EXPLICIT FORMULAS FOR THE EIGENFUNCTIONS OF THE N-BODY CALOGERO MODEL

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Abstract. We consider the quantum Calogero model, which describes N non-distinguishable quantum particles on the real line confined by a harmonic oscillator potential and interacting via two-body interactions proportional to the inverse square of the inter-particle distance. We elaborate a novel solution algorithm which allows us to obtain fully explicit formulas for its eigenfunctions, for arbitrary coupling parameter and particle number. We also show that our method applies, with minor changes, to all Calogero models associated with classical root systems.

1. Introduction

In this paper we elaborate a novel solution method for the N-body *Calogero model* defined by the Hamiltonian

$$H = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + x_j^2 \right) + 2\lambda(\lambda - 1) \sum_{j<k} \frac{1}{(x_j - x_k)^2}, \tag{1}$$

where $\lambda > 0$ is the coupling parameter, $x_j \in \mathbb{R}$ the particle coordinates, $\partial_{x_j} := \partial/\partial x_j$, and $N = 1, 2, 3, \ldots$ the particle number (we set the harmonic oscillator frequency $\omega > 0$ to 1 without loss of generality: this parameter can easily be introduced by scaling $x_j \to \sqrt{\omega} x_j$, $H \to \omega H$, etc.). As is well-known [Cal71,Sut72], this model has exact eigenfunctions of the form

$$\psi_{\mathbf{n}} = \psi_0 P_{\mathbf{n}}, \tag{2}$$

where

$$\psi_0(\mathbf{x}) = \prod_{j=1}^{N} \mathrm{e}^{-\frac{1}{2}x_j^2} \prod_{j<k} (x_k - x_j)^\lambda \tag{3}$$

is the groundstate eigenfunction and $P_{\mathbf{n}}(\mathbf{x})$ are polynomials which are symmetric, i.e., invariant under permutations of the particle coordinates. These polynomials are labeled by N-tuples $\mathbf{n} = (n_1, \ldots, n_N)$ of non-negative integers, $n_j \in \mathbb{N}_0$. Due to the permutation symmetry these labels can be restricted to partitions, i.e.,

$$n_1 \geq n_2 \geq \ldots \geq n_N \geq 0,$$

but we will *not* always make this restriction. The corresponding exact eigenvalues are given by the following remarkably simple formulas:

$$E_{\mathbf{n}} = 2(n_1 + n_2 + \cdots + n_N) + E_0, \quad E_0 = N(1 + \lambda(N-1)). \tag{4}$$

We refer to the $P_{\mathbf{n}}$ as *reduced polynomial eigenfunctions of the Calogero model*, and our aim is to derive explicit formulas for them. These polynomials are a

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natural many-variable generalization of the Hermite polynomials to which they reduce in the special case $N = 1$ [BF97]. Previous results on these functions [Cal69,Sut72,Per71,Gam75,BHV92,Kak96,DLM04] will be discussed in more detail below. We also mention that there has been considerable interest in many-variable generalizations of classical orthogonal polynomials in the mathematics literature; see e.g. [DX01, Mac95] and references therein.

Calogero found in his seminal paper [Cal71] the exact eigenvalues of a closely related model which differs from the one above only in its center of mass motion (for the convenience of the reader we discuss the precise relation of these two models in Appendix A). The eigenvalues of the Hamiltonian in (1) was given by Sutherland [Sut72], who also presented an algorithm for constructing the reduced eigenfunctions $P_{\mathbf{n}}$. This algorithm starts with the ansatz in (2), which converts the eigenvalue problem for H into a problem of diagonalizing a certain triangular matrix. Thus, the eigenvalues of H can be read off from the diagonal of this matrix, and the eigenfunctions are determined by certain recursion relations which truncate after a finite number of steps; see Section 2 for details. To our knowledge, these recursion relations have not been solved by a closed formula.

In Sutherland's paper [Sut72] the emphasis was on a translation invariant N-body model with a $1/\sin^2$-interaction which is also exactly solvable and the solution algorithm was elaborated in detail only for this so-called *Sutherland model*. In [Lan01] one of us presented an alternative algorithm to solve the Sutherland model which, different from Sutherland's, also can be generalized to the elliptic case; see [Lan05] and references therein. In the present paper we extend this solution algorithm to the Calogero model. We stress that the Calogero model is more complicated than the Sutherland model due to the presence of the harmonic oscillator potential, and this leads to various interesting and novel features. It is also interesting to note that, in our approach, the factorization of the eigenfunctions in (2) is a consequence, rather than an essential ingredient, of the method. Moreover, rather than constructing the eigenfunctions as linear combinations of the free boson eigenstates as Sutherland, we obtain a set of somewhat more complicated functions which lead to simpler recursion relations which we solve explicitly. This gives our main result: an explicit formula for the reduced polynomial eigenfunctions of the Calogero model.

We now briefly describe this result. For each fixed $\mathbf{x} \in \mathbb{R}^N$ and $\epsilon > 0$, let $\mathcal{C}_j$ denote the following set of nested circles in the complex plane:

$$\mathcal{C}_j: \; y_j = (\max_{1\leq k\leq N}(|x_k|) + \epsilon j)\mathrm{e}^{\mathrm{i}\varphi_j}, \quad -\pi \leq \varphi_j < \pi, \; j = 1, \ldots, N. \tag{5}$$

Using these curves as integration paths, define for each $\mathbf{n} \in \mathbb{N}_0^N$ the functions

$$f_{\mathbf{n}}(\mathbf{x}) := \prod_{j=1}^{N} \left(\oint_{\mathcal{C}_j} \frac{dy_j}{2\pi \mathrm{i} y_j} y_j^{n_j} \right) \frac{\prod_{j<k}(1 - y_j/y_k)^\lambda}{\prod_{j,k=1}^{N}(1 - x_j/y_k)^\lambda} \tag{6}$$

which are symmetric polynomials independent of $\epsilon > 0$; see Section 3. Our main result is a fully explicit formula for the functions $P_{\mathbf{n}}$ as linear superpositions of these functions $f_{\mathbf{n}}$. We use the natural basis elements $\mathbf{e}_j \in \mathbb{N}_0^N$ defined by $(\mathbf{e}_j)_k := \delta_{jk}$ and write $\delta_{\mathbf{n}}(\mathbf{m}) := \delta_{\mathbf{n},\mathbf{m}}$ for the Kronecker delta.

Theorem 1.1. *For* $\mathbf{n} \in \mathbb{Z}^N$ *let*

$$P_{\mathbf{n}} = \sum_{\mathbf{m}} \alpha_{\mathbf{n}}(\mathbf{m}) f_{\mathbf{m}}$$

with the functions $f_{\mathbf{m}}$ *defined in (6) and the coefficients*

$$\begin{aligned} \alpha_{\mathbf{n}}(\mathbf{m}) = \delta_{\mathbf{n}}(\mathbf{m}) + \sum_{s=1}^{\infty} \frac{1}{4^s s!} \sum_{j_1 \le k_1} \cdots \sum_{j_s \le k_s} \sum_{\nu_1, \ldots, \nu_s = 0}^{\infty} \\ \times \delta_{\mathbf{n}}(\mathbf{m} + \textstyle\sum_{r=1}^{s} \mathbf{E}_{j_r k_r}^{\nu_r}) \displaystyle\prod_{r=1}^{s} g_{j_r k_r}(\nu_r; \mathbf{n} - \textstyle\sum_{\ell=1}^{r} \mathbf{E}_{j_r k_r}^{\nu_r}), \end{aligned} \tag{7}$$

where we use the shorthand notations

$$\begin{aligned} g_{jk}(\nu; \mathbf{m}) = {} & 2\lambda(\lambda - 1)\nu(1 - \delta_{jk}) - \tilde{m}_j(\tilde{m}_j + 1)\delta_{\nu 0}\delta_{jk}, \\ & \tilde{m}_j = m_j + \lambda(N + 1 - j), \end{aligned} \tag{8}$$

and

$$\mathbf{E}_{jk}^{\nu} = (1 - \nu)\mathbf{e}_j + (1 + \nu)\mathbf{e}_k. \tag{9}$$

Then $P_{\mathbf{n}}$ *is a reduced polynomial eigenfunction of the Calogero model corresponding to the eigenvalue* $E_{\mathbf{n}}$ *in (4).*

(The proof will be given in Sections 3.1–3.4.)

It is important to note that the sums in (7) only contain a finite number non-zero terms. It is also remarkable that Theorem 1.1 is non-trivial already for the simplest case $N = 1$, as discussed in Section 3.5. In Section 3.6 we use this result to construct somewhat more complicated basis functions than the $f_{\mathbf{n}}$, leading to another explicit formula for the reduced polynomial eigenfunctions; see Theorem 3.1.

Observe that for $N > 1$, this result gives too many eigenfunctions: they are in Theorem 1.1 labeled by elements in $\mathbb{Z}^N$, but it is known that a complete set of eigenfunctions can be parameterized by partitions alone. Using the symbolic programming language MATHEMATICA we have checked for $N = 2$ that the $P_{\mathbf{n}}$ are non-zero eigenfunctions also for non-partitions $\mathbf{n}$, and we conjecture this to be true for all N. This over-completeness of our solution poses some interesting questions discussed in Remark 5.1.

Similarly as in Sutherland's algorithm [Sut72], we obtain the coefficients $\alpha_{\mathbf{n}}(\mathbf{m})$ by diagonalizing a certain triangular matrix, and they are therefore non-zero only for $\mathbf{m} \preceq \mathbf{n}$ in some partial ordering $\preceq$; see Section 2. However, this partial ordering is different from Sutherland's, and the matrix we get is simpler, which is why we can find its explicit eigenvectors.

As found by Olshanetsky and Perelomov [OP77], the model discussed so-far can be naturally associated with the root system A_{N-1}, and there are exactly solvable variants of the Calogero model related to all other root systems; see [OP83] for a comprehensive review. In particular, the Calogero models associated with the remaining classical root systems [OP83] can all be brought to the form of the B_N Hamiltonian

$$H_{B_N} = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + x_j^2 + \frac{\mu(\mu - 1)}{x_j^2} \right) + 4\lambda(\lambda - 1) \sum_{j<k} \frac{(x_j^2 + x_k^2)}{(x_j^2 - x_k^2)^2} \tag{10}$$

with two coupling parameters $\mu, \lambda > 0$. To demonstrate the generality of our solution method we show that the construction of eigenfunctions with minor changes goes through also in this case, and we thereby obtain explicit formulas for a many-variable generalization of the Laguerre polynomials. This adds support to our hope that the method can be used so solve any Calogero-Sutherland type model. We should mention that the B_N Calogero model also can be solved using Sutherland's method, of course.

