

Non-Rational 2D Quantum Gravity II. Target Space CFT

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We explore the formulation of non-rational 2D quantum gravity in terms of a chiral CFT on a Riemann surface associated with the target space. The CFT in question is constructed as the collective theory for a matrix chain, which is dual to a statistical height model on dynamical triangulations. The heights are associated with the sheets of the Riemann surface, which represents an infinite branched cover of the spectral plane. We consider two examples of height models: the SOS model and the semi-restricted SOS (SRSOS) model, in which the heights are restricted from below. Both models are described in the continuum limit by theories of 2D quantum gravity with conformal matter, perturbed by a thermal operator (1,3). We give a compact operator expression for the n -loop amplitudes as a collection of target space Feynman rules. The n -point functions of local fields are obtained by shrinking the loops. In particular, we show that the 4-point function of order operators in the SRSOS model coincides with the 4-point function of the “diagonal” world sheet CFT studied in [1].

1. Introduction

In a previous paper [1] we studied non-rational theories of 2D Quantum Gravity (QG), having continuous spectrum of the matter central charge $-\infty < c < 1$. We explored and generalized the ground ring approach to the construction of the tachyon correlation functions. The method was also tested for an unconventional model of 2D quantum gravity with interaction which induces matter charges corresponding to the diagonal of the Kac table. In this paper we develop a different technique for the computation of the tachyon correlators based on discrete realizations of the 2D gravity with non-rational conformal matter.

The simplest statistical model that leads to a non-rational CFT is the SOS height model, which we will consider on a triangular lattice. The local fluctuating variable in the SOS model is an integer height x , which can jump by ± 1 between neighboring sites. The Boltzmann weights are complex and depend on two parameters, the “background momentum” $p_0 \in [0, 1]$ and the temperature coupling $T > 0$. By restricting the heights of the SOS model to positive integers we obtain another height model which we call semi-restricted SOS, or SRSOS model. The Boltzmann weights of SRSOS model are real and anti-symmetric with respect to the reflection $x \rightarrow -x$. As we will consider the two models alongside, we denote by \mathbb{X} the target space of any of them,

$$\mathbb{X} = \begin{cases} \mathbb{Z} & \text{for SOS;} \\ \mathbb{Z}_+ & \text{for SRSOS.} \end{cases} \quad (1.1)$$

At the rational points the SRSOS model can be further restricted to the RSOS models, which have positive Boltzmann weights [2,3]. The RSOS models form the A -series of the ADE Pasquier models [4]. In this sense the SRSOS model is “universal cover” of the A -series of minimal models. Its target space can be identified with the Dynkin diagram of the A_∞ Lie algebra, considered as a certain limit of A_n . It is also possible to construct the non-rational extension of the D_n -series, with symmetric weights, which we discuss briefly in Appendix C. A common property of the height models is that they can be mapped to a gas of self-avoiding and mutually avoiding loops on the dual lattice. The observables are introduced by assigning special weights to the non-contractible loops, while the activity of the contractible loops

$$n = 2 \cos(\pi p_0) \quad (1.2)$$

is determined solely by the background momentum. The temperature coupling determines the “mass” of the loops.

At the critical temperature $T = T_c$ the large-distance behavior of the SOS model is that of a gaussian field with curvature term, while the SRSOS model is described by Virasoro CFT with charge reflection symmetry. Both CFT’s have the same central charge

$$c_{\text{critical}} = 1 - 6 \frac{p_0^2}{1+p_0}. \quad (1.3)$$

There is a set of primary fields which can be defined at microscopic level. These are the order operators, which are realized by the eigenfunctions of the adjacency matrix of the target space \mathbb{X} . Each order operator is characterized by a momentum p . The degenerate order operators, placed along the diagonal of the Kac table, have momenta $p = mp_0$, $m \in \mathbb{Z}_+$. The order operators are well defined also away from the critical temperature, where the long distance behavior of the height models is described by a perturbation with the thermal operator $\Phi_{1,3}$ [5][6][7][8].

The height models coupled to gravity are defined by generalizing the Boltzmann weights, originally defined on a flat lattice, to an arbitrary triangulated surface, according to the prescription in [9,10]. Then the path integral over two-dimensional geometries is discretized as a sum over all triangulations with given topology.

The SOS and SRSOS models have their dual formulations in terms of doubly infinite or semi infinite matrix chains, similar to the *ADE* matrix models [11]. The collective field theory for the matrix chain is that of a chiral boson on a Riemann surface representing an infinite branch cover of the spectral plane of the random matrices. The sheets of the Riemann surface are labeled by the points x of the target space. The point on the Riemann surface are enumerated by pairs (z, x) where $z \in \mathbb{C}$ is the projection on the spectral plane and $x \in \mathbb{X}$ labels the sheet. Thus each point of the Riemann surface corresponds to a boundary condition (FZZ-brane), with z being the complex boundary cosmological constant and x being the height at the boundary. At the rational points the Riemann surface represents the algebraic curve of the corresponding minimal model of 2D gravity [12].

We will be interested only in the genus zero amplitudes, which are well defined quantities even in a non-unitary theory. The disk and the annulus amplitudes are given respectively by the one-point and by the two-point functions of a free boson on the Riemann surface. The amplitudes with more than two boundaries are produced by the interaction terms. The latter, associated with the branch points, are needed to preserve the conformal invariance [13]. The operator solution of the Virasoro constraints yields a set of Feynman rules for calculating the loop correlation functions, which allows to evaluate any such function by a finite sum over all possible intermediate channels. In the case of rational *ADE* string theories these rules were formulated in [14].

We work out explicitly the example of the 4-point function and will compare the result to the prediction of the world-sheet CFT. In the case of the SOS model, our result reproduces at the critical point the Di Francesco-Kutasov formula [15] obtained for gaussian matter field with curvature term. In the case of the SRSOS model, our result for the 4-point function matches the one obtained in [1] for the “diagonal” perturbation of Liouville gravity. This perturbation is generated by the Liouville-dressed vertex operator of dimension zero obtained from the identity by charge reflection.

The paper is organized as follows. In sect 2 we give the definition of SOS and SRSOS models on a triangulated surface with curvature defects and their representation in terms of a gas of loops. The loop gas representation of the correlation functions of order operators is sketched in Appendix A. In sect. 3 we reformulate the height models on dynamical triangulations in terms of doubly or semi infinite matrix chains. In sect. 4 we give the solution for the disk amplitude in the continuum limit. In sect. 5 we construct the target space CFT as a bosonic field theory on the Riemann surface associated with the disk amplitude. The Feynman rules for calculating the n -loop amplitudes are obtained in sect. 6. In sect. 7 we work out the example of the 4-point function of order operators. Summary of the results is given in sect. 8.

2. The SOS and SRSOS models on a triangulation with curvature defects

2.1. Local Boltzmann weights and mapping to a loop gas

Let \mathcal{G} be a triangulated surface with the topology of a sphere with n boundaries. The triangulation is characterized by its vertices r , its links $l = \langle r_1 r_2 \rangle$ and its triangles $\Delta = \langle r_1 r_2 r_3 \rangle$ which are assumed equilateral. The curvature defects are associated with the points $r \in \mathcal{G}$ with coordination number c_r (the number of triangles having this point as a vertex) different than 6. The local curvature at such point is equal to the deficit angle,

$$\hat{R}_r = \frac{2\pi}{3}(6 - c_r). \quad (2.1)$$

Similarly, the boundary curvature at the points $r \in \partial\mathcal{G}$ is

$$\hat{K}_r = \frac{\pi}{3}(3 - c_r). \quad (2.2)$$

By Euler's relation the total curvature is

$$\sum_{r \in \mathcal{G}} \hat{R}_r + 2 \sum_{r \in \partial\mathcal{G}} \hat{K}_r = 4\pi(2 - n). \quad (2.3)$$

The SOS and SRSOS models on the triangulation \mathcal{G} are defined as follows. To each node $r \in \mathcal{G}$ we associate an integer height $x_r \in \mathbb{X}$, where the target space \mathbb{X} is defined in (1.1). The allowed height configurations are such that the heights of two nearest-neighbor points are either equal or differ by ± 1 . If one thinks of the target space as a one-dimensional graph (Fig. 1), then the allowed height configurations are the maps $\mathcal{G} \rightarrow \mathbb{X}$ which preserve the nearest neighborliness, i.e., such that points and links are mapped either to points or to links. Since the space \mathbb{X} is discrete, the momentum space \mathbb{P} is compact: $p \sim p + 2$. In the case of SRSOS there is additional reflection symmetry, $p \sim -p$.



Fig. 1: The target space of the SRSOS model as a one-dimensional graph.

The partition function $Z_{\mathcal{G}}(x_1, \dots, x_n)$ is the sum over all allowed maps $\mathcal{G} \rightarrow \mathbb{X}$ such that the boundaries have fixed heights x_1, \dots, x_n . Besides the standard local factors W_{Δ} associated with the faces of \mathcal{G} , there are extra weights W_{\bullet} that come from the curvature defects:

$$Z_{\mathcal{G}}(x_1, \dots, x_n) = \sum_{\mathcal{G} \rightarrow \mathbb{X}} \prod_{\langle r_1 r_2 r_3 \rangle} W_{\Delta}(x_{r_1}, x_{r_2}, x_{r_3}) \prod_r W_{\bullet}(x_r). \quad (2.4)$$

The Boltzmann weights are expressed in terms of the function

$$S_x = \begin{cases} \exp(-i\pi p_0 x) & \text{for SOS;} \\ \sin(\pi p_0 x) & \text{for SRSOS} \end{cases} \quad (2.5)$$

which plays the same role as Peron-Frobenius vector in the *ADE* height models. It is an eigenvector of the adjacency matrix of the graph \mathbb{X} with eigenvalue $2 \cos(\pi p_0)$. The background momentum $p_0 \in [0, 1]$ is assumed in this paper to be non-rational. By definition, the weight of a triangle Δ_{123} with heights x_1, x_2, x_3 is invariant under cyclic permutations and is non-zero only if the heights of each pair of points are either the same or adjacent. The possibilities are either $x_1 = x_2 = x_3$ or $x_1 = x_2 = x_3 \pm 1$, up to a cyclic permutation, and have weights

$$W_{\Delta}(x, x, x) = 1; \quad W_{\Delta}(x, x, x \pm 1) = \frac{1}{T} \left(\frac{S_{x \pm 1}}{S_x} \right)^{1/6}, \quad (2.6)$$

where T is a positive constant called temperature. The weight associated with the vertex $r \in \mathcal{G}$ with curvature defects are

$$W_{\bullet}(x_r) = \begin{cases} (S_{x_r})^{\hat{R}_r/4\pi} & \text{if } r \in \mathcal{G}; \\ (S_{x_r})^{\hat{K}_r/2\pi} & \text{if } r \in \partial\mathcal{G}. \end{cases} \quad (2.7)$$

While Boltzmann weights of the SOS model are complex for any $p_0 \neq 0$, those of the SRSOS model are real, but not always positive.

The height model is also described in terms of a *loop gas*. Let us represent the weights of the triangles with admissible heights symbolically as

$$\quad (2.8)$$

That is, whenever the three heights round the triangle are not the same, a line separates the two vertices with the same height from the vertex with different height. The line is

made by two segments at angle $\pi/3$, orthogonal to the edges they start from. Then each admissible height configuration determines a pattern of closed nonintersecting loops (Fig. 2). With our choice of boundary conditions the domain walls do not cross the boundary.

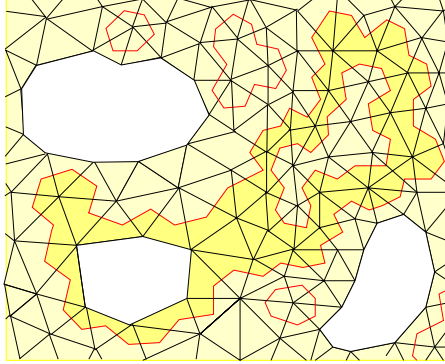


Fig. 2: A loop configuration on a triangulation with boundaries. The boundary of the highlighted domain has $n_b = 3$ components, two of them being dynamical loops.

The Boltzmann weights in (2.4) are such that the weight of each loop gas configuration factorizes to a product of the weights of the connected domains of constant height. The weight of a domain \mathcal{D} is topological – it depends on the height x of the domain only through its Euler characteristics, *i.e.*, the number of its boundaries:¹

$$\Omega_{\mathcal{D}}(x) = (S_x)^{2-n_b}, \quad n_b = \# \text{ boundaries of } \mathcal{D}. \quad (2.9)$$

The proof goes as follows. We can distribute the weights of the triangles that belong to two different domains, given by the second term of eqn. (2.6), in such a way that the weight of each domain \mathcal{D} is of the form $(S_x)^q$, where x is the height of the domain. It remains to evaluate the power q . It receives contributions from the vertices in the bulk inside the domain wall, from the vertices of the boundaries of \mathcal{G} (if any) that are also boundaries of \mathcal{D} , and from the triangles along the domain walls that delimit \mathcal{D} . If there are no domain walls at all, that is, $\mathcal{D} = \mathcal{G}$, then the total power of S_x is $q = 2 - n$ according to the Euler relation (2.3). If there are one or more domain walls, each domain wall contributes a factor $S_x^{\hat{K}}$, where \hat{K} is the total boundary curvature along it. Indeed, the weight of a domain wall is a product of factors $(S_x)^{\pm 1/6}$ that come from the triangles along it and account for the defects $\pm\pi/6$ along its edge. Applying again the Euler relation we find $q = 2 - n_b$, where n_b is the total number of (connected) boundaries of the domain, given by the number of boundaries of \mathcal{G} trapped in the domain plus the number of the domain walls.

¹ In the case of a lattice with higher genus, there would be also a factor $(S_x)^{-2}$ for each handle trapped in the domain.

As a consequence, the partition function (2.4) can be reformulated as a sum over all configurations of self and mutually avoiding loops on the dual lattice and heights $x_{\mathcal{D}}$ associated with the connected domains \mathcal{D} :

$$Z_{\mathcal{G}}(x_1, \dots, x_n) = \sum_{\substack{\text{loop configurations} \\ \text{on } \mathcal{G}}} T^{-L_{\text{tot}}} \sum_{\text{heights } x_{\mathcal{D}}} \prod_{\mathcal{D}} \Omega_{\mathcal{D}}(x_{\mathcal{D}}). \quad (2.10)$$

Here L_{tot} is the total length of the loops, which is equal to the total number of loop segments. The sum over the heights of the domains, $\{x_{\mathcal{D}}\}$, is such that the heights of the neighboring domains differ by ± 1 . In this way the mapping to a loop gas, originally established for a flat lattice [16], remains true also in the case of a lattice with curvature defects.

For the disk partition function ($n = 1$) the one can perform readily the sum over the heights but one using repeatedly the relation

$$\sum_{x'} A_{xx'} S_{x'} = 2 \cos \pi p_0 S_x. \quad (2.11)$$

In this way the sum over heights produces a factor $(2 \cos \pi p_0)^{\# \text{ loops}}$ times a factor S_{x_1} depending on the boundary height x_1 . Up to this last factor, this is the disk partition function of the $O(n)$ loop gas [16] with activity $n = 2 \cos \pi p_0$ per loop.

In the case of more than one boundary, the sum over the heights can be performed by Fourier transformation with the eigenvectors of the adjacency matrix

$$S_x^{(p)} = \begin{cases} e^{i\pi p x} & \text{for SOS;} \\ \sin(\pi p x) & \text{for SRSOS.} \end{cases} \quad (2.12)$$

In this case the loops can have different activities. The contractible loops, *i.e.*, the loops for which there is no topological obstacle for shrinking them to points, have activity $2 \cos \pi p_0$. A non-contractible loop contributes a factor $(2 \cos \pi p)$, where $p \in \mathbb{P}$ is the momentum associated with the loop. In Appendix A we give the loop gas representation of the correlation function of n order operators on the sphere, which is the Fourier transform of the the partition function (2.10) with boundaries of length zero.

