NONCOMMMUTATIVE BATALIN-VILKOVISKY GEOMETRY AND MATRIX INTEGRALS

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ABSTRACT. We describe matrix model associated naturally with any solution to quantum master equation of noncommutative Batalin-Vilkovisky geometry. In the simplest example we get the Kontsevich model of 2-dimensional gravity.

1. INTRODUCTION.

We describe matrix model associated naturally with solution to quantum master equation of noncommutative Batalin-Vilkovisky geometry. The simplest example is the Kontsevich model of 2-dimensional gravity.

Notations. We work in the tensor category of super vector spaces, over an algebraically closed field k, $char(k) = 0$. Let $V = V^{1|0} \oplus V^{0|1}$ be a super vector space. We denote by $\overline{\alpha}$ the parity of an element α and by ΠV the super vector space with inversed parity. Sometimes we may also assume that V has a differential, we leave to the interested reader to make the appropriate modifications, if they are not mentioned explicitely. For a finite group G acting on a vector space U, we denote via U^G the space of invariants with respect to the action of G.

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2. NONCOMMUTATIVE BATALIN-VILKOVISKY GEOMETRY.

2.1. **Even inner products.** Let $B : V^{\otimes 2} \to k$ be an even symmetric inner product on V:

$$B(x \otimes y) = (-1)^{\overline{x}+\overline{y}} B(y \otimes x)$$

The inner prodcut B corresponds to an even symplectic structure on ΠV. We introduced in [B1] the space $F = \bigoplus_n F_n$

$$(2.1) \qquad F_n = ((\Pi V)^{\otimes n} \otimes k[\mathbb{S}_n]')^{\mathbb{S}_n}$$

where $k[\mathbb{S}_n]'$ denotes the super $k-$vector space with the basis indexed by elements (σ, ρ_σ), where $\sigma \in \mathbb{S}_n$ is a permutation with i_σ cycles σ_α and $\rho_\sigma = \sigma_1 \wedge \ldots \wedge \sigma_{i_\sigma}$, $\rho_\sigma \in Det(Cycle(\sigma))$, $Det(Cycle(\sigma)) = Symm^{i_\sigma}(k^{0|i_\sigma})$, is one of the generators of

the one-dimensional determinant of the set of cycles of σ, i.e. ρ_σ is an order on the set of cycles defined up to even reordering, and $(\sigma, -\rho_\sigma) = -(\sigma, \rho_\sigma)$. The group $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]'$ by conjugation. The space F carries the naturally defined operator

$$\Delta(a_1 \otimes \ldots \otimes a_n \otimes (\sigma, \rho_\sigma)) = B(a_{n-1} \otimes a_n)(a_1 \otimes \ldots \otimes a_{n-2} \otimes (\pi_{n-1,n}(s_{n-1,n}\sigma), \rho_\delta)$$

where $s_{n-1,n}$ is the transposition interchanging $n-1$ and n, $\pi_{n-1,n}$ is the operator of erasing of elements $n-1$ and n from the cycles of the representation, and ρ_δ is the natural ordering on the cycles of resulting permutation $\pi_{n-1,n}(\sigma s_{n-1,n})$, see section 9 from loc.cit. The space F can be identified with the symmetric power of the parity transposed space of cyclic invariant tensors on ΠV

$$F = Symm(\oplus_j \Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

If one chooses a basis $\{a_i\}$ in ΠV, in which B has the form $B(a_i, a_j) = b_{ij}$ then operator Δ can be written as

$$\Delta = \sum_{ij} \gamma b_{ij} \frac{\partial^2}{\partial a_i \partial a_j}$$

where γ denotes the "dissection-gluing" operator acting on cycles, which corresponds to the to the symmetrisation of the operator $(\sigma, \rho_\sigma) \to (\pi_{n-1,n}(s_{n-1,n}\sigma), \rho_\delta)$ acting on pairs (σ, ρ_σ). The operator γ is odd since it changes the total number of cycles by one.

The operator Δ is of the second order with respect to the multiplication on F and by the general formula of Batalin-Vilkovisky geometry it defines the odd Lie bracket on F, which we described in loc.cit.,section 5, and which can be written as

$$\{x, y\} = \sum_{ij} \gamma b_{ij} \frac{\partial x}{\partial a_i} \frac{\partial x}{\partial a_j}$$

Let us denote by $\widetilde{\mathbb{S}}$ the $\mathbb{Z}/2\mathbb{Z}$-graded version of the twisted modular $\mathcal{K}-$operad $\widetilde{\mathfrak{s}}\Sigma\mathbb{S}[t]$. By (a $\mathbb{Z}/2\mathbb{Z}$-graded version) of the theorem 1 from [B1] the solutions of quantum master equation in F

$$d_V S + \hbar \Delta S + \frac{1}{2}\{S, S\} = 0,$$

$S = \sum_{2(g-1)+n>0} \hbar^g S_{g,n}$, are in one-to one correspondence with the structure of algebra over the $\mathbb{Z}/2\mathbb{Z}$-graded modular operad $\mathcal{F}_\mathcal{K}\widetilde{\mathbb{S}}$ on the vector space V with symetric inner product B. Its genus zero part corresponds to the structure of $A_\infty-$algebra with the even invariant inner product on V.