As mentioned in the first paragraph, various other explicit results for the reduced polynomial eigenfunctions of the Calogero model exist in the literature. Calogero obtained such results for the cases $N = 2, 3$ [Cal69]. By exploiting an underlying group structure of the Hamiltonian, Perelemov [Per71] for $N = 4$ and Gambardella for $N = 5$ [Gam75] obtained the eigenfunctions in terms of "raising" operators acting on the groundstate. More recently these operator solutions were generalized to all N [BHV92, Kak96]. We also mention that Desrosiers et.al. obtained explicit results for the eigenfunctions of a supersymmetric generalization of the Calogero model using a determinantal construction [DLM04]. Our results seem different and complementary to these.

The plan of the rest of this paper is as follows. In Section 2 we fix our notation and shortly review Sutherland's solution of the Calogero model [Sut72] and a simple variant thereof which, as we argue, is somewhat more natural. In Section 3 we present our solution of the A_{N-1} Calogero model and thereby prove Theorem 1.1. We also comment on the one-particle case, and we sketch a variant of our solution method which provides another explicit formula for the eigenfunctions. Our solution of the B_N Calogero model is presented in Section 4. We end with a few concluding remarks in Section 5. Some technical details are deferred to two appendices.

2. Sutherland's solution algorithm

In this section we fix our notation and, to put our work into context, briefly review Sutherland's solution of the Calogero model [Sut72]. We will actually discuss a somewhat simpler variant of this solution method, as explained below.

In the discussion below we make use of some notational conventions from the theory of partitions which we now recall; see e.g. [Mac95]. For partitions $\mathbf{n} = (n_1, n_2, \ldots, n_N)$, the non-zero n_i are called the *parts* of $\mathbf{n}$, and we use the short hand notation

$$|\mathbf{n}| := n_1 + n_2 + \ldots + n_N.$$

We also introduce a partial ordering of partitions: for two partitions $\mathbf{m}, \mathbf{n}$ we write

$$\mathbf{m} \leq \mathbf{n} \;\Leftrightarrow\; \sum_{k=1}^{j} m_k \leq \sum_{k=1}^{j} n_k \quad \forall j = 1, \ldots, N.$$

We will furthermore write $\mathbf{m} < \mathbf{n}$ if $\mathbf{m} \leq \mathbf{n}$ and $|\mathbf{m}| \neq |\mathbf{n}|$.

The starting point of Sutherland's algorithm is the observation that the function ψ_0 in (3) is the groundstate of the Hamiltonian H in (1), a fact which can be proved by a straightforward computation; see Remark 3.1. As previously mentioned, another key insight is that any eigenfunction of the Hamiltonian H in (1) can be factorized into a symmetric polynomial and the groundstate. This implies that such a symmetric polynomial is an eigenfunction of the differential operator

$$\tilde{H} := \psi_0^{-1} H \psi_0 - E_0 = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + 2x_j \partial_{x_j} \right) - 2\lambda \sum_{j<k} \frac{1}{x_j - x_k} (\partial_{x_j} - \partial_{x_k}). \tag{11}$$

The idea is now to construct these polynomials as linear combinations of the monomials

$$M_{\mathbf{n}} = \sum_{P \in S_N} x_{P(1)}^{n_1} \cdots x_{P(N)}^{n_N}, \tag{12}$$

where $\mathbf{n}$ is a partition of length N and S_N the permutation group of N elements. We note in passing that the standard normalization of these monomials is different in that the sum in (12) is restricted to the distinct permutations of the parts n_j (see e.g. [Mac95]), but for our purposes the normalization where one sums over all permutations is more convenient. To proceed we use the fact that

$$(-\partial_x^2 + 2x\partial_x) x^n = 2nx^n - n(n-1)x^{n-2}$$

as well as the identity

$$\begin{aligned}\frac{1}{x-y}(\partial_x - \partial_y)(x^n y^m + y^n x^m) = (n-m) &\sum_{k=1}^{n-m-1} x^{n-1-k} y^{m-1+k} \\ &- m(x^{n-1}y^{m-1} + y^{n-1}x^{m-1}),\end{aligned}$$

valid for all $x, y \in \mathbb{R}$ and $m, n \in \mathbb{N}_0$ such that $n \geq m$. A proof of this identity can be found in Appendix B. It follows that

$$\begin{aligned}\tilde{H} M_{\mathbf{n}} = 2|\mathbf{n}| M_{\mathbf{n}} - \sum_{j=1}^{N} n_j(n_j - 1) M_{\mathbf{n} - 2\mathbf{e}_j} \\ - \lambda \sum_{j<k} \sum_{\nu=0}^{\lfloor \frac{n_j - n_k}{2} \rfloor} (2 - \delta_{2\nu, n_j - n_k}) \left((1 - \delta_{\nu,0}) n_j - n_k \right) M_{\mathbf{n} - (\nu+1)\mathbf{e}_j + (\nu - 1)\mathbf{e}_k}\end{aligned} \tag{13}$$

where $\lfloor n/2 \rfloor = n/2$ or $(n-1)/2$ for even or odd integers n, respectively. Hence, the action of $\tilde{H}$ on the monomials $M_{\mathbf{n}}$ has triangular structure in the following sense:

$$\tilde{H} M_{\mathbf{n}} = 2|\mathbf{n}| M_{\mathbf{n}} + \sum_{\mathbf{m} < \mathbf{n}} b_{\mathbf{nm}} M_{\mathbf{m}}$$

for certain coefficients $b_{\mathbf{nm}}$ which can be determined from (13). This suggests that $\tilde{H}$ has eigenfunctions of the form

$$P_{\mathbf{n}} = M_{\mathbf{n}} + \sum_{\mathbf{m} < \mathbf{n}} u_{\mathbf{nm}} M_{\mathbf{m}} \tag{14}$$

with corresponding eigenvalues $\tilde{E}_{\mathbf{n}} = 2|\mathbf{n}|$. Indeed, inserting this result into the Schrödinger equation $\tilde{H} P_{\mathbf{n}} = \tilde{E}_{\mathbf{n}} P_{\mathbf{n}}$ and using the fact that the monomials $M_{\mathbf{m}}$ are linearly independent we obtain the following system of equations:

$$(\tilde{E}_{\mathbf{n}} - \tilde{E}_{\mathbf{m}}) u_{\mathbf{nm}} = b_{\mathbf{nm}} + \sum_{\mathbf{m} < \mathbf{k} < \mathbf{n}} u_{\mathbf{nk}} b_{\mathbf{km}}, \quad \mathbf{m} < \mathbf{n}. \tag{15}$$

It is important to note that $|\mathbf{m}| < |\mathbf{n}|$ for all $\mathbf{m} < \mathbf{n}$. Moreover, for each partition $\mathbf{n}$, there exists only a finite number of partitions $\mathbf{m} < \mathbf{n}$. Thus, (15) gives a well-defined recursion procedure for computing all coefficients $u_{\mathbf{nm}}$ in a finite number of steps.

As mentioned, the method described above is a somewhat simpler variant of Sutherland's original method [Sut72] who, instead of the monomials $M_{\mathbf{n}}$, used somewhat more complicated basis functions which we now describe. Let

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k} \tag{16}$$

denote the Hermite polynomial of order $n \in \mathbb{N}_0$, satisfying the differential equation

$$(-\partial_x^2 + 2x\partial_x) H_n(x) = 2n H_n(x).$$

Let $M_{\mathbf{n}}^{(H)}$ denote the symmetric polynomial

$$M_{\mathbf{n}}^{(H)}(\mathbf{x}) = M_{\mathbf{n}}(H_{n_1}(x_1), \ldots, H_{n_N}(x_N)) = \sum_{P \in S_N} H_{n_1}(x_{P(1)}) \ldots H_{n_N}(x_{P(N)}).$$

These symmetric polynomials are obviously eigenstates of the differential operator $\tilde{H}$ in (11) for the free case $\lambda = 0$ with eigenvalues $2|\mathbf{n}|$. The key identity is [Sut72]

$$\begin{aligned} &\frac{1}{x-y} (\partial_x - \partial_y) \left(H_n(x) H_m(y) + H_n(y) H_m(x) \right) \\ &= \sum_{r=1}^{n} \sum_{s=1}^{m} c_{rs}(n,m) \left(H_{n-r}(x) H_{m-s}(y) + H_{n-r}(y) H_{m-s}(x) \right) \end{aligned} \tag{17}$$

for all $n, m \in \mathbb{N}_0$ and certain real coefficients c_{rs}. Since a proof of this identity is not contained in Sutherland's paper [Sut72] we provide a sketch thereof in Appendix B. This identity shows that the action of $\tilde{H}$ on the symmetric polynomials $M_{\mathbf{n}}^{(H)}$ is triangular, which suggests that there are eigenfunctions $P_{\mathbf{n}} = M_{\mathbf{n}}^{(H)} + \sum_{\mathbf{m}<\mathbf{n}} v_{\mathbf{nm}} M_{\mathbf{m}}^{(H)}$ of $\tilde{H}$ with eigenvalues $2|\mathbf{n}|$ and a recursive procedure to compute all coefficients $v_{\mathbf{nm}}$ from $v_{\mathbf{nn}} = 1$, as above. However, the explicit formulas for the coefficients $c_{rs}(n,m)$ were not provided in [Sut72], and they indeed seem rather difficult to obtain: we neither found them in the literature, nor where we able to derive them.