2.2. Phase diagram and critical points

The critical thermodynamics of the SOS and SRSOS models is that of the $O(n)$ loop gas with $n = 2 \cos(\pi p_0)$. The phase diagram (Fig. 3) of the $O(n)$ model on the the honeycomb lattice (dual to the regular triangulation) was first established in [16]. At the critical temperature $T_c = 2 \cos \frac{\pi}{4} p_0$ the loop gas model is solvable and is described by a CFT with central charge

$$c_{\text{critical}} = 1 - 6 \frac{p_0^2}{1 + p_0}. \quad (2.13)$$

The critical point is also known as the *dilute phase* of the loop gas. For $T > T_c$ the theory has a mass gap. The low-temperature, or *dense*, phase $T < T_c$ is a flow to an attractive fixed point [17] at $T_c^{\text{dense}} = 2 \sin \frac{\pi}{4} p_0$, where the theory is again solvable and is described by a CFT with smaller central charge

$$c_{\text{dense}} = 1 - 6 \frac{p_0^2}{1 - p_0}. \quad (2.14)$$

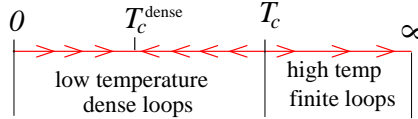


Fig. 3: Phase diagram of the loop gas on a regular (honeycomb) lattice.

The scaling behavior of the model in the vicinity of a critical point is described by an action of the form [8]

$$\mathcal{A} = \mathcal{A}_{\text{critical}} + \delta T \int \Phi_{1,3} \quad (2.15)$$

where $\delta T = T - T_c$ and $\Phi_{1,3}$ is the thermal operator with conformal dimensions

$$\Delta_{1,3} = \bar{\Delta}_{1,3} = \frac{1 - p_0}{1 + p_0}. \quad (2.16)$$

When the thermal operator $\Phi_{1,3}$ is added to the action, it generates a tension of the loops proportional to δT . For $\delta T > 0$ the deformation (2.15) describes, going from short to the long distance scales, the flow to a massive theory with mass gap

$$m \sim \delta T^{1/(2-2\Delta_{1,3})} = \delta T^{\frac{1+p_0}{4p_0}}. \quad (2.17)$$

When $\delta T < 0$ the deformation (2.15) describes a massless flow between two different CFT with central charges (2.13) and (2.14). In the CFT for the dense phase, describing the IR limit, the flow to the attractive critical point is generated by another, irrelevant, operator, $\Phi_{3,1}$.

2.3. Mapping to a gaussian field

- SOS model

It has been argued that at the critical points the large distance behavior of the SOS model on a lattice with curvature defects is that of a gaussian field $\chi(\sigma_1, \sigma_2)$ with a background charge proportional to the local curvature [16,18]. The product of the factors

(2.7) associated with the curvature defects yields the curvature-dependent term in the effective action

$$\mathcal{A}_{\text{gauss}} = \frac{1}{4\pi} \int d^2\sigma [(\nabla\chi)^2 + ie_0\chi]. \quad (2.18)$$

The two critical points are described by different values of the background charge e_0 , related to the background momentum p_0 by

$$e_0 = \pm \frac{p_0}{\sqrt{1 \pm p_0}}, \quad \text{with} \quad (+ \text{ for dilute, } - \text{ for dense}). \quad (2.19)$$

In the SOS model the order operators $\psi_p(x) = S_x^{(p)}/S_x$ have the form of plane waves

$$\psi_p(x) = e^{i\pi(p_0-p)x} = e^{2\pi iqx}. \quad (2.20)$$

At each of the critical points they can be identified with vertex operators of the gaussian field with conformal dimensions

$$\Delta_p = \frac{p^2 - p_0^2}{4(1 \pm p_0)} = \frac{q(q - p_0)}{1 \pm p_0} \quad (+ \text{ for dilute, } - \text{ for dense}). \quad (2.21)$$

- SRSOS model

In this case the correlation functions are constructed from vertex operators and screening charges. To get an intuition about how the vertex operators appear in the microscopic theory, let us notice that in the SRSOS model the effect of the curvature does not reduce to a pure phase factor, as it was the case for the SOS model. Here we follow the argument of [18]. After formally expanding, with S_x given by (2.5), one gets

$$\log S_x = -i\pi p_0 x - e^{2\pi i p_0 x} + \dots$$

The exponential term corresponds to a vertex operator (2.20) with $q = p_0 + \text{integer}$. The choice $q = p_0 + 1$ gives one of the screening charges added to the action (2.18), $e = \sqrt{1 + p_0}$.

3. SOS and SRSOS models on dynamical triangulations

When considered on dynamical triangulations, the heights models give the desired solvable discrete models of non-rational 2D quantum gravity. The partition function in the ensemble of triangulations with the topology of a sphere with n boundaries, or shortly n -loop amplitude, is defined as

$$Z_{(L_1, x_1), (L_2, x_2), \dots, (L_n, x_n)} = \sum_{\mathcal{G}_{L_1, \dots, L_n}} \kappa^A Z_{\mathcal{G}_{L_1, \dots, L_n}}(x_1, \dots, x_n). \quad (3.1)$$

The sum goes over all triangulations $\mathcal{G}_{L_1, \dots, L_n}$ with the above topology and fixed lengths (number of edges) L_1, \dots, L_n of the boundaries, and the ‘‘cosmological constant’’ κ coupled to the area A (the number of triangles) of the triangulation.

3.1. Dual description in terms of doubly infinite or semi infinite matrix chains

There is a dual, matrix, formulation of the height models on dynamical triangulations in terms of a doubly infinite (for SOS) or semi-infinite (for SRSOS) matrix chains. These matrix models are similar to the *ADE* matrix models [11], with some subtleties related to the fact that they describe non-unitary theories. The fluctuating variables are hermitian $N_x \times N_x$ matrices \mathbf{M}_x and the complex rectangular $N_x \times N_{x+1}$ matrices \mathbf{C}_x ($x \in \mathbb{X}$). The hermitian matrices \mathbf{M}_x are naturally associated with the sites x while the complex matrices \mathbf{C}_x are associated with the links $\langle x, x+1 \rangle$ of the target space. The partition function of the matrix chain is given by the integral

$$\mathcal{Z}[V] = \int [d\mathbf{M}] [d\mathbf{C}] [d\mathbf{C}^\dagger] \exp\left(-\frac{1}{g_s} \mathcal{A}\right), \quad (3.2)$$

$$\mathcal{A} = \sum_{x \in \mathbb{X}} S_x \operatorname{tr} \left(\frac{1}{2} \mathbf{M}_x^2 - \frac{1}{3} \kappa \mathbf{M}_x^3 + T \mathbf{C}_x^\dagger \mathbf{C}_x - \kappa \mathbf{C}_x \mathbf{C}_x^\dagger \mathbf{M}_x - \kappa \mathbf{M}_{x+1} \mathbf{C}_x^\dagger \mathbf{C}_x \right). \quad (3.3)$$

The perturbative expansion of the matrix integral is constructed according to the Feynman rules in Fig. 4, which correspond to the five terms in the action (3.3). The correlation functions in the matrix model can be expressed in terms of fat (double lined) graphs. We will be interested in the limit $N_x \rightarrow \infty$ for all $x \in \mathbb{X}$, in which limit only the planar graphs survive. Each planar graph is in one-to-one correspondence with a triangulated surface. The vertices of the triangulation (dual to the faces of the planar graph) have label $x \in \mathbb{X}$. Thus the sum over planar diagrams reproduces the path integral for the height model in the ensemble of triangulations.

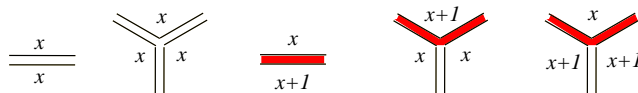


Fig. 4: Feynman rules for the SRSOS matrix model

In order to obtain the correct Boltzmann weights, we must tend the size of the matrices to infinity in a particular way:

$$g_s N_x \rightarrow S_x, \quad g_s \rightarrow 0. \quad (3.4)$$

Using the interpretation of the Feynman graphs as triangulated surfaces, one can easily see that the n -loop amplitude (3.1) is equal to the genus zero connected correlation function of a product of n traces

$$Z_{(L_1, x_1), (L_2, x_2), \dots, (L_n, x_n)} = \lim_{g_s \rightarrow 0} g_s^{2-n} \left\langle \left\langle \prod_{i=1}^n \operatorname{tr}(\mathbf{M}_{x_i}^{L_i}) \right\rangle \right\rangle. \quad (3.5)$$

The reader might object that the limit (3.4) does not exist, because S_x is not a positive number. The limit is in fact well defined only at the rational points of the SRSOS model, $p_0 = 1/(h + 1)$, with $h \in \mathbb{Z}_+$. In this case S_x is Peron-Frobenius vector for the adjacency matrix of the A_h Dynkin graph and all its components are positive. However, we can impose the condition (3.4) in a weaker sense as an analytical continuation from positive integer values.

As the observables (3.5) are associated with the \mathbf{M} -matrices, we can integrate out the \mathbf{C} -matrices in the partition function. After the integration the partition function (3.2) can be written, up to a factor equal to the volume of the symmetry group $\otimes_{x \in \mathbb{X}} SU(N_x)$, as an integral with respect to the eigenvalues $\lambda_{i,x}$, $i = 1, \dots, N_x$, of the hermitian matrices \mathbf{M}_x :

$$\mathcal{Z} \sim \int_{-\infty}^{\infty} \prod_{x \in \mathbb{X}} \prod_{i=1}^{N_x} d\lambda_{i,x} e^{-\frac{1}{g_s} S_x (\frac{1}{2} \lambda_{i,x}^2 - \frac{1}{3} \kappa \lambda_{i,x}^3)} \prod_{i < j}^{N_x} (\lambda_i^x - \lambda_j^x)^2 \prod_{i,j} \frac{1}{|T - \kappa(\lambda_{i,x} + \lambda_{j,x+1})|}. \quad (3.6)$$

The explicit dependence on the second coupling κ can be eliminated by a linear change of variables

$$\mathbf{M}_x \rightarrow \frac{1}{\kappa} \mathbf{M}_x + \frac{T}{2}, \quad g_s \rightarrow \kappa^2 g_s. \quad (3.7)$$

After that the partition function (3.6) takes the form

$$\mathcal{Z} \sim \int_{-\infty}^{\infty} \prod_{x,i} d\lambda_{i,x} e^{-\frac{1}{g_s} V_x(\lambda_{i,x})} \frac{\prod_{x;i < j} (\lambda_{i,x} - \lambda_{j,x})^2}{\prod_{i,j} |\lambda_{i,x} + \lambda_{j,x+1}|} \quad (3.8)$$

where the cubic potential depends only on the temperature T ,

$$V_x(z) \equiv S_x V(z), \quad V(z) = \frac{T(2-T)}{4} z + \frac{1-T}{2} z^2 - \frac{1}{3} z^3. \quad (3.9)$$

The coupling κ , the lattice cosmological constant, reappears in the definition of the planar limit, originally given by (3.4), which now becomes

$$g_s N_x \rightarrow \kappa^2 S_x, \quad g_s \rightarrow 0. \quad (3.10)$$

The eigenvalue integral (3.8) is not convergent. To make it convergent, we will drop the absolute value in the denominator and will understand the integrals over the eigenvalues as contour integrals in the complex plane. The new integral will have the same perturbative expansion as (3.8). We choose the integration contour \mathcal{C} so that it starts at $\lambda_{i,x} = -\infty$, passes near the origin and then goes to infinity along a direction in which the potential grows (Fig. 5a). This is a standard prescription used to define rigorously the matrix integrals that appear in 2D quantum gravity, see e.g. [19]. To avoid the poles of the

integrand at $\lambda_{x,i} = -\lambda_{x',j}$ we also require that the contour \mathcal{C} does not intersect the contour $\bar{\mathcal{C}}$ obtained from \mathcal{C} by a reflection $z \rightarrow -z$.

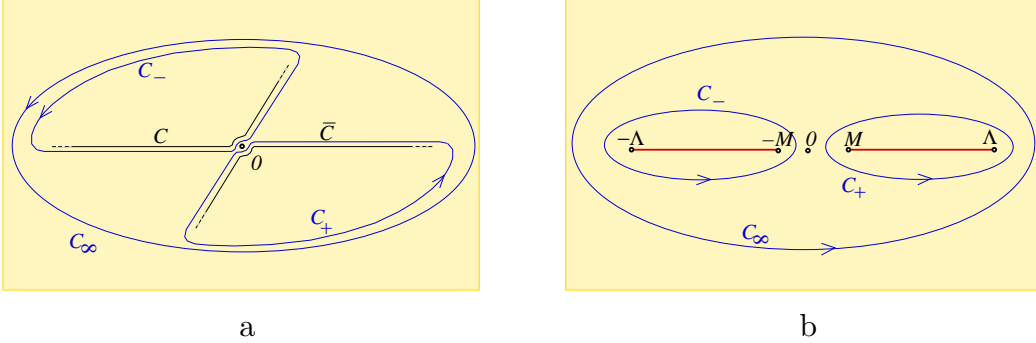


Fig. 5a. Integration contours in the complex plane of the eigenvalues for cubic potential

Fig. 5b. Integration contours relevant for the quasiclassical limit

There are several choices for the second branch of the contour \mathcal{C} , which however lead to the same perturbative expansion.

3.2. The matrix chain as a chiral conformal field theory

It is known that a large class of matrix models can be reformulated as conformal field theories [20][21][22][23][13][24]. The correspondence $\text{MM} \leftrightarrow \text{CFT}$ can be established if the measure in the eigenvalue integral can be produced by screening charge operators. We will show that this is the case for the integral (3.8) where the integration goes in the complex plane along the contour \mathcal{C} .

We associate with each point $x \in \mathbb{X}$ a chiral boson $\varphi_x(z)$ two-point correlator

$$\langle 0 | \varphi_x(z) \varphi_{x'}(z') | 0 \rangle = \delta_{x,x'} \log(z - z'), \quad (3.11)$$

where the right/left vacuum is annihilated by the negative/positive power part of the Laurent expansion of the fields $\partial\varphi_x(z)$ in z . We introduce two more bosonic fields,

$$\Gamma_x(z) = \varphi_x(z) - \varphi_{x+1}(-z), \quad (3.12)$$

and the conjugated field $\Phi_x(z)$, defined by

$$\langle 0 | \partial_z \Phi_x(z) \Gamma_{x'}(z') | 0 \rangle = \frac{\delta_{x,x'}}{z - z'}. \quad (3.13)$$

The field $\Phi_x(z)$ is related to $\varphi_x(z)$ and $\Gamma_x(z)$ by

$$\Phi_x(z) - \Phi_{x+1}(-z) = -\varphi_{x+1}(-z), \quad (3.14)$$

$$2\Phi_x(z) - \sum_{x'} A_{xx'} \Phi_{x'}(-z) = \Gamma_x(z), \quad (3.15)$$

where $A_{xx'}$ is the adjacency matrix of \mathbb{X} .

It is straightforward to check that the integrand of (3.8) can be written as Fock space expectation value of a product of vertex operators

$$E_x(z) =: e^{-\varphi_x(z)} : : e^{\varphi_{x+1}(x)} :. \quad (3.16)$$

Let $\langle \vec{N} |$ be the charged vacuum state defined by the asymptotics

$$\partial\Phi_x(z) = -\frac{N_x}{z} + \dots, \quad z \rightarrow \infty. \quad (3.17)$$

Then using the OPE of the vertex operators one finds

$$\prod_{x \in \mathbb{X}} \frac{\prod_{i \neq j} (\lambda_{i,x} - \lambda_{j,x})}{\prod_{i,j} (\lambda_{i,x} + \lambda_{j,x+1})} = \langle \vec{N} | \prod_{x,i} E_x(\lambda_{i,x}) | 0 \rangle. \quad (3.18)$$

To generate the external potential it suffices to replace the left vacuum by a coherent state

$$\langle B | = \langle \vec{N} | \exp\left(\sum_{x \in \mathbb{X}} H_x\right), \quad H_x = \frac{1}{g_s} \oint_{\mathcal{C}_\infty} \frac{dz}{2\pi i} V_x(z) \partial\Phi_x(z) \quad (3.19)$$

where the contour \mathcal{C}_∞ encircles both \mathcal{C} and $\bar{\mathcal{C}}$ (Fig. 4a). This is easily verified by applying the operator product expansion (3.13) to commute e^{H_x} to the right. The result is a factor $e^{-\frac{1}{g_s} V_x}$ per eigenvalue.