2.2. **Odd inner products.** Let $V = V^{1|0} \oplus V^{0|1}$ be a super vector space and $B : V^{\otimes 2} \to \Pi k$ be an odd symmetric inner product, $B(x \otimes y) = (-1)^{\overline{x}+\overline{y}}B(y \otimes x)$. By definition B is an odd symplectic structure on V. The space F in this situation has components

(2.2) $$F_n = (V^{\otimes n} \otimes k[\mathbb{S}_n])^{\mathbb{S}_n}$$

where $k[\mathbb{S}_n]$ is the $k-$vector space with the basis indexed by $\sigma \in \mathbb{S}_n$ and $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]$ by conjugation. The space F carries the naturally defined operator

$$\Delta(a_1 \otimes \ldots \otimes a_n \otimes \sigma = (-1)^{\sum_{i=1}^{n-2} \overline{a}_i} B(a_{n-1} \otimes a_n)(a_1 \otimes \ldots \otimes a_{n-2} \otimes (\pi_{n-1,n}(s_{n-1,n}\sigma))$$

where $s_{n-1,n}$ is the transposition interchanging $n-1$ and n, $\pi_{n-1,n}$ is the operator of erasing of elements $n-1$ and n from the cycles of the representation, see section

9 from loc.cit. The space F can be identified with the symmetric power of the space of cyclic invariant tensors on V

$$F = Symm(\oplus_i (V^{\otimes i})^{\mathbb{Z}/i\mathbb{Z}})$$

If one chooses a basis $\{a_i\}$ in V, in which B has the form $B(a_i, a_j) = b_{ij}$ then operator Δ can be written as

$$\Delta = \sum_{ij} \gamma b_{ij} \frac{\partial^2}{\partial a_i \partial a_j}$$

where γ denotes the "dissection-gluing" operator acting on cycles, which corresponds to the symmetrisation of the operator $\sigma \to \pi_{n-1,n}(s_{n-1,n}\sigma)$ acting on permutations. The operator γ is even in this case since in the definition of F the parity change on the space of cyclic invariant tensors is not involved.

The operator Δ is of the second order with respect to the multiplication on F and it defines the odd Lie bracket on F, which can be written as

$$\{x, y\} = \sum_{ij} \gamma b_{ij} \frac{\partial x}{\partial a_i} \frac{\partial x}{\partial a_j}$$

The odd analogue corresponds to the modular operad, which is natural to denote $\mathbb{S}$, the *untwisted* analogue of $\mathbb{Z}/2\mathbb{Z}$-graded version of $\widetilde{\mathfrak{s}\Sigma\mathbb{S}}[t]$. The spaces $\mathbb{S}((n))$ are simply the group algebras $k[\mathbb{S}_n]$ with the action of $\mathbb{S}_n$ by conjugation. The composition maps are defined by operators $\pi_{f,f'}(s_{f,f'}\sigma)$ as in section 9 of [B1]. The Feynman transform $\mathcal{F}\mathbb{S}[t]$ is a $\mathcal{K}-$twisted modular operad. The solutions of quantum master equation in F are in one-to one correspondence with the structure of algebra over the modular operad $\mathcal{F}\mathbb{S}[t]$ on the vector space V with *odd* symmetric inner product B. The genus zero part of $\mathcal{F}\mathbb{S}[t]$ action on V corresponds to the structure of $A_\infty-$algebra with the odd invariant inner product on V. It is straightforward generalisation of the theorem 2 from loc.cit. that the complexes $\mathcal{F}\mathbb{S}[t]((0, \gamma, \nu)))$ compute the cohomology of the compactified moduli spaces with coefficients in the local system $H^i(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu, \mathcal{L})$, $\mathcal{L} = Det(P_\Sigma)$, where P_Σ is the set of marked points on the Riemann surface Σ.

3. MATRIX INTEGRALS.

We showed in [B1] that solutions to quantum master equation corresponding to a twisted modular operad $\mathcal{P}$ define natural cocycles in the graph complex which is the Feynmann transform of $\mathcal{P}$. In this section we show that for $\widetilde{\mathbb{S}}$ and $\mathbb{S}$ these cocycles can be naturally interpreted as the terms of Feynman diagram expansions of matrix integrals of $\exp(S/\hbar)$.

