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Einstein Supergravity and New Twistor String Theories

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Abstract

A family of new twistor string theories is constructed and shown to be free from world-sheet anomalies. The spectra in space-time are calculated and shown to give Einstein supergravities with second order field equations instead of the higher derivative conformal supergravities that arose from earlier twistor strings. The theories include one with the spectrum of $N = 8$ supergravity, another with the spectrum of $N = 4$ supergravity coupled to $N = 4$ super-Yang-Mills, and a family with $N \geq 0$ supersymmetries with the spectra of self-dual supergravity coupled to self-dual super-Yang-Mills. The non-supersymmetric string with $N = 0$ gives self-dual gravity coupled to self-dual Yang-Mills and a scalar. A three-graviton amplitude is calculated for the $N = 8$ and $N = 4$ theories and shown to give a result consistent with the cubic interaction of Einstein supergravity.

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1 Introduction

The string theories in twistor space proposed by Witten and by Berkovits [1, 2, 3] give a formulation of $N = 4$ supersymmetric Yang-Mills theory coupled to conformal supergravity. They provide an elegant derivation of a number of remarkable properties exhibited by the scattering amplitudes of these theories, giving important results for super-Yang-Mills tree amplitudes in particular [4, 5]. However, in these theories the conformal supergravity is inextricably mixed in with the gauge theory so that, in computations of gauge theory loop amplitudes, conformal supergravity modes propagate on internal lines [6]. There appears to be no decoupling limit giving pure super-Yang-Mills amplitudes, and although there has been considerable progress in studying the twistor-space Yang-Mills amplitudes at loops (see e. g. [7] and references therein), the results do not follow from the known twistor strings. A twistor string that gave Einstein supergravity coupled to super-Yang-Mills would be much more useful, and might be expected to have a limit in which the gravity could be decoupled to give pure gauge theory amplitudes. (By Einstein supergravity, we mean a supergravity with 2nd order field equations for the graviton, in contrast to conformal supergravity which has 4th order field equations.) Indeed, it is known that MHV amplitudes for Einstein (super) gravity [8] have an elegant formulation in twistor space [1, 9, 10, 11], and it is natural to ask whether these can have a twistor string origin. In this paper, we propose new twistor string models which give Einstein (super) gravity coupled to Yang-Mills.

The new theories are constructed by gauging certain symmetries of the Berkovits twistor string. The structure of the theory is very similar to that of the Berkovits model, but the gauging adds new terms to the BRST operator so that the vertex operators have new constraints and gauge invariances. In this paper we construct a family of theories for which the world-sheet anomalies cancel, and find their spectra. We postpone a detailed discussion of the interactions and scattering amplitudes to a subsequent paper, but do show that there is a non-trivial cubic graviton interaction for two of the theories, so that at least these theories are non-trivial. The theories of [1, 2, 3] give target space theories that are anomalous in general, with the anomalies canceling only for 4-dimensional gauge groups. It is to be expected that these anomalies should arise from inconsistencies in the corresponding twistor string model, but the mechanism for this is as yet unknown [6]. If there are such inconsistencies in the Berkovits twistor string that only cancel in special cases, there should be similar problems for our theories, and this may rule out some of the models we construct, or restrict the choice of gauge group.

We find two classes of anomaly-free theories. The first is formulated in $N = 4$ super-twistor space. Gauging a symmetry of the string theory generated by one bosonic and four fermionic currents gives a theory with the spectrum of $N = 4$ Einstein supergravity coupled to $N = 4$ super-Yang-Mills with arbitrary gauge group, while gauging a single bosonic current gives a theory with the spectrum of $N = 8$ Einstein supergravity, provided the number of $N = 4$ vector multiplets is six. In the Yang-Mills sector, the string theory is identical to that of Berkovits, so that it gives the same tree level Yang-Mills amplitudes. Both theories have the MHV 3-graviton interaction (with two positive helicity gravitons and one negative helicity one) of Einstein gravity.

The gauging introduces new ghost sectors into our twistor string theories, and in the second family of string theories, gauging different numbers of bosonic and fermionic symmetries allows anomalies to be cancelled against ghost contributions for strings in twistor spaces with 3 complex bosonic dimensions and any number N of complex fermionic dimensions, corresponding to theories in four-dimensional space-time with N supersymmetries. We then find the spectrum of states arising from ghost-independent vertex operators. For $N = 0$, we find a theory with the bosonic spectrum of self-dual gravity together with self-dual Yang-Mills and a scalar, and for $N < 4$ we find supersymmetric versions of this self-dual theory. As twistor theory has been particularly successful in formulating self-dual gravity [12] and self-dual Yang-Mills [13], it seems fitting that these theories should emerge from twistor string theory. With $N = 4$, we find a theory whose spectrum is that of $N = 4$ Einstein supergravity coupled to $N = 4$ super-Yang-Mills with arbitrary gauge group. It is intriguing that some of the theories we find have similar structure to $\mathcal{N} = 2$ string theories [14].

One of the achievements of twistor theory was to give a general solution of the self-dual and conformally self-dual Einstein equations. Penrose's non-linear graviton construction [12] provides an equivalence between 4-dimensional space-times $\mathcal{M}$ with self-dual Weyl curvature and certain complex 3-folds, the curved projective twistor spaces $P\mathcal{T}$, providing an implicit construction of general conformally self-dual space-times. For flat space-time, the corresponding twistor space $\mathbb{PT}$ is $\mathbb{CP}^3$. In Euclidean signature, there is an elegant realisation of the twistor space $P\mathcal{T}$ corresponding to a space $\mathcal{M}$ with signature $++++$ as the projective primed spin-bundle over $\mathcal{M}$, the bundle of primed spinors $\pi_{A'}$ on $\mathcal{M}$ identified under complex scalings $\pi_{A'} \sim t\pi_{A'}$, so that it is a $\mathbb{CP}^1$ bundle over $\mathcal{M}$ [15]. For other signatures, the construction of curved twistor space $P\mathcal{T}$ is not quite so straightforward, and will be reviewed in section 3.

New twistor spaces, and hence new conformally self-dual space-times, can be constructed by deforming the complex structure of a suitable region of a given twistor space $P\mathcal{T}_0$ (such as a neighbourhood $\mathbb{PT}_0$ of a projective line in $\mathbb{CP}^3$). The complex structure of a space can be specified by a (1,1) tensor field J satisfying $J^2 = -1$ that is integrable, so that the Nijenhuis tensor $N(J)$ vanishes. Given the complex structure J_0 of $\mathbb{PT}_0$, one can construct a new complex structure

$$J = J_0 + \lambda J_1 + \lambda^2 J_2 + \dots \qquad (1.1)$$

as a power series in a parameter λ, imposing the conditions $J^2 = -1$ and $N(J) = 0$. In holomorphic coordinates for J_0, $J^2 = -1$ implies that J_1 decomposes into a section j of $\Lambda^{(0,1)} \otimes T^{(1,0)}$ and its complex conjugate on $P\mathcal{T}_0$. The linearised condition $N(J) = 0$ is equivalent to $\bar{\partial} j = 0$. Furthermore, j represents an infinitesimal diffeomorphism if $j = \bar{\partial}\alpha$ for some section α of $T^{(1,0)}$. Thus a deformation corresponds to an element of the first Dolbeault cohomology group on twistor space with values in the holomorphic tangent bundle. Moreover, the linearised deformations J_1 are unobstructed to all orders and determine the tangent space to the moduli space of complex structures if certain second cohomology groups vanish, which they do when $\mathbb{PT}_0$ is a small enough neighbourhood of a line.

Witten's twistor string [1] is a topological string theory on (super-)twistor space and has physical states corresponding to deformations of the complex structure of the target space $P\mathcal{T}_0$. The corresponding vertex operator constructed from J_1 is physical precisely when j represents an element of $H^1_{\bar{\partial}}(\mathbb{PT}_0)$. The twistor space string field theory action for Witten's theory has a term with a Lagrange multiplier imposing $N(J) = 0$ [6] and the corresponding term in the space-time action is

$$\int d^4x \sqrt{g} U^{ABCD} W_{ABCD}, \tag{1.2}$$

where W_{ABCD} is the anti-self-dual part of the Weyl tensor. If this were the complete gravity action, then U^{ABCD} would be a Lagrange multiplier imposing the vanishing of W_{ABCD}, so that the Weyl tensor would be self-dual. However, in addition there is a term $\int U^2$, which arises from D-instantons in Witten's topological B-model [6, 30]. Integrating out U gives the conformal gravity action $\int W^2$.

In split $++--$ space-time signature, there is a three real dimensional submanifold $P\mathcal{T}_{\mathbb{R}}$ of complex twistor space $P\mathcal{T}$. In the flat case, $\mathbb{PT}_{\mathbb{R}} \subset \mathbb{PT}$ is the standard embedding of $\mathbb{RP}^3 \subset \mathbb{CP}^3$, and the information about deformations of the complex structure is encoded in an analytic vector field f on $P\mathcal{T}_{\mathbb{R}}$. It was shown in [16] that conformally self-dual space-times in split signature can also be constructed by deforming the embedding of $\mathbb{PT}_{\mathbb{R}}$ to some $P\mathcal{T}_{\mathbb{R}}$ in $\mathbb{PT}$ instead of deforming the complex structure of some region in $\mathbb{PT}$ to give $P\mathcal{T}$. The deformations of the anti-self-dual conformal structure correspond to deformations of the embedding of $P\mathcal{T}_{\mathbb{R}}$ in $\mathbb{CP}^3$ and are determined at first order by a vector field f on $P\mathcal{T}_{\mathbb{R}}$, or more precisely by a section of the normal bundle to $P\mathcal{T}_{\mathbb{R}} \subset \mathbb{CP}^3$.

Berkovits' twistor string [2, 3] has open strings with boundaries on the real twistor space $\mathbb{PT}_{\mathbb{R}}$, and (conformal) supergravity physical states are created by an open string vertex operator constructed from a vector field f defined on $\mathbb{PT}_{\mathbb{R}}$, corresponding to deformations of the embedding of $\mathbb{PT}_{\mathbb{R}}$ in $\mathbb{PT}$.

There is an important variant of the Penrose construction that applies to the Ricci-flat case (in fact, this is the original non-linear graviton construction). A special case of the conformally self-dual spaces are those that are Ricci-flat, so that the full Riemann tensor is self-dual. The corresponding twistor spaces $P\mathcal{T}$ then have extra structure, as will be discussed in section 3. In particular, they have a fibration $P\mathcal{T} \to \mathbb{CP}^1$. The holomorphic one-form on $\mathbb{CP}^1$ pulls back to give a holomorphic one-form on $P\mathcal{T}$ which takes the form $I_{\alpha\beta}Z^\alpha dZ^\beta$ in homogeneous coordinates Z^α, for some $I_{\alpha\beta}(Z) = -I_{\beta\alpha}$ (Z) (which are the components of a closed 2-form on the non-projective twistor space $\mathcal{T}$). The dual bi-vector $I^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta}$ defines a Poisson structure and is called the *infinity twistor*.

Consider for example flat space-time $\mathcal{M} = \mathbb{R}^4$ in signature $++++$, which has conformal compactification S^4. The twistor space is $\mathbb{CP}^3$, which is a $\mathbb{CP}^1$ bundle over S^4: it is the projective primed spin bundle over the conformal compactification of $\mathcal{M}$. If conformal invariance is broken, then there is a distinguished point at infinity. Removing the point at infinity from S^4 to leave $\mathbb{R}^4$ amounts to removing the fibre over this point in the twistor space, leaving $\mathbb{PT}' = \mathbb{CP}^3 - \mathbb{CP}^1$, the projective primed spin bundle over $\mathbb{R}^4$. However, $\mathbb{PT}'$ is also a bundle over $\mathbb{CP}^1$ with fibres $\mathbb{C}^2$, the planes through the missing $\mathbb{CP}^1$. A projective line joining two points X^α and Y^β in twistor space can be represented by a bivector $X^{[\alpha}Y^{\beta]}$, and the infinity twistor is the bivector corresponding to the projective line over the point at infinity in S^4. Choosing a point at infinity, or an infinity twistor, breaks the conformal group down to the Poincaré group. For Minkowski space, the infinity twistor determines the light-cone at infinity in the conformal compactification. A similar situation obtains more generally: the infinity twistor breaks conformal invariance.

Self-dual space-times are obtained by seeking deformations of the complex structure of twistor space as before, but now Ricci-flatness in space-time places further restrictions on the deformations allowed. In the split signature picture, the vector field f on $\mathbb{RP}^3$ is required to be a Hamiltonian vector field with respect to the infinity twistor, so that in homogeneous coordinates we can write

$$f^\alpha = I^{\alpha\beta}\frac{\partial h}{\partial Z^\beta} \tag{1.3}$$

for some function h of homogeneity degree 2 on $\mathbb{RP}^3$. In the linearised theory, such a function h corresponds to a positive-helicity graviton in space-time via the Penrose transform, and the non-linear graviton construction gives the generalisation of this to the non-linear theory. In the Dolbeault picture, the tensor J_1 is given by a $(0,1)$-form j^α of the form

$$j^\alpha = I^{\alpha\beta}\frac{\partial h}{\partial Z^\beta} \tag{1.4}$$

where h is a $(0,1)$-form representing an element of $H^1(\mathbb{PT}', \mathcal{O}(2))$.

This suggests seeking a twistor string that is a modification of either the Berkovits or the Witten string theories which introduces explicit dependence on the infinity twistor, such that there are extra constraints on the vertex operators imposing that the deformation of the complex structure be of the form (1.3) or (1.4). Then the leading term in the action analogous to (1.2) should have a multiplier imposing self-duality, not just conformal self-duality, and further terms quadratic in the multiplier (from instantons in Witten's approach) could then give Einstein gravity. A formulation of Einstein gravity of just this form was discussed in [17].

We will present such a modification of the Berkovits twistor string here. The key ingredient is that the one-form corresponding to the infinity twistor is used to construct a current, and the corresponding symmetry is gauged. The resulting gauge-fixed theory is given by the Berkovits twistor string theory plus some extra ghosts, and there are extra terms in the BRST operator involving these ghosts. The dynamics and vertex operators are of the same form as for the Berkovits twistor string, but the extra terms in the BRST charge give extra constraints and gauge invariances for the vertex operators, including the constraint (1.3) that takes us from conformal gravity to Einstein gravity. Variants of the theory are obtained by also gauging some fermionic currents. The case of $N = 4$ is particularly interesting as in that case the spectrum is parity invariant and is that of $N = 4$ Einstein supergravity (together with $N = 4$ Yang-Mills). We expect that similar refinements of Witten's twistor string should also be possible.

A key difference between our models and the twistor strings of refs. [1, 2, 3] is that space-time conformal invariance is broken. The magnitude of the infinity twistor defines a length scale in space-time, and so determines the gravitational coupling κ. The theory has two independent coupling constants: the gravitational coupling κ, determined by the magnitude of the infinity twistor, and the Yang-Mills coupling g_{YM}, arising as in [6]. Then for the $N = 4$ theory there is a limit in which $\kappa \to 0$ and supergravity decouples from the super-Yang-Mills, so that, if the twistor string theory is consistent at loops, it will have a decoupling limit that gives $N = 4$ super-Yang-Mills loop amplitudes.

The plan of the paper is as follows. In section 2, relevant aspects of twistor theory are reviewed, including special features of different space-time signatures, super-twistor space, the Penrose transform and the infinity twistor. In section 3, the non-linear graviton construction of Penrose is reviewed, and its generalisations to bosonic spaces of split signature and to super-twistor spaces are given. In particular, we adapt [16] to the Ricci-flat case. In section 4, the Berkovits twistor string theory is reviewed. In section 5, the gauging of symmetries of so-called beta-gamma systems is studied. In section 6, this analysis is applied to the Berkovits twistor string, gauging various symmetry groups of the theory and calculating the world-sheet anomalies. In section 7, the conditions for anomaly cancellation are solved, and a number of anomaly-free bosonic and supersymmetric models is found. The spectra of these models are found in section 8, where they are compared to known (super)gravity theories. In section 9, we give a sample calculation of a nontrivial three point function in the theory giving $N = 4$ supergravity coupled to $N = 4$ super-Yang-Mills. Finally, in section 10 we discuss our results and the space-time theories that might emerge from our twistor strings.

Our conventions are those of Penrose, see for example [18], apart from our choice of sign of the helicity, which is opposite to that of Penrose.

2 Twistor space and the infinity twistor

2.1 Twistor space for flat complex space-time

We start by considering complexified flat space-time $\mathbb{C}^4$, and postpone the discussion of the real slices giving space-times of signature $(4,0)$, $(3,1)$ or $(2,2)$ to the next subsection. The twistor space $\mathbb{T}$ corresponding to flat complex space-time is also $\mathbb{C}^4$, with coordinates Z^α, $\alpha = 0,1,2,3$. We also use Z^α as homogeneous coordinates on projective twistor space $\mathbb{PT} = \mathbb{CP}^3$, which is obtained by identifying $Z^\alpha \sim \lambda Z^\alpha$ for complex $\lambda \neq 0$. The Z^α transform as a **4** under the complexified conformal group[4] $SL(4,\mathbb{C})$ and decompose into two-component spinors under the complexified Lorentz group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$:

$$Z^\alpha = (\omega^A, \pi_{A'})\,,$$

where $A = 0,1$ and $A' = 0', 1'$ are spinor indices for the two $SL(2,\mathbb{C})$ factors. Spinor indices are raised and lowered with $\epsilon_{AB} = \epsilon_{[AB]}$, $\epsilon_{01} = 1$, and its dual and primed counterparts.

Complex flat space-time $\mathbb{CM}$ is $\mathbb{C}^4$ with complex coordinates $x^{AA'}$ and complex-valued metric

$$ds^2 = \epsilon_{AB}\epsilon_{A'B'}dx^{AA'}dx^{BB'}. \tag{2.1}$$

A point $x^{AA'}$ in $\mathbb{CM}$ corresponds to a two dimensional linear subspace of $\mathbb{T}$ given by the incidence relation

$$\omega^A = x^{AA'}\pi_{A'}\,. \tag{2.2}$$

In the projective twistor space $\mathbb{PT}$, these two-dimensional subspaces determine projective lines (i.e. $\mathbb{CP}^1$'s), so that each point $x^{AA'}$ in $\mathbb{CM}$ corresponds to a $\mathbb{CP}^1$ in $\mathbb{PT}$.

However, some two-dimensional subspaces in $\mathbb{T}$ cannot be expressed in this way, and these correspond to 'points at infinity' in the conformal compactification $\widetilde{\mathbb{CM}}$ of $\mathbb{CM}$. The conformal compactification is obtained by adding a light cone at infinity $\mathscr{I}$ to $\mathbb{CM}$ [18]. The vertex i of the lightcone $\mathscr{I}$ at infinity corresponds to the subspace $\pi_{A'} = 0$, and other points of $\mathscr{I}$ correspond to two-dimensional subspaces lying in the three-spaces $\alpha^{A'}\pi_{A'} = 0$ in which one linear combination of the two components of π vanishes. There

[4]Strictly speaking, the complexified conformal group is $PGL(4,\mathbb{C}) = SL(4,\mathbb{C})/\mathbb{Z}_4$, as the centre $\mathbb{Z}_4$ acts trivially, but this $\mathbb{Z}_4$ will not play a role in this paper.

is then a one-to-one correspondence between points in compactified space-time $\widetilde{\mathbb{CM}}$ and two dimensional linear subspaces of $\mathbb{T}$, or projective lines in $\mathbb{CP}^3$.

A two dimensional linear subspace of $\mathbb{T}$ is determined by two vectors X^α, Y^α that lie in it, or equivalently by a simple bi-vector, that is a bi-vector $P^{\alpha\beta} = -P^{\beta\alpha}$ satisfying the simplicity condition

$$P^{[\alpha\beta}P^{\gamma\delta]} = 0 \tag{2.3}$$

which implies $P^{\alpha\beta} = X^{[\alpha}Y^{\beta]}$ for some X, Y. Then a point in compactified space-time corresponds to the linear subspace in $\mathbb{T}$ determined by a simple bi-vector $P^{\alpha\beta}$. As $P^{\alpha\beta}$ and $\lambda P^{\alpha\beta}$ ($\lambda \neq 0$) determine the same linear space, we are only interested in equivalence classes under scaling, so that the 6-dimensional space of bivectors $P^{\alpha\beta}$ is reduced to the space $\mathbb{CP}^5$ of scaling equivalence classes, and the simplicity condition selects a quadric in $\mathbb{CP}^5$. In this way, the conformal compactification $\widetilde{\mathbb{CM}}$ is represented as a complex 4-quadric in $\mathbb{CP}^5$ [18]. Instead of using a simple bi-vector, one can equivalently use the simple 2-form $P_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}P^{\gamma\delta}$ in $\mathbb{T}$ (where a simple 2-form is one satisfying $P_{[\alpha\beta}P_{\gamma\delta]} = 0$).

A point Z^α in twistor space corresponds to an 'α-plane' in $\mathbb{CM}$, which is a totally null self-dual 2-plane. This can be seen by regarding the incidence relation (2.2) as a condition on $x^{AA'}$ for fixed Z^α, the general solution of which is $x^{AA'} = x_0^{AA'} + \lambda^A\pi^{A'}$; this describes a 2-plane parametrised by λ^A. The two-form orthogonal to the two-plane is given by the symmetric bi-spinor $\pi_{A'}\pi_{B'}$, and is null and self-dual. In this way, the twistor space $\mathbb{PT}$ can be defined as the space of α-planes in $\mathbb{CM}$, and this formulation is useful as it generalises to curved space-times.

A standard tool for studying twistor correspondences is the double fibration of the bundle of primed spinors $\mathbb{S}$ over space-time and over twistor space

$$\begin{array}{ccc} & \mathbb{S} & \\ q \swarrow & & \searrow r \\ \mathbb{CM} & & \mathbb{T} \end{array} \tag{2.4}$$

Using coordinates $(x, \pi_{A'})$ on the spin bundle, q is the projection $q(x^{AA'}, \pi_{B'}) = x^{AA'}$, whose fibre at $x^{AA'}$ is the spin space at $x^{AA'}$. The other projection r takes $(x^{AA'}, \pi_{A'}) \in \mathbb{S}$ to the point $(\omega^{A'}, \pi_{B'}) = (x^{AA'}\pi_{A'}, \pi_{B'}) \in \mathbb{T}$. The fibre at $Z^\alpha = (x^{AA'}\pi_{A'}, \pi_{B'})$ is the set of all $(x, \pi_{A'}) \in \mathbb{S}$ with $Z^\alpha = (x^{AA'}\pi_{A'}, \pi_{B'})$, which is the 2-surface $(x^{AA'} + \lambda^A\pi^{A'}, \pi_{A'})$ parameterised by λ^A; this surface is the lift to the spin bundle of the α-plane corresponding to Z^α with tangent spinor $\pi_{A'}$. There is clearly a corresponding double fibration of the projective spin bundle $\mathbb{PS}$, but now over projective twistor space $\mathbb{PT}$. The Penrose transform can be understood in terms of this double fibration as pulling back objects from twistor space using r^* and then pushing them down to space-time using q_*.

The space $\mathbb{T}$ has various canonical structures. The space $\mathbb{T} - 0$ has a natural fibration over $\mathbb{PT}$ and the Euler homogeneity operator

$$\Upsilon = Z^\alpha \frac{\partial}{\partial Z^\alpha} \tag{2.5}$$

is a vector field which points up the fibres of the line bundle $\{\mathbb{T} - 0\} \to \mathbb{PT}$. We will represent objects on $\mathbb{PT}$ by their pull-backs to $\mathbb{T}$. Thus functions on $\mathbb{PT}$ are given by functions on $\mathbb{T}$ that are annihilated by Υ. The line bundle $\mathcal{O}(n)$ over $\mathbb{PT}$ has sections that are functions on $\mathbb{T}$ that are homogeneous of degree n, i. e. $\Upsilon f = nf$. Similarly, a form α on $\mathbb{PT}$ with values in $\mathcal{O}(n)$ pulls back to a form on $\mathbb{T}$ (which we will also denote by α) satisfying

$$\iota(\Upsilon)\alpha = \iota(\bar{\Upsilon})\alpha = 0, \qquad \mathcal{L}_{\bar{\Upsilon}}\alpha = 0, \qquad \mathcal{L}_{\Upsilon}\alpha = n\alpha, \tag{2.6}$$

where $\iota(\Upsilon)$ denotes the interior product (i. e. contraction) with Υ. We will denote the space of p-forms on $\mathbb{PT}$ with values in $\mathcal{O}(n)$ as $\Lambda^p(n)$.

We define the 3-form

$$\Omega = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta} Z^\alpha \mathrm{d}Z^\beta \wedge \mathrm{d}Z^\gamma \wedge \mathrm{d}Z^\delta\,, \qquad \epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\alpha\beta\gamma\delta]}\,, \qquad \epsilon_{0123} = 1\,. \tag{2.7}$$

This annihilates Υ (i.e. $\iota(\Upsilon)\Omega = 0$), but it does not descend to $\mathbb{PT}$, since it has homogeneity degree 4. However, it does so descend when multiplied by functions that are of homogeneity degree -4, and gives an isomorphism $\Lambda^{(3,0)}(\mathbb{PT}) \simeq \mathcal{O}(-4)$ (or alternatively defines a holomorphic section of $\Lambda^{(3,0)}(4)$). This also determines the holomorphic volume form $\mathrm{d}\Omega$ on $\mathbb{T}$:

$$d\Omega = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta} \mathrm{d}Z^\alpha \wedge \mathrm{d}Z^\beta \wedge \mathrm{d}Z^\gamma \wedge \mathrm{d}Z^\delta\,. \tag{2.8}$$

2.2 The infinity twistor

The conformal compactification $\widetilde{\mathbb{CM}}$ of space-time is invariant under the full conformal group. In order to break conformal invariance to conformal Poincaré invariance (i. e. the Poincaré group together with dilations), we choose a point in $\widetilde{\mathbb{CM}}$ to be the point i at infinity, and the complexified conformal Poincaré group is the subgroup of $SL(4,\mathbb{C})$ preserving this point. In particular, with a further choice of an origin 0 in $\widetilde{\mathbb{CM}}$, this chooses a Lorentz subgroup $SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \subset SL(4,\mathbb{C})$, and different choices of $i, 0$ lead to different conjugate Lorentz subgroups.