To write the Fock space representation of the eigenvalue integral (3.8) we define the screening operators

$$Q_x = \int_{\mathcal{C}} dz E_x(z). \quad (3.20)$$

Then we can write the eigenvalue integral in the form of a scalar product

$$\mathcal{Z} = \langle B | \Omega \rangle, \quad (3.21)$$

where $\langle B |$ is defined by (3.19) and

$$|\Omega\rangle = \exp\left(\sum_{x \in \mathbb{X}} Q_x\right) | 0 \rangle. \quad (3.22)$$

- *Correspondence between the observables in CFT and the matrix model*

The expectation value $\langle \cdot \rangle_{\text{MM}}$ in the matrix model becomes the vacuum expectation value in terms of Fock space. To any observable \mathcal{O}_{MM} in the matrix model one can associate an operator \mathcal{O} such that

$$\langle \mathcal{O}_{\text{MM}} \rangle_{\text{MM}} = \frac{\langle B | \mathcal{O} | \Omega \rangle}{\langle B | \Omega \rangle} \equiv \langle \mathcal{O} \rangle. \quad (3.23)$$

In particular, the resolvent of the matrix \mathbf{M}_x is represented in the CFT as

$$W_x(z) \equiv g_s \text{tr} \left(\frac{1}{z - \mathbf{M}_x} \right) \leftrightarrow -g_s \partial \Phi_x^{[+]}(z), \quad (3.24)$$

where $\partial \Phi^{[+]}(z)$ is the negative part of the Laurent series of $\partial \Phi(z)$, which annihilates the right vacuum, $\partial \Phi^{[+]}|0\rangle = 0$.

In other words, the resolvent is equal to (minus) the singular at $z = 0$ part of Laurent series of $\partial \Phi(z)$, which annihilates the right vacuum, $\partial \Phi^{[+]}|0\rangle = 0$.

The other important observable is the FZZ-brane, which measures the effect of suppressing the integral with respect to one of the eigenvalues of the matrix \mathbf{M}_x at assigning to it a complex value z . This matrix observable corresponds to the vertex operator (3.16),

$$e^{-\frac{1}{g_s} V_x(z)} \det(z - \mathbf{M}_x)^2 \prod_{x'} \det(z + \mathbf{M}_{x'})^{-A_{xx'}} \leftrightarrow E_x(z) =: e^{-\Gamma_x(z)} : . \quad (3.25)$$

In the limit $g_s \rightarrow \infty$ we can replace $\langle : e^{-\Gamma_x(z)} : \rangle = e^{-\langle \Gamma_x(z) \rangle}$. Thus the expectation value of the field (3.15) gives the effective potential of one eigenvalue taking value z in the mean field produced by the other eigenvalues,

$$\frac{1}{g_s} V_x(z) - 2 \text{tr} \log(z - \mathbf{M}_x) + \sum_{x'} A_{xx'} \text{tr} \log(z + \mathbf{M}_{x'}) \leftrightarrow \Gamma_x(z). \quad (3.26)$$

- *Virasoro constraints*

In the matrix model formulation, the n -loop amplitudes satisfy a set of loop equations that follow from the translational invariance of the matrix integration measure. The loop equations have been first derived from the combinatorics of planar graphs [9]. They have the form

$$\left\langle W_x(z)^2 + \frac{1}{2\pi i} \oint_{\mathcal{C}_-} \frac{d\zeta}{z - \zeta} W_x(\zeta) \left[\sum_{x'} A_{xx'} W_x(-\zeta) - \sum_n V'_x(\zeta) \right] \right\rangle_{\text{MM}} = 0. \quad (3.27)$$

The integration contour \mathcal{C}_- separates the eigenvalue contour \mathcal{C} from its reflection image $\bar{\mathcal{C}}$ (Fig. 4). It is also assumed that the point z is outside the integration contour.

In the Fock space representation, the loop equations are equivalent to the following set of operator identities satisfied by the right vacuum $|\Omega\rangle$:

$$\int_{\mathcal{C}_-} d\zeta \frac{[\partial \Gamma_x(\zeta)]^2}{z - \zeta} |\Omega\rangle = 0 + \dots \quad (x \in \mathbb{Z}_+). \quad (3.28)$$

The omitted terms vanish after multiplying with the left vacuum $\langle B|$. In other words for each x the singular at $z = 0$ part of the mode expansion of the energy-momentum tensor

$$T_x(z) \equiv \frac{1}{4} : [\partial\Gamma_x(z)]^2 : \quad (3.29)$$

annihilates the state $|\Omega\rangle$.

Proof: The operator $(\partial\Gamma_x)^2$ acts to the right vacuum as

$$\begin{aligned} T_x(z)|\Omega\rangle &= \frac{1}{2}\partial\Gamma_x(z) \int_{\mathcal{C}} d\zeta \left(2\frac{E_x(\zeta)}{z-\zeta} - \sum_{x'} A_{xx'} \frac{E_{x'}(\zeta)}{z+\zeta} \right) |\Omega\rangle \\ &= \int_{\mathcal{C}} d\zeta \frac{\partial}{\partial\zeta} \frac{E_x(\zeta)}{z-\zeta} - \sum_{x'} A_{xx'} \int_{\mathcal{C}} d\zeta \partial\Gamma_x(z) \frac{E_{x'}(\zeta)}{(z+\zeta)} |\Omega\rangle. \end{aligned} \quad (3.30)$$

The first term disappears after multiplying with the left vacuum since the potential is chosen so that it grows to $+\infty$ at the endpoints of \mathcal{C} . The second term is regular in the domain inside \mathcal{C}_- and therefore does not contribute to the integral (3.28).

- *Analytical properties of the collective field*

Arguing as in [9], we expect that in the quasiclassical limit $g_s \rightarrow 0$ the eigenvalues of the matrix \mathbf{M}_x condense on some interval $[-\Lambda, -M]$ on the negative real axis. The endpoints of the interval depend on the couplings T and κ . Up to exponentially small terms the contour \mathcal{C} can be restricted to the negative real axis, and the contour $\bar{\mathcal{C}}$ – to the positive real axis. Then the contour \mathcal{C}_- goes from $-\infty$ to 0 below the real axis and then back to $-\infty$ above the real axis (Fig. 5b).

The classical value of the resolvent

$$W_x^c(z) = g_s \left\langle \text{tr} (z - \mathbf{M}_x)^{-1} \right\rangle \quad (3.31)$$

is a meromorphic function with a branched cut along the eigenvalue interval and asymptotics at infinity as

$$W_x^c(z) \sim g_s \frac{N_x}{z} = S_x \frac{\kappa^2}{z}. \quad (3.32)$$

The classical Virasoro constraints (3.28) imply that $(\partial\Gamma_x)^2$ is analytic in the vicinity of the interval $[-\Lambda, -M]$. Since $\partial\Gamma_x$ is discontinuous across the cut, this means that

$$g_s \partial\Gamma_x^c(z) = \partial V_x(z) - 2W_x^c(z) - \sum_{x'} A_{xx'} W_{x'}^c(-z) \quad (3.33)$$

satisfies the boundary condition

$$\partial\Gamma_x^c(z + i0) + \partial\Gamma_x^c(z - i0) = 0, \quad z \in [-\Lambda, -M]. \quad (3.34)$$

The boundary condition (3.34), together with (3.32) and (3.33), determines uniquely the meromorphic function $W_x^c(z)$.²

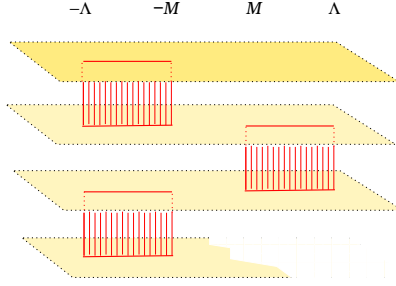


Fig. 6: Riemann surface of the classical solution $W^c(z)$. In the first (physical) sheet there is only one cut $[-\Lambda, -M]$ while all other sheets contain two symmetric cuts.

With the particular potential (3.9), the classical solution must be of the form

$$\Phi_x^c(z) = \frac{S_x}{S_1} \Phi^c(z), \quad \Gamma_x^c(z) = \frac{S_x}{S_1} \Gamma^c(z), \quad W_x^c(z) = \frac{S_x}{S_1} W^c(z) \quad (3.35)$$

and (3.34) becomes a condition for the single function $W^c(z)$:

$$W^c(z + i0) + W^c(z - i0) - 2 \cos \pi p_0 W^c(-z) - \partial V(z) = 0, \quad z \in [-\Lambda, -M]. \quad (3.36)$$

This is the same equation as in the $O(n)$ matrix model with $n = 2 \cos(\pi p_0)$ [25]. The solution of (3.34) for a general polynomial potential can be expressed in terms of Jacobi theta functions [26]. Its explicit form in the case of the cubic potential (3.9) can be found in [27]. The Riemann surface of the function $W^c(z)$ is sketched in Fig. 6.

It is consistent to assume that the quantum field has the same analytic properties as the classical solution and define its mode expansion at infinity in terms of a complete set of functions globally defined on the Riemann surface. This is the basic assumption for the iterative procedure for calculation of the n -loop functions perturbatively in g_s , known as method of moments, [28,26]. Our method can be considered as an operator equivalent of the method of moments. We thus assume that the boundary condition (3.34) actually holds at *operator* level,

$$\begin{aligned} \partial_z [\Phi_x(z + i0) + \Phi_x(z - i0) - \sum_{x'} A_{x,x'} \Phi_{x'}(-z)] &= \partial_z [\Gamma_x(z + i0) + \Gamma_x(z - i0)] \\ &= 0, \quad z \in [-\infty, -M], \end{aligned} \quad (3.37)$$

and should be understood as a condition on the n -point correlation functions of the collective field Φ .

² Note that the boundary condition (3.34) has a natural interpretation in the matrix model. It means that the classical distribution of the eigenvalues is such that the effective potential $\Gamma_x^c(z)$ is constant on the eigenvalue interval. Since the derivative of the effective potential has a discontinuity across the cut, one has to take the half-sum of its values on both sides.

3.3. Continuum limit, phase diagram and critical points

In the ensemble of dynamical triangulations the critical thermodynamics is controlled both by the temperature T and the bare cosmological constant κ . The continuum limit³ is achieved, for given T , at some critical value $\kappa_c(T)$. The singularity along the line $\kappa = \kappa_c(T)$ is due to the contribution of surfaces of diverging area. In the continuum limit, *i.e.*, near the critical line $\kappa = \kappa_c(T)$, the lattice can be approximated by a continuous worldsheet and the theory is described by perturbed Liouville gravity.

The behavior of the theory in the continuum limit depends on the value of T . As in the case of a flat lattice, there are three possible types of critical behavior, characterized by massive, dilute and dense loops [25][29]. The critical line $\kappa = \kappa_c(T)$ consists of two branches:

$$\kappa = \kappa_c(T) = \begin{cases} \kappa_c^I(T), & \text{if } T > T^*; \\ \kappa_c^{II}(T), & \text{if } T < T^*. \end{cases}$$

The first branch $\kappa = \kappa_c^I(T)$ describes high-temperature phase where the area of the graph diverges, while the loops remain finite. Its equation is given by $\partial\kappa/\partial M = 0$. Along this branch the critical behavior of the partition function is that of pure gravity with $c_{\text{matter}} = 0$. The second branch $\kappa = \kappa_c^{II}(T)$ describes the low-temperature phase of densely packed loops. It is determined by $M = 0$. Here the critical behavior is that of a 2D gravity with matter central charge given by (2.14). The two branches meet at the critical point $\kappa = \kappa^*, T = T^*$, where both the area of the graph and the length of the loops diverge. The double scaling limit at this point describes the phase of dilute loops with matter central charge (2.13).

The vicinity of the critical point T^*, κ^* is parametrized by the renormalized cosmological constant μ and the temperature coupling constant t , defined as

$$\mu \sim \kappa^* - \kappa, \quad t \sim T^* - T. \quad (3.38)$$

We will consider the more general case of finite t , where the matter field is not conformal invariant. Then the susceptibility $u = -4p_0\partial_\mu^2 Z$, where Z is the partition function on the sphere without boundaries, satisfies the transcendental equation [27]⁴

$$\mu = \frac{1}{2}u^{\frac{1}{p_0}} + \frac{1}{2-2p_0}tu^{\frac{1-p_0}{p_0}}. \quad (3.39)$$

This equation describes the flow between the dilute ($t = 0$) and the dense ($t \rightarrow \infty$) phases. The dimension of the coupling $t \sim \mu^{p_0}$ matches the gravitational dimension $\delta_{1,3} = 1 - p_0$, obtained from $\Delta_{1,3}$ by the KPZ scaling relation $\Delta = \frac{\delta(\delta+p_0)}{1+p_0}$ [30]. This is consistent with the conjecture that for finite t/μ^{p_0} the theory is described by a perturbation of the critical point $t = 0$ by the Liouville-dressed thermal operator $\Phi_{1,3}$. Eq. (3.39) is discussed in the context of perturbed Liouville gravity by Al. Zamolodchikov [31].

³ We do not call it scaling limit because the theory is scale invariant only at the critical points.

⁴ The normalization factor $4p_0$ in the definition of the susceptibility is fixed by the comparison with the 3-point function, see below.

3.4. The loop operator and the disk partition function in the continuum limit

The critical line $\kappa_c^{II}(T)$ is determined by the condition that the right edge of the eigenvalue distribution (after the shift (3.7)) hits the origin while the left end stays at a finite distance $\Lambda \sim T$. Therefore in the continuum limit $M \ll T$. If we introduce the renormalized length

$$\ell = \frac{2}{T}L, \quad (3.40)$$

then the loop operator $\text{tr} \mathbf{M}_x^L$ can be approximated, after the shift (3.7), by the exponential

$$\text{tr}[(\mathbf{M}_x + \frac{1}{2}T)^L] \approx e^{\frac{1}{2}\ell T \log T} \text{tr}(e^{\ell \mathbf{M}_x}). \quad (3.41)$$

Therefore up to trivial multiplicative factor the operator creating a boundary of length ℓ and height x is

$$W_x(\ell) = g_s \text{tr}(e^{\ell \mathbf{M}_x}). \quad (3.42)$$

The Laplace transform

$$W_x(z) = \int_0^\infty d\ell e^{-z\ell} W_x(\ell) = g_s \text{tr}\left(\frac{1}{z - \mathbf{M}_x}\right) \quad (3.43)$$

is the operator creating a boundary with a marked point with height x and (in general complex) boundary cosmological constant $\mu_B = z$. The n -loop amplitude (3.5) as a function of the boundary heights x_i and the boundary cosmological constants z_i is given by

$$Z(x_1, z_1; \dots; x_n, z_n) = \lim_{g_s \rightarrow 0} g_s^{2-n} \left\langle \left\langle \prod_{i=1}^n \text{tr} \log(z_i - \mathbf{M}_{x_i}) \right\rangle \right\rangle. \quad (3.44)$$

4. The collective field theory as a CFT on a Riemann surface

4.1. The classical solution in the scaling limit

In the following sections we will show how Virasoro conditions (3.28) can be solved directly in the continuum limit $\Lambda \rightarrow \infty$. The solution is completely determined by the classical value of the resolvent, whose explicit form we give below, and the target space \mathbb{X} of the height model. From now on we will concentrate on the SRSOS model,

$$\mathbb{X} = \mathbb{Z}_+, \quad S_x^{(p)} = \sin(\pi p x), \quad S_x = \sin(\pi p_0 x), \quad \psi_p(x) = \frac{\sin(\pi p x)}{\sin(\pi p_0 x)}.$$

We will give an operator solution of the Virasoro conditions (3.28). The n -loop amplitudes will be expressed as Fock space expectation values as in (3.23), but the left and right vacuum stated $\langle B|$ and $|\Omega\rangle$ will be given a new realization in terms of a chiral bosonic field

on this Riemann surface. The basic idea of this approach is that the collective field can be constructed perturbatively in the string coupling g_s as a bosonic field defined on the Riemann surface of the classical solution [13,24]. The operator solution yields a diagram technique similar to the one derived for the ADE matrix models [14].

In the continuum limit the classical solution is written in terms of the uniformization parameter τ by [27]

$$\begin{aligned} z &= M \cosh \tau, \\ \partial_z \Phi^c|_\mu &= -\frac{1}{2g_s} \left(\frac{M^{1+p_0}}{1+p_0} \cosh(1+p_0)\tau - t \frac{M^{1-p_0}}{1-p_0} \cosh(1-p_0)\tau \right), \\ \partial_\mu \Phi^c|_z &= \frac{1}{2g_s} \frac{M^{p_0}}{p_0} \cosh p_0 \tau. \end{aligned} \quad (4.1)$$

The expressions for the two derivatives are compatible if they lead to the same expression for the second derivative. From here we find, using the formulas

$$\partial_\mu = \partial_\mu M \left(\partial_M - \frac{1}{M \tanh \tau} \partial_\tau \right), \quad \partial_z = \frac{1}{M \sinh \tau} \partial_\tau$$

a transcendental equation for the modulus M :

$$2\mu = M^2 + \frac{1}{1-p_0} t M^{2-2p_0}. \quad (4.2)$$

It can be shown [27] that the modular parameter M is related to the string susceptibility u by

$$u = M^{2p_0}, \quad (4.3)$$

which relation implies the equation of state (3.39).

