum_{s=-\sqrt{n}}^{\sqrt{n}} (a^{-1})_s (a)_{k_j-s}(n-|s|) \\ &\quad + \sum_{\sqrt{n}<|s|\leq n} (a^{-1})_s (a)_{k_j-s}(n-|s|).\end{aligned} \tag{3.38}$$

After some preparation, the rightmost sum of the right-hand side of (3.38) can be estimated by the Cauchy-Schwarz inequality as before

$$\begin{aligned}\Big|\sum_{\sqrt{n}<|s|\leq n} (a^{-1})_s (a)_{k_j-s}(n-|s|)\Big| &\leq \sum_{\sqrt{n}<|s|\leq n} |(a^{-1})_s (a)_{k_j-s}||n-s| \\ &= \sum_{\sqrt{n}<|s|\leq n} \sqrt{|s|}|(a^{-1})_s| \sqrt{k_j - s}\,|(a)_{k_j-s}| \frac{(n-|s|)}{\sqrt{|s|(k_j-s)}} \\ &\leq \|a^{-1}\|_{B_2^{1/2}} \|a\|_{B_2^{1/2}} n^{1/4},\end{aligned} \tag{3.39}$$

where we used that

$$\frac{n-|s|}{\sqrt{|s|(k_j-s)}} \leq \sqrt{\frac{n-|s|}{|s|}} \leq n^{1/4},$$

for all $\sqrt{n} \le |s| \le n$.

Now consider the left sum of the right-hand side of (3.38).

$$\Big| \sum_{s=-\sqrt{n}}^{\sqrt{n}} (a^{-1})_s (a)_{k_j - s} (n - |s|) \Big| = n \Big| \sum_{s=-\sqrt{n}}^{\sqrt{n}} (a^{-1})_s (a)_{k_j - s} \Big| + \Big| \sum_{s=-\sqrt{n}}^{\sqrt{n}} |s| (a^{-1})_s (a)_{k_j - s} \Big|. \tag{3.40}$$

The second sum of the right-hand side of (3.40) can again be estimated by a Cauchy-Schwarz argument, from which it follows that it is of order $n^{-1/2}$. The first sum of the right-hand side of (3.40) can be dealt with by using Lemma 3.9 and therefore

$$\Big| \sum_{s=-\sqrt{n}}^{\sqrt{n}} (a^{-1})_s (a)_{k_j - s} (n - |s|) \Big| = \mathcal{O}(n^{1/4}), \tag{3.41}$$

for $n \to \infty$.

Inserting (3.39) and (3.41) in (3.38) and using (3.37) gives

$$\operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\widetilde{g^{(2)}}) P_n = \mathcal{O}(n^{-1/4}) \sum_{k_j > n} \alpha_{-j} = \mathcal{O}(n^{-1/4}),$$

for $n \to \infty$. This proves the statement. □

We are almost at the end of our proof. The final thing we need to show is that the dominant term in $\operatorname{Tr} T_n(a) F_n$ comes from $\operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\widetilde{g^{(2)}}) P_n$, which is small by the previous corollary.

Corollary 3.11.

$$\lim_{n \to \infty} \operatorname{Tr} T_n(a^{-1}) F_n = 0. \tag{3.42}$$

Proof. Since $W_n^2 = P_n$ and by (2.3) we find

$$\begin{aligned} \operatorname{Tr} T_n(a^{-1}) F_n &= \operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\tilde{b}) P_n + \operatorname{Tr} T_n(a^{-1}) W_n H(\tilde{a}) H(b) W_n \\ &= \operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\tilde{b}) P_n + \operatorname{Tr} W_n T_n(\widetilde{a^{-1}}) P_n H(\tilde{a}) H(b) W_n \\ &= \operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\tilde{b}) P_n + \operatorname{Tr} T_n(\widetilde{a^{-1}}) P_n H(\tilde{a}) H(b) P_n. \end{aligned}$$

