Non-Rational 2D Quantum Gravity: I. World Sheet CFT

I.K. Kostov[1] and V.B. Petkova[2]

[1] *Service de Physique Théorique, CNRS – URA 2306, C.E.A. - Saclay, F-91191 Gif-Sur-Yvette, France*

[2] *Institute for Nuclear Research and Nuclear Energy, 72 Tsarigradsko Chaussée, 1784 Sofia, Bulgaria*

We address the problem of computing the tachyon correlation functions in Liouville gravity with generic (non-rational) matter central charge $c < 1$. We consider two variants of the theory. The first is the conventional one in which the effective matter interaction is given by the two matter screening charges. In the second variant the interaction is defined by the Liouville dressings of the non-trivial vertex operator of zero dimension. This particular deformation, referred to as "diagonal", is motivated by the comparison with the discrete approach, which is the subject of a subsequent paper. In both theories we determine the ground ring of ghost zero physical operators by computing its OPE action on the tachyons and derive recurrence relations for the tachyon bulk correlation functions. We find 3- and 4-point solutions to these functional equations for various matter spectra. In particular, we find a closed expression for the 4-point function of order operators in the diagonal theory.

1. Introduction

The exact results in Liouville theory obtained in the last decade [1-8] allowed to improve some old techniques developed in $c \leq 1$ string theories (reviewed in [9-13]) and find new links between the world-sheet and matrix model descriptions. In particular, the fundamental OPE identities, used in [3,5,6,7,8] to evaluate various Liouville structure constants, are similar in nature to the ground ring relations in string theories [14-18], [19-22]. Recently the ground ring structure was reconsidered in [23,24,25], where it was applied to study the solitons, or D-branes, in $c \leq 1$ string theories.

In this work we generalize and exploit this approach to derive and solve finite difference equations for the tachyon correlators on the sphere. We recall that the ground ring is the ring with respect to the operator product expansions (OPE), modulo Q_{BRST}-exact terms, of the physical operators of zero ghost number. Physical operators of fixed ghost number, like the tachyons, represent modules under the action of the ring [14].

So far the technique has been tested in the simplest model of 2D string theory that can be considered as a marginal deformation by Liouville interaction of a two-component gaussian field action with background charges. One can perturb in a similar way the matter component of the gaussian field by the matter screening charges. As a result one obtains a gravitational analog of the Dotsenko-Fateev Coulomb gas construction. The correlation functions depend on two coupling constants: the Liouville coupling (cosmological constant) $\mu_{_L}$ and its matter counterpart $\mu_{_M}$, associated with the matter screening charge. Unlike most of the previous studies, which deal with the minimal string theories, we shall consider non-rational values of the matter central charge characterized by a real parameter b,

$$c = 1 - 6(\tfrac{1}{b} - b)^2 \,. \tag{1.1}$$

One of the motivations for this work was to compare the correlation functions in the continuous (world sheet), and the discrete (target space) approaches to the 2D quantum gravity. This is an old problem and there are few matrix model results on the correlation functions with a non-trivial matter. Such results are known only for the simplest examples of the rational, minimal theories, as the Ising model [26], recently reconsidered in [27]. Moreover, there is no matrix model to match the non-rational case, except the $O(n)$ matrix model [28], whose poor field content is too restricted. In a subsequent paper [29] we construct such a model, in which the matter degrees of freedom are parametrized by a semi-infinite discrete set, generalizing the ADE string theories [30]. In this matrix chain model – whose target space is the A_∞ Dynkin graph, one develops a finite diagram technique for the explicit computation of the n-loop amplitudes; see [31] for an early application of this technique in the rational case. Shrinking the loops, one extracts the n-point local correlation functions. However, this procedure is not unique on a fluctuating lattice. Moreover, it happens that the most natural definition of the local fields leads to a different interpretation of the matter screening than that in the conventional theory. Namely, the charge conservation condition involves only multiples of the matter background charge $e_0 = 1/b - b$, and the Liouville dressings of the "order parameter" fields on the diagonal of the infinite Kac table close under fusion. This has led us to introduce and study another variant of the $c < 1$ gravity in the continuum. Instead of the matter screening charges, *i.e.*, the tachyons of matter charge $-b$ and $1/b$, the interaction terms for the matter field are now generated by the two Liouville dressings of the vertex of charge $e_0 = 1/b - b$. The effective action of this "diagonal" string theory is no more a sum of Liouville and matter parts. Nevertheless it is possible to extend to this theory the ground ring technique and to find solutions of the corresponding functional equations. A class of these solutions reproduces the 4-point tachyon correlators found in the discrete approach.

The paper is organized as follows:

After some preliminaries collected in section 2, we determine in section 3 the generic tachyon 3-point function as a product of the $c > 25$ Liouville [1], [2] and a $c < 1$ matter OPE structure constants. The latter constant is given by an expression derived as in [3], which in particular reproduces the Coulomb gas OPE constant of [32]; see also [33].