3.1. **Odd inner product.** Let us assume that

$$S(a_i, \hbar) = \sum_{2(g-1)+n>0} \hbar^g S_{n,g}(a_i)$$

$S_{n,g} \in F_n$, be a solution to quantum master equation in the space (2.2). Let us consider the super vector space

$$M = Hom(V, End_k(U))$$

where U is a super vector space $\dim U < \infty$. If $V = k^{(d|f)}$, then $M = (End_k(U))^{\times d} \times (\Pi End_k(U))^{\times f}$. The supertrace gives a natural even symmetric inner product on the space $End_k(U)$:

$$Tr(X \otimes Y) = \Sigma_{\alpha\beta}(-1)^{\overline{\alpha}} X_\beta^\alpha Y_\alpha^\beta$$

$X, Y \in End_k(U)$, it is invariant with respect to the action of the group $GL(U)$. It gives an extension to M of the odd symmetric inner product on $Hom(V, k)$ dual to B. In coordinates it is given by

$$B^{-1}(X, Y) = \sum_{i,j,\alpha,\beta} (B^{-1})^{ij}(-1)^{\overline{\alpha}} X_{i,\beta}^\alpha Y_{j,\alpha}^\beta$$

Let us extend S to a function S_U on M in the obvious way, so that instead of each cyclically symmetric tensor we put the supertrace of the product of the corresponding matrices from $End_k(U)$

$$\sum a_{i_1} \otimes \ldots \otimes a_{i_k} \rightarrow Tr(X_{i_1} \cdot \ldots \cdot X_{i_k})$$

and the product of cyclic words goes to the product of traces.

We claim that the tensors which we associated with $S(a_i, \hbar)$ via the statistical sum model construction to a graph G, equipped with the marking corresponding to basis elements from $\mathbb{MS}^{dual}((G))$, are obtained naturally from the Feynman diagrams corresponding to the asymptotic expansion of the following integral

$$\log \int \exp \frac{1}{\hbar} \left(B^{-1}(X, \Xi) - \frac{1}{2} B^{-1}(X, X) + S_U(X) \right) dX/(2\pi\hbar)^{\frac{N}{2}}$$

(compare with the formula (0.1) from [GK]). This is immediate from the standard rules of the Feynman diagrammatics, in particular the combinatorics of the terms in $S_U(X)$ reproduce exactly the data associated with vertices in the complex $\mathbb{MS}^{dual}((G))$, i.e. symmetric product of cyclic permutations and an integer number.

3.2. **Even inner product.** Let now $S(a_i, \hbar) = \sum_{2(g-1)+n>0} \hbar^g S_{n,g}(a_i)$ is a solution to quantum master equation in the Batalin-Vilkovisky algebra 2.1. In this case the space M is the super vector space

$$M = Hom_k(\Pi V, \Pi End_k(U) \oplus End_k(U))$$

where U is a purely even vector space. The analogue of trace defines on the space $\Pi End_k(U) \oplus End_k(U)$ an *odd* symplectic structure, which after choosing a basis $\{e_\alpha\}$ in U can be written as

$$Tr(X \otimes Y) = \sum_{\alpha\beta} X_\beta^\alpha Y_\alpha^\beta$$

$X \in \Pi End_k(U), Y \in End_k(U)$. Notice that there are natural isomorphisms

$$\Pi End_k(U) = Hom_k(\Pi U, U) = Hom_k(U, \Pi U)$$

and also

$$Hom_k(\Pi U, \Pi U) = End_k(U)$$

Therefore elements of the space $\Pi End_k(U) \oplus End_k(U)$ can be naturally composed so that this space is in fact an associative algebra with odd invariant inner product.

We have an extension to M of the even symmetric inner product on $Hom(V, k)$ dual to B:

$$B^{-1}(X, Y) = \sum_{i,j,\alpha,\beta} (B^{-1})^{ij}(-1)^{\overline{\alpha}} X_{i,\beta}^\alpha Y_{j,\alpha}^\beta$$

$X, Y \in \Pi End_k(U) \oplus End_k(U)$.

We extend as above the solution S to a function S_U on M in the obvious way, so that instead of each cyclically symmetric tensor in $(\Pi V)^{\otimes k}$ we put the trace of the product of the corresponding elements from $(\Pi End_k(U) \oplus End_k(U))$

$$\sum a_{i_1} \otimes \ldots \otimes a_{i_k} \to Tr(X_{i_1} \cdot \ldots \cdot X_{i_k})$$

and the product of cyclic words goes to the product of traces.

Then the standard rules of the Feynman diagrammatics imply immediately that the asymptotic expansion of the integral

$$\log \int \exp \frac{1}{\hbar} \left(B^{-1}(X, \Xi) - \frac{1}{2} B^{-1}(X, X) + S_U(X) \right) dX/(2\pi\hbar)^{\frac{N}{2}}$$

$X, \Xi \in M$, give the statistical sum tensors on graphs which were associated with the solution S in [B1].

Example 1. *Let us consider the case of even one-dimensional space V. It is easy to see that in this case any linear combination of cyclic words of X^3, X^5,..., X^{2n+1} is a solution of quantum master equation. The corresponding matrix integral reproduces the matrix integrals of* [K].

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