We will only show that $\operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\tilde{b}) P_n \to 0$. The right term tends to 0 by the same arguments. Write

$$\begin{aligned} T_n(a^{-1}) P_n H(a) H(\tilde{b}) P_n = &\operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\tilde{b} - \widetilde{g^{(2)}} - 1) P_n \\ &+ \operatorname{Tr} T_n(a^{-1}) P_n H(a) H(\widetilde{g^{(2)}}) P_n. \end{aligned} \tag{3.43}$$

Since

$$\|H(\tilde{b} - \widetilde{g^{(2)}} - 1) P_n\|_2 \le \sqrt{n} \|b - g^{(2)} - 1\|_{\mathbb{L}_2} \le \sqrt{n} \left(\mathrm{e}^{\|g^{(2)}\|_W} - \|g^{(2)}\|_W - 1 \right),$$

and $\|g^{(2)}\|_W \leq A_1/\sqrt{n}$ it follows that

$$|\operatorname{Tr} T_n(a^{-1})P_n H(a)H(\widetilde{b} - \widetilde{g^{(2)}} - 1)P_n| \leq \|T_n(a^{-1})P_n H(a)H(\tilde{b} - \widetilde{g^{(2)}} - 1)P_n\|_1$$
$$\leq \|T_n(a^{-1})\|_\infty \|P_n H(a)\|_2 \|H(\tilde{b} - \widetilde{g^{(2)}} - 1)P_n\|_2 = \mathcal{O}(n^{-\frac{1}{2}}),$$

for $n \to \infty$. By combining this with (3.43) we see that it only remains to estimate $\operatorname{Tr} T_n(a^{-1})P_n H(a)H(\widetilde{g^{(2)}})P_n$, which was done in Corollary 3.10. This proves the statement. □

Now (1.12) follows from Corollary 3.11 and (3.23).

4 Proof of Theorem 1.1

We will now show how the condition $\sum |\alpha_j| < \infty$ can be made obsolete when we assume that $\alpha_{-j} = \overline{\alpha_j}$. Let $m \in \mathbb{N}$. We split X_n into two parts

$$X_n = X_{n,m} + Y_{n,m} = \sum_{0<|j|\leq m} \frac{\alpha_j}{\sqrt{\min(|k_j|, n)}} \operatorname{Tr} U^{k_j} + \sum_{|j|>m} \frac{\alpha_j}{\sqrt{\min(|k_j|, n)}} \operatorname{Tr} U^{k_j}. \tag{4.1}$$

Since both $X_{n,m}$ and $Y_{n,m}$ are real we find that

$$\left|\mathbb{E}[e^{iX_n}] - \mathbb{E}[e^{iX_{n,m}}]\right| = \left|\mathbb{E}[e^{i(X_{n,m}+Y_{n,m})} - \mathbb{E}[e^{iX_{n,m}}]\right| \leq \mathbb{E}[\left|e^{iY_{n,m}} - 1\right|]$$
$$\leq \mathbb{E}[|Y_{n,m}|] \leq \mathbb{E}[|Y_{n,m}|^2]^{1/2} = \left(\sum_{|j|>m} |\alpha_j|^2\right)^{1/2}. \tag{4.2}$$

In the last expression we used the fact that the elements $\frac{1}{\sqrt{\min(|k_j|,n)}} \operatorname{Tr} U^{k_j}$ are orthonormal with respect to the Haar measure on $U(n)$. It follows that

$$\limsup \left|\mathbb{E}[e^{i(X_{n,m}+Y_{n,m})} - \mathbb{E}[e^{iX_{n,m}}]\right| \leq \left(\sum_{|j|>m} |\alpha_j|^2\right)^{1/2}. \tag{4.3}$$

Since $\sum_{|j|\leq m} |\alpha_j| < \infty$, it follows by Theorem 1.2 and (1.5) that

$$\lim_{n\to\infty} \mathbb{E}[e^{iX_{n,m}}] = e^{-\sum_{j=1}^m |\alpha_j|^2}. \tag{4.4}$$