It is interesting to note that the recursion relations in (15) can be inverted to yield explicit formulas for the coefficients $u_{\mathbf{nm}}$: introducing $u_{\mathbf{nn}} = 1$ and a linear operator R acting on these coefficients as follows:

$$R u_{\mathbf{nm}} = \frac{1 - \delta_{\mathbf{n},\mathbf{m}}}{\tilde{E}_{\mathbf{n}} - \tilde{E}_{\mathbf{m}}} \left(b_{\mathbf{nm}} + \sum_{\mathbf{m}<\mathbf{k}<\mathbf{n}} u_{\mathbf{nk}} b_{\mathbf{km}} \right),$$

we can rewrite the recursion relations as

$$u_{\mathbf{nm}} = \delta_{\mathbf{n},\mathbf{m}} + R u_{\mathbf{nm}}.$$

It follows that they can be inverted according to

$$u_{\mathbf{nm}} = (1-R)^{-1} \delta_{\mathbf{n},\mathbf{m}} = \sum_{s=0}^{\infty} R^s \delta_{\mathbf{n},\mathbf{m}},$$

where it is important to note that the expansion of the geometric series is well-defined since it only contains a finite number of non-zero terms; see below. From the definition of the linear operator R given above and the fact that $\tilde{E}_{\mathbf{n}} - \tilde{E}_{\mathbf{m}} = 2(|\mathbf{n}| - |\mathbf{m}|)$ now follows that

$$u_{\mathbf{nm}} = \delta_{\mathbf{n},\mathbf{m}} + \frac{1-\delta_{\mathbf{n},\mathbf{m}}}{2(|\mathbf{n}-\mathbf{m}|)}\left(b_{\mathbf{nm}} + \sum_{s=1}^{\infty}\sum_{\mathbf{m}<\mathbf{k}_1<\cdots<\mathbf{k}_s<\mathbf{n}} \frac{b_{\mathbf{nk}_1}b_{\mathbf{k}_1\mathbf{k}_2}\dots b_{\mathbf{k}_s\mathbf{m}}}{2^s\prod_{r=1}^{s}(|\mathbf{n}|-|\mathbf{k}_r|)}\right). \tag{18}$$

The restrictions imposed by the inequality in the second sum clearly implies that this series representation for the coefficients $u_{\mathbf{nm}}$ only contains a finite number of non-zero terms. Also note that each term is well-defined. However, this formula is not very useful since the $b_{\mathbf{nm}}$ are not given by a simple formula. Indeed, to deduce these latter coefficients from (13) it is important to note that $\mathbf{n}-2\mathbf{e}_j$ is *not*, in general, a partition, e.g. $(3,2,2)-2\mathbf{e}_1=(1,2,2)$. A similar remark applies to the last term in (13). We therefore implicitly used an extension of the monomials $M_{\mathbf{n}}$ to non-partitions. To make this precise we introduce an ordering symbol as follows: for each $\mathbf{a}\in\mathbb{N}_0^N$ we let $p[\mathbf{a}]$ denote the corresponding partition obtained by permuting the elements of $\mathbf{a}$, e.g., $p[(3,1,0,4)]=(4,3,1,0)$. We can then define $M_{\mathbf{a}} := M_{p[\mathbf{a}]}$, which naturally extends the definition of the monomials $M_{\mathbf{n}}$ to non-partitions. Using this definition we deduce from (13) that

$$\begin{aligned} b_{\mathbf{nm}} = &-\sum_{j=1}^{N} n_j(n_j-1)\delta_{p[\mathbf{n}-2\mathbf{e}_j],\mathbf{m}} - \lambda\sum_{j<k}\sum_{\nu=0}^{\lfloor\frac{n_j-n_k}{2}\rfloor} \\ &\times\left(2-\delta_{2\nu,n_j-n_k}\right)\left((1-\delta_{\nu,0})n_j-n_k\right)\delta_{p[\mathbf{n}-(\nu+1)\mathbf{e}_j+(\nu-1)\mathbf{e}_k],\mathbf{m}}. \end{aligned}$$

Inserting this in (18) one hopes that, due to the Kronecker deltas, the sums simplify considerably. However, the appearance of the ordering symbol $p[\cdot]$ makes the resulting formula awkward to use. We therefore conclude that the Sutherland algorithm does not lead to simple explicit formulas for the eigenfunctions. The same difficulty arises in Sutherland's original algorithm described above.

3. Alternative solution algorithm

In this section we present our alternative method for solving the Calogero model defined by the Hamiltonian in (1) and, in particular, prove Theorem 1.1.

We will to a large extent use the notation introduced in the beginning of Section 2, with the important difference that elements $\mathbf{m},\mathbf{n}\in\mathbb{Z}^N$ now will be ordered as follows:

$$\mathbf{m}\preceq\mathbf{n} \;\Leftrightarrow\; \sum_{k=N+1-j}^{N} m_k \le \sum_{k=N+1-j}^{N} n_k,\quad \forall j=1,\dots,N.$$

3.1. A remarkable identity.

We start by proving a particular functional identity, which is the starting point for our construction.

Lemma 3.1. *Let* $c_N = 2(1-\lambda)N$ *and*

$$F(\mathbf{x},\mathbf{y}) = \frac{\prod_{j=1}^{N}\mathrm{e}^{-\frac{1}{2}(x_j^2-y_j^2)}\prod_{j<k}(x_k-x_j)^\lambda(y_k-y_j)^\lambda}{\prod_{j,k=1}^{N}(y_k-x_j)^\lambda}.$$

Then

$$H(\mathbf{x})F(\mathbf{x},\mathbf{y}) = [H(\mathbf{y})+c_N]F(\mathbf{x},\mathbf{y}), \tag{19}$$

where $H=H(\mathbf{x})$ *is the Hamiltonian in (1) and similarly for* $H(\mathbf{y})$.

Proof. We set $\mathcal{N} = 2N$, $X_j = x_j$, $X_{N+j} = y_j$, $m_j = +1$ and $m_{N+j} = -1$ for $j = 1, 2, \ldots, N$. Then $H(\mathbf{x}) - H(\mathbf{y}) = \mathcal{H}(\mathbf{X})$ with

$$\mathcal{H} = \sum_{j=1}^{\mathcal{N}} \left(-\frac{1}{m_j}\partial_{X_j}^2 + m_j X_j^2 \right) + \sum_{j<k} \frac{\lambda(m_j + m_k)(\lambda m_j m_k - 1)}{(X_j - X_k)^2} \tag{20}$$

and $F(\mathbf{x}, \mathbf{y}) = \Psi_0(\mathbf{X})$ with

$$\Psi_0(\mathbf{X}) = \prod_{j=1}^{\mathcal{N}} \mathrm{e}^{-\frac{1}{2}m_j X_j^2} \prod_{j<k} (X_k - X_j)^{m_j m_k \lambda}. \tag{21}$$

To prove the lemma will we show by explicit computation that

$$(\mathcal{H} - \mathcal{E}_0)\Psi_0(\mathbf{X}) = 0 \tag{22}$$

with the constant

$$\mathcal{E}_0 = \lambda \left(\sum_{j=1}^{\mathcal{N}} m_j \right)^2 + \sum_{j=1}^{\mathcal{N}} (1 - \lambda m_j^2) \tag{23}$$

and note that $\mathcal{E}_0 = c_N$. For that we introduce the operator

$$\mathcal{D} = \sum_{j=1}^{\mathcal{N}} \frac{1}{m_j} Q_j^+ Q_j^-$$

with

$$Q_j^{\pm} = \pm\partial_{X_j} + W_j, \quad W_j = -m_j X_j + \sum_{k \neq j} \frac{\lambda m_j m_k}{(X_j - X_k)}.$$

Note that $Q_j^- \Psi_0 = 0$ for all $j = 1, \ldots, \mathcal{N}$, and hence that

$$\mathcal{D}\Psi_0 = 0.$$

This implies the identity in (22) since $\mathcal{D} = \mathcal{H} - \mathcal{E}_0$, as can be shown by straightforward computations. Indeed,

$$\mathcal{D} = \sum_{j=1}^{N} \frac{1}{m_j} \left(-\partial_{X_j}^2 + W_j^2 + (\partial_{X_j} W_j) \right) = \mathcal{H} - \mathcal{R},$$

where the reminder terms

$$\mathcal{R} = \sum_{j=1}^{N} 1 + 2\sum_{k \neq j} \lambda m_j m_k \frac{X_j}{X_j - X_k} + \sum_{\substack{k,\ell \neq j \\ \ell \neq k}} \frac{\lambda m_j m_k m_\ell}{(X_k - X_j)(X_j - X_\ell)}$$

add up to the constant $\mathcal{E}_0$: upon symmetrization the double sum becomes independent of the X_j and equal to $\sum_{j \neq k} \lambda m_j m_k = \lambda(\sum_j m_j)^2 - \lambda \sum_j m_j^2$ and the triple sum vanishes, as can be seen by symmetrizing in the indices j, k, ℓ and using the identity

$$\frac{1}{(X_k - X_j)(X_j - X_\ell)} + \frac{1}{(X_k - X_j)(X_\ell - X_k)} + \frac{1}{(X_j - X_\ell)(X_\ell - X_k)} = 0.$$

$\square$

Remark 3.1. It is easy to see that we have, in fact, proved a more general result: the identity in (22) holds true for all $\mathcal{N} = 2, 3, \ldots$ and arbitrary real parameters m_j. Obviously, one particular consequence of this latter result is the fact that the function ψ_0 in (3) is the groundstate of the Hamiltonian H in (1). It is not difficult to see that we have proved a similar fact for a more general case: *if all $m_j > 0$ then $\mathcal{H}$ in (20) defines a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^{\mathcal{N}})$ with $\Psi_0(\mathbf{X})$ in (21) as groundstate.* This is true since Q_j^+ then is the Hilbert space adjoint of Q_j^-, implying that $\mathcal{D} = \mathcal{H} - \mathcal{E}_0$ is a sum of non-negative terms $(Q_j^-)^* Q_j^-/m_j$ and thus defines a unique non-negative self-adjoint operator via the Friedrichs extension; see e.g. [RS75]. We thus recover a known generalization of the Calogero model where the particles can have different masses $m_j > 0$ and such that its exact groundstate and groundstate energy can be computed exactly [For92,MMS03]. Other interesting special cases will be discussed in Remark 5.2.