The ground ring is discussed in section 4. Since its elements are built of vertex operators, such that both the matter and Liouville parts are labelled by degenerate Virasoro representations, one can compute their OPE with the tachyons using the free field Coulomb gas in the presence of integer number of screening charges. We determine the action of the two generators of the ground ring, a_- and a_+ , on a tachyon of arbitrary momentum, taking into account both Liouville and matter interactions. This amounts in the computation of the general 3-point function of two tachyons and one ring generator, the details are collected in Appendix A.1-2. The result confirms the expectation, see [24] for the rational case, that the deformed ring is isomorphic to $sl(2) \times sl(2)$ type fusion ring. In the "diagonal matter" string theory the deformed ground ring element $a_+ a_-$ generates a diagonal $sl(2)$ projection, preserving in particular the order parameter tachyons.

We use this action in section 5 to derive recurrence functional relations for the bulk tachyon 4-point correlators. The equations are written for the correlators satisfying the "chirality rule" in the terminology of [9] and they extend the ones previously obtained for the case of gaussian matter [17,18,25]. The contact 3-point terms in these relations require the computation of a set of free field 4-point functions containing a ring generator and an integrated tachyon, see sections A.3, A.4 of the Appendix. Appendix A.5 contains a computation of some chiral OPE constants relevant for the boundary ground ring, in particular the OPE coefficients of the boundary ground ring generators and a tachyon of generic momentum.

In the remaining four sections we look for solutions of the equations for the 4-point tachyon correlation functions. In section 6 we reproduce the correlators found in [34] for the case of gaussian matter field. The set of fixed chirality solutions are shown to serve as a local basis for another, fully symmetric in the momenta correlator, also described in [34]. We interpret this symmetry as locality requirement and discuss the relation between the two types of correlators and the respective equations they satisfy. In section 7 we solve the functional equations for a class of correlators such that the total matter (or total Liouville) charge can be compensated by integer number of screening charges. In section 8 we find correlators in the diagonal theory in which one or all four tachyons is degenerate, i.e., its momentum labels a degenerate $c<1$ Virasoro representation on the diagonal of the infinite Kac table. The case of degenerate fields in the conventional, non-diagonal theory, is considered in section 9. Here we find 4-point functions with one matter (or Liouville) degenerate and three generic momenta. A formula for the 4-point correlators with four degenerate fields is conjectured by analogy with the diagonal case.

The results for the 4-point tachyon correlators as functions of the momenta P_1, P_2, P_3, P_4 are summarized by a "partial wave expansion" formula sketched in Fig.1.

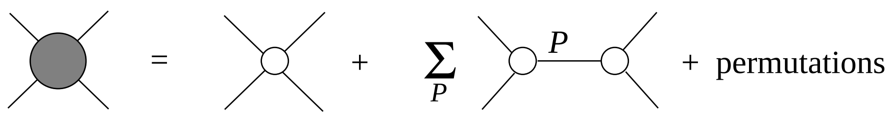

Fig.1 : The general structure of the 4-point function

The 4-point function is a sum of 1-particle irreducible (1pi) piece and a sum of the contributions of the three channels:

$$\mathcal{G}^{(\varepsilon)}_{P_1,P_2,P_3,P_4} \sim \hat{\mathcal{G}}^{1\text{pi}}_{P_1,P_2,P_3,P_4} - \varepsilon \sum_{P} \left(N_{P_1,P_2,P}\, P\, N^{'}_{-P,P_3,P_4} + \text{permutations} \right) , \qquad (1.2)$$

where ε is a sign determined by the chiralities. The form of the 1pi term $\hat{\mathcal{G}}^{1\text{pi}}$ and the interpretation of the 3-point "fusion multiplicities" in (1.2) depend on the considered spectrum of momenta. The case of gaussian matter in [34] corresponds to a single contribution in each of the three channels in (1.2). The formula (1.2) is symmetrised with the intermediate momentum εP replaced by $|P|$. This universal choice solves the locality requirement preserving the fusion rules, which in the conventional theory typically match those determined by the underlying $c<1$ (or $c>25$) local correlators. It is further supported by a recursive procedure extending the initial identities to equations for the local correlators. In the diagonal theory the term $\hat{\mathcal{G}}^{1\text{pi}}$ is proportional to the corresponding 4-point fusion multiplicity N_{P_1,P_2,P_3,P_4}, also expressed in terms of the 3-point vertices. The diagonal theory correlators are not a special case of those in the conventional string theory.

We conclude in section 10 with a summary of the results and a discussion on the open problems of this investigation.