The point i at infinity in $\widetilde{\mathbb{CM}}$ corresponds to a bi-vector $I^{\alpha\beta}$ up to scale which is simple,

$$I^{[\alpha\beta}I^{\gamma\delta]} = 0, \tag{2.9}$$

and which is called the *infinity twistor*. The infinity twistor can also be represented by the 2-form τ on $\mathbb{T}$ defined by

$$\tau = \frac{1}{2} I_{\alpha\beta} \mathrm{d}Z^\alpha \wedge \mathrm{d}Z^\beta,$$

where $I^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta}$. Choosing a point 0 in $\widetilde{\mathbb{CM}}$ to be the origin $x^\mu = 0$ corresponds to choosing a second two-form μ (dual to a simple bi-vector), and this can be chosen so that [5]

$$d\Omega = 4\mu \wedge \tau. \tag{2.10}$$

The choice of $i, 0$ in $\widetilde{\mathbb{CM}}$ selects an $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ subgroup of $SL(4,\mathbb{C})$ that preserves μ and τ separately, and this is the double cover of the rotation group $SO(4,\mathbb{C})$ preserving the origin $x = 0$ and the point at infinity in $\widetilde{\mathbb{CM}}$. It is natural to use 2-component spinor notation for this $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ subgroup, with $Z^\alpha = (\omega^A, \pi_{A'})$. Then

$$\tau = \frac{R}{2}\epsilon^{A'B'} \mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi_{B'}, \qquad \mu = \frac{1}{2R}\epsilon_{AB} \mathrm{d}\omega^A \wedge \mathrm{d}\omega^B \tag{2.11}$$

for some R. The corresponding space-time metric is

$$ds^2 = R^2 \epsilon_{AB}\epsilon_{A'B'} dx^{AA'} dx^{BB'}, \tag{2.12}$$

so that scaling the infinity twistor by R leads to a conformal scaling of the metric by R^2, and the scale of the infinity twistor determines the length scale in space-time. For the rest of the paper, we will set $R = 1$.

The infinity twistor determines the projective line I in $\mathbb{PT}$ corresponding to i by

$$Z^\alpha I_{\alpha\beta} = 0,$$

which in adapted coordinates is the line $\pi_{A'} = 0$, while the origin $x = 0$ corresponds to the line $\mu^A = 0$. Removing the light-cone at infinity $\mathscr{I}$ from $\widetilde{\mathbb{CM}}$ leaves complex space-time $\mathbb{CM}$ while removing the line I in $\mathbb{PT}$ corresponding to the infinity twistor gives the twistor space $\mathbb{PT}' = \mathbb{PT} - I$. As I is the $\mathbb{CP}^1 \subset \mathbb{PT}$ given by $\pi_{A'} = 0$, $\mathbb{PT}'$ consists of points $Z^\alpha = (\omega^A, \pi_{A'})$ in which at least one component of π is non-zero. For non-conformal theories, it is natural to use $\mathbb{PT}'$, and this (and its curved generalisations) is the twistor space that will be used in our constructions.

The infinity twistor determines a projection $\mathbb{T} \to \mathbb{S}_{A'}$ to $\mathbb{S}_{A'}$, the dual primed spinor space, given by $Z^\alpha = (\omega^A, \pi_{A'}) \to \pi_{A'}$. Projectively, this projection determines a fibration $\mathbb{PT}' \to \mathbb{CP}^1$. The infinity twistor $I^{\alpha\beta}$ defines a Poisson structure of homogeneity -2 by

$$\{f, g\}_I := I^{\alpha\beta} \frac{\partial f}{\partial Z^\alpha} \frac{\partial g}{\partial Z^\beta} = \epsilon^{AB} \frac{\partial f}{\partial \omega^A} \frac{\partial g}{\omega^B}\,.$$

[5] If no choice of origin is made, the two-form μ is defined by (2.10) up to the addition of multiples of $\mathrm{d}\pi_{A'}$.

We further define the one-form

$$k = I_{\alpha\beta} Z^\alpha dZ^\beta = \epsilon^{A'B'} \pi_{A'} \mathrm{d}\pi_{B'}, \tag{2.13}$$

for which $\tau = \frac{1}{2}dk = \frac{1}{2}\epsilon^{A'B'}\mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi_{B'}$; k is the pull-back of a holomorphic one-form on $\mathbb{CP}^1$ with weight 2 and will play a central role in our construction.

2.3 Twistor spaces for real space-times

We can choose a real slice $\mathbb{M} \subset \mathbb{CM}$ in such a way that the metric has signature $(p, 4-p)$ for $p = 0, 1, 2$, and the subgroup of the complexified conformal group that preserves the real slice is a real form of $SL(4, \mathbb{C})$. For Euclidean signature, Lorentzian signature, or split signature $(2, 2)$, the real conformal groups are $SU^*(4) = SL(2, \mathbb{H}) = Spin(5, 1)$, $SU(2, 2) = Spin(4, 2)$ and $SL(4, \mathbb{R}) = Spin(3, 3)$ respectively, where $\mathbb{H}$ denotes the quaternions.[6]

The conformal group acts on the twistor space $\mathbb{T} = \mathbb{C}^4$, with Z^α transforming as a complex Weyl spinor for $SO(6, \mathbb{C})$. For split signature, this representation is reducible: it decomposes into the direct sum of two copies of the real Majorana-Weyl representations of $Spin(3, 3)$, and it is possible to impose a reality condition on the twistors, giving the real twistor space $\mathbb{RP}^3$. However, for the other two signatures, the Weyl representation is irreducible so that twistors are necessarily complex.

We can characterise the real slices $\mathbb{M}$ of $\mathbb{CM}$ as fixed points of a complex conjugation $\tau : \mathbb{CM} \to \mathbb{CM}$ which, in local coordinates that are real on the appropriate real slice, are given by standard complex conjugation, $\tau(x^\mu) = (x^\mu)^*$. A point x^μ in $\mathbb{CM}$ is represented by a complex matrix $x^{AB'}$. The different conjugations can be expressed on this matrix as follows. For space-time of split signature, $\tau(x^{AB'}) = (x^{AB'})^*$ is the entry-by-entry complex conjugate, for Lorentzian signature $\tau(x^{AB'})$ is the hermitian conjugate $\tau(x) = x^\dagger$, while for Euclidean signature $\tau(x^{AB'}) = \hat{x}^{AB'}$, where $\hat{x} = \epsilon x^* \epsilon$ with ϵ the real anti-symmetric 2×2 matrix (given in terms of the Pauli matrix σ_2 by $\epsilon = i\sigma_2$).[7]

Complex conjugation $x \to \tau x$ in $\mathbb{CM}$ leads to a map on twistor space. In split signature and in Euclidean signature, τ sends α-planes to α-planes, but in Lorentz signature it sends α-planes to β-planes where β-planes are totally null 2-planes in $\mathbb{CM}$ that are anti-self-dual. The space of such β-planes together with tangent spinor λ_A, is dual twistor space $\mathbb{T}^*$ with coordinates $W_\alpha = (\lambda_A, \mu^{A'})$; a point in $\mathbb{T}^*$ corresponds to the β-plane in $\mathbb{CM}$ defined by the dual incidence relation $\mu^{A'} = x^{AA'}\lambda_A$. The complex conjugation τ on $\mathbb{CM}$ therefore induces a complex conjugation $\tau : \mathbb{T} \to \mathbb{T}$ in split signature and Euclidean signature, but in Lorentz signature, it determines an anti-holomorphic map $\tau : \mathbb{T} \to \mathbb{T}^*$.

[6] Again, we are ignoring factors of $\mathbb{Z}_4$ here.

[7] Note that in this definition, neither the map $x \to \bar{x}$ nor $x \to \epsilon x \epsilon$ are invariant under the $SO(4)$ rotation group, only the composition $x \to \epsilon \bar{x} \epsilon$ is.

We have the complex conjugate twistor space $\bar{\mathbb{T}}$ (i.e. $\mathbb{T}$ with the opposite complex structure) with coordinates $\bar{Z}^{\bar{\alpha}} = (Z^\alpha)^*$ on twistor space, and their counterparts on dual twistor space $\mathbb{T}^\dagger$ with coordinates $\bar{W}_{\bar{\alpha}} = (W_\alpha)^*$. For the real and split signature complex structure, τ is an isomorphism from $\bar{\mathbb{T}}$ to $\mathbb{T}$ and in the Lorentzian case it is a natural map from $\bar{\mathbb{T}}$ to $\mathbb{T}^*$, and this can be used to express conjugate twistors in $\bar{\mathbb{T}}$ in terms of twistors in $\mathbb{T}$ or $\mathbb{T}^*$, so that conjugate twistor indices are never needed explicitly. We now describe features of twistor geometry appropriate to each signature in more detail.

2.3.1 Lorentzian signature

In the case of Lorentzian signature, the conformal group $SU(2,2)$ preserves a Hermitian metric $\Sigma_{\alpha\bar{\beta}}$, and this defines the map $\tau : \bar{\mathbb{T}} \to \mathbb{T}^*$ under which $\bar{Z}^{\bar{\alpha}} = (Z^\alpha)^* \to \Sigma_{\alpha\bar{\beta}}\bar{Z}^{\bar{\beta}}$, so that each conjugate twistor can be identified with a dual twistor. Complex conjugation on $\mathbb{CM}$ leads to an anti-holomorphic map $Z^\alpha \to \bar{Z}_\alpha = \Sigma_{\alpha\bar{\beta}}\bar{Z}^{\bar{\beta}}$ from $\mathbb{T} \to \mathbb{T}^*$. The real Minkowski space-time $\mathbb{M}$ is the subspace of $\mathbb{CM}$ in which $x^{AB'}$ is Hermitian and is preserved by this conjugation. This is the standard case, discussed in detail in e. g. [18].

2.3.2 Split signature

For extensive discussions of the twistor correspondences in split signature see [19, 16]. Here we give a summary of the main ideas.

For split signature, the real space-time $\mathbb{M}$ is the subspace of $\mathbb{CM}$ with $x^{AB'}$ real. The ordinary complex conjugation on $\mathbb{CM}$ that preserves $\mathbb{M}$ is represented by the ordinary component-by-component complex conjugation on $\mathbb{T}$, viz. $Z^\alpha \to (Z^\alpha)^*$, that fixes the real slice $\mathbb{T}_\mathbb{R} = \mathbb{R}^4 \subset \mathbb{C}^4 = \mathbb{T}$ and hence $\mathbb{PT}_\mathbb{R} = \mathbb{RP}^3 \subset \mathbb{PT}$. Points of this real slice correspond to totally real α-planes in $\mathbb{M}$ and there is a totally real version of the twistor correspondence in which points in $\widetilde{\mathbb{M}}$ correspond to real projective lines (i.e. $\mathbb{RP}^1$s) in $\mathbb{PT}_\mathbb{R}$ via the incidence relation $\omega^A = x^{AA'}\pi_{A'}$ where now ω^A, $\pi_{A'}$ and $x^{AA'}$ are all real. Here $\widetilde{\mathbb{M}}$ is the conformal compactification of $\mathbb{M}$, which is $\widetilde{\mathbb{M}} = S^2 \times S^2/\mathbb{Z}_2$.

In order to use deformed twistor correspondences in split signature, we will also need to use the correspondence between $\mathbb{M}$ and the complex twistor space $\mathbb{PT}$. Each point $x \in \mathbb{M}$ corresponds to a complex line $L_x = \mathbb{CP}^1$ in $\mathbb{PT}$ that intersects the real slice $\mathbb{PT}_\mathbb{R}$ in a real line $L_{\mathbb{R}x} = \mathbb{RP}^1$. This real line divides L_x into two discs $D_x^\pm$, each with boundary $L_{\mathbb{R}x} \subset \mathbb{PT}_\mathbb{R}$. The space of such discs naturally defines a double cover $\widetilde{\widetilde{\mathbb{M}}}$ of conformally compactified Minkowski space $\widetilde{\mathbb{M}}$ (which is the space of all $L_{\mathbb{R}x} \subset \mathbb{PT}_\mathbb{R}$). In fact $\widetilde{\widetilde{\mathbb{M}}} = S^2 \times S^2$ with the conformal structure that is determined by the split signature product metric

$$g = \pi_1^* h - \pi_2^* h,$$

where h is the standard round metric on S^2 and $\pi_1, \pi_2 : S^2 \times S^2 \to S^2$ are the two factor projections. The conformal compactification $\widetilde{\mathbb{M}} = S^2 \times S^2/\mathbb{Z}_2$ is obtained from the double cover $\widetilde{\widetilde{\mathbb{M}}}$ by identifying under the $\mathbb{Z}_2$ that acts as the joint antipodal map on both S^2 factors.

$\widetilde{\widetilde{\mathbb{M}}}$ can be thought of as two copies $\mathbb{M}^\pm$ of $\mathbb{M}$ glued together across the double cover of the lightcone at infinity $\mathscr{I}$. With the choice of the infinity twistor, we have the fibration $\mathbb{PT}' = \mathbb{PT} - I \to \mathbb{CP}^1$ as above. The condition that $i\pi_{A'}\bar{\pi}^{A'}$ be positive, negative or zero defines $\mathbb{PT}_\pm$ and $\mathbb{PT}_0$. The holomorphic discs in $\mathbb{PT}_\pm$ project to $\pm i\pi_{A'}\bar{\pi}^{A'} > 0$ in $\mathbb{CP}^1$ and correspond respectively to points of $\mathbb{M}^\pm$, whereas the holomorphic discs in $\mathbb{PT}_0$ correspond to points of the double cover $\widetilde{\mathscr{I}}$ of $\mathscr{I}$. This will be important later for the Berkovits string, where the open string world-sheets are holomorphic discs. The moduli space of discs in twistor space gives $\widetilde{\widetilde{\mathbb{M}}}$ with two copies of space-time $\mathbb{M}$, and to get just one copy, the theory must be restricted to one in which the world-sheets are discs in one half of twistor space, say in $\mathbb{PT}_+$.

2.3.3 Euclidean signature

The anti-linear map $\tau : \mathbb{T} \to \mathbb{T}$ is given by the conjugation $Z^\alpha \to \hat{Z}^\alpha$ where, if $Z^\alpha = (\omega^A, \pi_{A'})$, then $\hat{Z}^\alpha = (\hat{\omega}^A, \hat{\pi}_{A'})$, with $\hat{\omega}^A = (\bar{\omega}^1, -\bar{\omega}^0)$ and $\hat{\pi}_{A'} = (\bar{\pi}_{1'}, -\bar{\pi}_{0'})$. The conjugation extends to multi-spinors and the real Euclidean space-time $\mathbb{M}$ is the subspace of $\mathbb{CM}$ preserved by this, $x^{AB'} = \hat{x}^{AB'}$. The conjugation $Z^\alpha \to \hat{Z}^\alpha$ is then the lift of the complex conjugation $x^\mu \to (x^\mu)^*$ on $\mathbb{CM}$ preserving real Euclidean slices. The conjugation $Z^\alpha \to \hat{Z}^\alpha$ is quaternionic in the sense that $\hat{\hat{Z}}^\alpha = -Z^\alpha$ so that it defines a complex structure that anticommutes with the standard one. It therefore has no fixed points (as $Z^\alpha = \hat{Z}^\alpha$ implies $Z^\alpha = -Z^\alpha$), and it is induced by the standard quaternionic conjugation on spinors: $\hat{\pi}_{A'} = (\bar{\pi}_{1'}, -\bar{\pi}_{0'})$ and similarly for ω^A.

The conformal compactification $\widetilde{M}$ of Euclidean $\mathbb{R}^4$ is given by adding a single point i at infinity to give S^4. The Euclidean signature correspondence is particularly straightforward since we have a fibration $\mathbb{PT} = \mathbb{CP}^3 \to S^4$ given by sending Z^α to the point in Euclidean space corresponding to the projective line through Z^α and $\hat{Z}^\alpha$ (this includes a line at infinity corresponding to $\pi_{A'} = 0$). The fibre over any point $x^{AA'}$ in S^4 is a $\mathbb{CP}^1$ with projective coordinates $\pi_{A'}$, and the corresponding point in $\mathbb{PT}$ is

$$(\omega^A, \pi_{A'}) = (x^{AA'}\pi_{A'}, \pi_{A'}). \tag{2.14}$$

Conversely, a point in $\mathbb{PT}$ with holomorphic coordinates $(\omega^A, \pi_{A'})$ is represented in local non-holomorphic coordinates $(x^{AA'}, \pi_{A'})$ by

$$(x^{AA'}, \pi_{A'}) = \left(\frac{\omega^A \hat{\pi}^{A'} - \hat{\omega}^A \pi^{A'}}{\pi_{A'}\hat{\pi}^{A'}}, \pi_{A'} \right). \tag{2.15}$$

The $\mathbb{CP}^1$ fibre at each point is the space of primed spinors $\pi_{A'}$, identified under scaling, so that $\mathbb{PT}$ is the projective primed spin bundle over S^4. Similarly, $\mathbb{T}-0$ is the bundle of primed spinors minus the zero section, and we can again use the formulae (2.14),(2.15).

To obtain $\mathbb{M}=\mathbb{R}^4$, we choose a point i on S^4 to be the point at infinity, and this corresponds to an infinity twistor I, specifying the $\mathbb{CP}^1$ fibre over i. Then the twistor space for $\mathbb{R}^4$ is given by removing this $\mathbb{CP}^1$, so that $\mathbb{PT}'=\mathbb{PT}-\mathbb{CP}^1$ is the projective spin bundle over $\mathbb{R}^4$. Choosing an infinity twistor and an origin chooses a subgroup $SU(2)\times SU(2)\subset SU^*(4)$ and a decomposition of Z^α into holomorphic coordinates $(\omega^A,\pi_{A'})$ transforming under this $SU(2)\times SU(2)$; in this frame, the twistor correspondence is given by (2.14),(2.15) on $\mathbb{T}'=\mathbb{T}-\{\pi_{A'}=0\}$ so that the point at infinity is $x^{AA'}=\infty$, corresponding to the 2-plane in $\mathbb{T}$ (or $\mathbb{CP}^1$ in $\mathbb{PT}$) given by $\{\pi_{A'}=0\}$.

2.4 The Penrose transform

The Penrose transform identifies fields of helicity $-n/2$ satisfying the massless wave equation on a suitable region $U\subset\mathbb{CM}$ with the cohomology group $H^1(\mathbb{PT}(U),\mathcal{O}(n-2))$ for $\mathbb{PT}(U)$ the corresponding subset of $\mathbb{PT}$. A Dolbeault representative of this group is a $(0,1)$-form α with values in $\mathcal{O}(n-2)$ such that $\bar\partial\alpha=0$, where α is defined modulo $\bar\partial g$ with g a smooth section of $\mathcal{O}(n-2)$. The corresponding massless space-time field of helicity $|n|/2$ for $n\le 0$ is given by the integral formula

$$\phi_{A'_1\ldots A'_{-n}}(x)=\int_{\omega^A=x^{AA'}\pi_{A'}}\pi_{A'_1}\ldots\pi_{A'_{-n}}\alpha\wedge\pi_{C'}\mathrm{d}\pi^{C'}. \qquad (2.16)$$

For $n\ge 0$, the massless space-time field of helicity $-n/2$ is given by

$$\phi_{A_1\ldots A_n}(x)=\int_{\omega^A=x^{AA'}\pi_{A'}}\left(\frac{\partial}{\partial\omega^{A_1}}\cdots\frac{\partial}{\omega^{A_n}}\alpha\right)\wedge\pi_{C'}\mathrm{d}\pi^{C'}. \qquad (2.17)$$

Alternatively, a Čech representative can be chosen for the cohomology class, and the space-time fields are then given by a contour integral formula. This can be implemented simply when it is possible to cover $\mathbb{PT}(U)$ by two open sets, V_0 and V_1 (this is the case for $\mathbb{PT}'$, for which we can take $V_0=\{\pi_{0'}\neq 0\}$ and $V_1=\{\pi_{1'}\neq 0\}$). Then the Čech cohomology class can be represented by a holomorphic function f of homogeneity $n-2$ on $V_0\cap V_1$. The analogues of the above formulae are then, for $n\le 0$,

$$\phi_{A'_1\ldots A'_{-n}}(x)=\oint_\Gamma\pi_{A'_1}\ldots\pi_{A'_{-n}}f\,\pi_{C'}\mathrm{d}\pi^{C'} \qquad (2.18)$$

and, for $n\ge 0$,

$$\phi_{A_1\ldots A_n}(x)=\oint_\Gamma\frac{\partial}{\partial\omega^{A_1}}\cdots\frac{\partial}{\omega^{A_n}}f\,\pi_{C'}\mathrm{d}\pi^{C'}. \qquad (2.19)$$

In both (2.18) and (2.19) the contour Γ is a suitable circle in $V_0 \cap V_1 \cap \{\omega^A = x^{AA'}\pi_{A'}\}$.

In split signature, instead of considering cohomology classes, we can consider smooth functions defined on $\mathbb{PT}_{\mathbb{R}}$ that are homogeneous of degree $n-2$ and apply the integral formulae (2.18) and (2.19), where now Γ is taken to be the real line $\{\omega^A = x^{AA'}\pi_{A'}\}$ in $\mathbb{PT}_{\mathbb{R}}$ for $x^{AA'}$ a point in real split signature Minkowski space. In the case of $n = 0$ this is known as the X-ray transform, and it is a classic theorem that these formulae define an isomorphism from functions on $\mathbb{PT}_{\mathbb{R}}$ to solutions of the ultrahyperbolic wave equation on $\mathbb{M}$ [20]. The close relationship between the Penrose transform and the X-ray transform was observed by Atiyah [21]. The connection between the X-ray transform and the Penrose transform can be understood naively by requiring f to be analytic, extending it to some complex neighbourhood of $\mathbb{PT}_{\mathbb{R}}$ and reinterpreting it as a Čech cohomology class. However there are a number of issues that this approach does not deal with; a full treatment of the relationship between the X-ray and Penrose transforms is given in [22, 23]. For the most part, it is this X-ray transform version of the Penrose transform that is used by Witten and Berkovits in [1, 2].

2.5 Super-twistor space

The superspace with N supersymmetries has space-time coordinates $x^{AA'}$ and anti-commuting coordinates $\theta^A_a, \tilde{\theta}^{aA'}$ where $a, b = 1, ..., N$. The latter are space-time spinors and transform as an N-dimensional representation of an R-symmetry group, which is $U(N)$ or $SU(N)$ for Lorentzian signature, $GL(N, \mathbb{R})$ or $SL(N, \mathbb{R})$ for split signature and $U^*(N)$ or $SU^*(N)$ for Euclidean signature.

The complexified superconformal group is $SL(4|N; \mathbb{C})$ and its real forms are $SU(2, 2|N)$ for Lorentzian signature, $SL(4|N; \mathbb{R})$ for split signature and $SU^*(4|N)$ for Euclidean signature. The group $SL(4|N; \mathbb{C})$ is realised on the space $\mathbb{C}^{4|N}$ with coordinates $Z^I = (Z^\alpha, \psi^a) \in \mathbb{C}^{4|N}$, consisting of the usual commuting coordinates Z^α as before and anti-commuting coordinates ψ^a, $a = 1, \ldots, N$. Super-twistor space $\mathbb{T}_{[N]}$ is the subset $\mathbb{C}^{4|N} - \mathbb{C}^{0|N}$ on which $Z^\alpha \neq 0$, and the projective super-twistor space $\mathbb{PT}_{[N]} = \mathbb{CP}^{3|N}$ is the space of equivalence classes under complex scalings [24]:

$$\mathbb{PT}_{[N]} = \mathbb{CP}^{3|N} = \{Z^I = (Z^\alpha, \psi^a) \in \mathbb{C}^{4|N} - \mathbb{C}^{0|N}\}/\{Z^I \sim \lambda Z^I \, , \lambda \in \mathbb{C}^\times\} \, .$$

Note that in this definition we have a fibration $\mathbb{PT}_{[N]} \to \mathbb{PT}$ given by $(Z^\alpha, \psi^a) \to Z^\alpha$. However, this fibration is not preserved by the action of the superconformal group $SL(4|N; \mathbb{C})$.

The $N = 4$ superspace is special for twistor theory because in that case there is a global holomorphic volume form on the projective super-twistor space,

$$\Omega_s = \Omega \mathrm{d}\psi_1 \mathrm{d}\psi_2 \mathrm{d}\psi_3 \mathrm{d}\psi_4 \, ,$$

with Ω the bosonic 3-form defined in (2.7). This has weight zero, since each $\mathrm{d}\psi_a$ has weight -1 according to the Berezinian integration rule $\int \psi_1 \mathrm{d}\psi_1 = 1$.