From the solution (4.1) we obtain for the classical effective potential

$$\partial_z \Gamma^c(\tau \pm i\pi) = \pm i \partial_z \varphi^c(\tau), \quad (4.4)$$

where

$$\begin{aligned} \partial_z \varphi^c(\tau) &:= 2 \sin(\pi x \partial_\tau) \partial_z \Phi^c(\tau) \\ &= \frac{\sin(\pi p_0)}{g_s} \left(\frac{M^{1+p_0}}{1+p_0} \sinh(1+p_0)\tau + t \frac{M^{1-p_0}}{1-p_0} \sinh(1-p_0)\tau \right). \end{aligned} \quad (4.5)$$

This function has two cuts, $[-\infty, -M]$ and $[M, \infty]$. The two signs in (4.4) correspond to the value of $\partial \Gamma^c$ above and below the left cut.

The field φ^c has vanishing imaginary part along the interval $z > M$, which implies that $\partial \Gamma_x^c(z)$ has vanishing real part along its left cut $z < -M$. Therefore it is expanded at the branch point $z = -M$ as a series of the half-integer powers of $z + M$. In the following we will need the explicit formula for this expansion,

$$\partial_z \Gamma_x^c(z) = -\frac{1}{M\sqrt{2}g_s} S_x \sum_{n \geq 1} \frac{\mu_n}{(2n-1)!!} \left(1 + \frac{z}{M}\right)^{n-1/2}. \quad (4.6)$$

The coefficients μ_n , $n = 1, 2, \dots$, which we call moments of the classical solution, are given by ⁵

$$\mu_n = 2 \left(M^{2+p_0} F_{n-1}(1+p_0) + t M^{2-p_0} F_{n-1}(1-p_0) \right), \quad (4.7)$$

where

$$F_n(g) := \frac{(\frac{1}{2} + g)_n (\frac{1}{2} - g)_n}{n!} \quad (4.8)$$

and $(y)_n$ is the Pochhammer symbol. In the particular case $p_0 = \frac{1}{2}$ (pure gravity) the series (4.6) truncates to the first two terms.

4.2. The collective field as an operator field on a Riemann surface

In the parametrization $z = M \cosh \tau$ the operator boundary condition (3.37) takes the form of a difference equation with respect to τ . Considered as a function of the uniformization variable τ , the operator field $\Phi_x(\tau) \equiv \Phi_x(z)|_{z=M \cosh \tau}$ must

- i) be entire function of τ ,
- ii) satisfy the orbifold condition

$$\Phi_x(\tau) = \Phi_x(-\tau), \quad (4.9)$$

- iii) satisfy the discrete Laplace equation in the $\{x, \tau\}$ space:

$$\sum_{x'} A_{x,x'} \Phi_{x'}(\tau) = \Phi_x(\tau + i\pi) + \Phi_x(\tau - i\pi). \quad (4.10)$$

The last condition can be conveniently written as a difference equation

$$(\cos \pi \partial_\tau - \cosh \partial_x) \Phi_x(\tau) = 0, \quad (4.11)$$

if the restriction to positive heights is imposed by requiring antisymmetry $\Phi_x = -\Phi_{-x}$.

The condition (4.11) means that the components $\{\Phi_x(z)\}_{x \in \mathbb{Z}_+}$ and $\{\Gamma_x(z)\}_{x \in \mathbb{Z}_+}$ of the collective field can be obtained by analytical continuation from a *single* holomorphic

⁵ The expansion coefficients can be obtained from the standard representation of the hypergeometric function (we use that $\sqrt{\frac{z+M}{2M}} = i \sinh \frac{\tau-i\pi}{2}$):

$$\frac{i \sinh(g(\tau-i\pi))}{2g} = \sqrt{\frac{z+M}{2M}} {}_2F_1\left(\frac{1}{2} + g, \frac{1}{2} - g, \frac{3}{2}; \frac{z+M}{2M}\right) = \frac{1}{\sqrt{2}} \sum_{n \geq 1} \frac{F_{n-1}(g)}{(2n-1)!!} \left(1 + \frac{z}{M}\right)^{n-1/2}.$$

field $\Phi(\tau) := \Phi_1(\tau)$. Indeed, (4.10) is equivalent to the defining functional relation of the Chebyshev polynomials of second kind

$$\begin{aligned}\Phi_x(\tau) &= U_{x-1}(\cos \pi \partial_\tau) \Phi_1(\tau) \\ &= \frac{\sin \pi x \partial_\tau}{\sin \pi \partial_\tau} \Phi(\tau).\end{aligned}\tag{4.12}$$

For the field $\Gamma_x(\tau)$ we have,

$$\begin{aligned}\Gamma_x(\tau \pm i\pi) &= 2\Phi_x(\tau \pm i\pi) - \Phi_{x+1}(\tau) - \Phi_{x-1}(\tau) \\ &= \pm 2i \sin \pi \partial_\tau \Phi_x(\tau) = \pm 2i \sin \pi x \partial_\tau \Phi(\tau) \\ &= \pm i \frac{\sin \pi x \partial_\tau}{\sin \pi \partial_\tau} \varphi(\tau)\end{aligned}\tag{4.13}$$

for real τ . Here we introduced for convenience a second field φ related to Φ by

$$\varphi(\tau) = 2 \sin(\pi \partial_\tau) \Phi(\tau).\tag{4.14}$$

Note that Φ is even, $\Phi(\tau) = \Phi(-\tau)$, while φ is odd, $\varphi(\tau) = -\varphi(-\tau)$.

4.3. Mode expansion and two-point correlator

We write the operator field Φ as a sum of positive and negative frequency parts,

$$\Phi = \Phi^+ + \Phi^-, \tag{4.15}$$

so that Φ^- as a function of z is analytic on the Riemann surface with the point $z = \infty$ removed, and Φ^+ is analytic in the vicinity of the point $z = \infty$. The two pieces are given, as functions of τ , by the spectral integrals

$$\begin{aligned}\Phi^-(\tau) &= \int_0^\infty \frac{d\nu}{\nu} \cosh \nu \tau \Phi_\nu^-, \\ \Phi^+(\tau) &= - \int_0^\infty \frac{d\nu}{\nu} e^{-\nu \tau} \Phi_\nu^+, \quad \text{Re}(\tau) > 0.\end{aligned}\tag{4.16}$$

Similarly, for the quantum field $\varphi(\tau)$ we find, using (4.14), the mode expansion

$$\begin{aligned}\varphi^-(\tau) &= \int_0^\infty d\nu \frac{2 \sin \pi \nu}{\nu} \sin \nu \tau \Phi_\nu^-, \\ \varphi^+(\tau) &= - \int_0^\infty d\nu \frac{2 \sinh \pi \nu}{\nu} e^{-\nu \tau} \Phi_\nu^+, \quad \text{Re}(\tau) > 0.\end{aligned}\tag{4.17}$$

The operator amplitudes Φ_ν^\pm are assumed to satisfy the canonical commutation relations

$$[\Phi_\nu^+, \Phi_{\nu'}^-] = \nu \delta(\nu - \nu') \tag{4.18}$$

and the left and right vacuum states are defined by

$$\langle 0_\infty | \Phi^- = 0, \quad \Phi^+ | 0_\infty \rangle = 0. \quad (4.19)$$

With this definition of the left and right vacuum states the field has zero expectation value. The expectation value (4.1) can be generated by replacing the left vacuum by an appropriate coherent state. The two-point correlator is given, for $\text{Re}\tau > \text{Re}\tau'$, by⁶

$$\begin{aligned} \langle 0_\infty | \Phi(\tau)\Phi(\tau') | 0_\infty \rangle &= \frac{1}{2} \log(\tau^2 - \tau'^2), \\ \langle 0_\infty | \varphi(\tau)\varphi(\tau') | 0_\infty \rangle &= 2 \sin \pi \partial_\tau \sin \pi \partial_{\tau'} \log(\tau^2 - \tau'^2), \end{aligned} \quad (4.20)$$

where we neglected an infinite constant term. From here we evaluate the two-point functions of the original field Γ_x and Φ_x :

$$\begin{aligned} \langle \Phi_x(\tau)\Phi_{x'}(\tau') \rangle &= \frac{\sin(\pi x \partial_\tau) \sin(\pi x' \partial_{\tau'})}{\sin(\pi \partial_\tau) \sin(\pi \partial_{\tau'})} \langle 0_\infty | \Phi(\tau)\Phi(\tau') | 0_\infty \rangle \\ &= \sum_{m=1-x}^{x-1} \sum_{m'=1-x'}^{x'-1} \langle 0_\infty | \Phi(\tau + i\pi m)\Phi(\tau' + i\pi m') | 0_\infty \rangle, \end{aligned} \quad (4.21)$$

$$\begin{aligned} \langle \Gamma_x(\tau + i\pi)\Gamma_{x'}(\tau' + i\pi) \rangle &= -\frac{\sin(\pi x \partial_\tau) \sin(\pi x' \partial_{\tau'})}{\sin(\pi \partial_\tau) \sin(\pi \partial_{\tau'})} \langle 0_\infty | \varphi(\tau)\varphi(\tau') | 0_\infty \rangle \\ &= -4 \sin(\pi x \partial_\tau) \sin(\pi x' \partial_{\tau'}) \langle 0_\infty | \Phi(\tau)\Phi(\tau') | 0_\infty \rangle. \end{aligned} \quad (4.22)$$

The two-point correlation functions (4.21) and (4.22) are diagonalized in the momentum space,

$$\Phi(\tau, p) = \sum_{x \in \mathbb{Z}_+} \sin \pi p x \Phi_x(\tau), \quad \varphi(\tau, p) = \pm \frac{1}{i} \sum_{x \in \mathbb{Z}_+} \sin \pi p x \Gamma_x(\tau \pm i\pi). \quad (4.23)$$

The two fields are related by (4.14),

$$\varphi(\tau, p) = 2 \sin \pi \partial_\tau \Phi(\tau, p). \quad (4.24)$$

Performing the sum in x in (4.23) we obtain the mode expansion of $\Phi(\tau, p)$ and $\varphi(\tau, p)$, which is a restriction of the spectral integral (4.16) to a discrete sum over the momenta $\nu = \pm p \pmod{2}$. For $\varphi(\tau, p)$ we find

$$\varphi(\tau, p) = \sum_{n \in \mathbb{Z}} \frac{1}{p + 2n} \left(\Phi_{|p+2n|}^- \sinh(|p + 2n|\tau) + \Phi_{|p+2n|}^+ e^{-|p+2n|\tau} \right). \quad (4.25)$$

⁶ The normalization of the field was fixed by the condition that short distance behavior at $\tau = \tau'$ is compatible with (3.11) and (3.14).

The two-point function of $\varphi(p, \tau)$ is, for $\text{Re}(\tau - \tau') > 0$,

$$\langle 0_\infty | \varphi(p, \tau) \varphi(p', \tau') | 0_\infty \rangle = \sum_{n \in \mathbb{Z}} \frac{e^{-|p+2n|(\tau-\tau')} - e^{-|p+2n|(\tau+\tau')}}{|p+2n|} \delta^{(2)}(p-p'). \quad (4.26)$$

Here and below $\delta^{(2)}$ denotes the periodic δ -function

$$\delta^{(2)}(p) := \sum_{n \in \mathbb{Z}} \delta(p+2n). \quad (4.27)$$

The time-ordered correlator

$$\begin{aligned} \delta^{(2)}(p-p') G(\tau, \tau'; p) &= \theta[\text{Re}(\tau - \tau')] \langle 0_\infty | \varphi(p, \tau) \varphi(p', \tau') | 0_\infty \rangle \\ &\quad + \theta[\text{Re}(\tau' - \tau)] \langle 0_\infty | \varphi(p', \tau') \varphi(p, \tau) | 0_\infty \rangle \end{aligned} \quad (4.28)$$

is further diagonalized as

$$\int_0^\infty d\tau d\tau' \sin(E\tau) \sin(E'\tau') G(\tau, \tau'; p) = 2\pi \delta(E-E') G(E, p), \quad (4.29)$$

$$G(E, p) = \sum_{n \in \mathbb{Z}} \frac{1}{E^2 + (p+2n)^2} = \frac{\pi \sinh \pi E}{E} \frac{1}{2 \cosh \pi E - 2 \cos \pi p}. \quad (4.30)$$

Finally, using the relations (4.23) and (4.24) we find the spectral integral for the two-point correlator (4.21):

$$\begin{aligned} \langle \Phi_{x_1}(\tau_1) \Phi_{x_2}(\tau_2) \rangle &= \frac{2}{\pi} \int_0^\infty dE \int_0^1 dp \cos E\tau_1 \cos E\tau_2 \sin \pi p x_1 \sin \pi p x_2 A(E, p) \\ A(E, p) &= \frac{\pi}{E \sinh \pi E} \frac{1}{2 \cosh \pi E - 2 \cos \pi p}. \end{aligned} \quad (4.31)$$

This is the well known expression for the annulus amplitude in momentum space [32,33,9]. The collective field $\Phi_x(\tau)$ creates Cardy type boundary state $|\tau, x\rangle$. The intermediate states in the spectral integral are Ishibashi states are characterized by two quantum numbers, the Liouville energy E and matter momentum p .

The spectral decomposition (4.31) is valid also for ADE string theories, with the integral over momenta replaced by a discrete sum over Coxeter exponents. The annulus amplitude for the A -series, or RSOS models coupled to gravity, was derived using the world sheet CFT approach in [34], see also [35] for an expression analogous to (4.21).

5. Operator solution of Virasoro constraints on the Riemann surface

The idea of the operator solution of Virasoro constraints (3.28) is that it is possible to introduce another Fock space, associated with the mode expansion of the fields $\Gamma_x(z)$ near the branch point $z = -M$. Then the problem reduces in the scaling limit to the already solved problem of perturbed $c_{\text{matter}} = -2$ gravity [36], where the operator solution of Virasoro constraints has been given in terms of a twisted boson. In this section we first define the mode expansion of the field Γ_x in the half-integer powers of $z + M$ and the associated bare Fock vacua, $\langle 0_{\text{tw}}|$ and $|0_{\text{tw}}\rangle$. Then we write down the solution of Virasoro constraints in terms of the corresponding oscillator modes. This will lead to the operator replacing the right vacuum $|\Omega\rangle$. Finally, we will construct the operator replacing the coherent state $\langle B|$. As a result we will obtain a diagram technique similar to the one obtained in [14] for the *ADE* models.

5.1. Mode expansion at the branch points $z = -M$

In the following we will use also the rescaled string interaction constant and moments defined by

$$\hat{g}_s = \frac{g_s}{\mu_1}, \quad \hat{\mu}_n = \frac{\mu_n}{\mu_1}. \quad (5.1)$$

For each x , the operator field $\Gamma_x(z)$ behaves near the branch point $z = -M$ as a twisted boson. Its mode expansion is given by a series of half-integer powers of $z + M = 2M \cosh^2 \frac{\tau}{2}$, now positive and negative,

$$\begin{aligned} \partial_z \Gamma_x(z) &= \frac{1}{M\sqrt{2}} \sum_{n \geq 0} (a_{n,x}^\dagger - \delta_{n,1} \frac{S_x}{\hat{g}_s}) \frac{(1 + \frac{z}{M})^{n-\frac{1}{2}}}{(2n-1)!!} \\ &+ \frac{1}{M\sqrt{2}} \sum_{n \geq 0} a_{n,x} (2n+1)!! (1 + \frac{z}{M})^{-n-\frac{3}{2}}. \end{aligned} \quad (5.2)$$

In this expansion we dropped all terms in the expansion of the classical value (4.6) except the first one, proportional to μ_1 .⁷ The rest of the classical background will be later reintroduced as a perturbation.