3.2. **Integral transformation.** The idea is now to apply to the identity in (19) an integral transform $\prod_{j=1}^N \left(\oint_{\mathcal{C}_j} dy_j (2\pi \mathrm{i} y_j)^{-1} \phi_{n_j}(y_j) \right)$ with the integration paths in (5) and certain functions $\phi_j(y_j)$ to be chosen such that this transform is well-defined. We observe that

$$F(\mathbf{x}, \mathbf{y}) = \psi_0(\mathbf{x}) \prod_{j=1}^N \left(\mathrm{e}^{\frac{1}{2} y_j^2} y_j^{\lambda(j-N-1)} \right) \frac{\prod_{j<k} (1 - y_j/y_k)^\lambda}{\prod_{j,k=1}^N (1 - x_j/y_k)^\lambda},$$

which shows that if we choose

$$\phi_j(y_j) = \mathrm{e}^{-\frac{1}{2} y_j^2} y_j^{\tilde{n}_j}, \quad \tilde{n}_j = n_j + \lambda(N + 1 - j) \tag{24}$$

with integers n_j, then this transformation is well-defined for all $\lambda > 0$. Indeed, for all $\mathbf{n} \in \mathbb{Z}^N$,

$$\hat{F}_{\mathbf{n}}(\mathbf{x}) := \prod_{j=1}^N \left(\oint_{\mathcal{C}_j} \frac{dy_j}{2\pi \mathrm{i} y_j} \mathrm{e}^{-\frac{1}{2} y_j^2} y_j^{\tilde{n}_j} \right) F(\mathbf{x}, \mathbf{y}) = \psi_0(\mathbf{x}) f_{\mathbf{n}}(\mathbf{x})$$

with $f_{\mathbf{n}}$ the functions defined in (6).

The application of this integral transform to the l.h.s. of the identity in (19) obviously gives $H\hat{F}_{\mathbf{n}}(\mathbf{x})$. To compute the integral transform of the r.h.s. we observe that

$$(-\partial_{y_j}^2 + y_j^2) \mathrm{e}^{-\frac{1}{2} y_j^2} y_j^{\tilde{n}_j - 1} = \mathrm{e}^{-\frac{1}{2} y_j^2} \left((2\tilde{n}_j - 1) y_j^{\tilde{n}_j - 1} - (\tilde{n}_j - 1)(\tilde{n}_j - 2) y_j^{\tilde{n}_j - 3} \right)$$

and that

$$\frac{1}{(y_j - y_k)^2} = \frac{1}{y_k^2 (1 - y_j/y_k)^2} = \sum_{\nu=1}^\infty \nu y_j^{\nu-1} y_k^{-\nu-1}$$

for all $|y_j| < |y_k|$. Using these two facts and the shorthand notation

$$\gamma = 2\lambda(\lambda - 1)$$

we obtain by straightforward computations that

$$H\hat{F}_{\mathbf{n}} = E_{\mathbf{n}} \hat{F}_{\mathbf{n}} - \sum_{j=1}^N (\tilde{n}_j - 1)(\tilde{n}_j - 2) \hat{F}_{\mathbf{n} - 2\mathbf{e}_j} + \gamma \sum_{j<k} \sum_{\nu=1}^\infty \nu \hat{F}_{\mathbf{n} - (1-\nu)\mathbf{e}_j - (1+\nu)\mathbf{e}_k}, \tag{25}$$

where we used that $\sum_{j=1}^N 2(2\tilde{n}_j - 1) + c_N = E_{\mathbf{n}}$, as given in (4).

3.3. **Construction of eigenfunctions.** Equation (25) shows that the action of H on the functions $\hat{F}_{\mathbf{n}}$ has triangular structure: $H\hat{F}_{\mathbf{n}}$ is a linear combination of functions $\hat{F}_{\mathbf{m}}$ with $\mathbf{m} \preceq \mathbf{n}$. Similarly as in the Sutherland algorithm, this suggests that the Calogero model has eigenfunctions of the form

$$\psi_{\mathbf{n}} = \alpha_{\mathbf{n}}(\mathbf{n})\hat{F}_{\mathbf{n}} + \sum_{\mathbf{m}\prec\mathbf{n}} \alpha_{\mathbf{n}}(\mathbf{m})\hat{F}_{\mathbf{m}} \tag{26}$$

with eigenvalues $E_{\mathbf{n}}$ and certain coefficients $\alpha_{\mathbf{n}}(\mathbf{m})$. Indeed, inserting this formula for $\psi_{\mathbf{n}}$ in (25), we obtain by straightforward computations that

$$H\psi_{\mathbf{n}} = \sum_{\mathbf{m}\preceq\mathbf{n}} \Bigg(E_{\mathbf{m}}\alpha_{\mathbf{n}}(\mathbf{m}) - \sum_{j=1}^{N} (\tilde{m}_j + 1)\tilde{m}_j \alpha_{\mathbf{n}}(\mathbf{m} + 2\mathbf{e}_j) + \gamma \sum_{j<k} \sum_{\nu=1}^{\infty} \nu\alpha_{\mathbf{n}}(\mathbf{m} + (1-\nu)\mathbf{e}_j + (1+\nu)\mathbf{e}_k) \Bigg) \hat{F}_{\mathbf{m}}.$$

We conclude that the validity of the Schrödinger equation $H\psi_{\mathbf{n}} = E_{\mathbf{n}}\psi_{\mathbf{n}}$ is implied by the recursion relations

$$2(|\mathbf{n}| - |\mathbf{m}|)\,\alpha_{\mathbf{n}}(\mathbf{m}) = \sum_{j\leq k} \sum_{\nu=0}^{\infty} g_{jk}(\nu;\mathbf{m})\alpha_{\mathbf{n}}(\mathbf{m} + \mathbf{E}_{jk}^{\nu}), \tag{27}$$

with $g_{jk}(\nu;\mathbf{m})$ and E_{jk}^{ν} defined in (8) and (9), respectively; we used $E_{\mathbf{n}} - E_{\mathbf{m}} = 2(|\mathbf{n}| - |\mathbf{m}|)$. We now construct an explicit solution of (27). The triangular structure of the eigenfunctions implies that $\alpha_{\mathbf{n}}(\mathbf{m}) = 0$ unless $\mathbf{m} \prec \mathbf{n}$ or $\mathbf{m} = \mathbf{n}$, and that we can set

$$\alpha_{\mathbf{n}}(\mathbf{n}) = 1$$

without loss of generality. This implies that the recursion relations in (27) can be written as follows:

$$\alpha_{\mathbf{n}} = \delta_{\mathbf{n}} + S\alpha_{\mathbf{n}},$$

where the operator S is defined by

$$(S\alpha_{\mathbf{n}})(\mathbf{m}) := \frac{1 - \delta_{\mathbf{n}}(\mathbf{m})}{2(|\mathbf{n}| - |\mathbf{m}|)} \sum_{j\leq k} \sum_{\nu=0}^{\infty} g_{jk}(\nu;\mathbf{m})\alpha_{\mathbf{n}}(\mathbf{m} + \mathbf{E}_{jk}^{\nu}) \tag{28}$$

for $\mathbf{m} \prec \mathbf{n}$, which allows us to suppress the common argument $\mathbf{m}$. This later equation can now be solved to yield

$$\alpha_{\mathbf{n}} = (1 - S)^{-1}\delta_{\mathbf{n}} = \sum_{s=0}^{\infty} S^s \delta_{\mathbf{n}},$$

where the latter expansion of the geometric series is well-defined since it only contains a finite number of non-zero terms, as shown below. Using (28) we deduce that

$$
\begin{aligned}
(S_{\mathbf{n}}^s\delta_{\mathbf{n}})(\mathbf{m}) &= \sum_{j_s\le k_s}\sum_{\nu_s=0}^{\infty}\frac{g_{j_sk_s}(\nu_s;\mathbf{m})}{2(|\mathbf{n}|-|\mathbf{m}|)}\sum_{j_{s-1}\le k_{s-1}}\sum_{\nu_{s-1}=0}^{\infty}\frac{g_{j_{s-1}k_{s-1}}(\nu_{s-1};\mathbf{m}+\mathbf{E}_{j_sk_s}^{\nu_s})}{2(|\mathbf{n}|-|\mathbf{m}+\mathbf{E}_{j_sk_s}^{\nu_s}|)}\\
&\quad\times\cdots\sum_{j_1\le k_1}\sum_{\nu_1=0}^{\infty}\frac{g_{j_1k_1}(\nu_1;\mathbf{m}+\sum_{\ell=2}^{s}\mathbf{E}_{j_\ell k_\ell}^{\nu_1})}{2(|\mathbf{n}|-|\mathbf{m}+\sum_{\ell=2}^{s}\mathbf{E}_{j_\ell k_\ell}^{\nu_\ell}|)}\,\delta_{\mathbf{n}}(\mathbf{m}+\textstyle\sum_{r=1}^{s}\mathbf{E}_{j_rk_r}^{\nu_r})\\
&=\sum_{j_1\le k_1}\cdots\sum_{j_s\le k_s}\sum_{\nu_1,\ldots,\nu_s}\prod_{r=1}^{s}\frac{g_{j_rk_r}(\nu_r;\mathbf{n}-\sum_{\ell=1}^{r}\mathbf{E}_{j_rk_r}^{\nu_r})}{2(|\mathbf{n}|-|\mathbf{n}-\sum_{\ell=1}^{r}\mathbf{E}_{j_\ell k_\ell}^{\nu_\ell}|)}\,\delta_{\mathbf{n}}(\mathbf{m}+\textstyle\sum_{r=1}^{s}\mathbf{E}_{j_rk_r}^{\nu_r})
\end{aligned}
$$

for all $\mathbf{m}\prec\mathbf{n}$. We now observe that

$$2(|\mathbf{n}|-|\mathbf{n}-\textstyle\sum_{\ell=1}^{r}\mathbf{E}_{j_\ell k_\ell}^{\nu_\ell}|)=4r$$

and thus obtain (7).

3.4. **Properties of the reduced eigenfunctions.** There remains to prove that the reduced eigenfunctions $P_{\mathbf{n}}$ in Theorem 1.1 indeed are well-defined symmetric polynomials. We do this in three steps: we first establish that the functions $f_{\mathbf{n}}$ are symmetric polynomials, then, that the $P_{\mathbf{n}}$ are finite linear combinations of the functions $f_{\mathbf{n}}$, and finally, that all the expansion coefficients $\alpha_{\mathbf{n}}(\mathbf{m})$ are well-defined.

A proof of the first fact can be found in [Lan01], but for the convenience of the reader we give the complete argument.