This paper is a detailed and extended presentation of the results announced in the short letter [35] and reported at conferences in Dubna, Bonn, Varna and Santiago de

Compostela. In the mean time, two papers appeared, which partially overlap with our results. Ref. [36] deals with the boundary ground ring relations in the minimal $c < 1$ gravity. In ref. [37] a different method for evaluating the bulk tachyon correlation functions in 2D gravity is developed. The class of 4-point correlators computed in [37], namely those containing one matter degenerate field, are of the type discussed in our section 9; see the text for a comparison. We have been also informed by V. Fateev about an unpublished recent work of him on the direct computation of some particular examples of such correlators.

2. Preliminaries: effective action, observables

2.1. Effective action

Consider Liouville gravity on the Riemann sphere. The effective action in the conformal gauge and locally flat reference metric $\hat{g}_{ab} = \delta_{ab}$ is a perturbation of the gaussian action

$$\mathcal{A}^{\text{free}} = \frac{1}{4\pi} \int d^2x \left[(\partial_a \phi)^2 + (\partial_a \chi)^2 + (Q\phi + ie_0\chi)\hat{R}\sqrt{\hat{g}} \right] + \frac{1}{\pi} \int d^2x \left[\mathbf{b}\partial_{\bar{z}}\mathbf{c} + \bar{\mathbf{b}}\partial_z \bar{\mathbf{c}} \right]. \tag{2.1}$$

Here ϕ is the Liouville field, χ is the matter field, and $\{\mathbf{b}, \mathbf{c}\}$ is a pair of reparametrization ghosts of scaling dimensions $\{2, -1\}$. The reference scalar curvature $\hat{R}$ is localized at the infinite point. The Liouville and matter fields background charges Q and e_0 are parametrized by a real constant b,

$$Q = \tfrac{1}{b} + b, \qquad e_0 = \tfrac{1}{b} - b\,. \tag{2.2}$$

We will consider the generic situation when b^2 is not a rational number. With the choice (2.2) the full central charge is

$$c_{\text{tot}} \equiv c_M + c_L + c_{\text{ghosts}} = \left[13 - 6(b^2 + \tfrac{1}{b^2})\right] + \left[13 + 6(b^2 + \tfrac{1}{b^2})\right] - 26 = 0. \tag{2.3}$$

We consider a marginal deformation of the gaussian action (2.1) by the Liouville interaction and its matter counterpart, which is one of the two screening charges in the $c < 1$ CFT,

$$\mathcal{A}_{\text{int}} = \int d^2x \left(\mu_{_L} e^{2b\phi} + \mu_{_M} e^{-2ib\chi} \right), \tag{2.4}$$

$$\tilde{\mathcal{A}}_{\text{int}} = \int d^2x \left(\tilde{\mu}_{_L} e^{2\phi/b} + \tilde{\mu}_{_M} e^{2i\chi/b} \right). \tag{2.5}$$

In fact the interaction depends on the type of the correlators to be computed. In some cases we shall take into account one or the two of the terms in the "dual" action, but it will be reduced to a source of integer number of screening charges. We shall refer to the above theory, in which the Liouville and matter parts of the action factorize, as the "conventional" $c < 1$ string theory. In sect. 4.1 we will also introduce another interaction which does not have this factorization property.

2.2. Vertex operators for the closed string tachyons

We shall consider the BRST invariant fields that correspond to the vertex operators $e^{2ie\chi}$ of the matter CFT. In general, the matter vertex operators should be dressed by Liouville vertex operators, $e^{2\alpha\phi}$,

$$V_{e,\alpha} = \gamma(1-\alpha^2+e^2)\;\; e^{2ie\chi}\, e^{2\alpha\phi}, \tag{2.6}$$

so that the dressed operator has conformal weight $(1,1)$ [38,39]:

$$\triangle_M(e) + \triangle_L(\alpha) = e(e-e_0) + \alpha(Q-\alpha) = 1. \tag{2.7}$$

To have simpler expressions we normalized by the "leg" factors as in [34], where we used the standard notation $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

The condition (2.7) is the on-mass-shell condition for the closed string tachyons propagating in the euclidean $2D$ target space. The simplest examples satisfying it are the four operators in which only one of the two vertex operators in (2.6) appears, *i.e.*, the sources of the Liouville and matter screening charges in (2.4), (2.5). In general the solutions can be parametrized by the tachyon target space momentum P and the chirality $\varepsilon = \pm 1$[1]. The matter and Liouville charges are expressed in terms of P and ε as

$$e = \tfrac{1}{2}(e_0 - P)\,, \quad \alpha = \tfrac{1}{2}(Q - \varepsilon P) = \varepsilon\, e + b^{\varepsilon}\,, \qquad \varepsilon = \pm 1. \tag{2.8}$$