The operators $a_{n,x}$ and $a_{n,x}^\dagger$ and the twisted left and right Fock vacua satisfy

$$[a_{n,x}, a_{n',x'}^\dagger] = \delta_{n,n'} \delta_{x,x'}. \quad (5.3)$$

$$\langle 0_{\text{tw}}| a_{n,x}^\dagger = 0, \quad a_{n,x} |0_{\text{tw}}\rangle = 0 \quad (n \geq 0, x \in \mathbb{Z}_+). \quad (5.4)$$

⁷ If we had dropped the whole classical field, Virasoro constraints would become singular and would have no solution [36].

The vacuum state $|0_{\text{tw}}\rangle$ can be thought of as the direct product of an infinite number of twist operators associated with the branch points of the Riemann surface [21,24]. The two-point function of the field $\Gamma_x(z)$ in the twisted vacuum is

$$\langle 0_{\text{tw}} | \Gamma_x(z) \Gamma_{x'}(z') | 0_{\text{tw}} \rangle = \delta_{x,x'} \ln \frac{\cosh \frac{\tau}{2} - \cosh \frac{\tau'}{2}}{\cosh \frac{\tau}{2} + \cosh \frac{\tau'}{2}}. \quad (5.5)$$

The creation operators $a_{k,x}^\dagger$ can be expressed by the generating function

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} (a_{n,x}^\dagger - \delta_{n,1} \frac{S_x}{\hat{g}_s}) = \sqrt{2} \oint_{\mathcal{C}_-} \frac{dz}{2\pi i} \frac{\partial_z \Gamma_x(z)}{\sqrt{1 + \frac{z}{M} - 2u}}. \quad (5.6)$$

where the contour \mathcal{C}_- encircles the cut $[-\infty, -M]$ of $\Gamma_x(z)$. Due to the vanishing of the real part of Γ_x along this cut the contour reduces to a closed circle around $-M$.

We will actually need the expansion of the field $\varphi(\tau, p)$, which is written in terms of the Fourier transformed creation and annihilation operators

$$\begin{aligned} a_k(p) &= \sum_{x \in \mathbb{Z}_+} \sin(\pi p x) a_{k,x} \\ a_k^\dagger(p) &= \sum_{x \in \mathbb{Z}_+} \sin(\pi p x) a_{k,x}^\dagger, \end{aligned} \quad (5.7)$$

where the canonical commutation relations have the form

$$[a_n(p), a_{n'}^\dagger(p')] = \delta_{n,n'} \delta^{(2)}(p - p'). \quad (5.8)$$

Then the twisted propagator in momentum space is given by replacing $\delta_{x,x'} \rightarrow \delta^{(2)}(p - p')$ in (5.5).

The formula (5.6) can be used to invert the expansion (5.2) for the creating operators, see Appendix B. We have in momentum space the operator identity

$$a_k^\dagger(p) - \delta_{k,1} \hat{g}_s^{-1} \delta^{(2)}(p - p_0) = F_k(\partial_\tau) \partial_\tau \varphi^\dagger(\tau, p) \Big|_{\tau=0} \quad (5.9)$$

where φ^\dagger is given by the Fourier transform of the first sum in (5.2) and the differential operator $F_k(\partial_\tau)$ is defined by (4.8), with $g \rightarrow \partial_\tau$,

$$F_n(\partial_\tau) := \frac{(\frac{1}{2} + \partial_\tau)_n (\frac{1}{2} - \partial_\tau)_n}{n!}.$$

We shall also use a weaker version of (5.9) also derived from (5.6) in which the negative mode part in (4.15) appears in the r.h.s.,

$$(a_k^\dagger(p) - \delta_{k,1} \hat{g}_s^{-1} \delta^{(2)}(p - p_0)) | 0_\infty \rangle = F_k(\partial_\tau) \partial_\tau \varphi^-(\tau, p) \Big|_{\tau=0} | 0_\infty \rangle. \quad (5.10)$$

5.2. Operator solution of Virasoro constraints on the Riemann surface

The right physical vacuum state will be constructed from the modes of the Z_2 -twisted boson. The conformal invariant physical vacuum $|\Omega\rangle$ must be of the form

$$|\Omega\rangle = \prod_{x \in \mathbb{Z}_+} \Omega_x |0_{\text{tw}}\rangle \quad (5.11)$$

where the operator Ω_x represents a formal series expansion in terms of the creation operators $a_{n,x}^\dagger$. The coefficients of the series are fixed by Virasoro constraints (3.28), where the stress-energy tensor (3.29) is expressed in terms of the oscillator modes associated with the expansion (5.2),

$$T_x(z) = \frac{1}{2} \circ [\partial \Gamma_x(z)]^2 \circ = \frac{1}{M^2} \sum_n L_{n,x} \left(1 + \frac{z}{M}\right)^{-n-2}, \quad (5.12)$$

where $\circ \circ$ means the normal ordering with respect to the mode expansion (5.2). We write down explicitly the first two of the generators L_n , $n \geq -1$,

$$\begin{aligned} L_{-1,x} &= \frac{1}{2} \left(\sum_{m=0} (a_{m+1,x}^\dagger - \delta_{m,0} \frac{S_x}{\hat{g}_s}) a_{m,x} + \frac{1}{2} (a_{0,x}^\dagger)^2 \right), \\ L_{0,x} &= \frac{1}{2} \left(\sum_{m=0} (2m+1) (a_{m,x}^\dagger - \delta_{m,1} \frac{S_x}{\hat{g}_s}) a_{m,x} \right) + \frac{1}{16}. \end{aligned} \quad (5.13)$$

We are looking for a perturbative in \hat{g}_s solution of the conditions of conformal invariance

$$L_{n,x} |\Omega\rangle = 0, \quad n \geq -1,$$

of the right physical vacuum state (5.11). The problem is identical (for each x) to the one of solving Virasoro constraints in pure gravity. The spectrum of operators in this theory is described by the flows of the KdV hierarchy [36,37]. An important property of the theory is that there is a conserved charge (ghost number) with a distinct background charge for each genus, so a specific correlation function can be non-zero for at most one genus. The explicit form of the operator Ω_x is

$$\Omega_x = \left(\frac{S_x}{\hat{g}_s} \right)^{-\frac{1}{24}} \exp \left(\sum_{n,g \geq 0} \left(\frac{S_x}{\hat{g}_s} \right)^{2-2g-n} \sum_{k_1 + \dots + k_n = n+3g-3} \frac{\langle \tau_{k_1} \dots \tau_{k_n} \rangle_g}{n!} a_{k_1,x}^\dagger \dots a_{k_n,x}^\dagger \right) \quad (5.14)$$

where $\langle \tau_{k_1} \dots \tau_{k_n} \rangle_g$ is the genus g correlation function in the topological ($c_{\text{matter}} = -2$) gravity.

Note: The generating function of these correlation functions can be written as

$$\begin{aligned} Z_{\text{tw}}(t) &:= \langle 0_{\text{tw}} | e^{\sum_{k \geq 0} t_{k,x} a_{k,x}} \Omega_x | 0_{\text{tw}} \rangle \\ &= \left(\frac{S_x}{\hat{g}_s} \right)^{-\frac{1}{24}} \exp \left(\sum_{n,g \geq 0} \left(\frac{S_x}{\hat{g}_s} \right)^{2-2g-n} \sum_{k_1 + \dots + k_n = n+3g-3} \frac{\langle \tau_{k_1} \dots \tau_{k_n} \rangle_g}{n!} t_{k_1} \dots t_{k_n} \right), \end{aligned} \quad (5.15)$$

$$\begin{aligned} \left(\frac{S_x}{\hat{g}_s} \right)^{2-2g-n} \langle \tau_{k_1} \dots \tau_{k_n} \rangle_g &= \frac{\partial^n}{\partial t_{k_1} \dots \partial t_{k_n}} \log \mathcal{Z}_x(t) |_{t=0} \\ &= \langle 0_{\text{tw}} | a_{k_1,x} \dots a_{k_n,x} \Omega_x | 0_{\text{tw}} \rangle |_{\text{conn}}. \end{aligned} \quad (5.16)$$

The Virasoro generators are realized as differential operators, i.e., $a_{k,x}^\dagger \rightarrow t_k$, $a_{k,x} \rightarrow \partial_{t_k}$. The formulas in [38] differ from (5.13) and (5.15), (5.16) by the rescaling $S_x/\hat{g}_s \rightarrow 1$.

We will restrict ourselves to the genus zero correlation functions and consider only the piece with $g = 0$ in the sum in (5.14), for which [39,38]

$$\langle \tau_{m_1} \dots \tau_{m_n} \rangle_0 = \frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!}, \quad m_1 + \dots + m_n = n - 3. \quad (5.17)$$

In momentum space we have, neglecting the overall constant,

$$|\Omega\rangle = \exp \left(\hat{g}_s^{n+2g-2} \sum_{n \geq 3} \frac{1}{n!} \int_{-1}^1 dp_1 \dots dp_n \sum_{k_1 + \dots + k_n = n-3} V_{k_1 \dots k_n}(p_1, \dots, p_n) a_{k_1}^\dagger(p_1) \dots a_{k_n}^\dagger(p_n) \right) |0_{\text{tw}}\rangle \quad (5.18)$$

with

$$V_{k_1 \dots k_n}(p_1, \dots, p_n) = \frac{(k_1 + \dots + k_n)!}{k_1! \dots k_n!} N(p_1, \dots, p_n), \quad (5.19)$$

$$N_{p_1, \dots, p_n} = \sum_{x \in \mathbb{Z}_+} S_x^2 \psi_{p_1}(x) \dots \psi_{p_n}(x). \quad (5.20)$$

5.3. Classical background and two-point correlator in terms of twisted bosons

The gaussian field $\partial \Gamma_x(\tau)$ is completely characterized by its vacuum expectation value (4.7) and its two-point correlator (4.22). In terms of the mode expansion (4.17), the left physical vacuum $\langle B|$ in (3.21) can be expressed as the coherent state associated with the vacuum $\langle 0_\infty|$ that generates a vacuum expectation value. Now we want to express it in terms of the mode expansion (5.2) and construct $\langle B|$ as a perturbation of the left twisted vacuum $\langle 0_{\text{tw}}|$. In this case the perturbation is not just a coherent state but rather Bogolyubov transformation, which changes also the two-point correlator.

$$\langle B| = \langle 0_{\text{tw}} | \exp \left(\frac{1}{2} \sum_{k,k' \geq 0} \sum_{x,x' \in \mathbb{Z}_+} D_{k,k'}^{x,x'} a_{k,x} a_{k',x'} - \frac{S_x}{\hat{g}_s} \sum_{k \geq 2} \sum_{x \in \mathbb{Z}_+} \hat{\mu}_k S_x a_{k,x} \right). \quad (5.21)$$

In momentum space with

$$\delta^{(2)}(p - p') D_{kn}(p) = \sum_{x \in \mathbb{Z}_+} \sin(\pi p x) \sin(\pi p' x') D_{kn}^{x,x'} \quad (5.22)$$

(5.21) is rewritten as

$$\langle B | = \langle 0_{\text{tw}} | \exp \left(\frac{1}{2} \sum_{k,k' \geq 0} \int_{-1}^1 dp D_{k,k'}(p) a_k(p) a_{k'}(p) - \frac{1}{\hat{g}_s} \sum_{k \geq 2} \hat{\mu}_k a_k(p_0) \right). \quad (5.23)$$

The left vacuum defined in this way has the properties

$$\langle B | (a_{n,x}^\dagger - \delta_{n,1} \frac{S_x}{\hat{g}_s}) | 0_{\text{tw}} \rangle = -\hat{\mu}_n \frac{S_x}{\hat{g}_s}, \quad (5.24)$$

$$\langle B | (a_{n,x}^\dagger - \delta_{n,1} \frac{S_x}{\hat{g}_s}) (a_{n',x'}^\dagger - \delta_{n',1} \frac{S_{x'}}{\hat{g}_s}) | 0_{\text{tw}} \rangle = \hat{\mu}_n \hat{\mu}_{n'} \frac{S_x S_{x'}}{\hat{g}_s^2} + D_{n,n'}^{x,x'}. \quad (5.25)$$

The first of these identities is equivalent to

$$\langle B | \Gamma_x(z) | 0_{\text{tw}} \rangle = \langle 0_\infty | \Gamma_x(z) | 0_\infty \rangle + \Gamma_x^c(z) = \Gamma_x^c(z) \quad (5.26)$$

The unknown kernel $D_{k,k'}^{x,x'}$ in the second identity (5.25) is determined requiring that ⁸

$$\langle B | \Gamma_x(z) \Gamma_{x'}(z') | 0_{\text{tw}} \rangle = \langle 0_\infty | \Gamma_x(z) \Gamma_{x'}(z') | 0_\infty \rangle + \Gamma_x^c(z) \Gamma_{x'}^c(z'). \quad (5.27)$$

From the expansion (5.2) we obtain

$$\langle B | \Gamma_x^\dagger(z) \Gamma_{x'}^\dagger(z') | 0_{\text{tw}} \rangle = \langle 0_\infty | \Gamma_x(z) \Gamma_{x'}(z') | 0_\infty \rangle - \langle 0_{\text{tw}} | \Gamma_x(z) \Gamma_{x'}(z') | 0_{\text{tw}} \rangle + \Gamma_x^c(z) \Gamma_{x'}^c(z') \quad (5.28)$$

where the twisted 2-point function is given in (5.5).

To calculate the kernel $D_{nn'}(p)$ in momentum space, we apply the operator identity (5.9) to the Fourier image of (5.28) and compare with the Fourier image of (5.25). As a result we obtain an expression for $D_{kk'}(p)$ in terms of the difference of the 2-point functions in (5.28). For the Fourier transform (5.22) the relation reads

$$D_{kk'}(p) = \partial_\tau \partial_{\tau'} F_k(\partial_\tau) F_{k'}(\partial_{\tau'}) D(\tau, \tau', p) |_{\tau=0=\tau'}, \quad (5.29)$$

⁸ This identity as well as (5.26) is consistent with the qualitative identification of the left vacuum $\langle B |$ with $\langle 0_\infty | e^{c\Phi_g^+}$, where the positive mode $\Phi_{\nu=g}^+$ generates the classical part $\Phi^c(\tau)$ of $\Phi^-(\tau)$. On the other hand we can identify the right vacua $|0_{\text{tw}}\rangle$ and $|0_\infty\rangle$.

where

$$\begin{aligned} \delta^{(2)}(p-p') D(\tau, \tau'; p) &= \langle 0_\infty | \varphi(\tau, p) \varphi(\tau', p') | 0_\infty \rangle \\ &\quad - \delta^{(2)}(p-p') \langle 0_{\text{tw}} | \varphi(\tau) \varphi(\tau') | 0_{\text{tw}} \rangle \end{aligned} \quad (5.30)$$

It is useful to note that the second term in (5.30) is equal to the first term at $p = \frac{1}{2}$:

$$\langle 0_{\text{tw}} | \varphi(\tau) \varphi(\tau') | 0_{\text{tw}} \rangle = G(\tau, \tau', p = \frac{1}{2}) = -\ln \frac{\sinh \frac{\tau}{2} - \sinh \frac{\tau'}{2}}{\sinh \frac{\tau}{2} + \sinh \frac{\tau'}{2}}. \quad (5.31)$$

The derivative of the first term in (5.30) computed from (4.26) takes, up to the δ -function factor, the simple form

$$\partial_\tau \langle 0_\infty | \varphi(\tau, p) \varphi(\tau', p) | 0_\infty \rangle = \frac{\cosh(|p| - 1)(\tau + \tau')}{\sinh(\tau + \tau')} - \frac{\cosh(|p| - 1)(\tau - \tau')}{\sinh(\tau - \tau')}. \quad (5.32)$$

The difference in (5.30) is smooth everywhere and can be expanded in Taylor series at the point $\tau = \tau' = 0$:

$$\begin{aligned} \partial_\tau D(\tau, \tau'; p) &= \partial_\tau (|p| - \frac{1}{2})(|p| - \frac{3}{2})\tau\tau' \left[1 + \frac{1}{12}(p^2 - 2|p| - \frac{3}{4})(\tau^2 + \tau'^2) \right. \\ &\quad \left. + \frac{1}{3}(p^4 - 4|p|^3 + \frac{5}{4}p^2 + \frac{11}{2}|p| + \frac{33}{16})(\frac{1}{36}\tau^2\tau'^2 + \frac{1}{120}(\tau^4 + \tau'^4)) + \dots \right], \end{aligned} \quad (5.33)$$

e.g.,

$$\begin{aligned} D_{00}(p) &= (|p| - \frac{1}{2})(|p| - \frac{3}{2}), \\ D_{01}(p) &= -\frac{1}{2}(|p| - \frac{1}{2})(|p| - \frac{3}{2})(|p| + \frac{1}{2})(|p| - \frac{5}{2}), \\ D_{11}(p) &= \frac{1}{3}(|p| - \frac{1}{2})(|p| - \frac{3}{2})(|p| + \frac{1}{2})(|p| - \frac{3}{2})(p^2 - 2p - \frac{9}{4}), \quad \text{etc.} \end{aligned} \quad (5.34)$$

5.4. Diagram technique for the correlation functions

Now we are in a position to compute of the n -loop amplitudes. The scaling limit of the amplitudes (3.44) is described as in (3.24) by the positive mode part of the field $\Phi(z)$. Thus the n -point function for $n \geq 3$ will be defined as

$$\langle \Phi(\tau_1, p_1) \dots \Phi(\tau_n, p_n) \rangle_{\text{genus zero}} = g_s^{2-n} \langle B | \Phi^+(\tau_1, p_1) \dots \Phi^+(\tau_n, p_n) | \Omega \rangle. \quad (5.35)$$

The field $\Phi^+(\tau, p)$ is the Fourier transform of the loop creation operator which is assumed to annihilate the twisted vacuum $\Phi^+(\tau, p) | 0_{\text{tw}} \rangle = 0$.