Lemma 3.2. *The functions $f_{\mathbf{n}}$ are homogeneous symmetric polynomials of degree $|\mathbf{n}|$ and non-zero only if*

$$n_j+\ldots+n_N\ge 0,\quad \forall j=1,\ldots,N-1.$$

For each $\mathbf{n}\in\mathbb{Z}^N$ and each partition $\mathbf{m}$, let

$$p_{\mathbf{nm}}=\sum\prod_{i<j}\prod_{r,s=1}^{N}(-1)^{\kappa_{ij}+\nu_{rs}}\binom{\lambda}{\kappa_{ij}}\binom{-\lambda}{\nu_{rs}},$$

where the second sum extends over all non-negative integers κ_{ij} and ν_{rs} such that

$$m_j=\sum_{l=1}^{N}\nu_{jl}\quad\text{and}\quad n_j=\sum_{l=1}^{j-1}\kappa_{lj}-\sum_{l=j+1}^{N}\kappa_{jl}+\sum_{l=1}^{N}\nu_{lj}.$$

Then

$$f_{\mathbf{n}}=\sum_{|\mathbf{m}|=|\mathbf{n}|}p_{\mathbf{nm}}M_{\mathbf{m}}. \tag{29}$$

Proof. Since $|y_j|<|y_k|$ and $|x_j|<|y_k|$ along the integration paths in $f_{\mathbf{n}}$, the terms in the fraction contained in its integral kernel can be expanded in binomial series in y_j/y_k and x_j/y_k, respectively. The integrals can then be computed using the residue theorem, and this yields

$$f_{\mathbf{n}}(\mathbf{x})=\sum(-1)^{\kappa_{ij}+\nu_{rs}}\binom{\lambda}{\kappa_{ij}}\binom{-\lambda}{\nu_{rs}}x_r^{\nu_{rs}},$$

where the last sum is to be taken over all non-negative integers κ_{ij} and ν_{rs} such that

$$n_j - \sum_{l=1}^{j-1} \kappa_{lj} + \sum_{l=j+1}^{N} \kappa_{jl} - \sum_{l=1}^{N} \nu_{lj} = 0. \tag{30}$$

Recalling the definition of the monomials $M_{\mathbf{m}}$ we deduce (29). To prove that the functions $f_{\mathbf{n}}$ are homogeneous of degree $|\mathbf{n}|$ note that the degree of each monomial $M_{\mathbf{m}}$ in the decomposition in (29) is given by

$$|\mathbf{m}| = \sum_{j,l=1}^{N} \nu_{jl} = \sum_{j=1}^{N} n_j \equiv |\mathbf{n}|.$$

It remains only to prove that the functions $f_{\mathbf{n}}$ are polynomials, i.e., that (30) only has a finite number of solutions. To this end consider the equation for $j = N$,

$$n_N = \sum_{l=1}^{N-1} \kappa_{lN} + \sum_{l=1}^{N} \nu_{lN}.$$

It is clear that it only has a finite number of solutions for each fixed n_N. Now observe that, for $j < N$, the possible values of κ_{jl}, $l \geq j+1$, are determined by the equations with larger values of j. Also observe that the equation for each fixed set of κ_{jl}, $l \geq j+1$, and n_j has only a finite number solutions. The statement thus follows by induction in j, starting with $j = N$. $\square$

Observe that either the symmetric polynomial $f_{\mathbf{m}}$ or the coefficient $\alpha_{\mathbf{n}}(\mathbf{m})$ is zero unless $m_j + \cdots m_N \geq 0$ for all $j = 1, \ldots, N-1$ and $\mathbf{m} \preceq \mathbf{n}$. Clearly only a finite number of $\mathbf{m} \in \mathbb{Z}^N$ fulfill these conditions. This proves the second fact: the reduced eigenfunctions are finite linear combinations of the symmetric polynomials $f_{\mathbf{n}}$. Also note that all sums in the explicit representation in (7) of the coefficients $\alpha_{\mathbf{n}}(\mathbf{m})$ truncate after a finite number of terms, and that they therefore are finite. It follows that the reduced eigenfunctions indeed are well-defined symmetric polynomials. This concludes the proof of Theorem 1.1.

3.5. **The one-particle case.** It is interesting to note that Theorem 1.1 is non-trivial already in the simplest case $N = 1$. Since the Calogero model for $N = 1$ reduces to the harmonic oscillator with well-known eigenfunctions given by the Hermite polynomials H_n (see e.g. 22.6.20 in [AS65]), Theorem 1.1 in this case implies that the functions P_n in (14) are equal to the Hermite polynomials up to normalization. Comparing with the standard definition of the Hermite polynomials in (16) we obtain that

$$H_n(x) = \frac{2^n n!}{(\lambda)_n} \sum_{s=0}^{\infty} (-1)^s \frac{(n+\lambda-2s)_{2s}}{4^s s!} \oint_{|y|>|x|} \frac{dy}{2\pi \mathrm{i} y} y^{n-2s} \frac{1}{(1-x/y)^{\lambda}},$$

where $(z)_n$ denotes the Pochhammer symbol

$$(z)_0 = 1, \quad (z)_n = z(z+1)\ldots(z+n-1),$$

defined for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Note that the series above truncates after a finite number of terms and thus is well-defined. This identity has an interesting interpretation. Observe that the Hermite polynomials can be generalized to arbitrary complex parameters a as follows:

$$H_a(x) := \sum_{s=0}^{\infty}(-1)^s \frac{(a-2s+1)_{2s}}{s!}(2x)^{a-2s}.$$

It is straightforward to verify that this series reduces to a Hermite polynomial when a is a non-negative integer, and that it satisfies the Hermite differential equation $(\partial_x^2 - 2x\partial_x + 2a)H_a(x) = 0$ in the sense of formal Laurent series. However, it is important to note that the series defining $H_a(x)$ does not converge anywhere in the complex plane but is only asymptotic unless a is a non-negative integer. Using this formal Laurent series we can formally rewrite the above identity as follows:

$$H_n(x) = \frac{n!}{2^{\lambda-1}(\lambda)_n}\oint_{|y|>|x|}\frac{dy}{2\pi\mathrm{i}}\, H_{n+\lambda-1}(x)\frac{1}{(y-x)^{\lambda}}$$

for any complex λ. For non-integer λ the r.h.s. can be made well-defined by exchanging the order of integration and summation. For integer $\lambda = m+1 > 2$ we can use the residue theorem to compute the integrals and recover the well-known identity

$$H_n(x) = \frac{n!}{2^m(m+n)!}\frac{d^m}{dx^m}H_{n+m}(x)$$

obeyed by the Hermite polynomials, and we therefore obtained an interesting generalization of this to the cases when n is not a non-negative integer. The integral transforms in our identity looks like a fractional integral transform which, as is well-known, shift parameters of hypergeometric functions; see e.g. Chapter 13 in [EWT53]. However, the details of our identity seem different.

3.6. **Alternative formulas for the eigenfunctions.** The results above can now be used to construct another explicit series representation for the eigenfunctions of the N-body Calogero model. For that we find it convenient to use a somewhat different normalization for the formal Laurent series H_a,

$$p_a(x) = 2^{-a}H_a(x) = \sum_{s=0}^{\infty} c_s(a)x^{a-2s}, \quad c_s(a) = (-1)^s\frac{(a-2s+1)_{2s}}{4^s s!}. \tag{31}$$

The idea is to apply a particular integral transform to the identity in (19) which differs from the one in Section 3.2 in that the simple powers $y_j^{\tilde{n}_j}$ in (24) are replaced by the formal Laurent series $p_{\tilde{n}_j}(y_j)$. This leads to recursion relations which are somewhat different from those in the proceeding discussion but also can be solved explicitly. The advantage is that this recursion becomes trivial in the free case $\lambda = 0$ but, as we will see, it becomes somewhat more complicated to deduce.

To obtain the recursion relations we need an explicit formula for $x^n p_a(x)$, $n \in \mathbb{Z}$, as a linear combination of $p_{a'}(x)$, $a' \leq a+n$.

Lemma 3.3. *Let $n \in \mathbb{Z}$ and $a \in \mathbb{C}$. Then*

$$x^n p_a(x) = \sum_{s=0}^{\infty} b_s(n,a)p_{a+n-2s}(x) \tag{32}$$

with

$$b_0(n,a) = c_0(a) = 1,$$

$$b_s(n,a) = c_s(a) - \sum_{j=0}^{s-1} b_j(n,a)c_{s-j}(a+n-2j), \quad s > 0.$$

and $c_s(a)$ as defined in (31).

Proof. Observe that the definition of p_a implies that

$$x^n p_a(x) = \sum_{s=0}^{\infty} c_s(a) x^{a+n-2s}$$

in the sense of formal Laurent series. It follows that

$$\begin{aligned} x^n p_a(x) &= c_0(a)p_{a+n}(x) + \sum_{s=1}^{\infty} \left(c_s(a) - c_0(a)c_s(a+n)\right) x^{a+n-2s} \\ &= b_0(n,a)p_{a+n}(x) + \sum_{s=1}^{\infty} \left(c_s(a) - b_0(n,a)c_s(a+n)\right) x^{a+n-2s}. \end{aligned}$$

The statement now follows by repeating this procedure of breaking off the leading term. □

Remark 3.2. Although it is not evident from the statement and proof of Lemma 3.3, the series in (32) truncates for non-negative integers n at $s = n$, i.e.,

$$b_s(n,a) = 0, \quad s > n, \quad n \in \mathbb{N}_0.$$

This can be deduced by observing that differential equation solved by p_a implies the three term recursion relation

$$2p_{a+1}(x) - 2xp_a(x) + ap_{a-1}(x) = 0.$$

From this the truncation of the series in (32) for non-negative integers n follows. This also shows that the series does not truncate for negative integers n.

Explicit formulas for the coefficients $b_s(n,a)$ can now be obtained by solving the recursion relations in Lemma 3.3.