We denote by $V^{\varepsilon}_{\alpha} := V_{\varepsilon\alpha - \varepsilon b^{\varepsilon},\alpha}$ the vertex operators (2.6) with α and e related by (2.8). To compare with the microscopic theory it is convenient to introduce also the alternative notation $\mathcal{V}^{(\varepsilon)}_P$,

$$\mathcal{V}^{(\pm)}_P \equiv V^{\pm}_{\alpha} = \;\gamma(\pm b^{\pm 1}\, P)\; e^{i(e_0-P)\chi + (Q\mp P)\phi}\,, \tag{2.9}$$

where we used the relation

$$1-\alpha^2+e^2 = b^{\varepsilon}(Q-2\alpha) = \varepsilon b^{\varepsilon} P\,. \tag{2.10}$$

We shall not restrict in general to the "physical" Seiberg bound $\varepsilon P = Q - 2\alpha > 0$ [40].[2]

The BRST invariant operators can be represented in two pictures: either as (1,1)-forms integrated over the world-sheet, or as $Q_{\rm BRST}$-closed 0-forms:

$$\mathcal{T}^{(\pm)}_P \equiv T^{\pm}_{\alpha} = \int \tfrac{d^2x}{\pi}\, V^{\pm}_{\alpha} \qquad \text{or} \qquad \mathcal{W}^{(\pm)}_P \equiv W^{\pm}_{\alpha} = \mathbf{c}\bar{\mathbf{c}}\; V^{\pm}_{\alpha}\,. \tag{2.11}$$

In the n-point tachyon correlators, $n-3$ vertex operators are integrated over the worldsheet, and three are placed, as usual, at arbitrary points, say $0, 1$ and ∞, and the ghost zero mode contribution $\langle \mathbf{c}_{-1}\mathbf{c}_0\mathbf{c}_1\rangle$ is normalized to 1. The correlation function should not depend on the choice of the three operators.

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[1] For simplicity we shall assume, unless stated otherwise, that the momenta are real, $P \in \mathbb{R}$.

[2] We recall that in the rational minimal gravity [41] the operators corresponding to the solutions of (2.7) with e and $e_0 - e$ are identified, and one is left with the two Liouville dressings, distinguished by the sign of $Q/2 - \alpha$. We shall not assume in general such identification.

2.3. Normalization of the couplings and duality transformations

It is also convenient to redefine the couplings in the effective action (2.4) according to the normalizations (2.9) of the vertex operators,

$$\begin{aligned} \mu_{_L} \int e^{2b\phi} &= \lambda_{_L}\, T_b^+ = \lambda_{_L}\, \mathcal{T}_{e_0}^{(+)}, \qquad \mu_{_M} \int e^{-2ib\chi} = \lambda_{_M}\, T_0^+ = \lambda_{_M}\, \mathcal{T}_Q^{(+)} \\ \tilde\mu_{_L} \int e^{\frac{2\phi}{b}} &= \tilde\lambda_{_L}\, T_{1/b}^- = \tilde\lambda_{_L}\, \mathcal{T}_{e_0}^{(-)}, \qquad \tilde\mu_{_M} \int e^{\frac{2i\chi}{b}} = \tilde\lambda_{_M}\, T_0^- = \tilde\lambda_{_M}\, \mathcal{T}_{-Q}^{(-)}, \end{aligned} \tag{2.12}$$

where the new coupling constants are related to the old ones by

$$\begin{aligned} \lambda_{_L} \;&=\; \pi\gamma(b^2)\, \mu_{_L}, \quad \tilde\lambda_{_L} \;= \pi\gamma(\tfrac{1}{b^2})\, \tilde\mu_{_L}, \\ \lambda_{_M} &= \pi\gamma(-b^2)\, \mu_{_M}, \quad \tilde\lambda_{_M} = \pi\gamma(-\tfrac{1}{b^2})\, \tilde\mu_{_M}. \end{aligned} \tag{2.13}$$

All correlation functions in Liouville theory are invariant (for fixed charges) w.r.t. the substitution [5]

$$\begin{aligned} b &\to 1/b, \\ \lambda_{_L} \to \tilde\lambda_{_L} &= \lambda_{_L}^{1/b^2}\,. \end{aligned} \tag{2.14}$$

As we will see in the next section, for a consistent description of the matter correlatation functions one should introduce the dual matter coupling constant so that the functions in the $c<1$ theory obey the symmetry

$$\begin{aligned} b &\to -1/b\,, \\ \lambda_{_M} \to \tilde\lambda_{_M} &= (\lambda_{_M})^{-1/b^2}\,. \end{aligned} \tag{2.15}$$