In order to evaluate the n -point amplitudes (5.35) we will also need the commutation relation between the loop operator $\Phi^+(p, \tau)$ and the creation operators a_n^\dagger . This can be done using the linear relation (5.10) between a_k^\dagger and $\varphi^-(p, \tau)$ and the expression for the commutator of φ^+ and φ^- , which is equal to the two-point correlator (4.26). Using the expression (5.32) for the τ -derivative of the propagator, we find

$$\langle 0_\infty | \varphi(\tau, p) a_k^\dagger(p) | 0_\infty \rangle = 2 \frac{(\frac{1}{2} - \partial_\tau)_k (\frac{1}{2} + \partial_\tau)_k}{k!} \frac{\cosh(1 - |p|)\tau}{\sinh \tau}. \quad (5.36)$$

Then from the relation (4.24) we get

$$F_k(p, \tau) = \left\langle 0_\infty | \Phi^+(\tau, p) a_k^\dagger(p) | 0_\infty \right\rangle = \frac{(\frac{1}{2} - \partial_\tau)_k (\frac{1}{2} + \partial_\tau)_k}{k!} \frac{\sinh(1 - |p|)\tau}{\sin \pi |p| \sinh \tau}. \quad (5.37)$$

We would like to write the leg factors directly for the local operators obtained by shrinking the boundaries to punctures. For this purpose we expand the loop operator Φ^+ as a power series in z . The allowed powers are of the form $z^{-\nu}$ with $\nu = |p + 2n|$, $n \in \mathbb{Z}$. The corresponding amplitudes create local operators on the world sheet with gravitational scaling dimensions $\frac{1}{2}(\nu - p_0)$. Therefore the leg factors for the local operators are given by the coefficients in the expansion of the leg factors for loops,

$$F_k(p, \tau) = \sum_{\nu=|p+2n|, n \in \mathbb{Z}} F_k(\nu) (z/M)^{-\nu}. \quad (5.38)$$

The coefficient corresponding to $\nu = p$ is obtained by replacing the last factor on the r.h.s. of (5.37) with M^p and substituting the derivative ∂_τ^2 by $\nu^2 = p^2$. Thus the n -tachyon correlation function $G(p_1, \dots, p_n)$ is obtained from the n -loop amplitude (5.35) by replacing

$$F_k(\tau, p) \rightarrow M^p F_k(p) = M^p \frac{(\frac{1}{2} - p)_k (\frac{1}{2} + p)_k}{k!}. \quad (5.39)$$

These are the leg factors for the order operators with $|p| < 1$.

Now we are able to evaluate the n -point function $G(p_1, \dots, p_n)$ of order operators by performing the necessary number of Wick contractions in (5.35). This amounts to a sum of Feynman diagrams composed by vertices, propagators, tadpoles and leg factors. The leg factors come from the commutators (5.37), while the tadpoles are associated with the action of a^\dagger on $\langle B |$ as in (5.24). The tadpoles are proportional to the normalized moments $\hat{\mu}_n = \mu_n/\mu_1$ and depend on t and μ through the dimensionless coupling

$$\hat{t} = \frac{t}{M^{2p_0}}. \quad (5.40)$$

The first moment μ_1 , eqn. (4.7), then reads

$$\mu_1 = 2M^{2+p_0}(1 + \hat{t}). \quad (5.41)$$

Since the string coupling constant enters only through the ratio $\hat{g}_s = g_s/\mu_1$, the terms retained in the perturbative expansion of an n -point function are only those with an overall factor μ_1^{n-2} . The vertices can be attached either to tadpoles or to the propagators or to leg factors. We summarize the Feynman rules in Fig. 7.

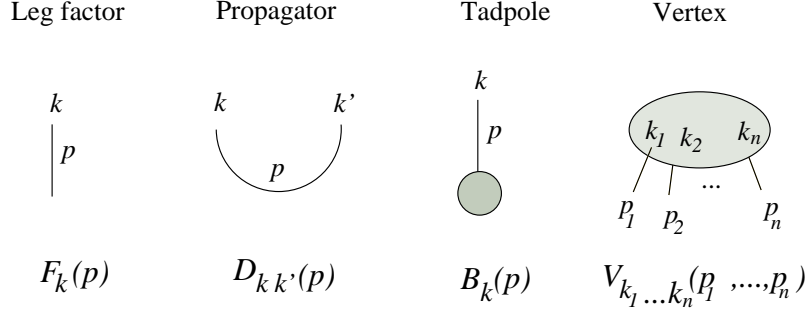


Fig. 7 : Feynman rules for the correlation functions

Feynman rules:

- External line factors (leg factors) $F_k(p)$, where

$$\begin{aligned} F_0(p) &= 1, \quad F_1(p) = \frac{1}{4} - p^2, \quad \dots, \\ F_k(p) &= \frac{(\frac{1}{2} - p)_k (\frac{1}{2} + p)_k}{k!}, \quad \dots \end{aligned} \tag{5.42}$$

- Propagator $D_{k,k'}(p)$:

$$\begin{aligned} D_{00}(p) &= (|p| - \frac{1}{2})(|p| - \frac{3}{2}) = -F_1(1 - |p|), \\ D_{01}(p) &= -F_2(1 - |p|) = D_{10}(p), \\ D_{11}(p) &= -2F_2(1 - |p|) - 2F_3(1 - |p|), \quad \dots \end{aligned} \tag{5.43}$$

- Tadpole $B_k(p)$:

$$\begin{aligned} B_0(p) &= B_1(p) = 0, \\ B_n(p) &= -\delta^{(2)}(p - p_0) \hat{\mu}_n; \\ \hat{\mu}_n &= \frac{F_{n-1}(1 + p_0) + \hat{t} F_{n-1}(1 - p_0)}{1 + \hat{t}}, \quad n \geq 2. \end{aligned} \tag{5.44}$$

- Vertices $V_{k_1, \dots, k_n}(p_1, \dots, p_n)$:

$$V_{k_1, \dots, k_n}(p_1, \dots, p_n) = \frac{(k_1 \dots + k_n)!}{k_1! \dots k_n!} N_{p_1, \dots, p_n}. \tag{5.45}$$

- An overall factor

$$\hat{g}_s^{n-2} \prod_i M^{p_i} = g_s^{n-2} \mu_1^{2-n} M^{p_1+\dots+p_n}. \quad (5.46)$$

We derived the Feynman rules for the case of the SRSOS model, where the momenta belong to the interval $0 < p < 1$. One can derive a similar set of Feynman rules for the n -point functions in the SOS model, where the momentum interval is $-1 < p < 1$. In both cases the multiplicities N_{p_1, \dots, p_n} are given by

$$N_{p_1, \dots, p_n} = \sum_{x \in \mathbb{X}} S_x^{2-n} S_x^{(p_1)} \dots S_x^{(p_n)}, \quad (5.47)$$

where $S_x^{(p)}$ is defined by (2.12).

The multiplicities (5.47) are periodic in $p \rightarrow p + 2$ and, in the case of the SRSOS model, also antisymmetric in $p \rightarrow -p$. The propagator $D_{k, k'}(p)$ is the same for both models. It has the symmetries $D_{k, k'}(p) = D_{k, k'}(-p) = D_{k, k'}(2 \pm p)$. The propagator and the vertices can be defined outside the interval $|p| < 1$ by periodicity in $p \rightarrow p + 2$.

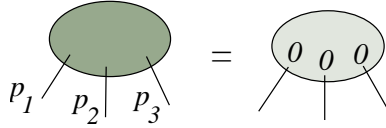


Fig. 8: The only Feynman diagram for the three-point function

The simplest example is the 3-point function (Fig. 8). The only term of order g_s in the exponent in $|\Omega\rangle$ which contributes to this correlator is the one with $\langle \tau_0 \tau_0 \tau_0 \rangle$, i.e., $k_1 = 0 = k_2 = k_3$. The corresponding leg factors for the local operators are $F_0(p_i) = 1$ and we obtain

$$G(p_1, p_2, p_3) = \frac{M^{p_1+p_2+p_3}}{\mu_1} V_{k_1, k_2, k_3}(p_1, p_2, p_3) = \frac{M^{p_1+p_2+p_3-2-p_0}}{1 + tM^{-2p_0}} N(p_1, p_2, p_3). \quad (5.48)$$

The 3-point function is proportional to the 3-point fusion multiplicity. In the case $p_1 = p_2 = p_3 = p_0$ it is equal to the third derivative of the partition function on the sphere. Indeed,

$$-\partial_\mu^3 Z = \frac{1}{4p_0} \partial_\mu u = \frac{M^{3p_0}}{\mu_1}.$$

This justifies the normalization of the susceptibility, $u = -4p_0 \partial_\mu^2 Z = M^{2p_0}$. To compare with the results from the world sheet CFT, we should adjust the normalizations of the couplings μ and t , as well as the one of the order operators.

6. The 4-point function in the SOS and SRSOS models

6.1. General formula for the 4-point function

The 4-point function is given by the sum of the three Feynman diagrams shown in Fig. 9. Each Feynman diagram stands for the sum of terms that differ by permutations of the external legs.

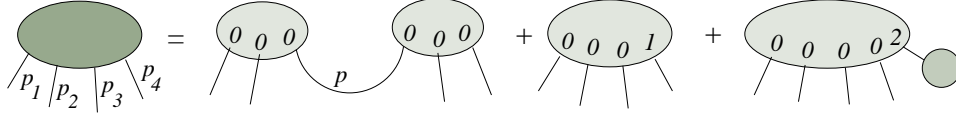


Fig. 9: The diagrams for the 4-point function

We will write the corresponding analytical expression in such a form that it holds both for the SOS and the SRSOS models. We assume that the momenta can have both signs and the momentum integration is done in the interval $-1 < p < 1$. For the SRSOS model the momentum space interval folds to $(0, 1)$ because of the symmetry of the Feynman rules under $p \rightarrow -p$. We have

$$\begin{aligned}
 G(p_1, p_2, p_3, p_4) &= M^{|p_1|+\dots+|p_4|} \mu_1^{-2} \times \\
 &\times \left[\int_{-1}^1 dp [F_0(p_1)F_0(p_2)N_{p_1 p_2 p} D_{00}(p)N_{-p p_3 p_4} F_0(p_3)F_0(p_4) + \text{perm.}] \right. \\
 &\quad + \mu_1^{-2} N_{p_1 p_2 p_3 p_4} [F_0(p_1)F_0(p_2)F_0(p_3)F_1(p_4) + \text{perm.}] \\
 &\quad \left. + \mu_1^{-2} N_{p_0 p_1 p_2 p_3 p_4} (-\hat{\mu}_2) F_0(p_1)F_0(p_2)F_0(p_3)F_0(p_4) \right]
 \end{aligned} \tag{6.1}$$

Then, using (5.42)-(5.45) we rewrite (6.1) as

$$\begin{aligned}
 G(p_1, p_2, p_3, p_4) &= M^{|p_1|+\dots+|p_4|} \mu_1^{-2} \left(\left[-\hat{\mu}_2 + \sum_{s=1}^4 \left(\frac{1}{4} - p_s^2 \right) \right] N_{p_1, p_2, p_3, p_4} \right. \\
 &\quad \left. + \int_{-1}^1 dp (N_{p_1, p_2, p} N_{-p, p_3, p_4} + N_{p_1, p_3, p} N_{-p, p_2, p_4} + N_{p_1, p_4, p} N_{-p, p_2, p_3}) \left(|p| - \frac{1}{2} \right) \left(|p| - \frac{3}{2} \right) \right).
 \end{aligned} \tag{6.2}$$

This is the general formula for the 4-point function in the continuum limit. The 4-point function depends on the scale parameter $\hat{t} = t/[M(t, \mu)]^{2p_0}$. It can be compared with the CFT results at the critical points $\hat{t} = 0$ and $\hat{t} \rightarrow \infty$, where the world sheet theory is described by Liouville gravity with matter CFT component with central charges respectively

(2.13) and (2.14). At the two critical points $\hat{\mu}_2$ is given by a numerical constant,

$$-\hat{\mu}_2 = \frac{3}{4} + p_0^2 + 2\frac{1-\hat{t}}{1+\hat{t}}p_0 \rightarrow \frac{3}{4} + p_0^2 \pm 2p_0, \quad (6.3)$$

while μ_1 scales as a power of $M(\mu, t)$:

$$\mu_1 \sim M^{2\pm p_0}. \quad (6.4)$$

Here the sign is $+$ for $\hat{t} = 0$ (dilute loops) and $-$ for $\hat{t} \rightarrow \infty$ (dense loops).

In the following we will write more explicit expressions using the concrete form of the fusion coefficients in the SOS and SRSOS models. To simplify the algebraic expressions we introduce an auxiliary quantity equal to the integral of a function $f(p)$ with respect to the intermediate momentum p and weighted by the fusion coefficients involving the four external momenta:

$$\rangle f(p) \langle := \int_{-1}^1 dp f(p) (N_{p_1 p_2 p} N_{-p p_3 p_4} + N_{p_1 p_3 p} N_{-p_2 p_4} + N_{p_1 p_4 p} N_{-p p_2 p_3 p}). \quad (6.5)$$

In particular $\rangle 1 \langle = 3N_{p_1 p_2 p_3 p_4}$. Then the general formula (6.2) can be written as

$$\begin{aligned} G(p_1, p_2, p_3, p_4) &= \frac{M^{|p_1|+\dots+|p_4|}}{\mu_1^2} \rangle \left[\frac{1}{3} \left(-\hat{\mu}_2 + \sum_{s=1}^4 \left(\frac{1}{4} - p_s^2 \right) \right) + \left(|p| - \frac{1}{2} \right) \left(|p| - \frac{3}{2} \right) \right] \langle \\ &= \frac{M^{|p_1|+\dots+|p_4|-2-2p_0}}{(1+\hat{t})^2} \left(2\frac{1-\hat{t}}{1+\hat{t}}p_0 + 4 + p_0^2 - \sum_{s=1}^4 p_s^2 + \rangle p^2 - 2|p| \langle \right) \end{aligned} \quad (6.6)$$

6.2. The 4-point function in the SOS model

We first evaluate the 4-point function for a simpler case, that of the SOS model on random triangulation, whose continuum limit is supposed to be described by Liouville gravity having a twisted boson as a matter field. In this case the wave functions are the plane waves (2.20) and momentum-space vertex (5.20) is a periodic delta-function, which describes $u(1)$ -type fusion rules:

$$N_{p_1, \dots, p_n} = \delta^{(2)} \left(\sum_{k=1}^n (p_k - p_0) + 2p_0 \right). \quad (6.7)$$

Since the fusion coefficients are delta-functions, the integral over the intermediate momenta yields $N_{p_1 p_2 p_3 p_4}$ times a sum over the three channels.