Anti-chiral super-Minkowski space $\mathbb{CM}^-_{[N]}$ with coordinates $x_+^{AA'}, \tilde{\theta}^{aA'}$ arises as the space of $\mathbb{CP}^{1|0}$s in $\mathbb{PT}_{[N]}$ via the incidence relations

$$(\omega^A, \pi_{A'}, \psi^a) = (x_+^{AA'}\pi_{A'}\,, \pi_{A'}, \tilde{\theta}^{aA'}\pi_{A'}), \tag{2.20}$$

where we have used $\pi_{A'}$ as homogeneous coordinates on $\mathbb{CP}^{1|0}$. Chiral super-Minkowski space $\mathbb{CM}^+_{[N]}$ with coordinates $x_-^{AA'}, \theta^A_a$ arises as the space of $\mathbb{CP}^{1|N}$s in $\mathbb{PT}_{[N]}$ via the incidence relations

$$(\omega^A, \pi_{A'}, \psi^a) = (x_-^{AA'}\pi_{A'} + \psi^a\theta^A_a\,, \pi_{A'}, \psi^a)\,, \tag{2.21}$$

where now we have used $(\pi_{A'}\,, \psi^a)$ as homogeneous coordinates on the $\mathbb{CP}^{1|N}$s. A point of full super-Minkowski space $\mathbb{CM}_{[N]}$ with coordinates $x^{AA'}, \theta^A_a, \tilde{\theta}^{aA'}$ arises from a choice of $\mathbb{CP}^{1|N}$ in $\mathbb{PT}_{[N]}$ together with a choice of $\mathbb{CP}^{1|0} \subset \mathbb{CP}^{1|N}$, so that full super-Minkowski space is the space of 'flags' $\mathbb{CP}^{1|0} \subset \mathbb{CP}^{1|N}$ in $\mathbb{PT}_{[N]}$ [24]. Taking (2.20) and (2.21) together we have $x_+^{AA'} = x_-^{AA'} + \tilde{\theta}^{aA'}\theta^A_a$ and it is usual to define $x^{AA'} = \frac{1}{2}(x_+^{AA'} + x_-^{AA'})$.[8]

The massless field formulae generalising (2.16) and (2.17) now give rise to superfields encoding supermultiplets. The easiest way to see this is to expand out an element $\mathcal{F}_n \in H^1(\mathbb{PT}_{[N]}(U), \mathcal{O}(n))$ as follows:

$$\mathcal{F}_n = f_{(n)} + f_{(n-1)a}\psi^a + f_{(n-2)a_1a_2}\psi^{a_1}\psi^{a_2} + f_{(n-3)a_1a_2a_3}\psi^{a_1}\psi^{a_2}\psi^{a_3} + \dots .$$

Here $f_{(n-k)\dots}$ has homogeneity degree $n-k$ so that its Penrose transform is a massless field of helicity $-(n-k-2)$ on space-time with skew-symmetric indices $a_1, \dots, a_k$, and it transforms as a k-th rank anti-symmetric tensor under the R-symmetry group.

It is possible to perform the transform on $\mathcal{F}_n$ to obtain a superfield directly on $\mathbb{CM}^{\pm}$, the $\pm$ depending on whether we integrate over $\mathbb{CP}^{1|0}$s or $\mathbb{CP}^{1|N}$ fibres. Particularly interesting examples are furnished by the cases of $n = \pm 2$ in the context of linearised $N = 4$ Einstein supergravity. We can define

$$H^+(x_-, \theta^A_a) \;=\; \oint_{\mathbb{CP}^{1|4}} \mathcal{F}_2(x_-^{AA'}\pi_{A'} + \psi^a\theta^B_a, \pi_{A'}, \psi^b)\pi_{A'}\mathrm{d}\pi^{A'}\,\mathrm{d}^4\psi \tag{2.22}$$

and

$$H^-(x_+, \tilde{\theta}^{aA'}) \;=\; \oint_{\mathbb{CP}^{1|0}} \mathcal{F}_{-2}(x_+^{AA'}\pi_{A'}, \pi_{A'}, \tilde{\theta}^{aA'}\pi_{B'})\pi_{A'}\mathrm{d}\pi^{A'}\,. \tag{2.23}$$

[8]To obtain standard conventions in Lorentz signature we must take $x^{AA'} = iy^{AA'}$ for real $y^{AA'}$; our conventions are adapted to split and Euclidean signature.

The integrand of (2.22) can be expanded in ψ^a using Taylor series in the anti-commuting coordinates and the variables ψ^a can be integrated out to yield a power series in θ^B_a; the standard Penrose transform in the form (2.18) can then be applied to the coefficients to yield a superfield on chiral super Minkowski space. Eq. (2.23) can be expanded as a Taylor series in $\tilde{\theta}^{aA'}$ to obtain a series whose coefficients can be integrated using (2.19) to obtain a superfield on anti-chiral super-Minkowski space $\mathbb{CM}^-_{[N]}$. This directly gives formulae for the full chiral and anti-chiral superfields for $N = 4$ supergravity in terms of the component fields.

In order to obtain an anti-chiral or a chiral superfield for other values of n or N, we need to either repeatedly differentiate $\mathcal{F}_n$ with respect to ω^A, or to multiply it by enough factors of $\pi_{A'}$. In the first case, this will reduce the homogeneity to -2 and enable us to apply (2.23) to obtain an anti-chiral superfield; in the second case, we arrange for homogeneity $N - 2$ and obtain a chiral superfield by applying (2.22).

As before, the space of $\mathbb{CP}^{1|0}$s (resp. $\mathbb{CP}^{1|N}$s or flags $\mathbb{CP}^{1|0} \subset \mathbb{CP}^{1|N}$) in $\mathbb{PT}_{[N]}$ is a conformal compactification of chiral (resp. anti-chiral or full) super Minkowski space on which the superconformal group acts. We will wish to break conformal invariance on super-twistor space by choosing points at infinity and a scale. There are three ways in which we can break superconformal invariance; we can choose points at infinity in either the chiral, anti-chiral or full Minkowski space, and these lead to different structures.

A choice of a point at infinity in chiral super-Minkowski space corresponds to a choice of a line I, a $\mathbb{CP}^{1|0}$, in $\mathbb{PT}_{[N]}$ and coordinates $(\omega^A, \pi_{A'}, \psi^a)$ can be chosen so that I is given by $\pi_{A'} = 0 = \psi^a$. This determines a projection $p_1 : \mathbb{PT}_{[N]} - I \to \mathbb{CP}^{1|N}$ given in homogeneous coordinates by

$$p_1 : (\omega^A, \pi_{A'}, \psi^a) \to (\pi_{A'}, \psi^a)\,.$$

The fibres of the projection are the $\mathbb{CP}^{2|0}$s through I.

If we choose a point in anti-chiral Minkowski space, then this gives a choice of a superline $I_{[N]} = \mathbb{CP}^{1|N}$ and we can then choose coordinates $(\omega^A, \pi_{A'}, \psi^a)$ so that $I_{[N]}$ is the set $\pi_{A'} = 0$. This, as before, leads to a fibration $p : \mathbb{PT}_{[N]} - I_{[N]} \to \mathbb{CP}^{1|0}$ given by

$$p_1 : (\omega^A, \pi_{A'}, \psi^a) \to \pi_{A'}$$

with fibres the $\mathbb{CP}^{2|N}$s through $I_{[N]}$.

The richest structure is obtained by choosing a vertex i of a super-light-cone at infinity $\mathscr{I}$ in the full conformally compactified super-Minkowski space (as opposed to one of its chiral versions). This is equivalent to the choice of a 'flag' $\mathbb{CP}^{1|0} \subset \mathbb{CP}^{1|N} \subset \mathbb{PT}_{[N]}$, i. e. the pair $I \subset I_{[N]}$. These lead to corresponding projections of $\mathbb{PT}'_{[N]} = \mathbb{PT} - I_{[N]}$

$$\mathbb{PT}'_{[N]} \xrightarrow{p_1} \mathbb{CP}^{1|N} \xrightarrow{p_0} \mathbb{CP}^{1|0}\,, \qquad Z^I = (\omega^A, \pi_{A'}, \psi^a) \to (\pi_{A'}, \psi^a) \to \pi_{A'}\,.$$

We will also be interested in the case in which there is only the projection $p : \mathbb{PT}'_{[N]} \to \mathbb{CP}^{1|0}$. We will see that this is a weaker structure and there will correspondingly be a larger class of deformations.

We can define the Poisson structure

$$\{f, g\}_I := I^{IJ} \frac{\partial f}{\partial Z^I} \frac{\partial g}{\partial Z^J} = \epsilon^{AB} \frac{\partial f}{\partial \omega^A} \frac{\partial g}{\omega^B}$$

as in the bosonic case, and p_0 can then be used to pull back the 1-form

$$I_{IJ} Z^I \mathrm{d}Z^J = \epsilon^{A'B'} \pi_{A'} \mathrm{d}\pi_{B'}$$

from $\mathbb{CP}^{1|0}$. These are special cases of more general correspondences between points of chiral Minkowski space and rank two bi-vectors $X^{IJ} = X^{[IJ]}$ up to scale, and between points of anti-chiral Minkowski space and simple (rank two) two-forms X_{IJ} up to scale. Alternative representations can be obtained by use of the volume form $\epsilon_{I_1 \ldots I_{4+N}}$ and its inverse on $\mathbb{T}_{[N]}$.

3 The non-linear graviton

3.1 The conformally anti-self-dual case

Penrose's non-linear graviton construction provides a correspondence between curved twistor spaces and conformally anti-self-dual space-times, and so gives a general construction of such space-times. This arises from nontrivial deformations of the flat twistor correspondence in which, on the one hand, the space-time is deformed from flat space to one with a curved conformal structure with anti-self-dual Weyl curvature, and, on the other, the complex structure of a region in twistor space is deformed away from that of a region in projective space. One cannot deform the complex structure of the whole of flat twistor space as $\mathbb{PT} = \mathbb{CP}^3$ is rigid and has no continuous deformations, so we instead consider deformations of $\mathbb{PT}'$, which is $\mathbb{CP}^3$ with a line removed. This has topology $\mathbb{R}^4 \times S^2$. We will find it convenient to start by describing the non-projective twistor space.

A curved twistor space $\mathcal{T}$ will be taken to be a 4-dimensional complex manifold equipped with a vector field Υ and a non-vanishing holomorphic 3-form Ω such that

- Υ gives $\mathcal{T}$ the structure of a line bundle over the space $P\mathcal{T} = \mathcal{T}/\{\Upsilon\}$ of orbits of Υ, for which Υ is the Euler vector field (in local coordinates (z, z_1, z_2, z_3) where (z_1, z_2, z_3) are coordinates on $P\mathcal{T}$ and z is a linear coordinate up the fibre, $\Upsilon = z\partial/\partial z$).

- Υ and Ω satisfy

$$\mathcal{L}_\Upsilon \Omega = 4\Omega\,, \qquad \iota(\Upsilon)\Omega = 0\,. \tag{3.1}$$

- $P\mathcal{T}$ contains a holomorphically embedded Riemann sphere that has the same normal bundle as a complex projective line in $\mathbb{CP}^3$.

The last condition is in fact rather mild and holds automatically not only for any twistor space that is constructed as described below from a conformally anti-self dual space-time, but also for any twistor space that is an arbitrary small deformation of such a twistor space. The space-time is reconstructed as the moduli space of such Riemann spheres; given one such sphere, Kodaira theory implies the existence of a full four-dimensional family [55].

The existence of the holomorphic volume form $\mathrm{d}\Omega$ implies that $\mathcal{T}$ is a non-compact Calabi-Yau space.[9] The global existence of Υ and Ω allows us to introduce local complex coordinates Z^α on $\mathcal{T}$ such that

$$\Upsilon = Z^\alpha \frac{\partial}{\partial Z^\alpha}\,, \qquad \Omega = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta} Z^\alpha \mathrm{d}Z^\beta \mathrm{d}Z^\gamma \mathrm{d}Z^\delta$$

as in the flat case, with $\epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\alpha\beta\gamma\delta]}$, $\epsilon_{0123} = 1$.

We now turn to the relation between curved twistor space and space-time. For complexified Minkowski space, a twistor corresponds to an α-plane, i. e. a totally null self-dual two-plane. In a curved complex space-time $\mathbb{C}\mathcal{M}$, which is a complex 4 manifold with a holomorphic metric g (so that locally the metric is $g_{\mu\nu}(x)dx^\mu dx^\nu$, depending on the complex coordinates x^μ but not their complex conjugates), α–plane elements in the tangent space are not generally integrable, i.e. one cannot in general find a two surface whose tangent planes are α-planes. A two-surface whose tangent plane is an α-plane at every point is called an α-surface. The nececessary and sufficient condition for there to exist α-surfaces through each α-plane element at every point is that the self-dual part of the Weyl curvature should vanish,

$$\tilde{\psi}_{A'B'C'D'} = 0. \tag{3.2}$$

If (3.2) holds, then the 3 complex dimensional curved twistor space $P\mathcal{T}$ is the space of such α–surfaces. An α-surface through x is specified by an α-plane in the tangent space at

[9] The second condition allows us to give a construction of $\mathcal{T}$ in terms of $P\mathcal{T}$ as the total space of the line bundle $\mathcal{T} = (\Lambda^{(3,0)})^{1/4}$ over $P\mathcal{T}$. This definition arises by analogy with the flat case, where $\Lambda^{(3,0)}$ is $\mathcal{O}(-4)$ because the holomorphic $(3,0)$-form Ω has weight 4 and so it needs to be multiplied by a weight -4 function to define a $(3,0)$-form. Since $\mathbb{T} - \{0\}$ is the total space of the line bundle $\mathcal{O}(-1)$ minus its zero-section, it is therefore the fourth root of $\Lambda^{(3,0)}$. With this definition of $\mathcal{T}$, the existence of Ω on $\mathcal{T}$ is tautological as $\mathcal{T}$ is a covering of the bundle of 3-forms and so Ω is the pull-back to $\mathcal{T}$ of the corresponding 3-form at that point. As the $(3,0)$-form Ω has weight 4, it is not a $(3,0)$-form on $\mathcal{PT}$, so that $\mathcal{PT}$ is not a Calabi-Yau space.

x, and this in turn is fixed by a choice of primed 'tangent' spinor $\pi_{A'}$ at x, up to complex scalings, so that the space of tangent vectors is given by $\pi^{A'}\lambda^A$ as λ^A varies.

A point in the non-projective twistor space $\mathcal{T}$ is determined by an α-surface in $\mathbb{C}\mathcal{M}$ and a tangent spinor $\pi_{A'}$ that is parallelly propagated over the α-surface using the Levi-Civita connection of any metric in the conformal class. It is a non-trivial fact that the parallel propagation of such a 'tangent' spinor over its α-surface is independent of the choice of conformal factor for the metric in the conformal class. A point in the projective twistor space $P\mathcal{T}$ is given by the α-plane together with $\pi_{A'}$ up to complex scalings of $\pi_{A'}$.

For Euclidean signature, we saw that in the flat case the twistor space $\mathbb{PT} = \mathbb{CP}^3$ is the projective spin bundle over compactified space-time S^4. This generalises, and for Euclidean signature, the curved twistor space $P\mathcal{T}$ for a conformally anti-self-dual space $\mathcal{M}$ is the projective spin bundle over $\mathcal{M}$, where the fibre at a point x is a $\mathbb{CP}^1$ with homogeneous coordinates given by the primed spinors $\pi_{A'}$ at x, while $\mathcal{T}$ is the corresponding non-projective spin bundle. In terms of coordinates $(x, \pi_{A'})$, $\Upsilon = \pi_{A'}\partial/\partial\pi_{A'}$ and $\Omega = \pi^{A'}D\pi_{A'} \wedge \pi_{B'}\pi_{C'}\epsilon_{BC}e^{BB'} \wedge e^{CC'}$ where D is the covariant exterior derivative with the Levi-Civita connection of some metric in the conformal class, and $e^{AA'}$ are the pull-backs from space-time to the spin bundle of the 'solder forms' $e_\mu^{AA'}dx^\mu$ constructed from a vielbein $e_\mu^{AA'}$. [10]

The famous result of Penrose [12] is that the space-time $\mathbb{C}\mathcal{M}$ together with its anti-self-dual conformal structure can be reconstructed from the complex structure of $\mathcal{T}$ together with (Υ, Ω) as described above, or from $P\mathcal{T}$ and its complex structure. The existence of the correspondence is preserved under small deformations, either of the complex structure on $P\mathcal{T}$, or of the anti-self dual conformal structure on $\mathbb{C}\mathcal{M}$. Thus one can attempt to construct anti-self-dual space-times by deforming, say, $\mathbb{PT}'$. The key idea is that a point $x \in \mathbb{C}\mathcal{M}$ corresponds to a Riemann sphere $\mathbb{CP}^1_x$ (the Riemann sphere with homogenous coordinates $\pi_{A'}$) in $\mathbb{PT}$ consisting of those α-surfaces through x. It follows from Kodaira theory that the moduli space of deformations of $\mathbb{CP}^1_x$ in $P\mathcal{T}$ is necessarily four dimensional, and naturally contains $\mathbb{C}\mathcal{M}$ as an open set (in general it is some analytic continuation of $\mathbb{C}\mathcal{M}$). Furthermore, this family of $\mathbb{CP}^1_x$s still survives after deformations of the complex structure of $P\mathcal{T}$.

If $\mathbb{C}\mathcal{M}$ arises as such a moduli space, an anti-self-dual conformal structure can be defined on $\mathbb{C}\mathcal{M}$ by declaring points x and y to be null separated if $\mathbb{CP}^1_x$ and $\mathbb{CP}^1_y$ intersect. The fact that the existence of such a correspondence survives deformations of the complex structure on $P\mathcal{T}$ means that, given one conformally anti-self-dual space-time, a family of new conformally self-dual space-times can be constructed by deforming the complex

[10] In this form, the construction makes sense for compact space-times of Euclidean signature with complicated topology: a celebrated result of Taubes is that Euclidean signature anti-self-dual conformal structures can be found on arbitrary compact 4-manifolds, possibly after performing a connected sum with a finite number of $\overline{\mathbb{CP}^2}$s, and so there are many nontrivial compact examples of twistor spaces.

structure of the corresponding curved twistor space $P\mathcal{T}$, and so the equations governing the deformation of the complex structure correspond to the field equations for conformal anti-self-dual gravity.

The data of the conformal structure on $\mathbb{C}\mathcal{M}$ is then encoded in the complex structure of $P\mathcal{T}$. There are two standard ways to represent the complex structure. The Dolbeault approach (cf. the introduction) is to regard $P\mathcal{T}$ as a real 6-manifold with an almost complex structure, i. e. a $(1,1)$-tensor J subject to the integrability condition that its Nijenhuis tensor $N(J)$ vanishes. We can equivalently encode J into a $\bar{\partial}$ operator, the restriction of the exterior derivative to the 1-forms $\Lambda^{(0,1)}$ in the $-i$ eigenspace of J. With this restriction, $N(J) = 0$ is equivalent to $\bar{\partial}^2 = 0$. The Čech approach is to consider $P\mathcal{T}$ as a 3 complex dimensional manifold formed by choosing a suitable open cover V_i, picking holomorphic coordinates on each V_i and then encoding the data of the manifold in the biholomorphic patching functions defined on the overlaps $V_i \bigcap V_j$. Both these points of view lead to a cohomological understanding of the deformation theory, the first via Dolbeault cohomology and the second via Čech cohomology. In either approach, the deformations of the complex structure are parametrised by $H^1(P\mathcal{T}, T^{(1,0)})$. If we consider linearised deformations of $\mathbb{PT}$, we obtain the following description of linearised conformal gravity.

We represent $f \in H^1(P\mathcal{T}, T^{(1,0)})$ by a $(0,1)$-form $f^\alpha(Z) = f^\alpha{}_{\bar{\beta}}(Z) d\bar{Z}^{\bar{\beta}}$ taking values in the bundle of holomorphic vector fields on $\mathcal{T}$, with the condition that f^α has homogeneity degree 1 and is defined up to the gauge freedom $f^\alpha \to f^\alpha + a(Z) Z^\alpha$ for some $(0,1)$-form $a(Z)$ of homogeneity zero. This freedom can be fixed by the requirement that $\partial f^\alpha / \partial Z^\alpha = 0$, which is the condition that the measure $\mathrm{d}\Omega$ is holomorphic for the deformed complex structure $\bar{\partial} + f(Z)^\alpha \partial / \partial Z^\alpha$. This implies that $f(Z)^\alpha \partial / \partial Z^\alpha$ is a deformation of $\mathbb{T}$ that preserves both Ω and $\mathrm{d}\Omega$.

The Penrose transform of f^α gives a helicity $+2$ field ψ_{ABCD} in space-time satisfying the field equation of linearised conformal gravity, which is the linearised Bach equation [25]:

$$\nabla^C_{A'} \nabla^D_{B'} \psi_{ABCD} = 0\,; \tag{3.3}$$

see [26, 27] for details.

Following [6] and [30], the negative helicity conformal graviton can be represented by an element $g \in H^1(\mathbb{PT}(U), \Lambda^1(-4))$. The pull-back of g to $\mathbb{T}$ gives a 1-form $g_\alpha(Z)\mathrm{d}Z^\alpha$ on $\mathbb{T}$, where $g(\Upsilon) = Z^\alpha g_\alpha = 0$ and the components g_α have weight -5. The Penrose transform of g_α gives a Weyl spinor $\tilde{\psi}_{A'B'C'D}$, now of helicity -2, satisfying

$$\nabla^{C'}_B \nabla^{D'}_A \tilde{\psi}_{A'B'C'D'} = 0\,. \tag{3.4}$$

The Penrose transform in this case is the opposite helicity to that of f^α, and can be derived using the methods of [27, 30]; it is discussed from a different point of view in [6], where g appears as the component $\psi^1\psi^2\psi^3\psi^4 g$ of the cohomology class b in $H^1(\mathbb{PT}_{[4]}, T^*)$ on super-twistor space, where T^* is the cotangent bundle.

3.1.1 Real space-times

The non-linear graviton construction cannot be applied to conformally curved Lorentzian space-times, as a real Lorentzian space-time satisfying (3.2) is conformally flat; the self-dual part of the Weyl curvature is the complex conjugate of the anti-self-dual part. However, it can be applied to the other two signatures by constructing a complex space-time and seeking a suitable real submanifold. The specialisation to Euclidean space-times gives the construction of general conformally anti-self-dual spaces. In this case, the twistor space is a $\mathbb{CP}^1$ bundle over space-time, so that the space-time is obtained from the twistor space by projection [15].

In split signature the non-linear graviton construction changes character, and there are two ways of constructing self-dual spaces [28, 16]; see also [19]. For flat space in this signature, there is a complex twistor space $\mathbb{PT} = \mathbb{CP}^3$ and a real subspace $\mathbb{PT}_{\mathbb{R}} = \mathbb{RP}^3$ fixed by the complex conjugation $\tau : Z \to Z^*$ inherited by twistor space from that on complex space-time, $x^\mu \to (x^\mu)^*$. There are two routes to the curved space generalisation. In the first, one deforms the complex structure of a region of the complex twistor space $\mathbb{PT} = \mathbb{CP}^3$ to obtain a curved twistor space $P\mathcal{T}$ as before, but in such a way as to preserve the complex conjugation. The fixed point set $P\mathcal{T}_{\mathbb{R}}$ of the conjugation defines an analogue of $\mathbb{PT}_{\mathbb{R}}$ in the deformed case and induces a complex conjugation on space-time that fixes a real slice of split signature. In the second, the complex twistor space $\mathbb{PT} = \mathbb{CP}^3$ is kept fixed but the real subspace is deformed from $\mathbb{PT}_{\mathbb{R}}$ to a subspace $P\mathcal{T}_{\mathbb{R}}$. Both approaches lead to considering deformations of the real twistor space from $\mathbb{PT}_{\mathbb{R}}$ to $P\mathcal{T}_{\mathbb{R}}$, but this is embedded in different complex spaces in the two cases. The two kinds of deformations are both locally encoded in the same cohomology classes on the real twistor space, but the second approach is better behaved globally and does not require analyticity of the space-time, so it is more powerful. However, it is the first approach that has been used to give a non-linear interpretation of the Berkovits string theory, in which open strings move in $P\mathcal{T}$ with boundaries lying in $P\mathcal{T}_{\mathbb{R}}$. In §4, we will propose a modification of the Berkovits string theory that corresponds to the second approach, in which there is a natural geometric interpretation of the vertex operators. In the first approach, points in space-time correspond to $\mathbb{CP}^1$'s in $P\mathcal{T}$ that are invariant under the conjugation, while in the second they correspond to discs in $\mathbb{PT}$ with boundary on $P\mathcal{T}_{\mathbb{R}}$.

We now describe the two constructions in more detail. In the first, the twistor space $P\mathcal{T}$ was the deformation of a region in flat twistor space in such a way that the complex conjugation $\tau : P\mathcal{T} \to P\mathcal{T}$ is preserved. We can construct such a twistor space starting with a real split signature space-time $\mathcal{M}$ that is real analytic.[11] The real analyticity can be used to find a complexification $\mathbb{C}\mathcal{M}$ of the real split-signature space $\mathcal{M}$. This can be found locally by allowing the coordinates to take complex values, and using the analyticity of the transition functions for the coordinates we can extend the charts and transition functions to construct a complex manifold $\mathbb{C}\mathcal{M}$ which contains $\mathcal{M}$ as a real slice (i.e. a slice fixed by complex conjugation of the coordinates we have just constructed). The analyticity of the metric implies that it can be extended to a holomorphic metric on $\mathbb{C}\mathcal{M}$. The complex non-linear graviton construction of §3.1 can be used locally on any suitable open set $U \subset \mathbb{C}\mathcal{M}$ to define a twistor space $P\mathcal{T}_U$ corresponding to U. The complex conjugation on space-time again sends α-planes to α-planes, inducing a complex conjugation on $P\mathcal{T}_U$ that fixes a real slice $P\mathcal{T}_{U\mathbb{R}}$ which is a totally real 3-dimensional submanifold of the complex twistor space. A point x in the real space-time $\mathcal{M}$ corresponds to a holomorphic Riemann sphere in the complex twistor space that intersects $P\mathcal{T}_{U\mathbb{R}}$ in a circle and cuts the Riemann sphere into two discs $D_x^{\pm}$. In the reverse direction, the complex twistor space can be used to reconstruct a complex conformally anti-self-dual space as before. This naturally has a complex conjugation that determines a real slice, on which the complex conformal structure restricts to give a real conformally anti-self-dual structure. In order to construct the global complex twistor space $P\mathcal{T}$, we first need to choose a suitable open cover $\{U_i\}$ of $\mathbb{C}\mathcal{M}$ and construct the twistor space $P\mathcal{T}_{U_i}$ for each open set; we then glue these twistor spaces together, identifying points in $P\mathcal{T}_{U_i}$ with those in $P\mathcal{T}_{U_j}$ whose corresponding α-surfaces coincide in $U_i \cap U_j$. However, this natural extension gives a $P\mathcal{T}$ which is a non-Hausdorff manifold [28]; see the appendix for a brief description of this space.