The duality transformation (2.14) (or (2.15), or their composition) relates the tachyon correlators to those of a conjugated theory, obtained by flipping the sign of e_0 (or Q, or both) respectively; thus effectively we can restrict the real parameter b to the region $(0,1)$. On the other hand, the composition of (2.14) with $\chi \to -\chi$ and $\lambda_{_M} \to \tilde\lambda_{_M}$ preserves the free action (2.1) and interchanges the two interaction actions (2.4), (2.5). The same effect yields the matter duality transformation (2.15), accompanied with $\phi \to -\phi$ and $\lambda_{_L} \to \tilde\lambda_{_L}$. In parameter space these duality transformations are formulated as

$$\{\, b, \lambda_{_L}, \lambda_{_M},\ P_i,\ \varepsilon_i\} \;\to\; \{\tfrac{1}{b}, \tilde\lambda_{_L}, \tilde\lambda_{_M}, -P_i, -\varepsilon_i\}\,, \tag{2.16}$$

$$\{\, b, \lambda_{_L}, \lambda_{_M},\ P_i,\ \varepsilon_i\} \;\to\; \{-\tfrac{1}{b},\ \tilde\lambda_{_L}, \tilde\lambda_{_M}, P_i, -\varepsilon_i\}. \tag{2.17}$$

3. The tachyon 3-point function as a product of Liouville and matter 3-point functions

In this simplest case the correlation function factorizes to a product of the matter and Liouville three-point OPE constants

$$G_3^{\varepsilon_1\varepsilon_2\varepsilon_3}(\alpha_1,\alpha_2,\alpha_3) = \left\langle W_{\alpha_1}^{\varepsilon_1} W_{\alpha_2}^{\varepsilon_2} W_{\alpha_3}^{\varepsilon_3} \right\rangle = \frac{C^{\rm Liou}(\alpha_1,\alpha_2,\alpha_3) C^{\rm Matt}(e_1,e_2,e_3)}{\prod_{j=1}^{3}\gamma(\alpha_j^2 - e_j^2)} \,. \tag{3.1}$$

The case of the 3-point function is unique in the sense that for $n > 3$ the factorization holds only before the integration over the $n-3$ moduli while for $n < 3$ there is a residual conformal symmetry which does not allow the direct evaluation.

3.1. The case $\sum e_i = e_0$

First we assume that $\sum e_i = e_0$, in which case the matter 3-point OPE constant is equal to one. For the Liouville 3-point OPE constant we take the DOZZ formula [1,2]

$$C^{\rm Liou}(\alpha_1\,,\,\alpha_2\,,\,\alpha_3) = \left(\lambda_{_L}^{1/b}\, b^{2e_0}\right)^{Q-\alpha_1-\alpha_2-\alpha_3} \frac{\Upsilon(b)\,\Upsilon(2\alpha_1)\,\Upsilon(2\alpha_2)\,\Upsilon(2\alpha_3)}{\Upsilon(\alpha_{123}-Q)\,\Upsilon(\alpha_{23}^1)\,\Upsilon(\alpha_{13}^2)\,\Upsilon(\alpha_{12}^3)} \tag{3.2}$$

with notation $\alpha_{12}^3 = \alpha_1+\alpha_2-\alpha_3\,, \alpha_{123} = \alpha_1+\alpha_2+\alpha_3$, etc. Here α_i and e_i are solutions (2.8) of the mass-shell condition (2.7). Imposing the constraint

$$\sum_i e_i = e_0 \quad \Leftrightarrow \quad \sum_i \varepsilon_i(\alpha_i - b^{\varepsilon_i}) = e_0 \tag{3.3}$$

and using the basic property of the function $\Upsilon = \Upsilon_b = \Upsilon_{1/b}$,

$$\Upsilon(x+b^{\varepsilon})/\Upsilon(x) = b^{\varepsilon(1-2b^{\varepsilon}x)}\,\gamma(x\,b^{\varepsilon})\,, \qquad \varepsilon = \pm 1 \tag{3.4}$$

one checks that the constant (3.2) reduces to

$$C^{\rm Liou}(\alpha_1\,,\,\alpha_2\,,\,\alpha_3) = \frac{\lambda_{_L}^{\frac{1}{b}(Q-\alpha_1-\alpha_2-\alpha_3)}}{\prod_{i=1}^{3} b^{\varepsilon_i}\,\gamma[b^{\varepsilon_i}\,(Q-2\alpha_i)]}\,, \qquad \text{for} \quad \sum_i \varepsilon_i(\alpha_i - b^{\varepsilon_i}) = e_0\,. \tag{3.5}$$

Thus one finds for the 3-point function (3.1), using (2.10),

$$G_3^{\varepsilon_1\varepsilon_2\varepsilon_3}(\alpha_1,\alpha_2,\alpha_3) = \frac{\lambda_{_L}^{\frac{1}{b}(Q-\alpha_1-\alpha_2-\alpha_3)}}{b^{\varepsilon_1+\varepsilon_2+\varepsilon_3}}. \tag{3.6}$$

- *Comment:*

To compare with the expressions in [34] and the perturbative Coulomb gas computation let us choose, say, the action (2.4). The constraint (3.3) combined with $\sum_{i=1}^{3}\alpha_i - Q = -sb$ for a positive integer s implies $2\alpha_3 = 1/b - sb$ or $2\alpha_3 = -(s-1)b$ for the choice of chiralities $^{(++-)}$ or $^{(--+)}$ respectively. (For the other two choices $^{(\pm\pm\pm)}$ the above two conditions are inconsistent.) The Coulomb gas computation of the *unnormalized* by the leg factors 3-point functions gives for generic values of α_1, α_2 a finite expression for the case $^{(++-)}$ and zero for the case $^{(--+)}$.[3] On the other hand in both cases the *normalized* by leg factors correlators are rendered finite by the overall zero mode divergence factor [34] and this corresponds to our expression (3.6). For the dual interaction (2.5) the roles of $^{(++-)}$ and $^{(--+)}$ are interchanged.