We would like to avoid the complications related to the compactness of the momentum space. This is why we assume that the external momenta are sufficiently small, so that

the periodic delta functions in (6.7) can be replaced by ordinary ones. Then we can easily perform the integration in (6.5), which leads to the following simple expression:

$$\begin{aligned} \langle p^2 \rangle &= N_{p_1 p_2 p_3 p_4} \left((p_1 + p_2 - p_0)^2 + \text{permutations} \right) \\ &= N_{p_1 p_2 p_3 p_4} \left(\sum_{s=1}^4 p_s^2 - p_0^2 \right). \end{aligned} \quad (6.8)$$

(For arbitrary external momenta we would obtain a more complicated expression, which is periodic in p_s and therefore not analytic.) Substituting this in (6.6) we get the final formula, in which the quadratic in the external momenta terms cancel:

$$\begin{aligned} G(p_1, p_2, p_3, p_4) &= \frac{M^{|p_1|+\dots+|p_4|-2-2p_0}}{(1+\hat{t})^2} N_{p_1 p_2 p_3 p_4} \times \\ &\times \left[2 + \frac{1-\hat{t}}{1+\hat{t}} p_0 - |p_1 + p_2 - p_0| - |p_1 + p_3 - p_0| - |p_1 + p_4 - p_0| \right]. \end{aligned} \quad (6.9)$$

At the critical point $t = 0$ this expression coincides with the well known formula for the 4-point correlator of gaussian matter field [15]. Indeed, using the map (2.19) we write the expression in the brackets as

$$Q - |P_1 + P_2 - e_0| - |P_1 + P_3 - e_0| - |P_1 + P_4 - e_0| \quad (6.10)$$

with

$$P_s = b p_s, \quad Q = b(2 + p_0) = \frac{1}{b} + b, \quad e_0 = b p_0 = \frac{1}{b} - b; \quad b = \frac{1}{\sqrt{1 + p_0}} \quad (t = 0). \quad (6.11)$$

The world sheet effective action for the point $t = 0$ is that of Liouville gravity with gaussian matter:

$$\mathcal{A} = \frac{1}{4\pi} \int d^2\sigma \left[(\partial_a \phi)^2 + (\partial_a \chi)^2 + (Q\phi + i e_0 \chi) \hat{R} \sqrt{\hat{g}} + \lambda_L e^{2b\phi} \right] + \text{ghosts}, \quad (6.12)$$

where $\lambda_L \sim \mu$. Therefore, at the critical point $t = 0$, (6.9) reproduces the CFT result of [15] for gaussian matter field. In the dense phase, $t \rightarrow \infty$, the formula (6.9) can be again interpreted as the 4-point function with gaussian matter, but with a smaller central charge (2.14). In this case we identify

$$P_s = -p_s b', \quad Q = (2 - p_0) b' = \frac{1}{b'} + b', \quad e_0 = -p_0 b' = \frac{1}{b'} - b'; \quad b' = \frac{1}{\sqrt{1 - p_0}} \quad (t \rightarrow \infty). \quad (6.13)$$

At finite t the world sheet theory does not factorize to a matter and Liouville components. It can be considered as a perturbation by the Liouville-dressed thermal operator $\Phi_{1,3}$ of the CFT describing the dilute phase.⁹

⁹ This perturbation is expected to lead to the periodicity of the correlation functions under $P \rightarrow P + 2b$ which matches the periodicity $p \rightarrow p + 2$ in the microscopic theory. We thank Al. Zamolodchikov for a discussion on this point.

6.3. Four-point function in the SRSOS model: generic momenta

Now we will consider the case of generic momenta in the SRSOS model. We assume again that the moments are sufficiently small so that the compactness of the momentum space is not felt. For generic (but sufficiently small) momentum $p > 0$ the wave functions

$$\psi_p(x) = \frac{\sin(\pi p x)}{\sin(\pi p_0 x)} \quad (6.14)$$

lead to a formal expression for the vertices (5.45), which does not seem to have a rigorous meaning, even as a distribution. It is therefore difficult to give a meaning of the general formula (6.1) for the 4-point function for generic momenta. We can however modify the definition of the observables to obtain a sensible formula for the 4-point function. Namely we consider the wave function for given momentum as a sum of two terms

$$\psi_p(x) = \psi_p^{(+)}(x) - \psi_p^{(-)}(x), \quad \psi_p^{(\pm)}(x) = \frac{e^{\pm i p x}}{e^{i \pi p_0 x} - e^{-i \pi p_0 x}}. \quad (6.15)$$

The two terms, $\psi_p^{(+)}$ and $\psi_p^{(-)}$, are analogs of the tachyons of positive and negative chirality considered in the world-sheet CFT analysis [1], and are related by $p \rightarrow -p$. With this replacement we pick up the analytic expression for the 4-point function valid in the infinitesimal vicinity of $p_1 = \dots = p_4 = p_0 = 0$.

In the case of irrational p_0 we can evaluate the ‘‘multiplicities’’ N_{p_1, \dots, p_n} by formally expanding the denominator in (6.15). Instead of specifying the chiralities, we will consider positive as well as negative values of the momenta p_s . Then we can write the fusion coefficients as a sum of δ -functions, *e.g.*

$$N_{p_1 p_2 p_3} \rightarrow \sum_{n=0}^{\infty} \delta(p_1 + p_2 + p_3 - (2n + 1)p_0) \quad (6.16)$$

$$N_{p_1 p_2 p_3 p_4} \rightarrow \sum_{m \geq 0} (m + 1) \delta(p_1 + p_2 + p_3 + p_4 - 2(m + 1)p_0), \quad (6.17)$$

which we substitute in the general formula (6.6). To write the final expression we must first evaluate the integral

$$\rangle_{-p^2} \langle = \sum_{n \geq 1} \delta\left(\sum_i p_i - 2n p_0\right) \times \sum_{k=0}^{n-1} \left((p_1 + p_2 - (2k + 1)p_0)^2 + \text{permutations} \right). \quad (6.18)$$

The expression multiplying the delta function simplifies to

$$\sum_{k=0}^{n-1} \left(\sum_{j=2}^4 (p_1 + p_j) - (2k + 1)p_0 \right)^2 = n \left(\sum_{i=1}^4 p_i^2 - p_0^2 \right) \quad (6.19)$$

where we used the identity (assuming $\sum p_s = 2np_0$)

$$\sum_{s=2}^4 (p_1 + p_s - np_0)^2 = \sum_{s=1}^4 p_s^2 - n^2 p_0^2. \quad (6.20)$$

Now we can write the explicit expression for the r.h.s. of (6.6), which is again linear in the momenta,

$$\begin{aligned} G(p_1, p_2, p_3, p_4) &= \frac{M^{p_1+\dots+p_4-2-2p_0}}{(1+\hat{t})^2} \sum_{n=1}^{\infty} \delta(p_{\text{tot}} - 2np_0) \times \\ &\times \left[n \left(2 + \frac{1-\hat{t}}{1+\hat{t}} p_0 \right) - \sum_{k=0}^{n-1} \sum_{s=2}^4 |p_1 + p_s - (2k+1)p_0| \right], \end{aligned} \quad (6.21)$$

where $p_{\text{tot}} = p_1 + p_2 + p_3 + p_4$ is the total momentum. Passing to world sheet CFT notations, we write (6.21) at the critical points as

$$G(P_1, P_2, P_3, P_4) \sim \sum_{n=1}^{\infty} \delta \left(\sum_1^4 P_i - 2ne_0 \right) \left[nQ - \sum_{k=0}^{n-1} \sum_{s=2}^4 |P_1 + P_s - (2k+1)e_0| \right]. \quad (6.22)$$

This expression is identical with the expressions for the 4-point tachyon correlation function obtained for the “diagonal perturbation” of the non-rational Liouville gravity [1]. More precisely (6.22) corresponds to the physical, symmetric in the four momenta correlators, extracted from the fixed chirality solutions of the functional relations. The intermediate states in (6.22) are spaced by $2e_0$ and can be interpreted in terms of insertions of a screening vertex operator (tachyon) with matter component $e^{2ie_0\chi}$. This amounts to adding an additional “diagonal perturbation” to the Liouville term:

$$\mathcal{A}_{\text{int}}^{t=0} = \int d^2\sigma (\lambda_L e^{2b\phi} + \lambda_D e^{2ie_0\chi} e^{2b\phi}). \quad (6.23)$$

In [1] the functional equations for the correlation functions were obtained by adding the action obtained by Liouville reflection, $\tilde{\mathcal{A}}_{\text{int}}^{t=0} = \tilde{\lambda}_L e^{2\phi/b} + \tilde{\lambda}_D e^{2ie_0\chi} e^{\frac{2}{b}\phi}$.

6.4. The four-point function of degenerate fields in the SRSOS model

Here we assumed that $p_i = m_i p_0$ are sufficiently small, so that the intermediate momenta are smaller than 1. The wave functions (6.14), restricted to the discrete spectrum $p \in p_0 \mathbb{Z}_+$,

$$\psi_m(x) = \frac{S_x^{(p)}}{S_x} = \frac{\sin(\pi p_0 m x)}{\sin(\pi p_0 x)}, \quad p = m p_0, \quad m \in \mathbb{N}, \quad (6.24)$$

form closed Pasquier-Verlinde-like fusion algebra:

$$\psi_{mp_0}\psi_{np_0} = \frac{1}{2}{}^{(m+n-|m-n|)} \sum_{k=1} \psi_{(m+n+1-2k)p_0}. \quad (6.25)$$

This is the algebra of the order operators. The multiplication rule in (6.25) coincides with the tensor product decomposition multiplicity of $sl(2)$ irreps of dimension m_i

$$N_{m_1, m_2, m_3} = \begin{cases} 1 & \text{if } |m_1 - m_2| + 1 \leq m_3 \leq m_1 + m_2 - 1 \\ & \text{and } m_1 + m_2 + m_3 = \text{odd} \end{cases}; \quad (6.26)$$

$$0 \quad \text{otherwise}$$

The 4-point multiplicity is correspondingly

$$N_{m_1 m_2 m_3 m_4} = \sum_{m>0} N_{m_1 m_2 m} N_{m m_3 m_4} \quad (6.27)$$

$$= \frac{1}{2} (\min(m_1 + m_2, m_3 + m_4) - \max(|m_1 - m_2|, |m_3 - m_4|)).$$

The n -point multiplicities are given by the general formula (5.20). The sum over $x \in \mathbb{Z}_+$ can be replaced, up to an infinite volume factor, by an integral w.r. to the compact variable $\theta = \pi p_0 x$:

$$N_{m_1, \dots, m_n} = \frac{1}{\pi} \int_0^{2\pi} d\theta \sin^2 \theta \prod_{i=1}^n \frac{\sin(m_i \theta)}{\sin \theta}. \quad (6.28)$$

To evaluate the sum in (6.2), we first rewrite the expression as

$$G(m_1, m_2, m_3, m_4) = \rangle - \frac{-\hat{\mu}_2 + \frac{1}{4} + 3 - p_0^2 \sum_{s=1}^4 m_s^2 + p_0^2 m^2 - 2p_0 m}{\mu_1^2} \langle \quad (6.29)$$

where the symbol $\rangle f(m) \langle$ is defined similarly as in (6.18):

$$\rangle f(m) \langle := \sum_m f(m) (N_{m_1 m_2 m} N_{m m_3 m_4} + N_{m_1 m_3 m} N_{m_2 m_4} + N_{m_1 m_4 m} N_{m m_2 m_3 m}). \quad (6.30)$$

It is straightforward to check that

$$\rangle m \langle = N_{m_1 m_2 m_3 m_4} \left(\sum_{s=1}^4 m_s - N_{m_1 m_2 m_3 m_4} \right) \quad (6.31)$$

$$\rangle m^2 \langle = N_{m_1 m_2 m_3 m_4} \left(\sum_{i=1}^4 m_i^2 - 1 \right).$$

Inserting this in (6.2) we obtain the following simple expressions for the 4-point function of order operators.

$$\begin{aligned}
G_{m_1 m_2 m_3 m_4} &= \frac{2}{\mu_1^2} \left(\left(2 + \frac{1-\hat{t}}{1+\hat{t}} p_0 \right) N_{m_1 m_2 m_3 m_4} - \sum_{m=1} [N_{m_1 m_2 m} (m p_0) N_{m m_3 m_4} + \text{perms.}] \right) \\
&= \frac{2}{\mu_1^2} N_{m_1 m_2 m_3 m_4} \left(2 + \frac{1+\hat{t}}{1-\hat{t}} p_0 - p_0 \sum_{i=1}^4 m_i + p_0 N_{m_1 m_2 m_3 m_4} \right).
\end{aligned} \tag{6.32}$$

In the two critical points, $\hat{t} = 0$ and $\hat{t} \rightarrow \infty$, the formula (6.32) for the 4-point function can be compared with the expression obtained in [1] for the CFT with the “diagonal perturbation” (6.23) using the ground ring relations in the Coulomb gas approach. We find agreement, up to an overall constant. In the case of finite temperature, at the moment there are no world sheet CFT results to compare with.

6.5. The 4-point function in ADE string theories

At rational values of the background momentum, $p_0 = 1/h$, $h \in \mathbb{Z}_+$, the integers m_i are restricted to the range $1 \leq m_i \leq h-1$ and the $sl(2)$ multiplicities (6.26), (6.28) are replaced by the Verlinde fusion multiplicities, i.e., the upper bound in (6.26) becomes $\min(m_1 + m_2 - 1, 2h - m_1 - m_2 - 1)$. The corresponding relations (6.31) do not hold true anymore in general, however the initial quadratic formula (6.2), derived in [14,23], survives and represents the tachyon 4-point function for $p_i = m_i p_0$. This formula extends to the ADE cases. The 3-point multiplicities $N_{m_1 m_2 m_3}$ of the ADE series are real numbers (square roots of rationals), defining the structure constants of the Pasquier algebra [4] associated with any ADE graph, with the numbers m_i corresponding to the set of exponents¹⁰.

The partition functions of the ADE string theories on a torus are calculated in [9]. They are obtained by summing only over the order fields propagating along one of the circles of the torus. Nevertheless they reproduce the expressions obtained from the world sheet CFT [41], and there are no configurations that are not taken into account. This can be explained by the particular choice of the time slice, which goes along a domain wall. The torus is thus obtained by identifying the boundaries of a cylinder with Dirichlet type boundary conditions.

The fact that the partition functions can be reconstructed only by taking into account the diagonal fields is also related to the fact that there is a world sheet CFT, the diagonal theory in [1], in which these fields form closed algebra. One can give the following heuristic explanation of that. In the case of rigid geometry an order operator at the point r is

¹⁰ These constants coincide with the relative scalar OPE coefficients of the ADE theories [40].

constructed by cutting out a circular domain having as a center this point and large compared to the lattice size. In general, there will be loops that enter and leave the domain, crossing its boundary. Therefore, when shrinking the domain to a point in the continuum limit, we will create also other operators as the thermal operators mentioned before. If we calculate a correlation function of two such operators, we will reproduce the standard OPE expansion of the diagonal fields ($r = s$) in CFT, which will include non-diagonal fields as well. Now consider the SRSOS model on a fluctuating lattice and repeat the procedure. In such a “liquid” lattice the symmetry is stronger and the notion of circular domain has no sense anymore. The domain surrounding the point r is formless; it has a single geometrical characteristics, its area and perimeter. We can restrict ourselves to the set of domains that do not cross any domain walls, i.e. with Dirichlet type boundary. Then the definition of the order operator by shrinking the boundary of the domain is such that in the 4-point function of order operators there will be only order operators in the intermediate channels. This is exactly what happens in the diagonal worldsheet string theory [1]. The definition of the order operators such that they form closed algebra is possible only in the theory coupled to gravity.

7. Summary and discussion

The theory we studied here – the SRSOS height model on random triangulations – is a good candidate for a generic microscopic realization of non-rational 2D gravity. At the rational values of the background momentum p_0 it describes the A -series of the ADE string theories. The model is characterized by semi-infinite target space, which we identified with the A_∞ Dynkin graph. The semi-infiniteness of the target space leads to identification of the positive and negative momenta, which implies a reflection property of the correlation functions. It is also possible to construct a model that generalizes the height models of the D series coupled to gravity. The construction of the D_∞ height model is sketched in Appendix B.

We formulated the SRSOS model as a conformal field theory on a Riemann surface representing an infinite branched cover of the complex plane. The points of the Riemann surface label the FZZ branes in the SRSOS model.

The calculation of the correlation functions is straightforward and it matches with the world sheet CFT predictions. Our results are compatible with the conjecture that the SRSOS model gives a microscopic realization of the “diagonal” CFT introduced in [1], in which the screening charges are the Liouville dressed identity operator (with momentum p_0) and its charge conjugated (with momentum $-p_0$).