In the second approach, we consider general anti-self-dual conformal structures on $S^2 \times S^2$. Recall that the conformal compactification of split signature flat space $\mathbb{R}^{2,2}$ is $S^2 \times S^2/\mathbb{Z}_2$, with double cover $S^2 \times S^2$. It turns out that there is only the conformally flat anti-self-dual conformal structure on $S^2 \times S^2/\mathbb{Z}_2$, while there is an infinite dimensional family of nontrivial such conformal structures on the double cover $S^2 \times S^2$ [16]. Real points in $S^2 \times S^2$ correspond to Riemann spheres that intersect the real subspace $\mathbb{PT}_{\mathbb{R}}$, dividing each sphere into two discs $D_x^{\pm}$. The best way to understand the twistor theory in this case is to focus on one of the two discs, say D_x^+, rather than the Riemann spheres.

In Euclidean space we were able to represent the twistor space $\mathbb{T}$ as the bundle of primed spinors $\mathbb{S}$ because we could solve the incidence relation $\omega^A = x^{AA'}\pi_{A'}$ with $x^{AA'} = (\omega^A \hat{\pi}^{A'} - \hat{\omega}^A \pi^{A'})/(\hat{\pi}^{B'}\pi_{B'})$ when $x^{AA'}$ was real. Thus the coordinate transformation between $(\omega^A, \pi_{A'})$ and $(x^{AA'}, \pi_{A'})$ is locally invertible and in fact globally invertible if $x^{AA'} = \infty$ is allowed. In the context of the double fibration (2.4), when the spin bundle

[11]This assumption is nontrivial as generic solutions will be non-analytic (this can be seen to follow from the second construction). Nevertheless, such non-analytic solutions can be approximated arbitrarily closely by analytic ones, and the construction captures the full functional freedom of these solutions.

$\mathbb{S}$ is restricted to the real slice $\mathbb{M}$, the projection r from $\mathbb{S}$ to $\mathbb{T}$ is one-to-one and identifies the spin bundle with the twistor space.

In split signature, with $\pi_{A'}$ complex, $x^{AA'} = (\omega^A \bar{\pi}^{A'} - \bar{\omega}^A \pi^{A'})/(\bar{\pi}^{B'}\pi_{B'})$ solves the incidence relation so that there is locally a one-to-one correspondence between the points in the bundle of complex spinors on $\mathcal{M}$ and twistor space. However, this fails where $\bar{\pi}^{B'}\pi_{B'}$ vanishes, i. e. when $\pi_{A'}$ is a complex multiple of a real spinor. This is because at real values of x and $\pi_{A'}$ there are real α-planes, and such planes correspond to points of $\mathbb{PT}_{\mathbb{R}}$. Indeed, the bundle $\mathbb{S}_{\mathbb{R}}$ of real spinors is foliated by the lifts of real α-planes to $\mathbb{S}_{\mathbb{R}}$, with the lifted α-plane through $(x, \pi_{A'})$ given by the α-plane through x with tangent spinor $\pi_{A'}$, i.e. the 2-surface in $\mathbb{S}_{\mathbb{R}}$ of the form $(x^{AA'} + \lambda^A \pi^{A'}, \pi_{A'})$ parameterised by λ^A. Thus, there is a one-to-one identification between $\mathbb{PS} - \{\bar{\pi}_{A'}\pi^{A'} = 0\}$ and points in $\mathbb{PT} - \mathbb{PT}_{\mathbb{R}}$, but $\mathbb{PT}_{\mathbb{R}}$ itself is a quotient of $\mathbb{PS}_{\mathbb{R}}$ by its foliation by α-planes.

The set $\mathbb{S}_0 = \{(x, \pi_{A'}) \in \mathbb{S} : \bar{\pi}_{A'}\pi^{A'} = 0\}$ is a co-dimension-1 hypersurface in $\mathbb{S}$ and divides $\mathbb{S}$ into two halves $\mathbb{S}_{\pm}$ on which $\pm i\bar{\pi}_{A'}\pi^{A'} \geq 0$ with common boundary $\mathbb{S}_0$. We define the corresponding bundles of projective primed spinors $\mathbb{PS}_{\pm}$ and $\mathbb{PS}_0$ by the same conditions on $\bar{\pi}_{A'}\pi^{A'}$. Working now on $S^2 \times S^2$ with a general anti-self-dual conformal structure, it is still possible to distinguish between $\mathbb{PS}_+$ and $\mathbb{PS}_-$ globally and we focus on one half, say $\mathbb{PS}_+$.[12] This is a bundle of discs over $\mathcal{M}$ with boundary $\mathbb{PS}_0$. It turns out that $\mathbb{PS}_+$ has an integrable complex structure and is naturally a complex manifold—in the conformally flat case, $\mathbb{PS}_+$ is $\mathbb{PT} - \mathbb{PT}_{\mathbb{R}}$. The boundary, $\mathbb{PS}_0$, is naturally foliated by the lifts of real α-surfaces in $\mathcal{M}$ as in the conformally flat case and the quotient is $P\mathcal{T}_{\mathbb{R}}$, the space of real α-planes. There is a natural way to glue $P\mathcal{T}_{\mathbb{R}}$ to the boundary of $\mathbb{PS}_+$ to obtain a smooth compact complex manifold which is a copy of $\mathbb{CP}^3$ topologically.[13] If the original space-time is smooth, it can be shown that this gluing can be performed in such a way that the twistor space has a smooth complex structure. If our anti-self-dual conformal structure on $S^2 \times S^2$ is a continuous deformation of the standard conformal structure, then this twistor space must be the standard $\mathbb{PT}$ because the complex structure on $\mathbb{CP}^3$ is rigid. However, the embedding of $P\mathcal{T}_{\mathbb{R}}$ into $\mathbb{PT}$ will be a deformation of the standard embedding of the real slice $\mathbb{PT}_{\mathbb{R}}$ inside $\mathbb{PT}$.

The original space-time together with its anti-self-dual conformal structure can be reconstructed as the moduli space of holomorphically embedded discs in $\mathbb{PT}$, with boundary in $P\mathcal{T}_{\mathbb{R}}$ in the appropriate topological class [16]. The central role played by discs in this approach makes open string theory seem rather natural.

Linearised deformations of the embedding of $\mathbb{PT}_{\mathbb{R}}$ in $\mathbb{PT}$ correspond to sections of the normal bundle to $\mathbb{PT}_{\mathbb{R}}$ over $\mathbb{PT}_{\mathbb{R}}$. These can be naturally represented as purely imaginary

[12] On $S^2 \times S^2/\mathbb{Z}_2$, it is not possible to distinguish between $\mathbb{PS}_+$ and $\mathbb{PS}_-$; the space-time is not simply connected and, as one traverses a non-contractible loop, $\mathbb{PS}_{\pm}$ interchange.

[13] This is done by considering the manifold with boundary $\mathbb{PS}_+ \cup \mathbb{PS}_0$ and compressing each horizontal lift of an α-plane to a point.

tangent vector fields on $\mathbb{PT}_{\mathbb{R}}$; they can be represented as vector fields on $\mathbb{T}_{\mathbb{R}}$ of the form $if^\alpha\partial/\partial Z^\alpha$, where f^α is real with homogeneity degree 1, defined up to $f^\alpha \to f^\alpha + Z^\alpha\Lambda$ for Λ of weight 0. This freedom can be fixed with the gauge choice $\partial_\alpha f^\alpha = 0$. The only such vector fields that give trivial deformations are the generators of $\mathrm{SL}(4,\mathbb{C})$.

The non-linear version of this is to define a submanifold $\mathcal{T}_{\mathbb{R}}$ in $\mathcal{T}$ by the constraint

$$Z^\alpha = X^\alpha + iF^\alpha(X^\alpha), \tag{3.5}$$

where $X^\alpha = Z^\beta + \bar{Z}^\beta$ is real and F^α is a real function of four real variables of homogeneity degree one. Given $P\mathcal{T}_{\mathbb{R}} \subset P\mathcal{T}$, there is some freedom in the choice of $\mathcal{T}_{\mathbb{R}}$ corresponding to the shift

$$Z^\alpha \to Z^\alpha = e^{i\theta(X)}\left(X^\alpha + iF^\alpha\right) \tag{3.6}$$

where θ is an arbitrary function of X^α of weight 0; this changes the non-projective real slice, but not the projective one. Infinitesimally, (3.6) induces

$$F^\alpha \to F^\alpha + \theta(X)X^\alpha + \ldots . \tag{3.7}$$

This freedom can be fixed by imposing that $\det\left(\delta^\alpha_\beta + i\partial_\alpha F^\beta\right)$ be real. This implies that

$$\partial_\alpha F^\alpha = \partial_\alpha F^{[\alpha}\partial_\beta F^\beta \partial_\gamma F^{\gamma]}, \tag{3.8}$$

which is an analogue of the Calabi-Yau condition on $\mathcal{T}$. Clearly, this is a non-linear generalization of the $\partial_\alpha f^\alpha = 0$ condition above.

Our primary interest in this paper will be in the second construction described above, but for completeness we give a discussion of the connection between the two approaches in an appendix.

3.2 The Ricci-flat case

We now return to complex space-time and suppose that the Ricci tensor vanishes in addition to $\tilde{\psi}_{A'B'C'D'} = 0$. This is the case if and only if the full Riemann curvature is anti-self-dual, and this is equivalent to the condition that the primed spin connection is flat, so that there exists a two complex dimensional vector space $\mathbb{C}^2$ of covariantly constant primed spinor fields.

We saw in §3.1 that each point in $\mathcal{T}$ corresponds to an α-surface in space-time with a non-vanishing parallelly propagated tangent spinor field $\pi_{A'}(x)$ defined over it. If the full Riemann curvature is self-dual, then a tangent spinor $\pi_{A'}(x)$ on an α-surface is naturally the restriction of a covariantly constant spinor field on the whole space-time and determined by a constant spinor $\pi_{A'} \in \mathbb{C}^2$, e. g. the value of the covariantly constant spinor field $\pi_{A'}(x_0)$ at some point x_0. Thus we have a projection $p : \mathcal{T} \to \mathbb{C}^2 - \{0\}$ that takes an α-plane with tangent spinor $\pi_{A'}(x)$ to $\pi_{A'}(x_0)$.

We can use this projection to characterise the twistor space for a Ricci-flat space-time. A non-projective twistor space is a a complex 4-manifold $\mathcal{T}$ satisfying the three conditions given in §3.1. Such a twistor space corresponds to a conformally anti-self-dual space-time, and for this to be Ricci-flat, the twistor space $\mathcal{T}$ must in addition have

- a projection $p : \mathcal{T} \to \mathbb{C}^2 - \{0\}$ such that $p_* \Upsilon = \pi_{A'} \partial / \partial \pi_{A'}$.

This condition arises because Υ generates scalings of the tangent spinors to α-planes.

The compatibility of Υ with the Euler vector field on $\mathbb{C}^2$ means that the projection descends to $p : P\mathcal{T} \to \mathbb{CP}^1$, giving a fibration over $\mathbb{CP}^1$ of the projective twistor space.[14] The fibres are two-dimensional complex manifolds (but have no linear structure in the curved case, although, as we will see, they do have certain symplectic and Poisson structures).

In order to clarify these conditions, we can introduce global coordinates $\pi_{A'}$ on the base $\mathbb{C}^2 - 0$ of the fibration $p : \mathcal{T} \to \mathbb{C}^2 - 0$ and use them to build local coordinates $(\omega^A, \pi_{A'})$ on $\mathcal{T}$. These coordinates will be homogeneous coordinates for $P\mathcal{T}$. As $\mathcal{T}$ is fibred over $\mathbb{C}^2 - 0$, the pull-back of the volume form gives a globally-defined two-form τ on $\mathcal{T}$ given by

$$\tau = \frac{1}{2} I_{\alpha\beta} \mathrm{d}Z^\alpha \wedge \mathrm{d}Z^\beta = \frac{1}{2} \epsilon^{A'B'} \mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi_{B'} ,$$

and a holomorphic 1-form

$$k = I_{\alpha\beta} Z^\alpha \mathrm{d}Z^\beta = \pi_{A'} \mathrm{d}\pi^{A'} \tag{3.9}$$

on $P\mathcal{T}$ (and $\mathcal{T}$) given by the pull-back of the holomorphic 1-form on $\mathbb{CP}^1$. We can now restrict our choice of coordinates ω^A so that

$$\mathrm{d}\Omega = \frac{1}{6} \epsilon_{\alpha\beta\gamma\delta} \mathrm{d}Z^\alpha \wedge \mathrm{d}Z^\beta \wedge \mathrm{d}Z^\gamma \wedge \mathrm{d}Z^\delta = 2\epsilon_{AB} \mathrm{d}\omega^A \wedge \mathrm{d}\omega^B \wedge \tau . \tag{3.10}$$

This can be expressed as the condition that we have a holomorphic $(2, 0)$ form μ on the fibres given in local coordinates by

$$\mu = \frac{1}{2} \epsilon_{AB} \mathrm{d}\omega^A \wedge \mathrm{d}\omega^B, \tag{3.11}$$

[14]Note that the existence of a projective twistor space with a projection to $\mathbb{CP}^1$ is not sufficient to reconstruct the projection $p : \mathcal{T} \to \mathbb{C}^2$ as, thinking of $\mathbb{C}^2 - 0$ as the total space of the $\mathbb{C}^*$ bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, $p^*\mathcal{O}(-1)$ will not in general be equivalent as a line bundle over $P\mathcal{T}$ to $\mathcal{T} \to P\mathcal{T}$. Given $p : P\mathcal{T} \to \mathbb{CP}^1$, in order to guarantee that there is a Ricci-flat metric in the conformal equivalence class, we need to require that $p^*\mathcal{O}(-1)$ is an equivalent line bundle to $\mathcal{T}$ as an independent condition.

where ϵ_{AB} is the constant alternating symbol (note that only contractions of this form with vertical vectors up the fibres are defined). Then

$$\mathrm{d}\Omega = 4\mu \wedge \tau\,, \qquad \Omega = 2\mu \wedge k\,. \tag{3.12}$$

Dually, there is a Poisson structure determined by a bi-vector $I^{\alpha\beta}$ and this is in turn given by ϵ^{AB}, the inverse of ϵ_{AB}, by

$$\{f, g\}_I := I^{\alpha\beta}\frac{\partial f}{\partial Z^\alpha}\frac{\partial g}{\partial Z^\beta} := \epsilon^{AB}\frac{\partial f}{\partial \omega^A}\frac{\partial g}{\partial \omega^B}.$$

Since $\mathrm{d}\Omega$ and τ are globally defined by construction, equation (3.12) implies that μ is globally defined up to the addition of multiples of $\mathrm{d}\pi_{A'}$. The Poisson structure $I^{\alpha\beta}$ is globally and unambiguously defined, as the relation $I^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta}$ determines it uniquely. We now consider the implications of the condition that these structures be globally defined. We introduce two coordinate patches: U_0 on which $\pi_{0'}$ does not vanish, and U_1 on which $\pi_{1'}$ does not vanish. We then introduce local coordinates 'up the fibres' of p, w_0^A on U_0 and w_1^A on U_1. These can be elevated to homogeneous coordinates on the respective patches by defining $\omega_0^A = \pi_{0'}w_0^A$ and $\omega_1^A = \pi_{1'}w_1^A$. The coordinates are related in the overlap by the patching relations

$$\omega_0^A = F^A(\omega_1^A, \pi_{A'})$$

for some transition function F^A, and these are required to be homogeneous: $F^A(\lambda\omega_1^A, \lambda\pi_{A'}) = \lambda F^A(\omega_1^A, \pi_{A'})$. This means that, as in the flat case, we can define the homogeneity operator $\Upsilon = Z_0^\alpha\partial/\partial Z_0^\alpha = Z_1^\alpha\partial/\partial Z_1^\alpha$.

The requirement that the Poisson structure be expressed in its normal form on each patch is that

$$\{f, g\}_I = I^{\alpha\beta}\frac{\partial f}{\partial Z_0^\alpha}\frac{\partial g}{\partial Z_0^\beta} = \epsilon^{AB}\frac{\partial f}{\partial \omega_0^A}\frac{\partial g}{\partial \omega_0^B} = I^{\alpha\beta}\frac{\partial f}{\partial Z_1^\alpha}\frac{\partial g}{\partial Z_1^\beta} = \epsilon^{AB}\frac{\partial f}{\partial \omega_1^A}\frac{\partial g}{\partial \omega_1^B}\,.$$

A similar condition arises for the μ and in both cases the condition amounts to the requirement

$$\epsilon^{AB} = \epsilon^{CD}\frac{\partial F^A}{\partial \omega_1^C}\frac{\partial F^B}{\partial \omega_1^D} \tag{3.13}$$

that the patching conditions preserve ϵ^{AB}.

Given a global $I^{\alpha\beta}$, the equation

$$\frac{1}{2}I^{\alpha\beta}\epsilon_{\alpha\beta\gamma\delta} = I_{\gamma\delta}$$

determines globally the scale of $\epsilon_{\alpha\beta\gamma\delta}$, and vice versa. Thus, the condition for Ricci flatness can be expressed as the condition that we have a global holomorphically defined simple bi-vector $I^{\alpha\beta}$ that determines a Poisson structure, and we will refer to this as the infinity twistor, as in the flat case.[15]

An infinitesimal deformation f^α of the complex structure is an element of $H^1(P\mathcal{T}, T^{(1,0)})$, represented either as a Čech cocycle or as a Dolbeault form. The condition that it preserves the Poisson structure $I^{\alpha\beta}$ is that it is a Hamiltonian vector field that can be expressed as

$$f^\alpha = I^{\alpha\beta}\frac{\partial h}{\partial Z^\beta}$$

for some $h \in H^1(P\mathcal{T}, \mathcal{O}(2))$. This is the linearised form of (3.13). Whereas the Penrose transform of a general f^α subject to the gauge equivalence under $f^\alpha \to f^\alpha + a(Z)Z^\alpha$ gives a spin-2 field ψ_{ABCD} satisfying the higher derivative equation (3.3), the Penrose transform of h gives a spin-2 field ψ_{ABCD} satisfying the usual spin-2 equations

$$\nabla^{AA'}\psi_{ABCD} = 0\,. \qquad (3.14)$$

3.2.1 Ricci-flat case in split signature

In the second of the two approaches to the split signature non-linear graviton construction, the complex twistor space is taken to be $\mathbb{PT} = \mathbb{CP}^3$, and conformally anti-self-dual space-times are constructed from deformations of a real slice $P\mathcal{T}_{\mathbb{R}}$, which is itself an arbitrary small deformation of the real subspace $\mathbb{RP}^3$. However, in the Ricci flat case, $P\mathcal{T}_{\mathbb{R}}$ is no longer an arbitrary deformation; instead it is subject to certain conditions as will now be explained.

Again we take $\mathbb{T}$ to have an infinity twistor $I^{\alpha\beta}$ defined on it, and this determines a projection from $\mathbb{T}' = \mathbb{T} - \{\pi_{A'} = 0\}$ to $\mathbb{C}^2 - 0$ given by $Z^\alpha \to \pi_{A'}$ together with the corresponding projection $p : \mathbb{PT}' \to \mathbb{CP}^1$. This should be compatible with the real slice in the sense that $P\mathcal{T}_{\mathbb{R}}$ should project to $\mathbb{RP}^1 \subset \mathbb{CP}^1$. Equivalently, $P\mathcal{T}_{\mathbb{R}}$ should lie inside the real codimension-1 hypersurface $\Sigma := p^{-1}(\mathbb{RP}^1) \subset \mathbb{PT}'$, which can also be defined by the equation $\pi_{A'}\bar{\pi}^{A'} = 0$ with $\bar{\pi}_{A'} = (\bar{\pi}_{0'}, \bar{\pi}_{1'})$ the standard complex conjugation. This is the analogue of the existence of the projection $p : P\mathcal{T} \to \mathbb{CP}^1$ and we need to express the second part of the condition for Ricci flatness in this context.

On $\mathbb{PT}'$ the line bundles $\mathcal{O}(n)$ of homogeneous functions of degree n are equal to the pull-backs of the corresponding line bundles from $\mathbb{CP}^1$. Thus, on Σ, the complex line bundles $\mathcal{O}(n)$ naturally have a fibrewise complex conjugation fixing the real sub-bundles $\mathcal{O}_{\mathbb{R}}(n)$, which are the pull-backs of the corresponding real sub-bundles of $\mathcal{O}(n)$ on $\mathbb{RP}^1$ (i. e. these real line sub-bundles are spanned by homogeneous polynomials of degree n in $\pi_{A'}$ with real coefficients).

[15] In fact, if we relax the simplicity condition, we obtain the condition that the space-time admits an Einstein metric for which the Ricci scalar can be non-zero.

The second condition necessary in order that $P\mathcal{T}_{\mathbb{R}} \subset \mathbb{PT}$ corresponds to a Ricci-flat anti-self-dual conformal structure is that the $\mathcal{O}(4)$-valued 3-form Ω, when restricted to $P\mathcal{T}_{\mathbb{R}}$, lies in $\mathcal{O}_{\mathbb{R}}(4)$, or equivalently that the restriction to $P\mathcal{T}_{\mathbb{R}}$ of the $\mathcal{O}(2)$-valued 2-form $\mu = \frac{1}{2}\mathrm{d}\omega^A \wedge \mathrm{d}\omega_A$ up the fibres is real. This can be stated geometrically by observing first that, on each 4 real-dimensional fibre of p over $\mathbb{RP}^1$, the form μ defines a complex symplectic form with values in $\mathcal{O}(2)$, and its imaginary part defines a real symplectic form ϖ with values in $\mathcal{O}_{\mathbb{R}}(2)$. Our requirement is then that on each fibre $p^{-1}(\pi_{A'})$ of p over $\mathbb{RP}^1$, the intersection of $P\mathcal{T}_{\mathbb{R}}$ with $p^{-1}(\pi_{A'})$ should be Lagrangian with respect to ϖ, i.e., $\varpi|_{P\mathcal{T}_{\mathbb{R}} \cap p^{-1}(\pi_{A'})} = 0$ for each $\pi_{A'}$. This will guarantee that μ is real on restriction to $P\mathcal{T}_{\mathbb{R}}$, since we have required that the restriction of its imaginary part ϖ vanishes; it then follows from equation (3.12) that Ω is real.

An infinitesimal deformation of $P\mathcal{T}_{\mathbb{R}}$ preserving this condition is therefore generated by a Hamiltonian vector field preserving μ, and so it is determined by a Hamiltonian function h which will be a global section of $\mathcal{O}_{\mathbb{R}}(2)$ defined over $P\mathcal{T}_{\mathbb{R}}$ (a finite deformation can then be obtained from a generating function).

To be more explicit, we can decompose ω^A into its real and imaginary parts, $\omega^A = \omega^A_R + i\omega^A_I$ where ω^A_R and ω^A_I are real; then $\varpi = 2\mathrm{d}\omega^A_R \wedge \mathrm{d}\omega_{IA}$. Assuming the deformation to be transverse to $\partial/\partial\omega^A_I$, we can express $P\mathcal{T}_{\mathbb{R}}$ in Σ, on which $\pi_{A'}$ is real, as the graph

$$\omega^A_I = F^A(\omega^A_R, \pi_{A'}) \,,$$

where F^A has homogeneity degree one. Then the Lagrangian condition is

$$\frac{\partial}{\partial \omega^A_R} F^A = 0 \,.$$

These conditions can be solved by introducing a smooth real function $H(\omega^A_R, \pi_{A'})$ on $\mathbb{T}_{\mathbb{R}}$ of homogeneity degree two and defining

$$F^A(\omega^A_R, \pi_{A'}) = \epsilon^{AB} \frac{\partial H}{\partial \omega^B_R} \,.$$

It can be seen that this automatically incorporates the condition (3.8).

Infinitesimally, a deformation of $\mathbb{PT}_{\mathbb{R}}$ to $P\mathcal{T}_{\mathbb{R}}$ is given by pushing $\mathbb{PT}_{\mathbb{R}}$ along the vector field

$$if^\alpha(Z^\beta_R)\frac{\partial}{\partial Z^B_I} = iI^{\alpha\beta}\frac{\partial h}{\partial Z^\alpha_R}\frac{\partial}{\partial Z^B_I} = i\epsilon^{AB}\frac{\partial h}{\partial \omega^A_R}\frac{\partial}{\partial \omega^B_I} \,,$$

where we have written $Z^\alpha = Z^\alpha_R + iZ^\alpha_I$ for Z^α_R and Z^α_I real, and $h = h(Z^\alpha_R)$ is the infinitesimal analogue of H. The vector field is understood to be a normal vector field to the real slice, so it can be taken to be imaginary.

As a final note, we observe that the hypersurface Σ divides $\mathbb{PT}$ into two halves $\mathbb{PT}^{\pm}$ according to $\pm i\pi_{A'}\bar{\pi}^{A'} > 0$. The holomorphic discs in $\mathbb{PT}$ with boundary on $\mathbb{RP}^3$ divide into those that lie entirely in Σ, and those that lie in one of $\mathbb{PT}^{\pm}$. Those in $\mathbb{PT}^{\pm}$ correspond to two distinct copies $\mathbb{M}^{\pm}$ of space-time $\mathbb{R}^4$, whereas those in Σ correspond to points at (null) infinity. We will wish to work with just one copy of space-time, so we discard $\mathbb{PT}^-$ and work only with the holomorphic discs in $\mathbb{PT}^+$ and hence just the one copy $\mathbb{M}^+$ of space-time.

3.2.2 Superspace, super-twistor space and anti-self-dual supergravity

We can consider deformations of super-twistor space $\mathbb{PT}'_{[N]}$ to obtain anti-self-dual solutions to the conformal supergravity equations. The formal definition of such a deformed complex supermanifold has been studied in the mathematics literature [51, 52]. Here we use the more general physics formulation in which both fermionic coordinates and fermionic constants are allowed. A supermanifold is constructed by patching together coordinate charts $\{U_i\}$ with coordinates $Z_i^I = (Z_i^\alpha, \psi_i^a)$ on each patch, where the Z_i^α are bosonic and the ψ_i^a fermionic. On the overlaps, the coordinates are related by patching functions

$$Z_i^I := (Z_i^\alpha, \psi_i^a) = P_{ij}^I(Z_j^J) := (P_{ij}^\alpha(Z_j^J), P_{ij}^a(Z_j^J)) ,$$

where P_{ij}^α is an even function, and P_{ij}^a is odd.[16] We also require that the matrices $\partial P_{ij}^I/\partial Z_j^J$ have non-zero super-determinant (in fact, it must be possible to choose coordinates so that it is equal to 1 in the $N = 4$ case for which the super-twistor spaces are super-Calabi-Yau; note that our projective twistor spaces are not Calabi-Yau for general N).