3.2. A general formula for the matter 3-point function

Now consider the case of arbitrary matter charges $e_1\,, e_2\,, e_3$, when the matter OPE constant is no more equal to one. The general $c<1$ matter 3-point OPE constant satisfies the identities

$$
\begin{aligned}
\frac{C^{\rm Matt}(e_1-b\,,\,e_2\,,\,e_3)}{C^{\rm Matt}(e_1\,,\,e_2\,,\,e_3)} &= \frac{1}{b^4\lambda_{_M}}\,\frac{\gamma(b(2e_1-b))\,\gamma(b\,2e_1)\,\gamma(b(e_{23}^1+b))}{\gamma(b(e_{123}-e_0))\,\gamma(b(e_{12}^3))\,\gamma(b(e_{13}^2))} \\
\frac{C^{\rm Matt}(e_1+\frac{1}{b}\,,\,e_2\,,\,e_3)}{C^{\rm Matt}(e_1\,,\,e_2\,,\,e_3)} &= \frac{b^4}{\tilde\lambda_{_M}}\,\frac{\gamma(-\frac{1}{b}(2e_1+\frac{1}{b}))\,\gamma(-\frac{1}{b}2e_1)\,\gamma(-\frac{1}{b}(e_{23}^1-\frac{1}{b}))}{\gamma(-\frac{1}{b}(e_{123}-e_0))\,\gamma(-\frac{1}{b}(e_{12}^3))\,\gamma(-\frac{1}{b}(e_{13}^2))}\,.
\end{aligned}
\tag{3.7}
$$

The change of variables $b\to -1/b\,, \lambda_{_M}\to\tilde\lambda_{_M}$ interchanges the two relations (3.7). These functional relations come from the locality requirement on the 4-point matter functions with one of the two simplest degenerate fields $e=\frac{b}{2}\,, e=-\frac{1}{2b}$ inserted. Their derivation is analogous to the one for the Liouville case in [3], where the DOZZ formula (3.2) was reproduced as the unique solution of the $c_L>25$ functional relations for positive, irrational b^2. Identifying the dual coupling constant as in (2.15), the solution of (3.7) is expressed in terms of the Liouville constant $C^{\rm Liou}$ in (3.2), with $\alpha_i=\varepsilon_i e_i + b^{\varepsilon_i}, i=1,2,3,$ [35][4]

$$
C^{\rm Matt}(e_1\,,\,e_2\,,\,e_3) = \frac{\lambda_{_M}^{-\frac{1}{b}(e_0-\sum_i e_i)}}{\prod_{i=1}^3 b^{\varepsilon_i}\,\gamma(b^{\varepsilon_i}(Q-2\alpha_i))}\;\frac{\lambda_{_L}^{\frac{1}{b}(Q-\sum_i\alpha_i)}}{C^{\rm Liou}(\alpha_1\,,\,\alpha_2\,,\,\alpha_3)}\,. \tag{3.8}
$$

The relation holds for any choice of the three signs ε_i. The overall constant is fixed by

$$
C^{\rm Matt}(e_1\,,\,e_2\,,\,e_3)=1 \quad \text{for} \quad \sum_i e_i = e_0\,, \tag{3.9}
$$

which is checked using (3.5).

[3] As a consequence, the derivatives with respect to μ_L, i.e, the (unnormalized) n-point tachyon correlators $G_n^{--++\cdots+}(\alpha_1,\alpha_2,\alpha_3,b,\cdots,b)$ are all vanishing.

[4] The derivation of the matter constant $C^{\rm Matt}(e_1,e_2,e_3)$ with a different choice of the normalization, has been carried out independently by Al. Zamolodchikov, published in [33].