Corollary 3.1. *Let $s \in \mathbb{N}_0$, $n \in \mathbb{Z}$ and $a \in \mathbb{C}$. Then*

$$\begin{aligned} b_s(n,a) &= (-1)^s \frac{(a-2s+1)_{2s}}{4^s s!} \\ &\quad + \sum_{j=1}^{s} (-1)^{j+s} \sum_{0 \leq s_j < \cdots < s} \frac{(a-2s_j+1)_{2s_j}(a+n-2s+1)_{2(s-s_j)}}{4^s s_j!(s_{j-1}-s_j)! \cdots (s-s_1)!}. \end{aligned} \tag{33}$$

Proof. Lemma 3.3 implies that

$$\begin{aligned} b_s(n,a) = c_s(a) + \sum_{j=1}^{s} (-1)^j \sum_{s_1=0}^{s-1} \sum_{s_2=0}^{s_1-1} \cdots \sum_{s_j=0}^{s_{j-1}-1} c_{s_j}(a) \\ c_{s_{j-1}-s_j}(a+n-2s_j) \ldots c_{s-s_1}(a+n-2s_1). \end{aligned}$$

The statement is now obtained by using the explicit form of the coefficients $c_s(a)$ and simple properties of the Pochhammer symbol. □

Suppose that $y_N > y_{N-1} > \ldots > y_1 > \max_k(x_k)$. It is then clear from the previous discussion that the product

$$\prod_{j=1}^{N} p_{\tilde{n}_j}(y_j) y_j^{\lambda(j-N-1)} \frac{\prod_{j<k}(1-y_k/y_j)^{\lambda}}{\prod_{j,k=1}^{N}(1-x_k/y_j)^{\lambda}}$$

is a formal multi-variable Laurent series in the variables y_j. Appealing to the residue theorem we define

$$f_{\mathbf{n}}^{(H)}(\mathbf{x}) := \prod_{j=1}^{N}\left(\oint_{\mathcal{C}_j}\frac{dy_j}{2\pi \mathrm{i} y_j} y_j p_{\tilde{n}_j-1}(y_j) y_j^{\lambda(j-N-1)}\right)\frac{\prod_{j<k}(1-y_j/y_k)^{\lambda}}{\prod_{j,k}(1-x_j/y_k)^{\lambda}} \tag{34}$$

as the coefficient of the term $(y_1\ldots y_N)^{-1}$ in this Laurent series; we use the superscript '(H)' to distinguish these from the analog functions defined in Section 3.2. This prescription amounts to interchanging the integrations and summations. Following the proof of Lemma 3.2 it is readily verified that the $f_{\mathbf{n}}^{(H)}$ are well-defined symmetric polynomials. Similarly as in Section 3.3 we now obtain that

$$\begin{aligned} H\hat{F}_{\mathbf{n}}^{(H)} =& E_{\mathbf{n}}\hat{F}_{\mathbf{n}}^{(H)} + 2\lambda(\lambda-1)\sum_{j<k}\sum_{\nu=1}^{\infty}\nu\sum_{r,s=0}^{\infty} b_r(\nu-1,\tilde{n}_j)b_s(-\nu-1,\tilde{n}_k) \\ &\times \hat{F}^{(H)}_{\mathbf{n}-(1+2r-\nu)\mathbf{e}_j-(1+2s+\nu)\mathbf{e}_k} \end{aligned}$$

for $\hat{F}_{\mathbf{n}}^{(H)}(\mathbf{x}) = \psi_0(\mathbf{x})f_{\mathbf{n}}^{(H)}(\mathbf{x})$; we used the fact that $hp_\nu = 2\nu p_\nu$, the functional identity in (19), and Lemma 3.3. As before we conclude that the action of H on the functions $\hat{F}_{\mathbf{n}}^{(H)}$ has triangular structure, and that there are eigenfunctions of the Calogero model which are of the form $\psi_{\mathbf{n}} = \beta_{\mathbf{n}}(\mathbf{n})\hat{F}_{\mathbf{n}}^{(H)} + \sum_{\mathbf{m}\prec\mathbf{n}}\beta_{\mathbf{n}}(\mathbf{m})\hat{F}_{\mathbf{m}}^{(H)}$. The Schrödinger equation $H\psi_{\mathbf{n}} = E_{\mathbf{n}}\psi_{\mathbf{n}}$ is now implied by the recursion relations

$$\beta_{\mathbf{n}} = \delta_{\mathbf{n}} + S^{(H)}\beta_{\mathbf{n}},$$

with

$$(S^{(H)}\beta_{\mathbf{n}})(\mathbf{m}) = \frac{1}{E_{\mathbf{n}}-E_{\mathbf{m}}}\sum_{j<k}\sum_{\nu,t,u=0}^{\infty} g_{jk}^{tu}(\nu;\mathbf{m})\beta_{\mathbf{n}}(\mathbf{m}+\mathbf{E}_{jk}^{tu,\nu}),$$

where we have introduced the notation

$$g_{jk}^{tu}(\nu;\mathbf{m}) = 2\lambda(\lambda-1)\nu b_t(\nu-1;\tilde{m}_j+1+2t-\nu)b_u(-1-\nu;\tilde{m}_k+1+2u+\nu) \tag{35}$$

and

$$\mathbf{E}_{jk}^{tu,\nu} = (1+2t-\nu)\mathbf{e}_j + (1+2u+\nu)\mathbf{e}_k. \tag{36}$$

Using

$$2(|\mathbf{n}| - |\mathbf{n}-\textstyle\sum_{\ell=1}^{r}\mathbf{E}_{j_\ell k_\ell}^{t_\ell u_\ell,\nu_\ell}|) = 4\displaystyle\sum_{\ell=1}^{r}(1+t_l+u_l),$$

computations similar to the ones in Section 3.2 lead to the following result:

Theorem 3.1. *For $\mathbf{n}\in\mathbb{Z}^N$ let*

$$P_{\mathbf{n}} = \sum_{\mathbf{m}}\beta_{\mathbf{n}}(\mathbf{m})f_{\mathbf{m}}^{(H)}$$

with the functions $f_{\mathbf{m}}^{(H)}$ defined in (34) and

$$\begin{aligned}\beta_{\mathbf{n}}(\mathbf{m}) = \delta_{\mathbf{n}}(\mathbf{m}) + \sum_{s=1}^{\infty}\sum_{j_1<k_1}\sum_{t_1,u_1,\nu_1=0}^{\infty}\cdots\sum_{j_s<k_s}\sum_{t_s,u_s,\nu_s=0}^{\infty} \\ \delta_{\mathbf{n}}(\mathbf{m}+\textstyle\sum_{r=1}^{s}\mathbf{E}_{j_rk_r}^{t_ru_r,\nu_r})\displaystyle\prod_{r=1}^{s}\frac{g_{j_rk_r}^{t_ru_r}(\nu_r;\mathbf{n}-\sum_{\ell=1}^{r}\mathbf{E}_{j_\ell k_\ell}^{t_\ell u_\ell,\nu_\ell})}{4\sum_{\ell=1}^{r}(1+t_l+u_l)}\end{aligned}$$

with the quantities given in (35), (36) and (33) above. Then $P_{\mathbf{n}}$ is a reduced polynomial eigenfunction of the Calogero model corresponding to the eigenvalue $E_{\mathbf{n}}$ in (4).

Remark 3.3. The polynomials $P_{\mathbf{n}}$ constructed here should be identical with the ones obtained in Theorem 1.1. We have checked this for $N = 2$ and various cases $\mathbf{n} \in \mathbb{Z}^N$ using MATHEMATICA.

4. Solutions for the remaining classical root systems

In this section we show that the constructions of the previous section can be adapted to the B_N variant of the Calogero model defined by the Hamiltonian in (10). The construction is very similar to the A_{N-1} case, and we therefore are rather sketchy and concentrate on the necessary changes. To simplify notation we denote corresponding quantities in the A_{N-1} and B_N cases by the same symbol, e.g. ψ_0, $F(\mathbf{x},\mathbf{y})$, $f_{\mathbf{n}}$ etc. have a different meaning here and in Section 3. However, since the parameter μ will play a special role, we will write the B_N Hamiltonian in (10) as H_μ. Moreover, the analog of the Hermite polynomials are the Laguerre polynomials denoted by the usual symbol $L_n^{(a)}(x)$; see e.g. [AS65].

4.1. A remarkable identity.

The analog of the key identity in Lemma 3.1 is

$$H_\mu(\mathbf{x})F(\mathbf{x},\mathbf{y}) = [H_{\lambda-\mu}(\mathbf{y}) + c_N]F(\mathbf{x},\mathbf{y}), \tag{37}$$

where $c_N = 2(1-\lambda)N$ is the same as in the A_{N-1} case but

$$F(\mathbf{x},\mathbf{y}) = \frac{\prod_{j=1}^N e^{-\frac{1}{2}(x_j^2-y_j^2)} x_j^\mu y_j^{\lambda-\mu} \prod_{j<k}(x_k^2-x_j^2)^\lambda (y_k^2-y_j^2)^\lambda}{\prod_{j,k=1}^N (y_k^2-x_j^2)^\lambda}. \tag{38}$$

It is important to note that this identity now involves two Hamiltonians with different coupling parameters μ and $\lambda-\mu$.

The proof is similar to the one of Lemma 3.1: by direct computation one can check that

$$\Psi_0(\mathbf{X}) = \prod_{j=1}^{\mathcal{N}} \left(X_j^{\mu_j} \mathrm{e}^{-\frac{1}{2}m_j X_j^2} \right) \prod_{j=1}^{\mathcal{N}} (X_k^2 - X_j^2)^{m_j m_k \mu} \tag{39}$$

and the differential operator

$$\mathcal{H} = \sum_{j=1}^{\mathcal{N}} \left(-\frac{1}{m_j}\partial_{X_j}^2 + m_j X_j^2 + \frac{\mu_j(\mu_j-1)}{m_j X_j^2} \right) + \sum_{j<k} \frac{\lambda}{(X_j^2-X_k^2)^2} \Big(2(\lambda m_j m_k - 1) \\ \times (m_k X_j^2 + m_j X_k^2) + (m_k(1+2\mu_j) - m_j(1+2\mu_k))(X_j^2 - X_k^2) \Big)$$

obey the identity in (22) with

$$\mathcal{E}_0 = \sum_{j=1}^{\mathcal{N}} (1+2\mu_j) + 2\lambda \left(\left(\sum_j m_j \right)^2 - \sum_j m_j^2 \right). \tag{40}$$

The identity to cancel the three-body terms is now

$$\frac{X_j^2}{(X_k^2-X_j^2)(X_j^2-X_l^2)} + \frac{X_k^2}{(X_l^2-X_k^2)(X_k^2-X_j^2)} + \frac{X_l^2}{(X_j^2-X_l^2)(X_l^2-X_k^2)} = 0.$$

Equation (37) is obtained from this as a special case, as before.

This also gives, as another important special case, the groundstate

$$\psi_0(\mathbf{x}) = \prod_{j=1}^{N} \left(\mathrm{e}^{-\frac{1}{2}x_j^2} x_j^{\mu} \right) \prod_{j<k} (x_k^2 - x_j^2)^{\lambda} \tag{41}$$

and corresponding groundstate energy E_0 of the B_N Calogero model, which are of course well-known; see e.g. [OP83]. We note in passing that we have obtained a generalization of the B_N Calogero model to particles with different masses, together with its exact groundstate and groundstate energy.