A complex supermanifold, e. g. $P\mathcal{T}_{[N]}$, is composed of an underlying ordinary complex manifold, $P\mathcal{T}$ (the 'body') with patching functions $P_{ij}^\alpha(Z_j^\beta, 0)$ with all anti-commuting coordinates and parameters set to zero, and a rank N vector bundle $E \to P\mathcal{T}$ (the 'soul') whose patching functions are $\partial P_{ij}^a/\partial\psi_j^b|_{\psi_j^b=0}$, again with all odd parameters set to zero. It is an important feature of generic complex supermanifolds that they are not in general obtained by simply reversing the Grassmann parity of the coordinates up the fibres of the vector bundle $E \to P\mathcal{T}$ (whereas this is the case for real supermanifolds). The higher derivatives of the patching functions with respect to odd variables encode information that cannot be gauged away.

One necessary restriction for a complex supermanifold to be a super-twistor space is the requirement that the ψ^a have homogeneity degree 1. One way of expressing this is to say that the bundle E should have degree $-N$ (i. e. first Chern class $-N$). As discussed earlier, the space $\mathbb{C}\mathcal{M}$ of rational curves in $P\mathcal{T}$ in the appropriate topological class will

[16]Here fermionic parameters are allowed in these functions.

be a space-time with anti-self-dual conformal structure. These rational curves will have deformations away from the body, and their moduli space $\mathbb{C}\mathcal{M}^+_{[N]}$ will be chiral superspace with body $\mathbb{C}\mathcal{M}$. The full superspace is obtained as the space of flags $\mathbb{CP}^{1|0} \subset \mathbb{CP}^{1|N}$ in $P\mathcal{T}_{[N]}$, with the chiral and anti-chiral superspaces arising as the space of $\mathbb{CP}^{1|0}$s and $\mathbb{CP}^{1|N}$s respectively. We are not aware of a full presentation of this construction in the literature, and to give one here would take us too far afield.

An infinitesimal deformation of $P\mathcal{T}_{[N]}$ can be obtained by varying the patching functions, and such an infinitesimal variation is given in local coordinates on the overlap of two coordinate charts by a tangent vector $f = f^\alpha \partial/\partial Z_i^\alpha + f^a \partial/\partial \psi_i^a$, where f^α is even and f^a is odd. To deform the complex structure, we use such a vector field on each overlap and a nontrivial deformation is defined modulo infinitesimal coordinate transformations on the open sets; thus the nontrivial deformations are parametrised by the cohomology group $H^1(\mathbb{PT}'_{[N]}, T^{(1,0)})$, where $T^{(1,0)}$ is (the sheaf of sections of) the holomorphic tangent bundle of the supermanifold. This group was studied in the case of $N = 4$ in [6] and the spectrum of $N = 4$ conformal supergravity was obtained (see the end of section 4). A similar analysis can be carried out for other values of N.

In order to obtain an anti-self-dual version of Einstein supergravity, we need to impose the supersymmetric analogues of the constraints imposed on a twistor space to obtain Ricci-flat anti-self-dual four-manifolds as described in §3.2. There is now some ambiguity because, in the supersymmetric case, the restriction to Poincaré invariance gives a projection to $\mathbb{CP}^{1|N}$ and hence also to $\mathbb{CP}^{1|0}$. In order to obtain a straightforward supermultiplet starting from helicity -2 and increasing to helicity $(N-4)/2$ in the linearised theory, we require that we have a projection

$$p_1 : P\mathcal{T}_{[N]} \to \mathbb{CP}^{1|N} \tag{3.15}$$

(and thence a further projection $p : P\mathcal{T}_{[N]} \to \mathbb{CP}^{1|0}$) and a global holomorphic volume form Ω_s with values in the pull-back of $\mathcal{O}(4-N)$ from $\mathbb{CP}^{1|0}$.

To make this more explicit, we introduce the non-projective super-twistor space $\mathcal{T}_{[N]}$, which as before can be defined as the total space of the pull-back of the line bundle $\mathcal{O}(-1)$ from $\mathbb{CP}^1$ using p. The projection p_1 then determines a projection $p : \mathcal{T}_{[N]} \to \mathbb{C}^{1|N}$. We can introduce coordinates $(\pi_{A'}, \psi^a)$, $A' = 0', 1'$, $a = 1\ldots, N$ on $\mathbb{C}^{1|N}$ and complete these to a local coordinate system Z^I on $\mathcal{T}_{[N]}$ by adjoining local coordinates ω^A $(A = 0, 1)$ of homogeneity degree 1.

In this case we can define 'infinity twistors' I_{IJ} and I^{IJ} on the non-projective twistor space $\mathcal{T}_{[N]}$ by setting

$$\begin{aligned} I_{IJ} \mathrm{d}Z^I \wedge \mathrm{d}Z^J &= \mathrm{d}\pi^{A'} \wedge \mathrm{d}\pi_{A'} \,, \\ I^{IJ} (\mathrm{d}\Omega_s)_{IJK_1\ldots K_{N+2}} \mathrm{d}Z^{K_1} \ldots \mathrm{d}Z^{K_{N+2}} &= I_{IJ} \mathrm{d}Z^I \wedge \mathrm{d}Z^J \Pi_{a=1}^N \mathrm{d}\psi^a \,. \end{aligned}$$

It is now straightforward to see that deformations of super-twistor space preserving these structures must be of the form

$$f^I \frac{\partial}{\partial Z^I} = I^{IJ} \frac{\partial h}{\partial Z^I} \frac{\partial}{\partial Z^J}\,, \qquad h \in H^1(P\mathcal{T}_{[N]}, \mathcal{O}(2))\,.$$

Such an h precisely describes an anti-self-dual supergravity multiplet, starting with helicity 2 and going down to helicity $(4-N)/2$; this will be discussed in more detail in section 8.3.

It is also possible to consider deformations of $\mathbb{PT}'_{[N]}$ that preserve less structure. For example, later we will consider the case where we only preserve the projection $p: P\mathcal{T}_{[N]} \to \mathbb{CP}^1$. In such cases, the space of possible deformations will be larger and correspond to more fields on space-time.

4 The Berkovits twistor string

4.1 The Berkovits open string theory

The Berkovits string is a theory of maps from the world-sheet Σ to a curved super-twistor space with coordinates $Z^I = (\omega^A, \pi_{A'}, \psi^a)$, $\tilde{Z}^I = (\tilde{\omega}^A, \tilde{\pi}_{A'}, \tilde{\psi}^a)$. In the following, we will find it useful to use a notation that can handle different signatures and different reality properties in a unified way. There are three different cases that we will consider:

(i) Z^I are complex coordinates on a complex super-twistor space $\mathcal{T}$ and $\tilde{Z}^I$ are the complex conjugate coordinates $\tilde{Z} = (Z)^*$,

(ii) $Z^I, \tilde{Z}^I$ are independent real coordinates on a space $\mathcal{T}_{\mathbb{R}} \times \mathcal{T}_{\mathbb{R}}$ for some real twistor space $\mathcal{T}_{\mathbb{R}}$,

(iii) $Z^I, \tilde{Z}^I$ are independent complex coordinates on a space $\mathcal{T} \times \mathcal{T}$ for some complex twistor space $\mathcal{T}$.

For space-times of signature $++++$ or $+++-$, the twistors are necessarily complex, while for signature $++--$ either complex or real twistors can be used. In the flat case, $Z^I, \tilde{Z}^I$ are complex conjugate coordinates on $\mathbb{C}^{4|4}$, real coordinates on $\mathbb{R}^{4|4} \times \mathbb{R}^{4|4}$, or complex coordinates on $\mathbb{C}^{4|4} \times \mathbb{C}^{4|4}$; then we write $Z^I = (\omega^A, \pi_{A'}, \psi^a)$, $\tilde{Z}^I = (\tilde{\omega}^A, \tilde{\pi}_{A'}, \tilde{\psi}^a)$. For open strings in any of the three cases, the boundary of the world-sheet $\partial\Sigma$ is constrained to map to the submanifold defined by $Z = \tilde{Z}$. For case (i) with complex Z, this is the real submanifold $P\mathcal{T}_{\mathbb{R}}$ that arose in §3.1.1.

We use world-sheet coordinates $\sigma, \tilde{\sigma}$ with world-sheet metric $ds^2 = 2d\sigma d\tilde{\sigma}$. For Euclidean world-sheet signature, $\sigma, \tilde{\sigma}$ are complex conjugate coordinates $\tilde{\sigma} = \sigma^*$ while for Lorentzian world-sheet signature, $\sigma, \tilde{\sigma}$ are independent real null coordinates.

The fields include maps $Z^I(\sigma,\tilde{\sigma}), \tilde{Z}^I(\sigma,\tilde{\sigma})$ from the world-sheet to super-twistor space and these are world-sheet scalar fields. The action is

$$S = \int d^2\sigma \left(Y_I \tilde{\partial} Z^I + \tilde{Y}_J \partial \tilde{Z}^J - \tilde{A} J - A \tilde{J} \right) + S_C, \tag{4.1}$$

where $Y_I, \tilde{Y}_I$ are conjugate momenta of conformal dimensions $(1,0)$ and $(0,1)$ respectively and $\partial = \partial/\partial\sigma$, $\tilde{\partial} = \partial/\partial\tilde{\sigma}$. The world-sheet gauge fields $A, \tilde{A}$ couple to currents

$$J = Y_I Z^I, \qquad \tilde{J} = \tilde{Y}_I \tilde{Z}^I, \tag{4.2}$$

so that there is a local symmetry

$$Z^I \to t Z^I, \qquad Y_I \to \frac{1}{t} Y_I, \qquad \tilde{Z}^I \to \tilde{t} \tilde{Z}^I, \qquad \tilde{Y}_I \to \frac{1}{\tilde{t}} \tilde{Y}_I,$$

$$\tilde{A} \to \tilde{A} + \frac{1}{t} \tilde{\partial} t, \qquad A \to A + \frac{1}{\tilde{t}} \partial \tilde{t}. \tag{4.3}$$

This symmetry ensures that the theory projects to one defined on a projective twistor space $\mathcal{PT}$, $P\mathcal{T}_{\mathbb{R}} \times P\mathcal{T}_{\mathbb{R}}$ or $P\mathcal{T} \times P\mathcal{T}$.

The action is real for Euclidean world-sheets if one chooses case (i) above, all variables are complex, and the tilde operation is complex conjugation, so that for any field Φ, $\tilde{\Phi} = \Phi^*$. For Lorentzian world-sheets the action is real if all variables are real, requiring signature $++--$, and $\Phi, \tilde{\Phi}$ are independent real variables. For Euclidean world-sheets the parameter t is complex and the gauge symmetry (4.3) is $GL(1,\mathbb{C})$ while for Lorentzian world-sheets $t, \tilde{t}$ are independent real parameters and the gauge group is $GL(1,\mathbb{R}) \times GL(1,\mathbb{R})$. For the case of Lorentzian world-sheets in which $\Phi, \tilde{\Phi}$ are independent real variables, 'Wick rotation' gives a theory on Euclidean world-sheets in which $\Phi, \tilde{\Phi}$ become independent complex variables, leading to case (iii) above, and it is the action of this theory that is used in the Euclidean path integral.

The term S_C in (4.1) is the action for an additional matter system which is a conformal field theory with Virasoro central charges $c_C = \tilde{c}_C$ and currents j^r and $\tilde{j}^r$, for $r = 1, \ldots \dim G$. Here G is some group whose Kač-Moody algebra is generated by the currents. The Kač-Moody central charges are denoted by $k = \tilde{k}$ and the group G becomes a Yang-Mills gauge group in space-time.

Open strings are included in the model with the boundary conditions

$$Z^I = \tilde{Z}^I, \qquad Y_I = \tilde{Y}_I, \qquad j^r = \tilde{j}^r \tag{4.4}$$

on $\partial\Sigma$. For complex Z with $\tilde{Z} = Z^*$, the string endpoints lie in a real subspace $\mathcal{T}_{\mathbb{R}}$ of $\mathcal{T}$, which projects onto a real subspace $\mathcal{PT}_{\mathbb{R}}$ of $\mathcal{PT}$. In the flat case, this is $\mathbb{RP}^{3|4} \subset \mathbb{CP}^{3|4}$ and (4.4) breaks the $SL(4|4;\mathbb{C})$ symmetry to $SL(4|4;\mathbb{R})$. This boundary condition is natural for the case of split space-time signature $++--$, where the real subspace plays a natural and important role, as was discussed in §2.3 and §3.1.1. As the interpretation of the results for other signatures is less clear, we will restrict ourselves to the split space-time signature $++--$ in what follows. For independent real $Z, \tilde{Z}$ and split space-time signature, the ends of the strings lie in the diagonal $\mathcal{PT}_{\mathbb{R}}$ in $\mathcal{PT} = \mathcal{PT}_{\mathbb{R}} \times \mathcal{PT}_{\mathbb{R}}$. For the flat twistor space $\mathbb{PT} = \mathbb{RP}^{3|4} \times \mathbb{RP}^{3|4}$, the endpoints lie in the diagonal $\mathbb{RP}^{3|4}$, breaking the conformal symmetry from $SL(4|4;\mathbb{R}) \times SL(4|4;\mathbb{R})$ to the diagonal subgroup. In either case, the boundary theory lives on a real twistor space $\mathcal{PT}_{\mathbb{R}}$ (which is $\mathbb{RP}^{3|4}$ in the flat case) and the scaling symmetry is broken to $GL(1,\mathbb{R})$ by the boundary conditions.

Quantisation gives the usual conformal gauge ghosts (b, c) and $(\tilde{b}, \tilde{c})$ together with $GL(1)$ ghosts (u, v) and $(\tilde{u}, \tilde{v})$ (v and $\tilde{v}$ have conformal dimensions $(0, 0)$, while u and $\tilde{u}$ have dimensions $(1, 0)$ and $(0, 1)$). Variables $\tilde{\phi}$ with a tilde are right-moving ($\partial\tilde{\phi} = 0$), while those without are left-moving ($\tilde{\partial}\phi = 0$). The matter stress-energy tensor is

$$\begin{aligned} T^m &= Y_I \partial Z^I + T^C \\ \tilde{T}^m &= \tilde{Y}_I \tilde{\partial} \tilde{Z}^I + \tilde{T}^C, \end{aligned} \tag{4.5}$$

where T^C and $\tilde{T}^C$ are the left and right-moving stress-energy tensors for the current algebra. The stress-energy tensor for the ghosts is

$$\begin{aligned} T^{gh} &= b\partial c + \partial(bc) + u\partial v \\ \tilde{T}^{gh} &= \tilde{b}\tilde{\partial}\tilde{c} + \tilde{\partial}(\tilde{b}\tilde{c}) + \tilde{u}\tilde{\partial}\tilde{v}. \end{aligned} \tag{4.6}$$

The open string theory is defined by the boundary conditions (4.4) on the twistor variables, together with additional boundary conditions on the ghosts:

$$c = \tilde{c}, \qquad b = \tilde{b}, \qquad v = \tilde{v}, \qquad u = \tilde{u}. \tag{4.7}$$

The BRST charges are

$$\begin{aligned} Q &= \oint d\sigma\, (cT + vJ + cu\partial v + cb\partial c) \\ \tilde{Q} &= \oint d\tilde{\sigma} \left(\tilde{c}\tilde{T} + \tilde{v}\tilde{J} + \tilde{c}\tilde{u}\tilde{\partial}\tilde{v} + \tilde{c}\tilde{b}\tilde{\partial}\tilde{c} \right) \end{aligned} \tag{4.8}$$

and they are nilpotent provided the additional matter system has Virasoro central charge $c_C = 28$; this value cancels the contributions $c = -26$ of the (b, c) system and $c = -2$ of the (u, v) system to the conformal anomaly. There is no $GL(1) \times GL(1)$ anomaly because of cancellation between bosons and fermions.

The physical open string states are BRST cohomology classes represented by vertex operators that are $GL(1)$ neutral and are dimension one primary fields with respect to the

Virasoro and Kač-Moody generators (4.5), (4.6) and (4.2). The super-Yang-Mills vertex operators are the dimension one operators constructed with Kač-Moody currents of the auxiliary matter system [2]:

$$V_\phi = j_r \phi^r(Z), \tag{4.9}$$

where the $\phi^r(Z)$ are any Lie-algebra-valued functions that are invariant under scalings of Z^I (i. e. any Lie-algebra-valued functions on $\mathbb{RP}^{3|4}$) and have conformal weight zero. The dimension one vertex operators [6]

$$V_f = Y_I f^I(Z), \qquad V_g = g_I(Z)\partial Z^I \tag{4.10}$$

are $GL(1)$-invariant provided the functions f^I carries $GL(1)$ charge $+1$ (i. e. it is in $\mathcal{O}(1)$) and g_I carries $GL(1)$ charge -1 (i. e. it is in $\mathcal{O}(-1)$). They will be physical if the f^I and g_I satisfy

$$\partial_I f^I = 0, \qquad Z^I g_I = 0. \tag{4.11}$$

Changing f^I, g_I by

$$\delta f^I = Z^I \Lambda, \qquad \delta g_I = \partial_I \chi, \tag{4.12}$$

gives operators in the same BRST cohomology class as those given in (4.10), so that (4.12) are gauge invariances giving physically equivalent states [2, 6]. The vertex operators (4.10) give the states of conformal supergravity [6].

Since f^I has $GL(1)$ charge 1, the vector field

$$f = f^I \frac{\partial}{\partial Z^I} \tag{4.13}$$

on $\mathcal{T}$ is invariant under scaling, and the first equivalence relation in (4.12) means that f can be interpreted as a vector field on $\mathcal{PT}$ [6]. The first constraint in (4.11) means that f is a volume-preserving vector field. The second constraint in (4.11) means that the one-form

$$g = g_I dZ^I \tag{4.14}$$

is well-defined on $\mathcal{PT}$ [6]. The second gauge equivalence in (4.12) means that g is an abelian gauge field.

The functions $\phi^r(Z)$ in (4.9) are superfields which can be expanded in terms of ordinary functions on twistor space with values in the line bundles $\mathcal{O}(0), \mathcal{O}(-1), \mathcal{O}(-2)$, $\mathcal{O}(-3)$, $\mathcal{O}(-4)$. By the Penrose transform, these represent fields of helicities $(1, \frac{1}{2}, 0, -\frac{1}{2}, -1)$ with the correct R-symmetry representations to describe $N = 4$ super-Yang-Mills states [1, 2].

Likewise, the spectrum of Minkowski space helicity states associated with the vertex operators (4.10) follows from the expansions of the superfields $f^I(Z)$ and $g_I(Z)$ in powers of ψ [6]. The analysis of [6] shows that, taking (4.11,4.12) into account, $f^A(Z)$ and $f^{A'}(Z)$ each describe the helicity states $(+2, +\frac{3}{2}, +1, +\frac{1}{2}, 0)$ of an $N = 4$ supergravity multiplet (with the correct R-symmetry representations) while $f^a(Z)$ describe the helicity states $(+\frac{3}{2}, +1, +\frac{1}{2}, 0, -\frac{1}{2})$ of (four) gravitino multiplets. Similarly, $g_A, g_{A'}$ give two supergravity multiplets with negative helicities $(0, -\frac{1}{2}, -1, -\frac{3}{2}, -2)$ and g_a give (four) gravitino multiplets $(+\frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2})$. Taken together, the space-time fields described by the vertex operators V_f and V_g given in (4.10) can be identified with the physical states of $N = 4$ conformal supergravity.

4.2 Generalised boundary conditions

In split signature, the non-linear graviton can be constructed from deformations of a real subspace $P\mathcal{T}_{\mathbb{R}}$ in a fixed flat twistor space $\mathbb{PT}$, as was reviewed in §3.1.1. This suggests a modification of the Berkovits string model in which, for the case (i) of complex Z, the strings live in $\mathbb{PT}$ and the open string boundaries are constrained to lie in the general subspace $P\mathcal{T}_{\mathbb{R}}$ defined in terms of functions F^α by (3.5) instead of the real subspace defined by the condition $Z = Z^*$. We then consider a string theory in which the boundary condition $Z^I = \tilde{Z}^I$ is replaced with

$$Z^I - \tilde{Z}^I = \hat{F}^I(Z^J + \tilde{Z}^J) \tag{4.15}$$

for some function $\hat{F}^I$ of homogeneity degree one. There is a gauge freedom in the definition of F, which can be multiplied by a function of homogeneity degree 0 (see also the discussion following equation (3.5)). This can be fixed by imposing the condition that $\mathrm{sdet}(\delta^I_J + \partial_J \hat{F}^I) = \mathrm{sdet}(\delta^I_J - \partial_J \hat{F}^I)$ where sdet denotes the super-determinant. This is the condition that the Calabi-Yau forms $\mathrm{d}\Omega$ in Z^α and in $\tilde{Z}^\alpha$ agree. The corresponding boundary conditions for Y are found by requiring the surface term in the variation of the action to vanish. Varying the action (4.1) gives terms proportional to the field equations together with a surface term

$$\int_{\partial\Sigma} (Y_I \delta Z^I - \tilde{Y}_I \delta \tilde{Z}^I) = \frac{1}{2} \int_{\partial\Sigma} \left[(Y_I - \tilde{Y}_I)(\delta Z^I + \delta \tilde{Z}^I) + (Y_I + \tilde{Y}_I)(\delta Z^I - \delta \tilde{Z}^I) \right], \tag{4.16}$$

where the boundary $\partial\Sigma$ is specified by $\sigma + \tilde{\sigma} = 0$. Using equation (4.15), this will vanish if the boundary conditions for Y are modified to become

$$Y_J - \tilde{Y}_J = -\hat{F}^I{}_{,J}(Y_I + \tilde{Y}_I)\,. \tag{4.17}$$

In the cases (i) or (iii) above in which $\tilde{Z}^\alpha$ and Z^α are independent quantities, the deformation of the boundary condition amounts to a deformation of the location of the diagonal subspace inside $\mathbb{PT}_{\mathbb{R}} \times \mathbb{PT}_{\mathbb{R}}$ or $\mathbb{PT} \times \mathbb{PT}$ where the world-sheet boundary is constrained to lie. In the complex case (ii) in which Z is complex and $\tilde{Z} = (Z)^*$ and the boundary is the real axis $\sigma = \sigma^*$, it is useful to write $\hat{F} = iF$ so that (4.15) becomes

$$Z^I - \bar{Z}^I = iF^I(Z^J + \bar{Z}^J)\,, \tag{4.18}$$

where $\mathrm{sdet}(\delta^I_J + i\partial_J F^I)$ is constrained to be real (in order to fix the gauge freedom). This is a supersymmetric version of (3.5), and the boundary condition (4.17) becomes

$$Y_J - \bar{Y}_J = -iF^I{}_{,J}(Y_I + \bar{Y}_I)\,. \tag{4.19}$$

With these boundary conditions, the worldsheets of degree 1 correspond to points of the compactified space-time $S^2 \times S^2$, and this has the non-trivial split signature anti-self-dual conformal structure determined by F^I. The construction of §3.1.1 then suggests that the geometric interpretation of the vertex operator $V_f = Y_I f^I$ should be that f^I determines an infinitesimal variation in F^I, and so deforms the boundary conditions.

Next we turn to the interpretation of the vertex operator $V_g = g_I \partial Z^I$. If one adds a boundary term

$$\int_{\partial\Sigma} G_I(Z^J + \tilde{Z}^J)\partial(Z^I + \tilde{Z}^I) \tag{4.20}$$

to the action (4.1), for some $G_I = G_I(Z^J + \tilde{Z}^J)$, then the condition that the surface term in the variation of the action vanishes is now

$$Y_J - \tilde{Y}_J = -\hat{F}^I{}_{,J}(Y_I + \tilde{Y}_I) + 2G_{[I,J]}\partial(Z^J + \tilde{Z}^J)\,, \tag{4.21}$$

so that the surface term leads to a modification of the boundary conditions for Y. Then the vertex operator $g_I \partial Z^I$ corresponds to a deformation of G_I.

The quantisation of the string models based on the generalised boundary conditions (4.15) and (4.21) will be discussed elsewhere.

5 Gauged β-γ systems

5.1 1-form symmetries

The system (sometimes referred to as a β-γ system)

$$S = \int d^2\sigma Y_I \tilde{\partial} Z^I, \tag{5.1}$$

where the Z^I are coordinates on some manifold (or supermanifold) M, has recently been discussed in [32, 33]. The Berkovits twistor string has kinetic terms of this form, with super-twistor space as the target space. If $k^i = k^i_I dZ^I$ are 1-forms on M labeled by an index i, $i = 1, \dots p$, then the chiral currents

$$K^i = k^i_I \partial Z^I \tag{5.2}$$

are conserved:

$$\tilde{\partial} K^i = 0 \tag{5.3}$$

and generate a symmetry with parameters $\alpha_i(\sigma)$ satisfying $\tilde{\partial}\alpha_i = 0$,

$$\delta Z^I = 0, \qquad \delta Y_I = k^i_I \partial\alpha_i + 2\alpha_i k^i_{[I,J]} \partial Z^J. \tag{5.4}$$

The rigid symmetry with constant parameters was discussed in [32]. Both bosonic and fermionic local symmetries can be considered, and below we consider models with d bosonic currents and n fermionic currents and $p = d + n$. The currents K^i commute, so they satisfy an abelian Kač-Moody algebra with vanishing central charge:

$$[K^i(\sigma), K^j(\sigma')] = 0. \tag{5.5}$$

This can be promoted to a local symmetry by coupling to gauge fields $\tilde{B}_i$ to give the action

$$S = \int d^2\sigma \left(Y_I \tilde{\partial} Z^I - \tilde{B}_i K^i \right), \tag{5.6}$$

which is invariant under (5.4) and

$$\delta \tilde{B}_i = \tilde{\partial}\alpha_i \tag{5.7}$$

for general local parameters $\alpha_i(\sigma, \tilde{\sigma})$. Gauge-fixing and introducing ghosts s_i and anti-ghosts r^i gives the action

$$S = \int d^2\sigma \left(Y_I \tilde{\partial} Z^I + r^i \tilde{\partial} s_i \right), \tag{5.8}$$

and the BRST charge

$$Q = \oint d\sigma \, s_i K^i \tag{5.9}$$

is nilpotent.