Hence

$$\limsup_{n\to\infty} \left|\mathbb{E}[e^{i(X_{n,m}+Y_{n,m})} - e^{-\sum_{j=1}^\infty |\alpha_j|^2}\right| \leq \limsup_{n\to\infty} \left|\mathbb{E}[e^{i(X_{n,m}+Y_{n,m})} - \mathbb{E}[e^{iX_{n,m}}]\right| \tag{4.5}$$
$$+ \limsup_{n\to\infty} \left|\mathbb{E}[e^{iX_{n,m}}] - e^{-\sum_{j=1}^m |\alpha_j|^2}\right| + \left|e^{-\sum_{j=1}^m |\alpha_j|^2} - e^{-\sum_{j=1}^\infty |\alpha_j|^2}\right| \tag{4.6}$$
$$\leq \left(\sum_{|j|>m} |\alpha_j|^2\right)^{1/2} + \left|e^{-\sum_{j=1}^m |\alpha_j|^2} - e^{-\sum_{j=1}^\infty |\alpha_j|^2}\right|. \tag{4.7}$$

If we let $m \to \infty$ the right-hand side tends to zero.

5 Some comments on more general n-dependence

The n-dependence in the symbols we consider is of a special type. Let U be a $n \times n$ unitary matrix randomly chosen with respect to the Haar measure. Consider the random variable X_n by

$$X_n(U) = \sum_{|j|>0} \frac{\alpha_j(n)}{\sqrt{\min(|k_j(n)|, n)}} \operatorname{Tr} U^{k_j(n)}, \tag{5.1}$$

where $\alpha_j(n)$ now also depends on n. Again we assume that for each n we have that $\alpha_j(n) = \overline{\alpha_{-j}(n)}$, $\{k_j(n)\}_{j\in\mathbb{N}}$ is a sequence of mutually distinct positive integers and $k_{-j}(n) = -k_j(n)$. Define

$$\sigma_n^2 = 2\sum_{j=1}^{\infty} |\alpha_j(n)|^2, \tag{5.2}$$

and assume that $\sigma_n \to \sigma$ as $n \to \infty$ for some σ. A natural question is now under what conditions it is still true that

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n}] = \mathrm{e}^{-t\sigma^2/2}. \tag{5.3}$$

Since then X_n converges to a complex normal with mean zero and variance σ^2. Although, it is known in some cases that it is true, it will not hold in general.

We will illustrate the subtleties that are involved by an explicit example inspired on [12]. Let f be a C^∞ function with support within $[-\pi, \pi]$ and let $0 < \gamma \leq 1$. Define $k_j(n) = j$ and

$$\alpha_j(n) = \frac{\sqrt{\min(|j|, n)}}{2\pi n^\gamma} \hat{f}(j/n^\gamma), \tag{5.4}$$

for all j and n. Here $\hat{f}$ stands for the Fourier transform of f. We assume that

$$\hat{f}(0) = \int_{\mathbb{R}} f(x)\ \mathrm{d}x = 0. \tag{5.5}$$

The random variable X_n can now be rewritten as

$$X_n(U) = \sum_{\mu=1}^{n} f(n^\gamma \theta_\mu). \tag{5.6}$$

Since f has compact support X_n only depends on a few eigenvalues, for which θ_μ is close to zero. If $0 < \gamma < 1$, then it is true that $X_n \to N(0, \sigma^2)$, where

$$\sigma^2 = \frac{1}{4\pi^2} \int |y||\hat{f}(y)|^2\ \mathrm{d}y, \tag{5.7}$$

assuming that the latter is finite. This is proved by Soshnikov [12].

However, the result does not longer hold for $\gamma = 1$. This case is considered by Hughes and Rudnick in [9] and for the classical compact groups other then $U(n)$ in [10]. In these works the authors analyzed the limiting behavior of the moments

$\mathbb{E}(X_n^m)$ for $m \in \mathbb{N}$ and proved that in general the limiting value of the moments depend on f and are certainly not Gaussian moments. Hence a result like (5.3) can not hold. However, if supp $\hat{f} \subset [-2/m, 2/m]$ then the m-th moment does converge to the m-th moment of the normal distribution with mean zero and variance

$$\sigma^2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \min(|y|, 1) |\hat{f}(y)|^2 \, \mathrm{d}y. \tag{5.8}$$

This phenomenon is called mock-Gaussian behavior in [9].

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