4.2. **Integral transformation.** The kernel of the integral transform is now taken to be

$$\prod_{j=1}^{N} \frac{\mathrm{e}^{-\frac{1}{2}y_j^2} y_j^{2\tilde{n}_j+\mu-\lambda}}{2\pi \mathrm{i} y_j}$$

with the same $\tilde{n}_j$ as in (24). Note that we need to restrict to even integers $2n_j$ to get non-zero results; see (43) below.

Applying the resulting integral transform to the identity in (37) straightforward computations lead to

$$\begin{aligned} H_\mu \hat{F}_{\mathbf{n}} = E_{\mathbf{n}} \hat{F}_{\mathbf{n}} &- \sum_{j=1}^{N} 2(2(\tilde{n}_j + \mu - \lambda) - 1)(\tilde{n}_j - 1) \hat{F}_{\mathbf{n}-\mathbf{e}_j} \\ &+ 4\lambda(\lambda - 1) \sum_{j<k} \sum_{\nu=1}^{\infty} (2\nu - 1) \hat{F}_{\mathbf{n}-(1-\nu)\mathbf{e}_j - \nu \mathbf{e}_k} \end{aligned}$$

with

$$E_{\mathbf{n}} = 4(n_1 + n_2 + \cdots + n_N) + E_0, \quad E_0 = N(1 + 2\mu + 2\lambda(N-1)) \tag{42}$$

and $\hat{F}_{\mathbf{n}} = \psi_0 f_{\mathbf{n}}$,

$$f_{\mathbf{n}}(\mathbf{x}) = \prod_{j=1}^{N} \left(\oint_{\mathcal{C}_j} \frac{dy_j}{2\pi \mathrm{i} y_j} y_j^{2n_j} \right) \frac{\prod_{j<k} (1 - y_j^2/y_k^2)^{\lambda}}{\prod_{j,k} (1 - x_j^2/y_k^2)^{\lambda}}. \tag{43}$$

Note that the function $f_{\mathbf{n}}$ by a change of variables, $x_j^2 \to x_j$ and $y_j^2 \to y_j$ for all $j = 1, \ldots, N$, becomes identical with the function $f_{\mathbf{n}}$ used in the A_{N-1} case. Thus Lemma 3.2 directly extends to the B_N case.

4.3. **Construction of eigenfunctions.** The action of the B_N Hamiltonian on the functions $\hat{F}_{\mathbf{n}}$ is triangular, which suggests that the Hamiltonian in (10) has eigenfunctions of the same form as in (26) with eigenvalues $E_{\mathbf{n}}$ in (42). As before, the Schrödinger equation $H\psi_{\mathbf{n}} = E_{\mathbf{n}}\psi_{\mathbf{n}}$ is implied by the recursion relations in (27), where now

$$\begin{aligned} g_{jk}(\nu; \mathbf{m}) = 4\lambda(\lambda - 1)(2\nu - 1)(1 - \delta_{\nu,0})(1 - \delta_{jk}) \\ - 2(2(\tilde{m}_j + \mu - \lambda) + 1)\tilde{m}_j \delta_{\nu,0} \delta_{jk} \end{aligned} \tag{44}$$

and

$$\mathbf{E}_{jk}^{\nu} = (1 - \nu)\mathbf{e}_j + \nu \mathbf{e}_k. \tag{45}$$

With these substitutions the arguments in Section 3 go through as they stand, and we obtain the analog of Theorem 1.1 for the B_N case. To summarize, *the reduced polynomial eigenfunctions of the B_N Calogero Hamiltonian in (10) are given by the* *functions $P_{\mathbf{n}}$ in Theorem 1.1, where (6), (8) and (9) have to be replaced by (43), (44) and (45), respectively. The corresponding eigenvalues are given by (42).*

4.4. **The one-particle case.** The B_N Calogero Hamiltonian in (10) for $N=1$ reduces to $-\partial_x^2 + x^2 + \mu(\mu-1)x^{-2}$ with well-known eigenfunctions given by the Laguerre polynomials with non-degenerate eigenvalues; see e.g. 22.8.18 in [AS65]. Comparing this with our eigenfunctions we can conclude that, in the case $N=1$, the polynomials $P_n(x)$ constructed above are proportional to the Laguerre polynomials $L_n^{(\mu-1/2)}(x^2)$. By straightforward computations we obtain that

$$L_n^{(\mu-1/2)}(x) = \mathrm{e}^{\mathrm{i}\pi(1-\lambda)}\frac{\Gamma(n+\lambda)}{(\lambda)_n}\oint_{|y|>|x|}\frac{dy}{2\pi\mathrm{i}}\, L_{n+\lambda-1}^{(\mu-\lambda+1/2)}(x)\frac{1}{(y-x)^{\lambda}},$$

where we have made the substitutions $y^2 \to y$ and $x^2 \to x$ as well as extended the definition of the Laguerre polynomials to arbitrary complex values a as follows:

$$L_a^{(\mu-1/2)}(x) = \sum_{k=0}^{\infty}\mathrm{e}^{\mathrm{i}\pi(a-k)}\frac{(a+\mu+1/2-k)_k}{\Gamma(a+1-k)k!}\,x^{a-k}$$

with the Gamma function Γ; note that $L_a^{(\mu-1/2)}(x)$ is a formal Laurent series obeying the Laguerre differential equation $(x\partial_x^2+(\mu+1/2-x)\partial_x+a)L_a^{(\mu-1/2)}(x)=0$, and it reduces to a Laguerre polynomial when a is a non-negative integer; see e.g. 22.3.9 in [AS65]. As in Section 3.5 the r.h.s. in the previous equation has to be interpreted by exchanging integrations and summations as well as computing the integrals of the individual terms using the residue theorem. For integer values $\lambda = m+1 > 2$ we recover the well-known classical identities

$$L_n^{(\mu-1/2)}(x) = (-1)^m\frac{d^m}{dx^m}L_{n+m}^{(\mu-1/2-m)}(x).$$

As in Section 3.5, the general case is similar to known identities involving fractional integral transforms; see e.g. Chapter 13 and 16.6.(5) in [EWT53].

Using these results it is straightforward to extend our alternative formulas for the eigenfunctions in Section 3.6 to the B_N Calogero model.

5. Concluding remarks

In the present paper we extended a solution method for the Sutherland model [Lan01, Lan05] to the A_{N-1} and B_N Calogero models. Below we discuss various open questions and mention interesting complimentary results. In particular, in Remark 5.1 we comment on the issue of whether our solution is complete or not, and in Remark 5.2 we point out further identities generalizing our key results in (19) and (37). In the concluding Remark 5.3 we sketch an interesting alternative interpretation of our method.

Remark 5.1. The main difference between Sutherland's method [Sut72] and ours is that he expands the reduced eigenfunctions of the Calogero model in the monomials $M_{\mathbf{n}}$ defined in (12), whereas we obtain the reduced eigenfunctions as linear combinations of the more complicated functions $f_{\mathbf{n}}$ in (6).

The monomials $M_{\mathbf{n}}$ labeled by partitions $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are a basis of the space of symmetric polynomials, and it is therefore obvious that the eigenfunctions obtained by Sutherland's method are complete.

On the other hand, the functions $f_{\mathbf{n}}$ and also our eigenfunctions are labeled by unrestricted N-tuples $\mathbf{n} \in \mathbb{Z}^N$, and it therefore seems that we are working with overcomplete sets of functions. While this is the reason why we can get more explicit formulas, it also makes the questions of completeness of our solution more complicated. It would therefore be interesting to extend our results to the higher differential operators commuting with the Hamiltonians since this might shed some light on this important issue.

It is interesting to note that the very same functions $f_{\mathbf{n}}$ appeared as building blocks in all the models we so far have solved by our method: in the A_{N-1} Calogero model they appear as functions of the variables x_j (see (6)), in the B_N Calogero model as functions of x_j^2 (see (43)), and for the trigonometric Sutherland case we found the very same functions but in the variables $z_j = \mathrm{e}^{\mathrm{i}x_j}$ [Lan01].

We finally mention that we convinced ourselves that, for $\lambda = 1$, the $f_{\mathbf{n}}$ are up to a difference in sign identical with the Schur polynomials.

Remark 5.2. The key to our method was the identity in Lemma 3.1, but in its proof we obtained a more general result, given in (20)–(23), which has other interesting special cases, as discussed in Remark 3.1. We now point out further interesting identities which can be obtained from this general result.

For example, it is possible to generalize Lemma 3.1 by allowing the particle numbers N and M in the $\mathbf{x}$- and $\mathbf{y}$-variables to be different: choosing $\mathcal{N} = N + M$, $X_j = x_j$, $m_j = +1$ and $X_{N+k} = y_k$, $m_{N+k} = -1$ for $j = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, M$ we obtain the identity

$$H_N(\mathbf{x})F_{N,M}(\mathbf{x},\mathbf{y}) = [H_M(\mathbf{y}) + c_{N,M}]F_{N,M}(\mathbf{x},\mathbf{y}) \tag{46}$$

for the function

$$F_{N,M}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{N} \mathrm{e}^{-\frac{1}{2}x_j^2} \prod_{J=1}^{M} \mathrm{e}^{\frac{1}{2}y_J^2} \frac{\prod_{1\le j<k\le M}(x_k - x_j)^\lambda \prod_{1\le J<K\le M}(y_K - y_J)^\lambda}{\prod_{j=1}^{N}\prod_{K=1}^{N}(y_K - x_j)^\lambda} \tag{47}$$

of $N + M$ variables and the constant

$$c_{N,M} = \lambda(N - M)^2 + (N + M)(1 - \lambda); \tag{48}$$

$H_N(\mathbf{x})$ in (46) is the Calogero Hamiltonian in (1) and similarly for $H_M(\mathbf{y})$, where we now also have to indicate the particle numbers and variables. Another interesting family of identities is obtained by choosing $\mathcal{N} = N + M$, $X_j = x_j$, $m_j = +1$ and $X_{N+k} = y_k$, $m_{N+k} = 1/\lambda$ for $j = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, M$. This yields the identity

$$H_N(\mathbf{x})\tilde{F}_{N,M}(\mathbf{x},\mathbf{y}) = [-\lambda H_{M,1/\lambda}(\mathbf{y}) + \tilde{c}_{N,M}]\tilde{F}_{N,M}(\mathbf{x},\mathbf{y})$$

for the function

$$\begin{aligned}\tilde{F}_{N,M}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{N} \mathrm{e}^{-\frac{1}{2}x_j^2} \prod_{J=1}^{M} \mathrm{e}^{-\frac{1}{2}y_J^2/\lambda} \prod_{1\le j<k\le M}(x_k - x_j)^\lambda \\ \times \prod_{1\le J<K\le M}(y_K - y_J)^{1/\lambda} \prod_{j=1}^{N}\prod_{K=1}^{N}(y_K - x_j)\end{aligned} \tag{49}$$

and the constant

$$\tilde{c}_{N,M} = (N\lambda + M/\lambda)^2 + N(1 - \lambda) + M(1 - 1/\lambda); \tag{50}$$

the Calogero Hamiltonian for the variables $\mathbf{y}$ is now given by

$$H_{M,1/\lambda}(\mathbf{y}) = \sum_{J=1}^{N} \left(-\partial_{y_J}^2 + y_J^2/\lambda^2\right) + 2(1/\lambda)((1/\lambda)-1) \sum_{J<K} \frac{1}{(y_J - y_K)^2}. \tag{51}$$

We thus recover a well-known duality between Calogero models with reciprocal coupling constants.