For the vertex operator $V_f = f^I Y_I$,

$$[Q, V_f] = (\partial s_i) f^I k^i_I + 2 s_i f^I k^i_{[I,J]} \partial Z^J = \partial(s_i f^I k^i_I) - s_i [\mathcal{L}_f k^i]_I \partial Z^I \tag{5.10}$$

and so $f^I Y_I$ is BRST invariant provided

$$f^I k_I^i = 0, \qquad f^I k_{[I,J]}^i = 0, \tag{5.11}$$

while the integrated vertex operator $\int V_f$ is invariant (up to a surface term) provided the Lie derivative of k^i with respect to the vector field f vanishes,

$$\mathcal{L}_f k^i = 0. \tag{5.12}$$

Changing the vertex operator $g_I \partial Z^I$ by a BRST exact term leads to the symmetry

$$\delta g_I = \eta_i k_I^i \tag{5.13}$$

for any $\eta_i(Z)$, since $\eta_i k_I^i \partial Z^I = \{Q, \eta_i r^i\}$.

This can be generalised to the case in which the one-forms k^i are not globally-defined[17] but are local sections of a bundle [34]. For example, the k^i might be a local section of the co-frame bundle, i.e. a local basis for the cotangent bundle T^*M. If M is a bundle over some E, the k_i could be a local section of the co-frame bundle of E (or rather the pull-back of this co-frame bundle). We will be interested mainly in the case in which M is projective (super-)twistor space, and is a bundle over E where E is $\mathbb{CP}^1$ or $\mathbb{CP}^{1|N}$. Given an open cover $\{U_r\}$ of M, suppose there is a set of 1-forms k_r^i in each patch U_r, with

$$k_r^i = (L_{rs})^i{}_j k_s^j \tag{5.14}$$

in the overlaps $U_r \cap U_s$, and transition functions L_{rs} in $GL(d|n)$ if the k_r^i consist of d bosonic one-forms and n fermionic ones. The k_r^i are then sections of a bundle X over M, and we can introduce a connection one-form $(\hat{B}_r)_i = (\hat{B}_r)_{iI} dZ^I$ with transition functions

$$(\hat{B}_r)_{iI} = (L_{rs}^{-1})_i{}^j (\hat{B}_s)_{jI} + \partial_I \hat{\alpha}_i \tag{5.15}$$

for the bundle $\hat{X}$ whose structure group is the group of fibre translations (with parameters $\hat{\alpha}_i$). Then the gauged theory is well-defined provided the gauge fields $\tilde{B}_i$ are taken to be connections on the pull-back of $\hat{X}$ to a bundle over the world-sheet, by a similar construction to that given in [34]. The theory is locally the same as that described above.

5.2 1-form symmetries and scale symmetry

A natural generalisation of the construction of the last section would be to consider a set of vector fields $V_j = V_j^I(Z)\partial/\partial Z^I$ on M, and construct the currents $V_j^I Y_I$. A necessary

[17] As emphasised by E. Witten, a geometrically clearer formulation of the construction and of its generalisation can be given in terms of the distribution (i. e. the sub-bundle of the cotangent bundle T^*M of M) generated by the k^i. In particular, the distribution does not depend on the choice of basis for the one-forms k^i.

condition for the current algebra to close is that the V_j are closed under the Lie bracket, so that they generate the action of a group L on M. In certain circumstances, the corresponding symmetries can be gauged, resulting in a theory on the quotient space M/L. Thus the gauging leads to replacing M with M/L, and gauging symmetries from vectors and 1-forms on M is equivalent to gauging symmetries from 1-forms only on M/L. There is then no loss of generality in considering general M without gauging the symmetries generated by vector fields on M. However, it will be useful to consider the case of the Euler vector field

$$\Upsilon = Z^I \frac{\partial}{\partial Z^I} \tag{5.16}$$

generating the one-dimensional group L_S of scale transformations. Gauging the symmetries from 1-forms and Υ on M is then the same as gauging 1-forms alone on the projective space $PM = M/L_S$, but using the formulation on M will be useful for the Berkovits twistor string.

Suppose the one-forms k^i have scaling weights h_i under the action of (5.16), so that for each i

$$\mathcal{L}_\Upsilon k^i = h_i k^i \tag{5.17}$$

where $\mathcal{L}_\Upsilon$ is the Lie derivative with respect to Υ, and have constant vertical projections, so that $\iota(\Upsilon)k^i = e^i$ for some constants e^i, i.e.

$$Z^I k_I^i = e^i. \tag{5.18}$$

If $h_i = 0, e^i = 0$, then k^i is horizontal and is the pull-back of a form on PM, the projective space given by taking the quotient by the action of the scalings generated by Υ. Then the current $J = Y_I Z^I$ has the commutation relations

$$[J(\sigma), K^i(\sigma')] = h_i K^i(\sigma)\delta(\sigma - \sigma') + e^i \delta'(\sigma - \sigma') \tag{5.19}$$

for each i. If $Z^I = (Z^\alpha, Z^a)$ and $Y_I = (Y_\alpha, Y_a)$ where Y_α, Z^α with $\alpha = 1, ..., D$ are bosonic β-γ systems and Y_a, Z^a with $a = 1, ..., N$ are fermionic b-c systems, then

$$[J(\sigma), J(\sigma')] = \delta'(\sigma - \sigma')(D - N). \tag{5.20}$$

Then the currents J, K^i generate a Kač-Moody algebra which is non-abelian if the weights h_i are not all zero and which has central charges e^i, $D - N$. If the e^i were not constant, the algebra would not close and one would need to introduce the e^i as extra generators.

This symmetry can be gauged by introducing gauge fields $\tilde{A}, \tilde{B}_i$ only if $e^i = 0$, so that the k^i are all horizontal; it will now be assumed that this is the case. The gauged action is

$$S = \int d^2\sigma \left(Y_I \tilde{\partial} Z^I - \tilde{A} J - \tilde{B}_i K^i \right), \tag{5.21}$$

which is invariant under the gauge transformations given by (5.4) together with

$$\delta\tilde{A} = 0 \tag{5.22}$$

and

$$\delta\tilde{B}_i = \tilde{\partial}\alpha_i - h_i\tilde{A}\alpha_i. \tag{5.23}$$

It is also invariant under the scaling symmetry

$$Z^I \to tZ^I, \qquad Y_I \to \frac{1}{t}Y_I, \qquad \tilde{A} \to \tilde{A} + \frac{1}{t}\tilde{\partial}t \qquad \tilde{B}_i \to t^{-h_i}\tilde{B}_i. \tag{5.24}$$

Introducing ghosts v, s_i and anti-ghosts u, r^i, the BRST charge is now

$$Q = \oint d\sigma \left(vJ + \sum_i [s_i K^i - v h_i s_i r^i] \right). \tag{5.25}$$

The ghost s_i is a world-sheet scalar with scaling weight $-h_i$ (transforming as $s_i \to t^{-h_i} s_i$ under $GL(1)$) while the antighost r^i has world-sheet conformal dimension one and scaling weight h_i. Then Q^2 is proportional to $\int \kappa v \partial v$, where

$$\kappa = D - N - \sum_i \epsilon_i (h_i)^2 \tag{5.26}$$

with $\epsilon_i = 1$ for bosonic symmetries (with α_i a bosonic parameter) and $\epsilon_i = -1$ for fermionic symmetries (with α_i a fermionic parameter). The constant κ is the central charge for the Kač-Moody algebra generated by the currents

$$J_{gf} = J - \sum_i h_i s_i r^i \tag{5.27}$$

which generate scalings of the gauge-fixed action, and quantum consistency (cancellation of the anomaly in the scaling symmetry) requires $\kappa = 0$ [18].

6 Gauging the Berkovits twistor string

The formalism of the previous section will now be applied to the Berkovits twistor string, generalised to a target space $\mathcal{T}$ that is a supermanifold with D bosonic dimensions and

[18] It was pointed out to us by E. Witten that, if a global and everywhere nonzero function w exists on M then the last term (involving the scaling weigths h_i) in the anomaly (5.26) can be eliminated by adding to the BRST operator Q a term proportional to $\oint \partial v \log w$. This is natural in the formulation in terms of the distribution generated by the one-forms k^i rather than that using a specific choice of k^i adopted here.

N fermionic ones; the flat twistor space is $\mathbb{C}^{D|N}$, $\mathbb{R}^{D|N} \times \mathbb{R}^{D|N}$ or $\mathbb{C}^{D|N} \times \mathbb{C}^{D|N}$. The case of physical interest is $D = 4$, and we will see that, remarkably, this value is selected by anomaly cancellation in some of the models.

We saw in §3.2 that the twistor space $\mathcal{T}$ for a Ricci-flat space-time is fibred over $\mathbb{C}^2 - 0$, so that $P\mathcal{T}$ is fibred over $\mathbb{CP}^1$, and this in particular implies the existence of the 1-form k given by (3.9), corresponding to an infinity twistor. In the flat case, this requires working with $\mathbb{PT}' = \mathbb{CP}^3 - \mathbb{CP}^1$, which has such a fibration, whereas the full twistor space $\mathbb{CP}^3$ does not. In the supersymmetric case, $P\mathcal{T}$ is fibred over $\mathbb{CP}^{1|0}$ or $\mathbb{CP}^{1|N}$, and in the latter case a local basis of N fermionic 1-forms on $\mathbb{CP}^{1|N}$ pull back to N locally defined fermionic 1-forms k^a on super-twistor space. In this section we will assume that the target space $\mathcal{T}$ is equipped with a set of 1-forms k^i and gauge the corresponding symmetries. In the following sections, we will suppose that these 1-forms arise from a fibration of the super-twistor space that follows from the condition for a Ricci-flat space-time, and find that the gauging restricts the physical states of the string theory so that they can be associated with deformations of the super-twistor space preserving the fibration structure, and hence the Ricci-flatness.

Given a set of 1-forms $k^i = k^i_I(Z)dZ^I$ and $\tilde{k}^i = \tilde{k}^i_I(\tilde{Z})d\tilde{Z}^I$ of weights $h_i, \tilde{h}_i$ there are currents

$$K^i = k^i_I \partial Z^I, \qquad \tilde{K}^i = \tilde{k}^i_I \tilde{\partial} \tilde{Z}^I. \tag{6.1}$$

These are conserved Kač-Moody currents for the free theory given by (4.1) with $A = \tilde{A} = 0$. For the case of Euclidean world-sheets, in which $\tilde{\sigma} = \sigma^*$ and $\tilde{Z} = Z^*$, the currents $\tilde{K}^i$ are the complex conjugates of the K^i. For the other cases, the $\tilde{K}^i$ and the K^i are independent currents satisfying $K^i = \tilde{K}^i$ on the boundary as a result of the boundary conditions (4.4).

We assume that the 1-forms satisfy

$$Z^I k^i_I = 0\,, \qquad \tilde{Z}^I \tilde{k}^i_I = 0 \tag{6.2}$$

so that the central charges $e^i, \tilde{e}^i$ vanish and gauging is possible. Then gauging the symmetries generated by $K^i, \tilde{K}^i$ gives the action

$$S = \int d^2\sigma \left(Y_I \tilde{\partial} Z^I + \tilde{Y}_J \partial \tilde{Z}^J - \tilde{A} J - A \tilde{J} - B_i \tilde{K}^i - \tilde{B}_i K^i \right) + S_C, \tag{6.3}$$

and this is invariant under (5.4), (5.22),(5.23) together with the corresponding symmetries with parameter $\tilde{\alpha}, \tilde{t}$. For open strings, the boundary conditions (4.4) are imposed as before.

Under the symmetries with parameter $\alpha, \tilde{\alpha}$, the action changes by a total derivative term

$$\delta S = \int d^2\sigma (\partial - \tilde{\partial}) \left(\alpha K - \tilde{\alpha} \tilde{K} \right) \tag{6.4}$$

and with the boundary conditions (4.4), this vanishes for gauge transformations in which the parameters satisfy

$$\alpha = \tilde{\alpha} \tag{6.5}$$

on the boundary.

Gauge-fixing by choosing conformal gauge and requiring all gauge fields to vanish introduces the ghosts (u, v) and $(\tilde{u}, \tilde{v})$ of the Berkovits string, together with the ghost system (r^i, s_i) of the last section and its conjugate system $(\tilde{r}^i, \tilde{s}_i)$. The open string theory is defined by the boundary conditions (4.4) on the twistor variables and

$$c = \tilde{c}, \qquad b = \tilde{b}, \qquad v = \tilde{v}, \qquad u = \tilde{u}, \qquad r^i = \tilde{r}^i, \qquad s_i = \tilde{s}_i \tag{6.6}$$

on the ghosts.

The BRST operators are

$$\begin{aligned} Q &= \oint d\sigma \left(cT + vJ + s_i K^i + cu\partial v + cb\partial c + cr^i \partial s_i - \sum_i v h_i s_i r^i \right) \\ \tilde{Q} &= \oint d\tilde{\sigma} \left(\tilde{c}\tilde{T} + \tilde{v}\tilde{J} + \tilde{s}_i \tilde{K}^i + \tilde{c}\tilde{u}\tilde{\partial}\tilde{v} + \tilde{c}\tilde{b}\tilde{\partial}\tilde{c} + \tilde{c}\tilde{r}^i \tilde{\partial}\tilde{s}_i - \sum_i \tilde{v}\tilde{h}_i \tilde{s}_i \tilde{r}^i \right). \end{aligned} \tag{6.7}$$

In Q^2, there are two potentially non-zero terms: a conformal anomaly term proportional to $C \int c\partial^3 c$, where C is the Virasoro central charge, and a gauge anomaly term proportional to $k \int v\partial v$, where k is the Kač-Moody central charge. The Virasoro central charge is

$$C = D - N + c_C - 28 - 2(d - n), \tag{6.8}$$

where $D - N$ comes from the YZ system, c_C is the central charge of the auxiliary matter system S_C, the contribution $-28 = -26 - 2$ comes from the bc and uv systems, and $-2(d - n)$ comes from the (r^i, s_i) system consisting of d fermionic ghosts and n bosonic ones. The Kač-Moody central charge is

$$k = D - N - \sum_i \epsilon_i (h_i)^2, \tag{6.9}$$

where $\epsilon_i = 1$ for bosonic symmetries (with α_i bosonic) and $\epsilon_i = -1$ for fermionic symmetries (with α_i fermionic).

The gauge anomaly cancels if $\kappa = 0$. If $\kappa \neq 0$, one might attempt to cancel the anomaly against a contribution from the matter system S_C. If the matter system S_C has a current J_C generating a $GL(1)$ Kač-Moody symmetry with central charge κ_C, and S_C is chosen to contain the coupling $\tilde{A}J_C$, then

$$k = D - N + \kappa_C - \sum_i \epsilon_i (h_i)^2. \tag{6.10}$$

However, this is likely to lead to problems from mixing between the auxiliary matter system and the twistor space sector, and its most natural interpretation would be as a change in the definition of the twistor space. We therefore restrict ourselves to solutions with

$$D - N - \sum_i \epsilon_i (h_i)^2 = 0, \tag{6.11}$$

so that no resort to such a compensating coupling is needed.

There will be similar anomalies with coefficients $\tilde{C}, \tilde{k}$ from $\tilde{Q}$. Quantum consistency requires $C = \tilde{C} = 0$ and $k = \tilde{k} = 0$. In the next section, some string theories in which these anomalies cancel will be considered.

7 World-sheet anomaly cancellation in twistor strings

7.1 No supersymmetry

Consider first the bosonic case in which $N = 0, n = 0$, so that the twistor space $\mathcal{PT}$ is an ordinary (bosonic) complex manifold of dimension $D-1$. The Penrose construction of the non-linear graviton for $D = 4$ requires the projective twistor space $\mathcal{PT}$ to be fibred over $\mathbb{CP}^1$. We then restrict ourselves to twistor spaces in which $\mathcal{PT}$ is fibred over $\mathbb{CP}^1$ (or in the real case, to spaces $\mathcal{PT}_{\mathbb{R}} \times \mathcal{PT}_{\mathbb{R}}$ with $\mathcal{PT}_{\mathbb{R}}$ fibred over $\mathbb{RP}^1$). Then there is a holomorphic 1-form on $\mathbb{CP}^1$, given by $\epsilon^{A'B'}\pi_{A'} \wedge d\pi_{B'}$ where $\pi_{A'}$ are homogeneous coordinates on $\mathbb{CP}^1$, and its pull-back to $\mathcal{PT}$ is

$$k = I_{\alpha\beta} Z^\alpha dZ^\beta \tag{7.1}$$

with $I_{\alpha\beta}$ the dual of the infinity twistor. This in turn pulls back to a 1-form on (non-projective) twistor space $\mathcal{T}$, again given by (7.1). This 1-form has weight $h = 2$. Gauging the symmetry generated by this 1-form then gives the Kač-Moody central charge $k = D - h^2 = D - 4$, which vanishes precisely when D takes the value $D = 4$ needed for Penrose's twistor space, and no κ_C is needed. Then from (6.8) with $D = 4, d = 1$, we find

$$C = c_C - 26 \tag{7.2}$$

so the matter system can be taken to be a critical bosonic string with $c_C = 26$.

7.2 N supersymmetries, $P\mathcal{T}$ fibred over $\mathbb{CP}^{1|N}$

Suppose now that there are N fermionic dimensions, and the projective twistor space is fibred over $\mathbb{CP}^{1|N}$ (or $\mathbb{RP}^{1|N} \times \mathbb{RP}^{1|N}$). On $\mathbb{CP}^{1|N}$, a section of the co-frame bundle gives one bosonic one-form and N fermionic ones. The bosonic 1-form is the globally-defined k given in (7.1), while the N locally-defined fermionic one-forms k^a are of the form

$$k^a = d\psi^a + e^a_{A'} d\pi^{A'} \tag{7.3}$$

and are of weight $h_a = 1$. Here $e^a_{A'}$ satisfies

$$\pi^{A'} e^a_{A'} = -\psi^a, \tag{7.4}$$

so that the k^a satisfy $\iota(\Upsilon)k^a = 0$. In a patch where $\pi^{A'}\rho_{A'} \neq 0$ for some fixed spinor $\rho_{A'}$, this can be solved by

$$e^a_{A'} = -\frac{\psi^a \rho_{A'}}{\pi^{B'}\rho_{B'}} \tag{7.5}$$

so that

$$k^a = \pi^{A'}\rho_{A'}\, d\left(\frac{\psi^a}{\pi^{B'}\rho_{B'}}\right). \tag{7.6}$$

These forms pull back to one-forms (k, k^a) on $P\mathcal{T}$ and $\mathcal{T}$, so they can be used in the construction of the last section. The k^a are only locally-defined, but the gauging is still defined globally, as discussed at the end of §5.1. Now from (6.9), the Kač-Moody central charge k is independent of N and

$$\kappa = D - 4, \tag{7.7}$$

so that anomaly cancellation again selects $D = 4$. Then (6.8) gives

$$C = c_C - (26 - N), \tag{7.8}$$

so that the matter system should be chosen to have $c_C = 26 - N$.

7.3 General weights

The form (7.1) is of weight $h = 2$, but a 1-form of general weight h can be made by multiplying by a function $w(Z)$ of weight $h-2$ (so that w is a section of $\mathcal{O}(h-2)$) to give

$$\hat{k} = w(Z) I_{IJ} Z^I dZ^J. \tag{7.9}$$

Similarly, multiplying (7.3) by a $w^a(Z)$ that is a section of $\mathcal{O}(h_a - 1)$ gives for each a

$$\hat{k}^a = w^a(Z)(d\psi^a - e^a_{A'} d\pi^{A'}) \tag{7.10}$$

which is of weight h_a.

Introducing such factors gives many formal anomaly-free solutions for which the central charges (6.8) and (6.9) vanish. For example, choosing all $\hat{k}^a$ to be of equal weights h', the conditions are

$$\begin{aligned} 0 &= D - N + c_C - 30 \\ 0 &= D - N - h^2 + N(h')^2. \end{aligned} \tag{7.11}$$

In the bosonic case $N = 0$, the only solution with $D = 4$ is the model with $h = 2$ and matter central charge $c_C = 26$ discussed in §7.1; however, formally there are higher dimensional solutions of (7.11) with

$$h^2 = D, \qquad\qquad c_C = 30 - D. \tag{7.12}$$

For the case $D = 4$ with N fermionic currents,

$$\begin{aligned} c_C &= 26 + N \\ h^2 - N(h')^2 &= 4 - N. \end{aligned} \tag{7.13}$$

For $h' = 1$, there are solutions with $h = 2$ and $c_C = 26 + N$ (including an $N = 4$ model which is distinct from the $N = 4$ model with $c_C = 22$ discussed in §7.2), and there are additional solutions of (7.13) with $h' > 1$. It is straightforward to find further anomaly-free solutions corresponding to currents of general weights h, h_a.

7.4 Weightless forms

An important special case of the construction with general w, w^a consists in choosing w of weight -2 and all the w^a of weight -1, which gives forms $\hat{k}, \hat{k}^a$ all with weights 0. Then (6.9) gives the same constraint $D = N$ as for the Berkovits string, and with $D = 4$ this selects $N = 4$. If one gauges $\hat{k}$ and n of the $\hat{k}^a$ with $0 \leq n \leq N$, then the central charge is

$$C = c_C - 30 + 2n. \tag{7.14}$$

There are two models of particular interest with $D = N = 4$, that with $n = 0$ and that with $n = 4$.

If w is chosen to depend on $\pi_{A'}$ only, then the one-form $\hat{k}$ is closed, $d\hat{k} = 0$. In a patch where $\pi^{A'}\rho_{A'} \neq 0$ for some fixed spinor $\rho_{A'}$ with k^a given by (7.6), choosing $w^a = (\pi^{A'}\rho_{A'})^{-1}$ for each a gives

$$\hat{k}^a = d\left(\frac{\psi^a}{\pi^{B'}\rho_{B'}}\right) \tag{7.15}$$

which automatically satisfies $d\hat{k}^a = 0$. More generally, for any $w^a(\pi)$ on $\mathbb{CP}^1$ of weight -1, we can choose $\hat{k}^a = d(\psi^a w^a)$ (with no sum over a).

A potential problem with this construction is that functions $w(Z), w^a(Z)$ of negative weights can have singularities. For example, for weight -1, $w' = (\pi^{A'}\rho_{A'})^{-1}$ is singular on the surface $\pi^{A'}\rho_{A'} = 0$ on which $\pi^{A'} = \lambda\rho^{A'}$ for arbitrary parameter λ. A function $w(Z)$ of weight h on $\mathbb{CP}^1$ will have $-h$ singularities if $h < 0$, and it is not clear how to define the construction at these singularities.

For the case of real twistor space with $Z, \tilde{Z}$ independent and real, there are non-singular functions of negative weights. For example, a function of weight -2 on $\mathbb{RP}^1$ is given by

$$w(\pi) = \frac{1}{M^{A'B'}\pi_{A'}\pi_{B'}} \tag{7.16}$$

where $\pi_{A'}$ are real homogeneous coordinates for $\mathbb{RP}^1$, and this is non-singular if the constant symmetric real matrix $M_{A'B'}$ is positive definite, since the point $\pi^{A'} = 0$ is excluded. This can then be pulled back to a non-singular function of weight -2 on any space that is fibred over $\mathbb{RP}^1$. For a real twistor space given by a region of $\mathbb{RP}^{3|4} \times \mathbb{RP}^{3|4}$, or more generally one that is of the form $P\mathcal{T}_{\mathbb{R}} \times P\mathcal{T}_{\mathbb{R}}$ for some real $P\mathcal{T}_{\mathbb{R}}$ that is fibred over $\mathbb{RP}^1 \times \mathbb{RP}^1$, non-singular functions $w(\pi), \tilde{w}(\tilde{\pi})$ can be constructed in this way, and they can be used to construct well-defined one-forms $\hat{k}(Z), \hat{\tilde{k}}(\tilde{Z})$ of weight $h = \tilde{h} = 0$. A function w' of weight -1 can be defined as $w' = \sqrt{w}$ as w is positive.

For the complex case, $w(Z)$ can be chosen to be non-singular in a holomorphic disc with boundary on the real subspace, so that it is non-singular on the embedding of the open string world-sheet in super-twistor space. For a twistor space $\mathbb{PT}$ fibred over $\mathbb{CP}^1$, w can be chosen as

$$w = \frac{1}{(\rho_1^{A'}\pi_{A'})(\rho_2^{B'}\pi_{B'})} \tag{7.17}$$

for some fixed complex spinors $\rho_1^{A'}, \rho_2^{A'}$. Then each singularity lies in a plane $\rho^{A'}\pi_{A'} = 0$. Recall that twistor space divides into two parts $\mathbb{PT}^{\pm}$ with $\pm i\pi_{A'}\bar{\pi}^{A'} \geq 0$ and that these two parts correspond to two copies of space-time. To obtain just one copy of space-time, we choose $\mathbb{PT}^+$, say, as the twistor space, and the space of holomorphic discs in this part of twistor space with boundary on $\mathbb{PT}_{\mathbb{R}}$ gives a complete copy of space-time. If we take both $\rho_1^{A'}, \rho_2^{A'}$ to lie in $\mathbb{PT}^-$, then $w(Z)$ is non-singular on $\mathbb{PT}^+$ and the gauging of the twistor string is well-defined for world-sheets that are discs in $\mathbb{PT}^+$.