The formula (3.8) is obtained alternatively as analytic continuation of the particular (thermal) Dotsenko-Fateev constant computed with one of the matter screening charges, in full analogy with the derivation of (3.2) in [1,2]. For $\sum_i e_i - e_0 = mb - \frac{n}{b}$, n, m non-negative integers, the expression (3.8) for the matter structure constant is finite for generic b^2 and reproduces the 3-point Dotsenko-Fateev constant in (B.10) of [32], times the powers $(-\mu_{_M})^m(-\tilde{\mu}_{_M})^n$. In other words, in the Coulomb gas range of the three parameters e_i (3.8) can be looked as a compact representation of the DF constant. Introducing the function

$$\hat{\Upsilon}_b(x) := \frac{1}{\Upsilon_b(x+b)} = \frac{1}{\Upsilon_b(-x+\frac{1}{b})} = \hat{\Upsilon}_b(e_0 - x) = \hat{\Upsilon}_{\frac{1}{b}}(-x) \tag{3.10}$$

we can rewrite (3.8) in a form analogous to the DOZZ formula (3.2) [5]

$$C^{\rm Matt}(e_1\,,\,e_2\,,\,e_3) = \left(\lambda_{_M}^{1/b}\, b^{2Q}\right)^{e_1+e_2+e_3-e_0}\ \frac{\hat{\Upsilon}(0)\,\hat{\Upsilon}(2e_1)\,\hat{\Upsilon}(2e_2)\,\hat{\Upsilon}(2e_3)}{\hat{\Upsilon}(e_{123}-e_0)\,\hat{\Upsilon}(e_{23}^1)\,\hat{\Upsilon}(e_{13}^2)\,\hat{\Upsilon}(e_{12}^3)}\,. \tag{3.11}$$

The functional relations (3.4) are replaced by

$$\hat{\Upsilon}(x-b)/\hat{\Upsilon}(x) = \gamma(bx)\, b^{1-2bx}\,, \quad \hat{\Upsilon}(x+\tfrac{1}{b})/\hat{\Upsilon}(x) = \gamma(-\tfrac{1}{b}x)\, b^{-1-2x/b}\,. \tag{3.12}$$

The logarithm of the function $\hat{\Upsilon}(x)$ admits an integral representation as the one for the logarithm of Υ, with Q replaced by e_0 (so that it is invariant under the change $b \to -1/b$),

$$\log \hat{\Upsilon}_b(x) = -\int_0^\infty \frac{dt}{t}\left(\big(\frac{e_0}{2}-x\big)^2 e^{-t} - \frac{\sinh^2(\frac{e_0}{2}-x)\frac{t}{2}}{\sinh b\frac{t}{2}\,\sinh\frac{t}{2b}} \right) = \log \hat{\Upsilon}_{-\frac{1}{b}}(x)\,, \tag{3.13}$$

which converges (for $b > 0$) in the strip $-b < {\rm Re}\ x < \frac{1}{b}$.

- *Examples:*

$$\begin{aligned} C^{\rm Matt}(\tfrac{b}{2}, e, e_0 - e + \tfrac{b}{2}) &= -\mu_{_M}\,\pi\, \frac{\gamma((2e-e_0)b)}{\gamma(b^2)\,\gamma(2eb)}\,, \\ C^{\rm Matt}(-\tfrac{1}{2b}, e, e_0 - e - \tfrac{1}{2b}) &= -\tilde{\mu}_{_M}\,\pi\, \frac{\gamma((e_0-2e)\frac{1}{b})}{\gamma(\frac{1}{b^2})\,\gamma(-2e\frac{1}{b})}\,. \end{aligned} \tag{3.14}$$

- The Liouville three-point OPE constant satisfies the reflection property [2]

$$C^{\rm Liou}(\alpha_1\,,\,\alpha_2\,,\,\alpha_3) = \frac{1}{b^2}\,\lambda_{_L}^{\frac{Q-2\alpha_1}{b}}\ \frac{\gamma(\frac{1}{b}(2\alpha_1-Q))}{\gamma(b(Q-2\alpha_1))}\, C^{\rm Liou}(Q-\alpha_1\,,\,\alpha_2\,,\,\alpha_3)\,. \tag{3.15}$$

---

[5] This expression and the function (3.10) have been earlier considered, see e.g. [42], without discussion of the relation to the $c < 1$ DF constant.

This identity has been used to write the r.h.s. of (3.8) in various equivalent ways corresponding to the different choices (2.8) of the relation between e_i and α_i. Analogously,

$$C^{\rm Matt}(e_1\,,\,e_2\,,\,e_3) = b^2\,\lambda_{_M}^{\frac{2e_1-e_0}{b}}\;\frac{\gamma(b(2e_1-e_0))}{\gamma(\frac{1}{b}(2e_1-e_0))}\;C^{\rm Matt}(e_0-e_1\,,\,e_2\,,\,e_3). \tag{3.16}$$

In particular for $e_1=e_2=e, e_3=0$ (3.16) implies

$$C^{\rm Matt}(e\,,e\,,0) = b^2\,\lambda_{_M}^{\frac{2e-e_0}{b}}\;\frac{\gamma(b(2e-e_0))}{\gamma(\frac{1}{b}(2e-e_0))}\,. \tag{3.17}$$

The last formula reproduces the 2-point constant found in [43]. As pointed out in [43], the choice $e_3=0$ does not force $e_1=e_2$, or $e_1=e_0-e_2$; see also [42] for a different solution in the case $c=1$ and a different interpretation of the identity operator which avoids this problem, not essential for our purposes.[6] The construction of a consistent non-rational matter theory for arbitrary momenta is beyond the scope of this paper.