It is interesting to note that corresponding identities also exist in the B_N case, but we do not write them down since they can be obtained from our general result quoted in (39)–(40) in the same way as explained in the A_{N-1} case: to obtain the analog of (47)–(48) fix the parameters as in the A_{N-1} case and, in addition, set $\mu_j = \mu$, $\mu_{N+k} = \lambda - \mu$, and for the analog of (49)–(51) the additional parameters are to be fixed as $\mu_j = \mu$, $\mu_{N+k} = \frac{1}{2}(3/\lambda - 1)$.

With these identities one can obtain many more explicit formulas for the eigenfunctions of the Calogero models using our method. It is interesting to note that similar identities were previously found also in the Sutherland model and its elliptic generalization using quantum field theory techniques [Lan04], and we believe that such identities should exist also for other integrable many-body systems.

Remark 5.3. We now sketch an interesting alternative interpretation of our method which, when explored in more detail, could shed more light on the questions discussed in Remark 5.1.[1] Suppose that we want to construct eigenfunctions of the Calogero Hamiltonian in (1) of the following form

$$\chi_{\mathbf{n}} = \chi_0 \check{P}_{\mathbf{n}}, \quad \chi_0(\mathbf{x}) = \prod_{j=1}^{N} \mathrm{e}^{-\frac{1}{2}x_j^2}$$

with $\check{P}_{\mathbf{n}}$ linear combinations of arbitrary monomials

$$\mathbf{x}^{\tilde{\mathbf{n}}} := x_1^{\tilde{n}_1} x_2^{\tilde{n}_2} \cdots x_N^{\tilde{n}_N},$$

where the $\tilde{n}_j = n_j + s_j$ are integers shifted by common amounts s_j which, at this point, are arbitrary. If we ignore the issue of square integrability it is easy to construct such eigenfunctions: with the ansatz

$$\check{P}_{\mathbf{n}}(\mathbf{x}) = \mathbf{x}^{\tilde{\mathbf{n}}} + \sum_{\mathbf{m} \prec \mathbf{n}} \alpha_{\mathbf{n}}(\mathbf{m}) \mathbf{x}^{\tilde{\mathbf{m}}}$$

we find by straightforward computations that the Schrödinger equation $H\chi_{\mathbf{n}} = \check{E}_{\mathbf{n}}\chi_{\mathbf{n}}$ is equivalent to $\check{E}_{\mathbf{n}} = |\mathbf{n}| + |\mathbf{s}|$, $|\mathbf{s}| = s_1 + s_2 + \ldots + s_N$, and the recursion relations in (27) for the coefficients $\alpha_{\mathbf{n}}(\mathbf{m})$. As shown in Section 3.3, these relations can be easily solved, and their explicit solution is given in Theorem 1.1. It is important to note that the construction of these eigenfunctions $\chi_{\mathbf{n}}$ is not restricted to the Calogero model, but one can easily generalize it to construct similar eigenfunctions for non-integrable systems like the generalized Calogero model where the particles have different masses. However, these eigenfunctions are completely uninteresting from a physical point of view: the series in the previous formula do not converge and are only asymptotic. The fact that makes the Calogero model special is that *there exists an operator $\mathcal{F}$ which maps these unphysical eigenfunctions to physical ones*,

[1]We would like to thank Vadim Kuznetsov for useful discussions on this point.

and this operator is given by the function $F(\mathbf{x},\mathbf{y})$ in Lemma 3.1 in the following sense: for an asymptotic series

$$\chi(\mathbf{x}) = \chi_0(\mathbf{x}) \sum_{\mathbf{m}\preceq\mathbf{n}} \alpha(\mathbf{m})\mathbf{x}^{\tilde{\mathbf{m}}}$$

let

$$\mathcal{F}(\chi)(\mathbf{x}) = \sum_{\mathbf{m}} \alpha(\mathbf{m}) \prod_{j=1}^{N} \left(\oint_{\mathcal{C}_j} \frac{dy_j}{2\pi \mathrm{i} y_j} \right) F(\mathbf{x},\mathbf{y})\chi_0(\mathbf{y})\mathbf{y}^{\tilde{\mathbf{m}}}$$

with the integration paths defined in (5). Note that this map is well-defined if we set $s_j = \lambda(N+1-j)$, and then the functions

$$\mathcal{F}(P_{\mathbf{n}}\chi_0) = \psi_0 f_{\mathbf{n}}$$

are equal to the building blocks of our solution given in (3) and (6). Moreover, since Lemma 3.1 implies that

$$H\mathcal{F}(\chi) = \mathcal{F}([H + c_N]\chi),$$

we conclude that

$$\psi_{\mathbf{n}} := \mathcal{F}(\chi_{\mathbf{n}})$$

is an eigenfunction of the Calogero Hamiltonian with eigenvalue $E_{\mathbf{n}} = \check{E}_{\mathbf{n}} + c_N$.

Obviously the generalized identities pointed out in Remark 5.2 provide operators which, similarly, transform unphysical eigenfunctions of the M-variable Calogero model to physical N-variable ones, and thus our results in this paper give the further explicit formulas for the eigenfunctions mentioned in Remark 5.2.

We note that the operator $\mathcal{F}$ is similar to the Q-operator which has appeared in a separation-of-variables approach to the Sutherland model [KMS03].

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Appendix A. The relation to Calogero's original model

We give here a brief account of the relation between the Hamiltonian in (1) and the following one studied by Calogero [Cal69, Cal71]:

$$H_{Cal} = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{j<k} \left(\omega^2 (x_j - x_k)^2 + 2\lambda(\lambda-1)\frac{1}{(x_j-x_k)^2} \right).$$

It seems well known that they differ only in their center of mass motion[2] but we have been unable to find a discussion of this in the literature.

The simplest way to see this is to verify the following identity:

$$H_{Cal} + \omega^2(x_1 + x_2 + \cdots + x_N)^2 = H \quad \text{for } N\omega^2 = 1.$$

[2] We thank F. Calogero for explaining this to us.

Since H_{Cal} is translational invariant it is possible to use center of mass coordinates with $r_0 = (x_1 + x_2 + \cdots + x_N)/\sqrt{N}$ and other coordinates r_j, $j = 1, 2, \ldots, N-1$, linearly independent of r_0 and

$$H_{Cal} = -\frac{1}{N}\partial_{r_0}^2 + H_{CM}$$

were H_{CM} only depends on the r_j with $j > 0$. We thus conclude that

$$H = -\frac{1}{N}\partial_{r_0}^2 + r_0^2 + H_{CM},$$

where we used that $N\omega^2 = 1$. This makes manifest that the difference between the two Hamiltonians H and H_{Cal} lies only in the center of mass motion: in the former it is trapped in a harmonic oscillator potential, and in the later it is free. Thus the Hamiltonian H_{Cal} has discrete spectrum only if its center of mass motion is fixed, i.e., if only H_{CM} is considered. One the other hand, H has purely discrete spectrum even if the center of mass is not fixed, and it is therefore simpler to work with.

It is instructive compute H_{CM} explicitly. For $N = 3$ the center of mass coordinates are r_0 above and

$$r_1 = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad r_2 = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3).$$

By a straightforward computation follows that

$$H_{CM} = -\partial_{r_1}^2 - \partial_{r_2}^2 + 3\omega^2(r_1^2 + r_2^2) + 2\lambda(\lambda - 1)\left(\frac{1}{2r_1^2} + 4\frac{r_1^2 + 3r_2^2}{(r_1^2 - 3r_2^2)^2}\right).$$

This computation extends straightforwardly to the general case by a generalization of the center of mass coordinates to arbitrary N. The latter can be found in Reference [Cal71], for example.

Appendix B. Proof of the identity in (17)

In this appendix we sketch a proof of the identity in (17). It should be noted though that the proof does not give the explicit form of the coefficients $c_{r,s}$, which seemingly is rather complicated.

The key ingredient in the proof is the following:

Lemma B.1. *Let $n, m \in \mathbb{N}_0$ and $n \geq m$. Then*

$$\frac{1}{x-y}(\partial_x - \partial_y)(x^n y^m + y^n x^m) = (n-m)\sum_{k=1}^{n-m-1} x^{n-1-k}y^{m-1+k} - mx^{n-1}y^{m-1} - my^{n-1}x^{m-1}.$$

Proof. The case $n = m$ is easily verified. Now suppose that $n \geq m+1$. Then

$$\frac{1}{x-y}(\partial_x - \partial_y)(x^n y^m + y^n x^m) = n\frac{x^m y^m}{x-y}(x^{n-m-1} - y^{n-m-1}) - m\frac{x^{m-1}y^{m-1}}{x-y}(x^{n-m+1} - y^{n-m+1}).$$

Expand the fractions in geometric series to obtain

$$n\sum_{k=0}^{n-m-2} x^{n-2-k}y^{m+k} - m\sum_{k=0}^{n-m} x^{n-1-k}y^{m-1+k}.$$

Collect terms of equal degree to deduce the statement. □

The next step is to use the series representation of the Hermite polynomials and apply Lemma B.1 to each term separately. Recall that, for each $n \in \mathbb{N}_0$, the Hermite polynomial H_n is a polynomial of degree n. Hence, this procedure clearly gives a polynomial of degree at most $n-1$ in the variables x and y. The validity of the identity in (17) now follows from the fact that as n runs through the integers less than or equal to m, for some $m \in \mathbb{N}_0$, the H_n form a basis for the space of polynomials with degree at most m.

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