In the complex case with $\tilde{Z} = Z^*$, the cancellation of the surface term in the variation (6.4) requires that $w\alpha = \tilde{w}\tilde{\alpha} = (w\alpha)^*$ on the boundary. If $w(Z)$ is real on the real axis $Z = Z^*$, this gives the boundary condition $\alpha = \tilde{\alpha}$ as before, but if w is a complex function on the real axis, then the boundary conditions of α and hence of the ghosts s are modified. However, in the case of Euclidean world-sheet, in which Z and $\tilde{Z}$ are independent complex variables, the boundary condition is $Z = \tilde{Z}$ and it is possible that $w(Z), \tilde{w}(\tilde{Z})$ can be chosen so that $w(Z) = \tilde{w}(\tilde{Z})$ on the boundary with $w(Z)$ non-singular on the holomorphic disc, and the boundary condition on α is $\alpha = \tilde{\alpha}$.

The models in which the zero-weight one-form (7.9) or the one-forms (7.9), (7.10) are gauged are then well-defined both for the real case, and for the complex case with independent complex coordinates $Z, \tilde{Z}$. The models depend on an arbitrary function w, or on the functions w and w^a, but these only enter into the BRST charge. It will be seen in the next section that the spectrum is independent of w, w^a, provided these functions are chosen to have no zeroes or poles; tree-level amplitudes at degree zero are also independent of the choice of w, w^a, as will be checked explicitly in an example in §9.

8 Spectra of the twistor string theories

8.1 Physical vertex operators

In this section, we will investigate the constraints and gauge invariances for the vertex operators V_f, V_g, V_ϕ for each of the anomaly-free theories of the last section, and obtain the ghost-independent part of the BRST cohomology. We will discuss the ghost-dependent vertex operators elsewhere.

The gauged twistor string is constructed on a twistor space with a set of 1-forms $k^i = k^i_I dZ^I$ with weights h_i defined by (5.17) and satisfying

$$Z^I k^i_I = 0. \tag{8.1}$$

The vertex operator $V_f = Y_I f^I(Z)$ is physical provided

$$\partial_I f^I = 0, \qquad f^I k^i_I = 0, \qquad f^I k^i_{[I,J]} = 0 \tag{8.2}$$

for each i. However, the gauge invariance (4.12) is now modified, as

$$\{Q, u\} = J + \sum_i h_i r^i s_i. \tag{8.3}$$

If all the weights h_i vanish, then ΛJ is BRST trivial for any $\Lambda(Z)$ of zero weight, and

$$\delta f^I = Z^I \Lambda \tag{8.4}$$

changes V_f by a BRST trivial term. However, if any of the weights h_i are non-zero, then the extra ghost terms in (8.3) mean that (8.4) is not a symmetry. This is just as well, as the constraints (8.2) are only invariant under (8.4) if all the h_i are zero.

The vertex operator $V_g = g_I(Z)\partial Z^I$ is physical provided

$$Z^I g_I = 0, \tag{8.5}$$

and it has the gauge invariances

$$\delta g_I = \partial_I \chi, \qquad \delta g_I = \eta_i k_I^i \tag{8.6}$$

for any $\chi(Z)$ and any $\eta_i(Z)$ of weights $-h_i$.

The Yang-Mills vertex operator $V_\phi = j_r \phi^r(Z)$ receives no further constraints from the gauging. In the following the spectrum will be analysed for the anomaly-free strings of the last section in the flat case. The twistor space is $\mathbb{PT}'_{[N]} = \mathbb{PT}_{[N]} - I$ and results from removing the appropriate (super)line I (which is $I = \mathbb{CP}^{1|0}$ or $I_{[N]} = \mathbb{CP}^{1|N}$ in the complex case, and $\mathbb{RP}^{1|0} \times \mathbb{RP}^{1|0}$ or $\mathbb{RP}^{1|N} \times \mathbb{RP}^{1|N}$ in the real case) from $\mathbb{CP}^{3|N}$ or $\mathbb{RP}^{3|N} \times \mathbb{RP}^{3|N}$. The vertex operators live on the boundary of the world-sheet, which in turn lies in $\mathbb{RP}^{3|N}$.

8.2 Self-dual gravity without supersymmetry

Consider first the bosonic $N = 0$ theory of §7.1 with the one-form

$$k = I_{\alpha\beta} Z^\alpha dZ^\beta \tag{8.7}$$

on the twistor space $\mathbb{PT}' = \mathbb{CP}^3 - \mathbb{CP}^1$ (or $\mathbb{PT}'_{\mathbb{R}} = \mathbb{RP}^3 - \mathbb{RP}^1$ in the real case), so that

$$k_\alpha = -I_{\alpha\beta} Z^\beta, \qquad k_{[\alpha,\beta]} = -I_{\alpha\beta}. \tag{8.8}$$

The coordinates on twistor space are $Z^\alpha = (\omega^A, \pi_{A'})$ and

$$k = \epsilon^{A'B'} \pi_{A'} d\pi_{B'}. \tag{8.9}$$

Then $f^I = (f^A, f_{A'})$ are of degree one and the constraints (8.2) imply

$$\frac{\partial f^A}{\partial \omega^A} = 0, \qquad f_{A'} = 0, \tag{8.10}$$

which in turn imply

$$f^A = \epsilon^{AB} \frac{\partial h}{\partial \omega^B} \tag{8.11}$$

for some twistor function $h(Z)$ homogeneous of degree 2. Via the twistor transform, this corresponds to a space-time field of helicity 2 satisfying the field equations of linearised Einstein gravity [26].

The 1-form $g = g_\alpha \mathrm{d}Z^\alpha$ in the vertex operator $g_\alpha \partial Z^\alpha$ satisfies $Z^\alpha g_\alpha = 0$, which means that g_α is defined on the projective twistor space, and moreover it follows from (8.6) that it is defined up to two gauge freedoms:

$$g_\alpha \to g_\alpha + \partial_\alpha \chi\,, \qquad g_\alpha \to g_\alpha + I_{\alpha\beta} Z^\beta \eta\,. \tag{8.12}$$

The four components of g_α are subject to one constraint and two gauge invariances, and the remaining degree of freedom is conveniently represented by a function $\tilde{f}$ of homogeneity degree -2 defined by

$$\tilde{h} = I^{\alpha\beta}\partial_\alpha g_\beta = \epsilon^{AB}\partial_A g_B, \tag{8.13}$$

which is invariant under the two gauge transformations given in (8.12). This function of degree -2 corresponds to a space-time scalar field. Finally, the Yang-Mills vertex operator with functions $\phi_r(Z)$ of degree zero gives states of helicity $+1$ in the adjoint of the gauge group G.

Thus the spectrum of this theory consists of a state of helicity $+2$, a scalar state of spin 0 and $dim(G)$ states of helicity $+1$. Note that the state of spin zero could come from a scalar field or a 2-form gauge field. An interacting theory with this spectrum is self-dual gravity coupled to self-dual Yang-Mills and a scalar (or 2-form gauge field), and this has covariant field equations but no covariant action. In the absence of the scalar, the field equations would be

$$R = *R, \qquad F = *F, \tag{8.14}$$

where R is the curvature 2-form, F is the Yang-Mills field strength and $*$ denotes the Hodge duality operation. Finding out whether this interacting theory arises, and finding the form of the scalar coupling, requires investigating the interactions arising from string amplitudes. This will be discussed elsewhere.

8.3 Supergravity with N supersymmetries

Consider next the case of §7.2, with projective twistor space $\mathbb{PT}'_{[N]}$ of dimension $3|N$ (given by $\mathbb{CP}^{3|N} - \mathbb{CP}^{1|0}$, or $\mathbb{RP}^{3|N} - \mathbb{RP}^{1|0}$ in the real case) that is fibred over $\mathbb{CP}^{1|N}$, and the gauging associated with the bosonic one-form (8.9) and the N fermionic one-forms

$$k^a = d\psi^a - e^a_{A'} d\pi^{A'}. \tag{8.15}$$

The vector field f^I decomposes as $f^I = (f^\alpha, f^a) = (f^A, f_{A'}, f^a)$ and the conditions (8.2) imply

$$\frac{\partial f^A}{\partial \omega^A} = 0, \qquad f^{A'} = 0, \qquad f^a = 0, \tag{8.16}$$

and again

$$f^A = \epsilon^{AB} \frac{\partial h}{\partial \omega^B} \tag{8.17}$$

for some super-twistor function $h(Z)$ homogeneous of degree 2.

Consider first the case $N = 4$. Then $h(Z)$ has an expansion

$$h(Z^I) = g(Z^\alpha) + \chi_a(Z^\alpha)\psi^a + A_{ab}(Z^\alpha)\psi^a\psi^b + \Lambda^d(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c + \varphi(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c\psi^d, \tag{8.18}$$

where $Z^\alpha = (\omega^A, \pi_{A'})$ are the coordinates on bosonic twistor space. This gives twistor fields $g, \chi_a, A_{ab}, \Lambda_{abc}, \varphi$ in $\mathcal{O}(2), \mathcal{O}(1), \mathcal{O}(0), \mathcal{O}(-1), \mathcal{O}(-2)$ respectively. Via the twistor transform, these correspond to space-time fields of helicities $2, 3/2, 1, 1/2, 0$ in the $SL(4, \mathbb{R})$ representations $(\mathbf{1}, \mathbf{4}, \mathbf{6}, \mathbf{4}', \mathbf{1})$ respectively. We then obtain the following positive helicity fields in space-time: a graviton $g_{\mu\nu}$, four gravitini χ_a^μ, six helicity one fields A_{ab}^μ, four helicity half fields Λ_{abc} and a scalar φ. These satisfy the field equations of linearised $N = 4$ supergravity.

For general N, one again has an expansion

$$h(Z^I) = g(Z^\alpha) + \chi_a(Z^\alpha)\psi^a + A_{ab}(Z^\alpha)\psi^a\psi^b + ... \tag{8.19}$$

terminating with a term of order ψ^N, giving twistor fields in $\mathcal{O}(2), \mathcal{O}(1), .., \mathcal{O}(2 - N)$ corresponding to space-time fields of helicities $2, 3/2...., 2 - (N/2)$ in the $SL(N, \mathbb{R})$ representations $(\mathbf{1}, \mathbf{N}, \mathbf{N(N-1)/2}, ..., \mathbf{N}', \mathbf{1})$ respectively.

For the vertex operator $g_I \partial Z^I$, $g_I = (g_A, g_{A'}, g_a)$ and the symmetry (8.6) with the one-forms k^a can be used to set $g_a = 0$. Then (8.13) again defines a function of homogeneity degree -2 that is invariant under the remaining symmetries, and gives rise to the conjugate multiplet to the one obtained from f. For $N = 4$, this is

$$\tilde{h}(Z^I) = \tilde{g}(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c\psi^d + \tilde{\chi}^d(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c + \tilde{A}_{ab}(Z^\alpha)\psi^a\psi^b + \tilde{\Lambda}_a(Z^\alpha)\psi^a + \tilde{\varphi}(Z^\alpha), \tag{8.20}$$

giving twistor functions $\tilde{g}, \tilde{\chi}_a, \tilde{A}_{ab}, \tilde{\Lambda}_{abc}, \tilde{\varphi}$ in $\mathcal{O}(-6), \mathcal{O}(-5), \mathcal{O}(-4), \mathcal{O}(-3), \mathcal{O}(-2)$ corresponding to helicities $-2, -3/2, -1, -1/2, 0$ with multiplicities $1, 4, 6, 4, 1$ respectively. For general N, this gives twistor fields in $\mathcal{O}(-2 - N), .., \mathcal{O}(-3), \mathcal{O}(-2)$ corresponding to helicities $-N/2, ..., -1/2, 0$.

Finally, the Yang-Mills sector is represented by a function of degree zero in super-twistor space, corresponding to helicities $1, 1/2, ..., -N/2$ in the $SL(N,\mathbb{R})$ representations $(\mathbf{1}, \mathbf{N}, \mathbf{N(N-1)/2}, ..., \mathbf{N'}, \mathbf{1})$, and for $N > 4$, there are higher-spin fields with helicities less than -1.

For $N = 4$ this is the spectrum of $N = 4$ supergravity coupled to $N = 4$ super-Yang-Mills. For $N < 4$, this is a self-dual supergravity theory coupled to self-dual Yang-Mills. Interacting self-dual supergravity theories in 2+2 dimensions have been discussed in [35, 36, 37, 38, 39, 40]. For $N > 4$, we find multiplets with spins greater than two, and with more than one state of helicity -2. Free theories can be written down for all these spectra, but the possibilities for interactions are more limited. However, there is the intriguing possibility of self-dual interactions for these theories, as the usual higher-spin inconsistencies are absent for certain self-dual theories. The possibility of interactions will be discussed in section 10.

8.4 $N = 8$ supergravity

Consider the theory of §7.4 formulated in $N = 4$ super-twistor space with the gauging for the single weightless 1-form

$$\hat{k} = w(Z) I_{IJ} Z^I dZ^J, \tag{8.21}$$

where w is of degree -2. We need only assume a fibration over $\mathbb{CP}^{1|0}$, so that the flat twistor space can be taken to be $\mathbb{PT}'_{[4]} = \mathbb{CP}^{3|4} - \mathbb{CP}^{1|4}$ (or the real analogue thereof). We choose $w = w(\pi)$ so that $\hat{k}$ is closed, $d\hat{k} = 0$. For real twistors $Z, \tilde{Z}$, the function w could be chosen as in (7.16), and for complex ones as in (7.17).

Starting with the vector field f^I, we work through the various conditions and gauge equivalences as follows. In this case, the constraints (8.2) are weaker than in §8.3 as $d\hat{k} = 0$, but there is now a gauge invariance of the type (8.4) since the form has weight $h = 0$. We set $f^I = (f^\alpha, f^a) = (f^A, f_{A'}, f^a)$. We fix the gauge freedom $f^I \to f^I + Z^I \Lambda$ from equation (8.4) by requiring that

$$\frac{\partial f^A}{\partial \omega^A} = 0, \tag{8.22}$$

which in turn implies

$$f^A = \epsilon^{AB} \frac{\partial h}{\partial \omega^B} \tag{8.23}$$

for some twistor function $h(Z)$ homogeneous of degree 2. This has the expansion (8.18) and gives the space-time fields of helicities $2, 3/2, 1, 1/2, 0$ of the positive helicity $N = 4$ supergravity multiplet.

For $w = w(\pi)$, (8.21) implies $\hat{k}_{[I,J]} = 0$, so that the constraints (8.2) give

$$\partial_I f^I = 0 \qquad w(\pi) f^{A'} \pi_{A'} = 0, \tag{8.24}$$

implying that $f_{A'} = \pi_{A'}\lambda$ for some λ of homogeneity degree -1. The function λ can be understood to be determined in terms of the f^a by the condition $\partial_I f^I = 0$ (cf. eq. (8.2)) and so λ does not represent any independent degrees of freedom. We expand the f^a to obtain

$$f^e = \chi^e(Z^\alpha) + A^e_a(Z^\alpha)\psi^a + \Lambda^e_{ab}(Z^\alpha)\psi^a\psi^b + \varphi^{ea}(Z^\alpha)\epsilon_{abcd}\psi^b\psi^c\psi^d + \tilde{\Lambda}^e(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c\psi^d. \tag{8.25}$$

We have used the same symbol as in equation (8.18) to denote fields of the same helicity. Eq. (8.25) gives four gravitino multiplets, each with states of helicities $3/2, 1, 1/2, 0, -1/2$, and so leads to a further four gravitini, sixteen helicity one fields, twenty four helicity one half fields, sixteen scalars and four helicity minus one half fields.

The 1-form $g = g_I \mathrm{d}Z^I$ in the vertex operator $g_I \partial Z^I$ satisfies $Z^I g_I = 0$, which means that g_I is defined on the projective twistor space; moreover g_I is defined up to two gauge freedoms:

$$g_I \to g_I + \partial_I \chi\,, \qquad g_I \to g_I + w I_{IJ} Z^J \eta\,. \tag{8.26}$$

We define a gauge-invariant function $\tilde{h}$ of homogeneity degree -2 by (8.13) and this again gives rise to the conjugate supergravity multiplet with helicities -2,$-3/2$,-1,$-1/2$,0 and multiplicities $1, 4, 6, 4, 1$ respectively.

The fermionic components g_a contribute further states to the spectrum. In order to see this and find the full spectrum, we write $g_I = (g_\alpha, g_a) = (g_A, g^{A'}, g_a)$. The gauge freedom $g_I \to g_I + \partial_I \chi$ can be fixed by imposing the gauge condition $g^{A'}\pi_{A'} = 0$. This implies $g^{A'} = \pi^{A'}\xi$ for some ξ which can then be set to zero by use of the gauge freedom $\delta g_I = I_{IJ} Z^J \eta$. Consider next the two degrees of freedom in g_A. One is the component $\omega^A g_A$, which is determined in terms of the g_a by the final constraint $Z^I g_I = 0$ (cf. (8.5)) and so is not an independent degree of freedom. This leaves one degree of freedom represented by the gauge-invariant function $\tilde{h}$ given by (8.13), corresponding to the negative helicity $N = 4$ supergravity multiplet.

The remaining components g_a are unconstrained and, together with $\tilde{h}$, determine the gauge fixed conponents of g_I. The g_a can be expanded as

$$g_e = \tilde{\chi}_e(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c\psi^d + \tilde{A}^d_e(Z^\alpha)\epsilon_{abcd}\psi^a\psi^b\psi^c + \tilde{\Lambda}_{eab}(Z^\alpha)\psi^a\psi^b + \tilde{\varphi}(Z^\alpha)_{ea}\psi^a + \Lambda(Z^\alpha)_e\,. \tag{8.27}$$

This gives four negative helicity gravitino multiplets, conjugate to those from f^a.

Note that the spectrum is independent of the choice of $w(\pi)$. Combining all the positive and negative helicity states, we obtain a spectrum consisting of a graviton $h_{\mu\nu}$, 8 gravitini, 22 vector fields, 32 spin-half fields Λ_{abc} and 34 scalars. This is six $N = 4$ vector multiplets short of the full $N = 8$ supergravity spectrum. In addition, the Yang-Mills vertex operator gives vector multiplets in the adjoint of some group G. If G is six-dimensional, then the spectrum of $N = 8$ supergravity is obtained.

8.5 $N = 4$ supergravity coupled to super-Yang-Mills

Consider the theory of §7.4 formulated in $N = 4$ super-twistor space with the gauging for the weightless 1-form

$$\hat{k} = w(Z) I_{IJ} Z^I dZ^J, \tag{8.28}$$

where w is of degree -2, and the four weightless 1-forms

$$\hat{k}^a = w'(Z)(d\psi^a - e^a_{A'} d\pi^{A'}) \tag{8.29}$$

where w' is of degree -1. We assume a fibration over $\mathbb{CP}^{1|4}$, so that the flat twistor space can be taken to be $\mathbb{PT}'_{[4]} = \mathbb{CP}^{3|4} - \mathbb{CP}^{1|0}$ (or the real analogue thereof). It will be assumed that w, w' are chosen so that $\hat{k}, \hat{k}^a$ are closed, and that they have no zeroes or poles on the boundary space defined by the boundary condition $Z = \tilde{Z}$ (which is $\mathbb{RP}^{3|4}$ for the Lorentzian world-sheet theory). It was shown in the previous subsection that the constraints from $\hat{k}$ imply that the vertex operator V_f is determined by a function $h(Z)$ of degree 2 and four functions f^a of degree 1, while V_g is given in terms of a function $\tilde{h}(Z)$ of degree -2 and four functions g_a of degree -1. The constraints $f^I \hat{k}^a_I = 0$ from the fermionic 1-forms give

$$w' f^a = 0; \tag{8.30}$$

this implies that $f^a = 0$ as w' is chosen to have no zeroes on $Z = \tilde{Z}$, while the symmetry $\delta g_I = \eta_a \hat{k}^a_I$ can be used to set $g_a = 0$. In this way the gravitino multiplets are eliminated, leaving the twistor functions $h(Z)$ of degree 2 and $\tilde{h}(Z)$ of degree -2, and this gives the spectrum of $N = 4$ supergravity. In addition, the vertex operators V_ϕ give the spectrum of $N = 4$ super-Yang-Mills with gauge group G, so the spectrum of $N = 4$ supergravity coupled to $N = 4$ super-Yang-Mills is obtained.

9 Amplitudes for $N = 8$ and $N = 4$ supergravity

The scattering amplitudes for the Berkovits string, calculated from open string correlation functions with vertex operators V_f, V_g, V_ϕ inserted on the world-sheet boundary, give rise to nontrivial scattering amplitudes and hence to interactions for the space-time fields [2, 3, 6]. The n-point tree-level amplitude is given by the formula [2, 3]

$$\sum_d \left\langle cV_1(\sigma_1)cV_2(\sigma_2)cV_3(\sigma_3)\int d\sigma_4 V_4(\sigma_4)\ldots\int d\sigma_n V_n(\sigma_n)R\right\rangle_d \tag{9.1}$$

where V_i are any of the vertex operators V_f, V_g, V_ϕ and $\langle\ldots\rangle_d$ is the correlation function on a disc of degree d, corresponding to a gauge instanton on the disc with a topologically non-trivial configuration for the gauge field A characterised by the integer d [3]. The coordinates are written as $Z^I = \rho\hat{Z}^I$, where ρ is a scale factor (which is complex for complex Z), and a BRST-invariant operator R is

$$R = \delta(\rho - 1)v + \ldots \tag{9.2}$$

This has the property that it gives an insertion of the zero-mode of the ghost v, so that the integration over v is non-zero, and regulates the integral over ρ. (Changing the insertion point σ_0 changes $R(\sigma_0)$ by a BRST exact term, so that the amplitude is independent of σ_0.) Integrating out ρ, v leaves an amplitude defined on a 'small Hilbert space' of $GL(1)$-neutral states independent of the v zero-mode, giving results defined on the projective twistor space [3].

Consider now the new theories based on weightless forms of §7.4, §8.4, §8.5, corresponding to $N = 8$ supergravity or $N = 4$ supergravity coupled to super-Yang-Mills. These new string theories are similar to the Berkovits string, and the twistor fields Y, Z have the same world-sheet dynamics and the same vertex operators. However, there is an additional ghost sector and the extra terms in the BRST operator give extra constraints and extra gauge invariances for the twistor wave-functions f^I, g_I, while there are no further constraints or invariances for the Yang-Mills wave-functions ϕ_r. In the $N = 8$ theory, there is an extra anti-commuting ghost s of conformal weight zero, which has one zero mode on the disc, so that one insertion of the s zero-mode is needed to obtain a non-zero amplitude. For any BRST-invariant vertex operator cV, scV is also BRST-invariant, so that a non-zero amplitude is given by replacing e.g. $cV_1(\sigma_1)$ with $scV_1(\sigma_1)$ in (9.1). Upon integrating over the s zero-mode, the amplitude (9.1) is recovered. For the $N = 4$ theories of section 8.5, there is in addition one zero-mode for each of the four commuting ghosts s^a, and the integral over these can be handled by choosing appropriate pictures for the vertex operators V_i. A convenient choice is to replace $cV_1(\sigma_1)$ with $s\delta^4(s^a)cV_1(\sigma_1)$ in (9.1). Again, on integrating out the ghost zero modes s, s^a, the formula (9.1) is recovered.

As a result, after integrating out the zero-modes of the new ghosts, the tree-level correlation functions for the $N = 4$ and $N = 8$ theories of §8.4 and §8.5 have the same form as for the Berkovits string in [2, 3, 6] when written in terms of f^I, g_I, ϕ_r. However, in our case these wave-functions are subject to further constraints and have further gauge invariances. As we have seen, these can be used to write f^I, g_I in terms of the unconstrained wave-functions $h, \tilde{h}$ (defined by (8.17),(8.13)) for the $N = 4$ theory, or $h, \tilde{h}, f^a, g_a$ for the $N = 8$ theory. These are wave-functions for supergravity and matter systems whose field equations are of 2nd order in space-time derivatives for bosons (1st order for fermions), not those for conformal supergravity with 4th order equations for bosons. When written in terms of $h, \tilde{h}$ or $h, \tilde{h}, f^a, g_a$, the scattering amplitudes of the new twistor strings should then give interactions for Einstein gravitons and matter. These will be systematically investigated and compared with known gravity amplitudes elsewhere, but it is straightforward to see that non-vanishing amplitudes are obtained in certain examples, confirming that these theories have non-trivial interactions, and moreover we can compare these with the known MHV gravity amplitudes.