The SRSOS model gives formulas only in the physical regions of momenta, for which the tachyons describe local observables on the world sheet. The expression is given by

a sum of terms associated with intermediate channels. Each such term is a polynomial of the momenta of the particles involved. These polynomials are higher order than the polynomials that occur in the iteration procedure of Di Francesco and Kutasov [15]. In the case of non-rational theory there are relations that allow to lower the degree of the polynomials. Using them we have transformed the discrete model 4-point functions into expressions comparable with the CFT solutions of the ring relations. This property does not seem to extend to the case of the minimal matter theory with rational b^2 - a case which is especially difficult to be analyzed in the CFT approach. Thus we expect that the 4-point functions in this case are given by expressions similar to the initial quadratic formula (6.2). The discrete approach provides as well the analogs of the ADE correlators, which cannot be treated in the CFT approach, as the bulk ground ring relations, expressing the fundamental isospin 1/2 fusion relations, do not apply directly to the non-diagonal D, E cases.

In this paper we restricted ourselves to the correlation functions of order operators. In fact our general formula (6.32) is the generating function of the correlators of 4 order operators and an arbitrary number of Liouville dressed thermal operators $\Phi_{1,3}$, which are given by the expansion coefficients of the thermal coupling t . The meaning of the results for finite temperature will be discussed in detail elsewhere [42].

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Appendix A. Loop gas representation of the correlation functions

A.1. Correlation functions of order operators in the SOS model

The order, or vertex, operator in the SOS model is defined as the effect of inserting the wave function

$$\hat{\psi}_p(x) = e^{i\pi(p_0-p)x} = e^{2iqx} \tag{A.1}$$

at some point of the planar graph. We will refer to p as a target space momentum and to $q = \frac{1}{2}(p_0 - p)$ as electric charge. The correlation functions of the vertex operators (A.1) can be formulated as the partition function of the loop gas with some of the loop weights

modified. The rule is that the weight of a loop encircling a total charge $q^{\text{in}} = \frac{1}{2}(p_0 - p^{\text{in}})$ is equal to

$$2 \cos(\pi p^{\text{in}}) = 2 \cos[\pi(p_0 - 2q^{\text{in}})]. \quad (\text{A.2})$$

Since the reference point (the infinite point for the plane) can be put anywhere, this rule is consistent only if it gives the same result with q^{in} replaced with the total charge q^{out} outside the loop. This is guaranteed by the charge conservation condition $q^{\text{in}} + q^{\text{out}} = p_0$ or, in terms of momenta, $p^{\text{in}} + p^{\text{out}} = 0$.

- *One-point function*

There is only one vertex operator, $\hat{\psi}_{-p_0}(x) = e^{2i\pi p_0 x}$, with non-trivial one-point function. This operator is related to the identity operator by the charge reflection

$$q \rightarrow p_0 - q \quad \text{or} \quad p \rightarrow -p. \quad (\text{A.3})$$

The charge of this operator compensates the total background charge $-p_0$ for the sphere. The correlation function $\langle \hat{\psi}_{-p_0} \rangle$ is equal to the partition function of loop gas with weight $2 \cos(\pi p_0)$ per loop.

- *Two-point function*

The two points can be connected by a line that intersects each loop at most once. The two-point function $\langle \hat{\psi}_p \hat{\psi}_{-p} \rangle$ is obtained by changing the weight of the non-contractible loops, i.e., those that intersect the line, to $2 \cos \pi p$.

- *n-point function ($n \geq 3$)*

Consider the n -point function

$$G_{p_1, \dots, p_n} = \langle \hat{\psi}_{p_1} \hat{\psi}_{p_2} \cdots \hat{\psi}_{p_n} \rangle, \quad \sum_i p_i = (n-2)p_0 \text{ modulo } 2 \quad (\text{A.4})$$

where p_1, \dots, p_{n-1} are positive and p_n is negative (“chirality rule”). The n points where the operators are inserted can be connected by a set of oriented lines that form a tree, with the condition that the tree intersects each loop at most once. An example for $n = 4$ is given in Fig. 10, where the fourth point is at infinity. To each line of the tree we associate a charge. The charges associated with the external lines are those introduced by the order operators. The charges associated with the internal lines are determined by the charge neutrality of the vertices of the tree. In terms of momenta the neutrality condition for a vertex with k lines is written as $\sum_l (p_l - p_0) + 2p_0 = 0$ modulo 2. Then the loops crossed by a line l of the tree change from $2 \cos(\pi p_0)$ to $2 \cos(\pi p_l)$, where $p_l = p_0 - 2q_l$ is the momentum associated with this line. The above argument hold unchanged if instead of point-like insertions one considers finite boundaries.

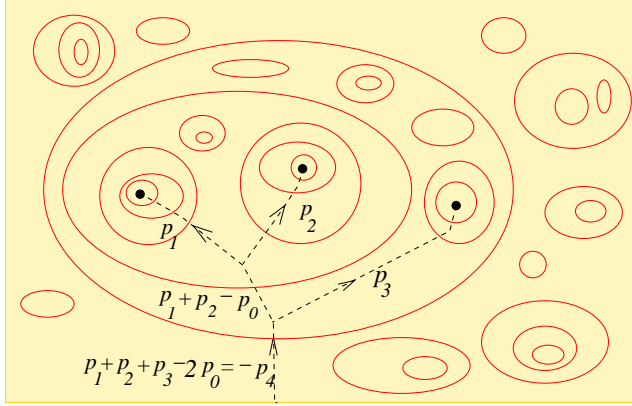


Fig. 10: A loop configuration in presence of four operator insertions.

A.2. Correlation functions of order operators in the SRSOS model

Unlike the SOS model with $p_0 \neq 0$, the SRSOS model has a well defined *partition function* on the sphere without boundaries. Summing over all height configurations yields a factor $2 \cos \pi p_0$ per loop. Thus the partition function of the SRSOS model on the sphere coincides with the one for the $O(n)$ model with $n = 2 \cos(\pi p_0)$.

The order operators are introduced as insertions of the wave functions

$$\psi_p(x) = \frac{\sin(\pi p x)}{\sin(\pi p_0 x)}, \quad p > 0. \quad (\text{A.5})$$

If we restrict the spectrum of momenta to

$$p = m p_0, \quad m = 1, 2, \dots \quad (\text{A.6})$$

then the functions (A.5) generate the $su(2)$ character ring.

- *One-point function*

The only non-trivial one-point function is that of the identity operator, $p = p_0$. One can perform the sum of the heights by repeatedly applying the relation (2.11) starting with the domains that do not contain loops. As a result, the expectation value $\langle \psi_{p_0} \rangle$ is equal to the partition function of the loop gas on a sphere with one marked point, with weight $2 \cos(\pi p_0)$ for each loop.

- *Two-point function*

The insertion of two operators (A.5) at two points changes the weights of the non-contractible loops to $2 \cos \pi p$. The weights of these loops follow from the identity (2.11) with p_0 replaced by p . This two-point function is defined for any real value of p .

- *n-point functions* ($n \geq 3$)

The n -point functions in the SRSOS model are well defined when the momenta belong to the spectrum (A.6),

$$G_{m_1, \dots, m_n} = \langle \psi_{p_1} \psi_{p_2} \dots \psi_{p_n} \rangle, \quad p_i = m_i p_0. \quad (\text{A.7})$$

One can again describe the weights by drawing a tree connecting the n points and such that each loop is intersected at most once. The momenta m_1, \dots, m_n in (A.7) are associated with the external lines of the tree. Unlike the SOS case, here the momenta associated with the internal branches are not determined uniquely. When calculating the correlation function with $n > 3$, a sum over these momenta is to be performed. The weights in the momentum space are associated with the vertices of the tree. If m_{l_1}, \dots, m_{l_k} are the momenta associated with a k -vertex, then the weight of this vertex is given by the $su(2)$ multiplicity $N_{m_{l_1}, \dots, m_{l_k}}$, which is the Fourier image of the weight (2.9) with $n = k$.

It follows from the explicit form of the $su(2)$ multiplicities that if the n external momenta satisfy the neutrality condition

$$m_1 + \dots + m_{n-1} - m_n = (n - 2), \quad (\text{A.8})$$

then the internal momenta are uniquely determined and the n -point correlation function (A.7) coincides with the n -point function in the SOS model with $n - 1$ positive momenta, $p_i = m_i p_0$ ($1 \leq k \leq n - 1$) and one negative momentum, $p_n = -m_n p_0$. Thus the n -point functions in the SRSOS model satisfying (A.8) are identical to the corresponding amplitudes in the SOS model that satisfy the ‘‘chirality rule’’.

In the general case eqn. (A.8) is replaced by

$$m_1 + \dots + m_{n-1} - m_n = (n - 2) + 2k, \quad k \in \mathbb{Z}_+. \quad (\text{A.9})$$

The contribution of given loop configuration to the n -point function can be written as a sum of SOS n -point functions with k extra operators $\hat{\psi}_{-p_0}(x) = e^{i\pi p_0 x}$, associated with the domains containing the vertices of the tree.

Appendix B. Expression for the operators $a_{k,x}^\dagger$ in terms of the collective field

We will use the expression (5.6) of the creation operators $a_{n,x}^\dagger$ as contour integrals of the effective potential $\Gamma_x(z)$. The operators $a_{n,x}^\dagger$ is proportional to the coefficients in the expansion of the effective potential in the half-integer powers of $z + M$. In the τ parametrization

$$\sqrt{1 + \frac{z}{M}} = \sqrt{2} \cosh \frac{\tau}{2}.$$

Instead of expanding $\Gamma_x(z)$ at $z = -M$ we will evaluate the integrals (5.2) by deforming the contour \mathcal{C}_- to the contour \mathcal{C}_+ going around the second cut of $\Gamma_x(z)$ along the interval

$[M, \infty]$ as in Fig. 5b. Deforming the contour \mathcal{C}_- to \mathcal{C}_+ (with clockwise orientation) we get, up to an infinite constant,

$$\frac{1}{\sqrt{2}\hat{g}_s} \sum_{n=0}^{\infty} \frac{u^n}{n!} (a_{n,x}^\dagger - \delta_{k,1} \frac{S_x}{\hat{g}_s}) = \oint_{\mathcal{C}_-} \frac{dz}{2\pi i} \frac{\partial_z \Gamma_x(z)}{\sqrt{1 + \frac{z}{M} - 2u}} = \oint_{\mathcal{C}_+} \frac{dz}{2\pi i} \frac{\partial_z \Gamma_x(z)}{\sqrt{1 + \frac{z}{M} - 2u}} \quad (\text{B.1})$$

The denominator in the last integral is regular and we express the discontinuity of the field $\Gamma_x(\tau(z))$ in terms of $\Phi(\tau(z))$. To evaluate the integral we go to the τ parametrization

$$\frac{1}{\sqrt{2}\hat{g}_s} \sum_{n=0}^{\infty} \frac{u^n}{n!} (a_{n,x}^\dagger - \delta_{k,1} \frac{S_x}{\hat{g}_s}) = 2 \int_0^\infty \frac{d\tau}{\pi} \frac{\partial_\tau \sin \pi x \partial_\tau \cos \pi \partial_\tau \Phi^-(\tau)}{\sqrt{1 + \cosh \tau - 2u}} \quad (\text{B.2})$$

In this step we have retained the negative frequency part of the field, i.e., (B.2) is understood to hold applied on the right vacuum $|0_\infty\rangle$. The projections (B.2) are seemingly non-local, but they are actually given by polynomials of the derivatives of the holomorphic field $\varphi(\tau)$ at $\tau = \pm i\pi x$. To see that we evaluate the integral for each Fourier mode in the expansion (4.16) of $\varphi^-(\tau)$ using the integral formula for the Legendre function of first kind [43]:

$$\int_0^\infty \frac{d\tau}{\pi} \frac{\cosh \nu \tau}{\sqrt{\cosh \tau + 1 - 2u}} = \frac{P_{-\frac{1}{2}+\nu}(1-2u)}{\sqrt{2} \cos \pi \nu}, \quad (\text{B.3})$$

$$P_{-\frac{1}{2}+\nu}(1-2u) = \sum_{k=0}^{\infty} \frac{u^k}{k!} \frac{(\frac{1}{2} + \nu)_k (\frac{1}{2} - \nu)_k}{k!}, \quad k = 1, 2, \dots \quad (\text{B.4})$$

The integral over ν amounts in replacing $\nu \rightarrow \partial_\tau$ and setting $\tau = 0$, i.e., we get

$$a_{n,x}^\dagger - \delta_{k,1} \frac{S_x}{\hat{g}_s} = 2 \frac{(\frac{1}{2} + \partial_\tau)_k (\frac{1}{2} - \partial_\tau)_k}{k!} \partial_\tau \sin \pi x \partial_\tau \Phi^-(\tau) \Big|_{\tau=0} \quad (\text{B.5})$$

or, transforming to the momentum space and using the shorthand notation (4.8) for the differential operator,

$$\begin{aligned} (a_k^\dagger(p) - \delta_{k,1} \hat{g}_s^{-1} \delta^{(2)}(p - p_0)) |0_\infty\rangle &= \hat{g}_s F_k(\partial_\tau) \partial_\tau \varphi^-(p, \tau) \Big|_{\tau=0} |0_{\text{tw}}\rangle \\ &= i F_k(\partial_\tau) \sinh \tau \partial \Gamma^-(\tau + i\pi, p) \Big|_{\tau=0} |0_\infty\rangle. \end{aligned} \quad (\text{B.6})$$

On the other hand assuming that we can identify $|0_\infty\rangle$ with the right twisted vacuum $|0_{\text{tw}}\rangle$, we can replace in the r.h.s. $\partial \Gamma^-(\tau + i\pi)$ with the twisted negative mode part. Using that the decomposition (5.2) takes the form

$$\partial \Gamma_x^\dagger(\tau + i\pi) = i \frac{1}{M} \sum_{n \geq 0} (a_{n,x}^\dagger - \delta_{n,1} \frac{S_x}{\hat{g}_s}) \frac{(-2)^{n-1} (\sinh \frac{\tau}{2})^{2n-1}}{(2n-1)!!} \quad (\text{B.7})$$

and plugging (B.7) in the identity (B.6) we see that it implies that the operators F_k act as projectors for $\tau = 0$, namely,

$$F_k(\partial_\tau) \sinh \tau \frac{2^{n-1} (-1)^n (\sinh \frac{\tau}{2})^{2n-1}}{(2n-1)!!} \Big|_{\tau=0} = \delta_{k,n}, \quad n \geq 0 \quad (\text{B.8})$$

This property is also checked independently to hold true, which in turn ensures the identification of the right vacua above.

Appendix C. The D_∞ model

The target space of the D_∞ model is labeled by the positive integers $x \in \mathbb{Z}_+$ and the two extremities of the ‘fork’ $x = 0$ and $x = \bar{0}$ (Fig. 11). The adjacency matrix is given by

$$\begin{aligned} A^{xx'} &= \delta_{x,x'+1} + \delta_{x,x'-1}, & x, x' \in \mathbb{N}, \\ A_{0x} &= A_{\bar{0}x} = \delta_{1,x}, & x \in \mathbb{N}. \end{aligned} \tag{C.1}$$

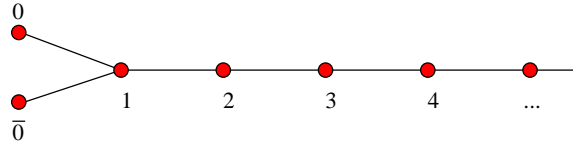


Fig. 11: The target space of the D_∞ model

Its eigenvectors $S_x^{(p)}$, $0 \leq p \leq \pi$, are given by

$$S_x^{(p)} = \begin{cases} \cos(\pi px) & \text{if } x \in \mathbb{Z}_+; \\ \frac{1}{2} & \text{if } x = 0 \text{ or } x = \bar{0}. \end{cases}, \quad 0 \leq p \leq \pi. \tag{C.2}$$

The zeroth eigenvalue $A(p) = 0$ or $p = \frac{1}{2}$ is twice degenerated. The second eigenvector, which we denote by $\bar{S}_x^{(1/2)}$, has components

$$\bar{S}_x^{(1/2)} = \begin{cases} \frac{1}{2} & \text{if } x = 0; \\ -\frac{1}{2} & \text{if } x = \bar{0}; \\ 0 & \text{if } x \in \mathbb{Z}_+. \end{cases} \tag{C.3}$$

For $p_0 = \frac{1}{2n+2}$ the D_∞ target space can be restricted to the D_n Dynkin graph.

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