We now check this for tree-level amplitudes at degree zero by first calculating amplitudes in terms of f^I, g_I using the procedure described in [6, 3], and then writing these in terms of the $h, \tilde{h}$ defined by (8.17) and (8.13). The Yang-Mills amplitudes are the same as for the Berkovits string. At degree zero, the amplitudes $\langle V_g V_g V_g \rangle$, $\langle V_f V_g V_g \rangle$ vanish automatically. Now consider the amplitude $\langle V_{f_1} V_{f_2} V_{g_3} \rangle$. Following the procedure given in [6], we obtain the formula

$$\langle V_{f_1} V_{f_2} V_{g_3} \rangle = \int_{\mathbb{RP}^{3|4}} \Omega_s \, f_1^I f_2^J \partial_{[I} g_{3J]}, \tag{9.3}$$

where Ω_s is the volume form on $\mathbb{RP}^{3|4}$. Briefly, this formula follows upon identifying the open string worldsheet with the upper-half complex plane, inserting open string vertex operators on the real axis, and evaluating the correlation function $\langle V_{f_1}(\sigma_1) V_{f_2}(\sigma_2) V_{g_3}(\sigma_3) \rangle$ of three vertex operators given in terms of the f^I and g_I by $V_f = Y_I f^I(Z)$ and $V_g = \partial Z^I g_I(Z)$. This correlation function is computed by taking contractions and using the OPE

$$Z^I(\sigma_1) Y_J(\sigma_2) \sim \frac{\delta^I_J}{\sigma_1 - \sigma_2}. \tag{9.4}$$

The contractions give rise to a factor of $(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1)$ in the denominator that cancels an identical factor in the numerator coming from the integral over zero-modes of the conformal ghost c. The result is then integrated over the space of zero-modes of the fields $Z^I(\sigma)$, which are just constant maps from the disc to twistor space, giving an integral over $\mathbb{RP}^{3|4}$. To obtain the formula (9.3), one also needs to integrate certain terms by parts and use the fact that $\partial_I f^I = 0$. Furthermore, it can be checked that, for our vertex V_f with

$$f_i^I = (\epsilon^{AB} \frac{\partial h_i}{\partial \omega^B}, 0, 0), \qquad i = 1, 2, 3, \tag{9.5}$$

the formula for the remaining amplitude $\langle V_{f_1} V_{f_2} V_{f_3} \rangle$ given in [6] (eq. (5.10) of that paper) yields

$$\langle V_{f_1} V_{f_2} V_{f_3} \rangle = \frac{1}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1)} \times \int_{\mathbb{RP}^{3|4}} \Omega_s \left(\epsilon^{AB} \epsilon^{CD} \epsilon^{EF} - \epsilon^{CB} \epsilon^{ED} \epsilon^{AF} \right) \frac{\partial h_1}{\partial \omega^E \partial \omega^B} \frac{\partial h_2}{\partial \omega^A \partial \omega^D} \frac{\partial h_3}{\partial \omega^C \partial \omega^F} . \tag{9.6}$$

We now focus on the amplitudes between two positive helicity and one negative helicity graviton states so we consider the case in which the wave functions are given in terms of functions $h, \tilde{h}$. We choose

$$f_1^I = (\epsilon^{AB} \frac{\partial h_1}{\partial \omega^B}, 0, 0) , \qquad f_2^I = (\epsilon^{AB} \frac{\partial h_2}{\partial \omega^B}, 0, 0) , \qquad g_{3I} = (g_{3A} \Pi_{a=1}^4 \psi^a, 0, 0) , \tag{9.7}$$

where h_1, h_2 and g_{3A} are functions of the bosonic twistor coordinates Z^α alone, g_{3A} has weight -5 and

$$\epsilon^{AB} \frac{\partial}{\partial \omega^A} g_{3B} = \tilde{h}_3 , \tag{9.8}$$

where $\tilde{h}_3$ has homogeneity degree -6. Performing the integrals over the odd variables, the integral (9.3) now becomes

$$\langle V_{f_1} V_{f_2} V_{g_3} \rangle = \int_{\mathbb{RP}^3} \Omega_s \, \epsilon^{AB} \left(\frac{\partial}{\partial \omega^A} h_1 \right) \left(\frac{\partial}{\partial \omega^B} h_2 \right) \tilde{h}_3 \tag{9.9}$$

where Ω is the volume form on $\mathbb{RP}^3$. We now take h_1, h_2, and $\tilde{h}_3$ to be momentum eigenstates with momenta $P_i^{AA'} = p_i^A p_i^{A'}$, $i = 1, 2, 3$:

$$h_i = \exp \left(\frac{\omega^A P_{iAA'} \alpha^{A'}}{\pi_{B'} \alpha^{B'}} \right) \left(\frac{\pi_{A'} \alpha^{A'}}{p_{1B'} \alpha^{B'}} \right)^3 \delta(\pi_{A'} p_1^{A'}) \tag{9.10}$$

for $i = 1, 2$ and

$$\tilde{h}_3 = \exp \left(\frac{\omega^A P_{iAA'} \alpha^{A'}}{\pi_{B'} \alpha^{B'}} \right) \left(\frac{\pi_{A'} \alpha^{A'}}{p_{1B'} \alpha^{B'}} \right)^{-5} \delta(\pi_{A'} p_1^{A'}) . \tag{9.11}$$

Here $\alpha_{A'}$ is a fixed spinor on which the representatives (9.10) and (9.11) in fact do not depend (see e. g. [41, 1]). The integral (9.9) can now be done; after some delta-function manipulations, this yields the standard formula for the three point MHV amplitude for gravity in split signature (or in Lorentz signature with complex momenta) [9, 10, 11]:

$$\langle V_{f_1} V_{f_2} V_{g_3} \rangle = \delta^4(P_1 + P_2 + P_3) \frac{\left(p_{1A} p_2^A\right)^6}{\left(p_{3B} p_1^B\right)^2 \left(p_{2C} p_3^C\right)^2} . \tag{9.12}$$

Thus the new $N = 4$ and $N = 8$ twistor string theories each have at least one non-trivial interaction, and this gives precisely the helicity $(++-)$ 3-graviton interaction of Einstein gravity.

Under scaling the infinity twistor $I^{IJ} \to RI^{IJ}$, $\epsilon^{AB} \to R\epsilon^{AB}$, so that if f^I, g_I are kept fixed, then $h \to R^{-1}h$ and $\tilde{h} \to R\tilde{h}$. Then the amplitude scales as R^{-1}, so that R^{-1} sets the strength of the gravitational coupling.

10 Discussion

In this paper, a number of new twistor string theories have been constructed. They were shown to be free from perturbative world-sheet anomalies, and the ghost-independent part of the spectra in space-time have been found. The full BRST cohomology including ghost-dependent vertex operators will be discussed elsewhere. The key questions that remain are whether these give fully consistent quantum theories, and whether they have non-trivial interactions. We have seen in section 9 that non-vanishing 3-point supergravity amplitudes are obtained in the $N = 4$ and $N = 8$ cases, so these theories have non-trivial interactions. Other amplitudes for these theories, and those for the other theories, will be discussed elsewhere.

The string theories giving the $N = 4$ and $N = 8$ theories involve arbitrary functions w, w' of homogeneity -2 and -1 respectively. These can be chosen to be non-singular for the theory with Lorentzian world-sheet and independent real coordinates $Z, \tilde{Z}$ (with target space $\mathbb{RP}^{3|4} \times \mathbb{RP}^{3|4}$ in the flat case) and for the Wick-rotated version of this with Euclidean world-sheet and independent complex coordinates $Z, \tilde{Z}$ (with target space $\mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4}$ in the flat case). There is also a theory with Euclidean world-sheet obtained from this by setting $\tilde{Z} = Z^*$ (with target space $\mathbb{CP}^{3|4}$ in the flat case); in this case, we can choose w, w' to be non-singular on the disc but complex on the boundary, resulting in a modification of the boundary conditions for the ghosts, or we can choose w, w' to be real on the boundary but singular on the disc. With the latter choice, however, the gauging of the weightless one-forms may be problematic. The $N = 4$ and $N = 8$ theories then arise from the real theory with Lorentzian world-sheet and real $Z, \tilde{Z}$, while the amplitudes are calculated using the Euclidean version of this.

The Berkovits twistor string gives a theory of $N = 4$ superconformal gravity coupled to $N = 4$ super-Yang-Mills for any gauge group that can arise as a current algebra of a $c = 28$ conformal field theory. However, it is known that $N = 4$ superconformal gravity coupled to $N = 4$ super-Yang-Mills has an $SU(4)$ (or $SL(4, \mathbb{R})$ in split signature) R-symmetry anomaly that cancels only if G is 4-dimensional [42, 43], so $G = SU(2) \times U(1)$ or $U(1)^4$. This is so for the theory with minimal kinetic term $\int W^2$, but a similar result is expected to apply for the theory with non-minimal kinetic term $\int e^{-2\Phi} e^{2W}$ arising from the twistor string [6]. This suggests that the Berkovits string may only be consistent at loops for special gauge groups, and that there are constraints and potential inconsistencies that have not yet been found. In [6], it was suggested that these may come from open string tadpole cancellation. At loops, there may be interactions with a closed string sector, and further issues could arise from closed strings. (Closed string vertex operators are constructed from products of left-moving and right-moving vertex operators, so that one might expect the closed string spectrum to be related to the tensor product of the open string spectrum with itself. The twistor space spectrum appears to be the tensor product of that for open strings, but it is not clear what this means for the space-time spectrum, as the conventional Penrose transform does not apply to non-holomorphic fields $\Phi(Z, \tilde{Z})$.)

The new string theories described here have the same form as the Berkovits string, but with extra terms in the BRST operator. It is therefore to be expected that for these theories, too, there will be further constraints that will eliminate some models. We do not understand these constraints from the string theory perspective, but some clues might be obtained from the corresponding space-time theories. The new theories have different symmetries from those of conformal supergravity (for example, they do not have a gauged R-symmetry or a conformal symmetry) and so they will have different anomalies, and different constraints from anomaly cancellation. Interestingly, there are supersymmetric theories which can be defined in 2+2 dimensions that have no analogue in 3+1 dimensional space-time, and the spectra of some of these arise here.

First, the theory of section 8.2 has the spectrum of self-dual gravity coupled to self-dual Yang-Mills and a scalar (or 2-form gauge field). Consistent non-linear interactions are possible classically for this theory, with field equations given by some scalar-dependent modification of (8.14). There is no covariant action for such field equations, but there are non-covariant actions of the type proposed by Plebanski [44]. The theory is a chiral one in $2+2$ dimensional space-time, and so it is prone to potential anomalies. An interacting theory of self-dual gravity coupled to self-dual Yang-Mills in $2+2$ dimensions arises from the $\mathcal{N}=2$ string [14], and this is believed to be a consistent quantum theory (however, see [45, 46]). This suggests the intriguing possibility that the $N=0$ twistor string found here could be dual to an $\mathcal{N}=2$ string theory. A string theory with the spectrum of self-dual gravity coupled to self-dual Yang-Mills and a 2-form gauge field is given by the $\mathcal{N}=2$ string whose target space is generalised Kähler [53]; this is obtained by coupling the $(2,2)$ supersymmetric sigma-model with torsion [54] to $\mathcal{N}=2$ world-sheet supergravity. The theories of section 8.3 with $N<4$ give supersymmetric extensions of this bosonic theory with self-dual supergravity coupled to self-dual super-Yang-Mills and N supersymmetries, and these could be consistent non-trivial theories if the $N=0$ theory is.

For $N=4$, we have two twistor theories, both of which have the spectrum of $N=4$ supergravity coupled to $N=4$ super-Yang-Mills. One is the theory of §8.3 with $N=4$ (for any gauge group that can arise as a current algebra of a $c=22$ conformal field theory) and the other is the theory of §8.5. However, there are a number of different supersymmetric theories with this spectrum, and the question we now turn to is which of these arises in the twistor string. Consider first the Yang-Mills sector, for which there is the free theory and two possible interacting supersymmetric theories. For $N = 4$ Yang-Mills, there is the standard non-chiral theory, which can be rewritten in the Chalmers-Siegel form [47] with Yang-Mills kinetic term $\int EF + E^2$ where E is a self-dual 2-form and $F = dA + A^2$ is the usual Yang-Mills field strength. There is also Siegel's chiral theory with Yang-Mills kinetic term $\int EF$ [48]. This is sometimes called a self-dual theory, but it has the same spectrum as the usual super-Yang-Mills theory. It differs from the usual theory in that the interactions are chiral, i.e. they are not symmetric under the parity transformation interchanging positive and negative helicities, and the action is linear in the negative helicity fields (such as E). The full non-chiral $N = 4$ super-Yang-Mills theory is obtained in the Berkovits string, and the same is true for our $N = 4$ theory as it is the same as that of Berkovits in the Yang-Mills sector.

The supergravity sector has the spectrum of $N = 4$ Einstein supergravity, and we have seen that it has at least one non-trivial interaction. Just as for Yang-Mills, there is the possibility of either the standard non-chiral theory or of one with chiral interactions. A formulation of Einstein gravity with chiral interactions was discussed in [40, 17]. The fields consist of a vierbein ${e_\mu}^a$ (the analogue of the Yang-Mills connection A) and an independent Lagrange multiplier field ω_μ^{ab} which is anti-self-dual in the Lorentz indices ab (the analogue of the anti-self-dual Lagrange multiplier field E). The multiplier ω_μ^{ab} imposes the constraint that the anti-self-dual part of the Levi-Civita spin-connection $\Omega(e)$ constructed from e vanishes, so that the corresponding curvature is self-dual. An $N = 4$ supersymmetric version of this theory was given by Siegel [40], with component action given by truncating the $N = 8$ component action of ref. [40].

To determine whether the free, chiral or the non-chiral interacting $N = 4$ supergravity arises from the two $N = 4$ string theories requires further analysis of the scattering amplitudes, and we will return to this elsewhere. However, the theory of §8.5 has the usual non-chiral Yang-Mills interactions and has a non-trivial cubic gravitational coupling, so it is presumably the full Yang-Mills theory coupled to either chiral or non-chiral $N = 4$ supergravity. The usual non-chiral interacting $N = 4$ supergravity coupled to Yang-Mills theory has no anomalies, but it is expected to have ultra-violet divergences. Nonetheless, it has a limit in which gravity decouples to leave $N = 4$ super-Yang-Mills, and this is believed to be a consistent ultra-violet finite field theory. The theory of chiral $N = 4$ supergravity coupled to $N = 4$ super-Yang-Mills is likely to have better ultra-violet behaviour than the full supergravity (and might conceivably be finite) and it has a similar decoupling limit so that, whichever supergravity theory arises, there should be a decoupling limit giving pure $N = 4$ super-Yang-Mills amplitudes. This limit in the twistor theory is given by scaling the infinity twistor so that $I^{IJ} \to 0$. Then from (1.3), for any supergravity wave-function h, the corresponding f^α will vanish and so any amplitude involving h will vanish. It will be interesting to check that this leads to a full decoupling of gravity at all orders in perturbation theory. There is then the intriguing possibility that this twistor string can give $N=4$ super-Yang-Mills in this limit.

For the $N=4$ supergravity and Yang-Mills theories, a relation with $\mathcal{N}=2$ strings has also been suggested in [48, 49], and again there is the possibility of a link between our twistor strings and an $\mathcal{N}=2$ string theory. A relation between Siegel's $N=4$ supersymmetric $\mathcal{N}=2$ string and a different twistor string theory was suggested in [50].

Next, consider the theory of section 8.4, giving the spectrum of $N=4$ supergravity plus four $N=4$ gravitino multiplets, together with super-Yang-Mills (for any gauge group that can arise as a current algebra of a $c=26$ conformal field theory). There are then 8 gravitini of helicity $+3/2$ and 8 gravitini of helicity $-3/2$, so that the theory should be an $N=8$ supergravity theory. Again, there is the possibility of either a theory with chiral interactions, or a non-chiral one. (There is also the possibility of a free theory.) If it is a standard non-chiral $N=8$ supergravity, the total number of vector fields should be 28 and this requires the number of Yang-Mills multiplets to be six. This suggests that, if the twistor string gives a consistent non-chiral theory, there must be a constraint fixing the number of vector multiplets to be 6. The Berkovits string is expected to have a constraint fixing the number of vector multiplets to be 4, to cancel the anomalies of conformal supergravity, and both constraints could arise in the same, as yet unknown, way. Alternatively, the theory arising could be Siegel's chiral $N=8$ supergravity [40], in which the negative helicity fields appear linearly. In [40], Siegel argued that the $\mathcal{N}=2$ string gives $N=4$ chiral Yang-Mills from the open string sector and $N=8$ chiral supergravity from the closed string sector, and that the chirality of the interactions implied that the supergravity and super-Yang-Mills fields do not couple, so that one can consistently have $N=8$ chiral supergravity and an arbitrary number of $N=4$ chiral Yang-Mills multiplets. It will be interesting to see whether either of these interacting $N=8$ supergravity theories arise here. If the space-time theories arising from the perturbative string theory are chiral supergravities, then it is possible that non-perturbative effects could give rise to the non-chiral interactions, as they do for Yang-Mills in Witten's topological twistor string [1].

Finally, for the models of section 8.3 with $N>4$, the spectrum is chiral with states of spin greater than 2, and with more than one state of spin 2. It is believed that there are no chirally-symmetric theories with spins higher than 2 or with more than one graviton which have non-trivial interactions, but the no-go theorems do not apply to chiral theories. Consider first the $N>4$ Yang-Mills theories, with helicities $1, 1/2, ..., -N/2$ in the $SL(N,\mathbb{R})$ representations $(\mathbf{1}, \mathbf{N}, \mathbf{N(N-1)/2}, ..., \mathbf{N'}, \mathbf{1})$, and all in the adjoint of the Yang-Mills gauge group, so that for $N>4$ there are negative helicity states of spin greater than one. The field equation for a free massless field $\Phi_{A'_1 A'_2 \ldots A'_n}$ of helicity $-n/2$ is

$$\nabla^{BA'_1}\Phi_{A'_1 A'_2 \ldots A'_n} = 0. \tag{10.1}$$

For a field in a representation of the gauge group, the corresponding field equation is (10.1) where ∇ is the Yang-Mills covariant derivative. For $n \geq 2$ this is consistent only if the Yang-Mills connection is self-dual,

$$F_{A'B'} = 0. \tag{10.2}$$

The chiral $N = 4$ theory is of this type, with self-dual Yang-Mills coupled to a field $E_{A'B'}$ with field equation of the form (10.1). There are then consistent chiral interactions for the $N > 4$ Yang-Mills multiplets of this type provided the Yang-Mills equation is the self-duality condition (10.2). It remains to investigate whether such interactions can be supersymmetric, and we will return to this elsewhere. For $N > 4$, the chirality of the spectrum will mean that it is unlikely that there will be a covariant action.

Similar considerations apply to the $N > 4$ supergravities arising from the twistor strings, in which there are negative helicity states of spin greater than two. The field equation for a free massless field of helicity $-n/2$ is again (10.1), but with ∇ denoting the gravitational covariant derivative. In curved space, this has an integrability condition for $n > 2$ (the Buchdahl constraint) given by

$$\tilde{\psi}_{A'B'C'D'} = 0 \tag{10.3}$$

where $\tilde{\psi}_{A'B'C'D'}$ is the anti-self-dual part of the Weyl-curvature. For Lorentzian signature, this would imply that space-time is conformally flat, but for Euclidean or split signatures, non-trivial conformally self-dual spaces are possible. A free field of helicity $-n/2$ can then be consistently coupled to conformally self-dual gravity. Self-dual supergravities for $N \leq 8$ have been given in [40], and it is to be expected that these can be coupled to the free supermultiplet with helicities $0, -1/2, ..., -N/2$. Such theories could provide consistent interactions for the space-time theory arising from the $N \leq 8$ twistor strings, with the self-dual supergravity fields arising from the twistor field f and the negative helicity multiplet from the twistor field g. For $N > 8$ supergravity, just as for $N > 4$ Yang-Mills, there are consistent interactions that can be written down and it remains to be seen whether these can be supersymmetric.

Much remains to be done to investigate the interactions of the theories presented in this paper. It would be interesting to find and analyse super-twistor space actions, following [30, 31], and to seek corresponding modifications of Witten's topological twistor string that gave similar results. It is conceivable that some of the strings found here give free theories, and that others may be inconsistent. However, it is encouraging that suitable interacting supersymmetric space-time theories exist for many of the cases, and interesting that the interactions are typically chiral for $N \neq 4, 8$. However, the most promising theories are the $N = 4$ theory giving an interacting theory of supergravity coupled to super-Yang-Mills, and the one giving $N = 8$ supergravity. The $N = 4$ theory has a decoupling limit giving pure Yang-Mills, opening the prospect of a twistor string formulation of super-Yang-Mills loop amplitudes.

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A Appendix: relation between split signature constructions

In this appendix, we continue our discussion in §3.1.1 of two distinct twistor constructions for space-times of split signature. In the first construction, we obtained a deformed twistor space $\mathcal{PT}$ with a complex conjugation $\tau : \mathcal{PT} \to \mathcal{PT}$ whose fixed point set defined a real slice $\mathcal{PT}_{\mathbb{R}}$, whereas in the second we considered a deformation $\mathcal{PT}_{\mathbb{R}}$ of the real slice $\mathbb{PT}_{\mathbb{R}}$ inside $\mathbb{PT}$. Although the first construction is perhaps more intuitive, the second is more powerful and has a better conceptual fit with the Berkovits open twistor string model, so we will derive the first construction from the second. We will assume that we have obtained a twistor space $\mathcal{PT}$ by suitably gluing together the twistor spaces for small open sets in space-time, with the assumption that the space-time is $S^2 \times S^2$ globally and admits an analytic conformal structure. This space is non-Hausdorff, and we give a brief description of it here.

The second construction starts from the data of $\mathcal{PT}_{\mathbb{R}} \subset \mathbb{PT}$ determined by equation

(3.5):

$$Z^\alpha - \bar{Z}^\alpha = iF^\alpha(Z^\beta + \bar{Z}^\beta)\,. \tag{A.1}$$

With the assumption of analyticity, F^α can be analytically continued to become a holomorphic function $F^\alpha(Z^\beta)$ on a neighbourhood containing $\mathbb{T}_\mathbb{R}$ (initially, $F^\alpha(Z^\beta)$ was defined only for real values of Z^α). Thus equation (A.1) will make sense when $\bar{Z}^\alpha$ is replaced by $\tilde{Z}^\alpha$ where $\tilde{Z}^\alpha$ is close to, but not necessarily equal to $\bar{Z}^\alpha$. This gives the equation

$$Z^\alpha - \tilde{Z}^\alpha = iF^\alpha(Z^\beta + \tilde{Z}^\beta), \tag{A.2}$$

where now $\tilde{Z}^\alpha$ is an independent variable that is no longer the complex conjugate of Z^α. For F^α sufficiently small, this equation can be solved for $\tilde{Z}^\alpha$ in terms of Z^β as

$$\tilde{Z}^\alpha = P^\alpha(Z^\beta) \tag{A.3}$$

for some invertible functions P^α. Since (3.5) was defined for $Z^\alpha \in \mathcal{T}_\mathbb{R}$ and $\tilde{Z}^\alpha \in \overline{\mathcal{T}_\mathbb{R}}$, the analytic continuation (A.2) will be defined for Z^α in some neighbourhood V of $\mathcal{T}_\mathbb{R} \subset \mathbb{PT}_+$ and, from the reality properties of (3.5), the P^α will map V holomorphically onto the complex conjugate set $\bar{V} \subset \mathbb{PT}_-$. It follows from this definition that the real slice $P\mathcal{T}_\mathbb{R}$ is given by the subset of V on which $\tilde{Z}^\alpha = \bar{Z}^\alpha$, since (A.2) then reduces to (3.5).

We will construct $P\mathcal{T}$ by gluing together two copies of $\mathbb{CP}^3$ using $P^\alpha(Z^\beta)$. We now take Z^α to be holomorphic coordinates on one copy of $\mathbb{CP}^3$, denoted $\mathbb{PT}_+$, and $\tilde{Z}^\alpha$ to be coordinates on another copy denoted $\mathbb{PT}_-$. We construct $P\mathcal{T}$ by interpreting equation (A.3) as a patching relation for constructing a complex manifold by gluing the neighbourhood $V \subset \mathbb{PT}_+$ to $\bar{V} \subset \mathbb{PT}_-$. We note, however, that this global description is not Hausdorff. Furthermore, the full space $P\mathcal{T}$ admits a complex conjugation τ which interchanges $\mathbb{PT}_+$ and $\mathbb{PT}_-$ so that τ maps the point $Z^\alpha \in \mathbb{PT}_+$ to the point $\tilde{Z}^\alpha = \bar{Z}^\alpha \in \mathbb{PT}_-$ and vice-versa. In order to see that this is well defined, we need to check that it is compatible with the patching (A.3); if $Z^\alpha \in V$ then $\tau(Z^\alpha)$ is the point in $\mathbb{PT}_-$ with $\tilde{Z}^\alpha = \bar{Z}^\alpha$, but Z^α is identified with $\tilde{Z}^\alpha = P^\alpha(Z^\beta)$ in $\mathbb{PT}_-$ whose conjugate point is $Z^\alpha = \overline{P^\alpha(Z^\beta)}$ in $\mathbb{PT}_+$. For τ to give the same point in each case, we need to see that $\bar{Z}^\alpha = P^\alpha(\overline{P^\beta(Z^\gamma)})$. This follows from the fact that (A.3) is equivalent to (A.2) and F^α is a real function for real values of its argument, so that its analytic continuation satisfies $\overline{F^\beta(Z^\alpha + \tilde{Z}^\alpha)} = F^\beta(\bar{Z}^\alpha + \bar{\tilde{Z}}^\alpha)$. Thus (A.2) implies

$$\bar{\tilde{Z}}^\alpha - \bar{Z}^\alpha = iF^\alpha(\bar{Z}^\beta + \bar{\tilde{Z}}^\beta)$$

and this equation is the same as (A.2) except that the role of Z^α has been taken by $\bar{\tilde{Z}}^\alpha$ and that of $\tilde{Z}^\alpha$ by $\bar{Z}^\alpha$. Thus we have $\bar{Z}^\alpha = P^\alpha(\bar{\tilde{Z}}^\beta) = P^\alpha(\overline{P^\beta(Z^\gamma)})$ as desired.

Given a holomorphic disc D_x in $\mathbb{PT}_+$ with boundary on $P\mathcal{T}_\mathbb{R}$, we can define the Riemann sphere $\mathbb{CP}^1_x = D_x \cup \tau(D_x)$ in $P\mathcal{T}$ since τ fixes $P\mathcal{T}_\mathbb{R}$ and hence glues the boundary of D_x to that of $\tau(D_x)$. It is a standard theorem in complex analysis that this embedding will actually be holomorphic along ∂D_x as well as over the interiors of D_x and $\tau(D_x)$.

We can carry out the non-linear graviton construction on $P\mathcal{T}$ and construct the space $\mathbb{C}\mathcal{M}$ of Riemann spheres in $P\mathcal{T}$ in the same family as $\mathbb{CP}^1_x$. This will be four complex dimensional as before, and admit a holomorphic conformal structure that is anti-self-dual. The anti-holomorphic involution τ on $P\mathcal{T}$ takes Riemann spheres to Riemann spheres, and so it induces a complex conjugation on $\mathbb{C}\mathcal{M}$ that preserves the conformal structure; thus it fixes a real slice $\mathcal{M} \subset \mathbb{C}\mathcal{M}$ on which the conformal structure is real. The points of the real slice correspond to Riemann spheres in $P\mathcal{T}$ that are mapped to themselves by the anti-holomorphic involution. Such Riemann spheres contain an equatorial circle that is fixed by the involution, and which must lie in the fixed points $P\mathcal{T}_{\mathbb{R}}$ in $P\mathcal{T}$. Thus such a Riemann sphere corresponds to a pair of holomorphic discs in $P\mathcal{T}$ with common boundary on $P\mathcal{T}_{\mathbb{R}}$ and conversely a disc D gives rise to the Riemann sphere $D \cup \tau(D)$ as described above.

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