3.3. The tachyon 3-point function for generic momenta

Having evaluated the matter three-point function (3.8), we insert it together with (3.2) in (3.1) and obtain a simple expression for the tachyon 3-point function. If we factorize the dependence on the coupling constants $\lambda_{_L}\,,\lambda_{_M}$ and the powers of b, denoting the remaining momentum-dependent factor by N_{P_1,P_2,P_3}, we get

$$\begin{aligned}\mathcal{G}^{\varepsilon_1\varepsilon_2\varepsilon_3}(P_1\,,\,P_2\,,\,P_3) \;&=\; \frac{\lambda_L^{\frac{1}{2b}(\sum_i\varepsilon_i P_i - Q)}\;\lambda_M^{\frac{1}{2b}(e_0-\sum_i P_i)}}{b^{\varepsilon_1+\varepsilon_2+\varepsilon_3}}\,N_{P_1,P_2,P_3}\\ &= b^{-\sum_i\varepsilon_i}\,\lambda_L^{\frac{1}{b}(Q-\sum_i\alpha_i)}\;\lambda_M^{-\frac{1}{b}(e_0-\sum_i e_i)}\,,\end{aligned} \tag{3.18}$$

or $N_{P_1,P_2,P_3}=1$. This reduces to (3.6) for $\lambda_{_M}=1$. The simple 3-point function (3.18) satisfies reflection properties inherited from those of its Liouville (3.15) and matter (3.16) parts. That is, in the expectation value (3.1) given by (3.18) the following identities hold:

$$\mathcal{W}_P^{(+)} = b^{-2}\,\lambda_{_L}^{P/b}\;\mathcal{W}_P^{(-)} = b^{-2}\,\lambda_{_M}^{-P/b}\;\mathcal{W}_{-P}^{(-)} = \left(\lambda_{_L}/\lambda_{_M}\right)^{P/b}\;\mathcal{W}_{-P}^{(+)}. \tag{3.19}$$

[6] We recall that the DF constants [32], [44] for rational b^2 are nonvanishing for some values in the minimal spectrum, but beyond the restrictions of the fusion rules. In that sense these constants by themselves do not determine the fusion. However, due to certain identities satisfied by the constants and the fusing matrix elements, there occur cancellations in the block expansion of the local 4-point functions, so that each channel is consistent with the fusion rules; see e.g. [45].

Notice however that the relations (3.19) do not necessarily hold within the 4-point functions, as will be discussed below.

• Given the expression (3.18) for the 3-point constant, the 2-point tachyon correlators are conventionally defined [1] as integrals of 3-point ones over some of the interaction constants. For example, the 2-point tachyon function for $e_1 + e_2 = e_0$ is determined from $G_3^{++-}(b, \alpha, \alpha) = -\partial_{\lambda_L} G_2^{+-}(\alpha, \alpha)$, $\alpha = e_1 + b = Q/2 - P/2$, i.e.

$$\begin{aligned} G_2^{+-}(\alpha, \alpha) = \mathcal{G}_2^{+-}(P, -P) &= -\frac{\lambda_L^{P/b}}{P} = -\frac{\lambda_L^{\frac{1}{b}(Q-2\alpha)}}{(Q-2\alpha)} \\ &= (Q-2\alpha)\, b^2 \lambda_L^{\frac{1}{b}(Q-2\alpha)} \langle 0|\mathbf{c}(z)\partial_z \bar{\mathbf{c}}(\bar{z})\partial_{\bar{z}} W_\alpha^+(z,\bar{z}) W_{Q-\alpha}^+(z',\bar{z}')|0\rangle \,. \end{aligned} \tag{3.20}$$

The composition of matter times Liouville reflections reproduces up to a sign the correlator $G_2^{+-}(e, e)$ determined analogously from $G_3^{++-}(0, \alpha, Q-\alpha) = -\partial_{\lambda_M} G_2^{+-}(\alpha, Q-\alpha)$.

The same convention leads to the partition function Z and its dual $\tilde{Z}$ defined as

$$\begin{aligned} -\partial^3_{\lambda_L} Z(\lambda_L, \lambda_M, b) = < W_b^+ W_b^+ W_b^+ > \ &\Rightarrow\ Z(\lambda_L, \lambda_M, b) = -\frac{b}{Qe_0}\, \lambda_L^{\frac{Q}{b}} \lambda_M^{\frac{-e_0}{b}} \,, \\ -\partial^3_{\tilde{\lambda}_L} \tilde{Z}(\lambda_L, \lambda_M, b) = < W^-_{\frac{1}{b}} W^-_{\frac{1}{b}} W^-_{\frac{1}{b}} > \ &\Rightarrow \tilde{Z}(\lambda_L, \lambda_M, b) = \frac{1}{b\,Qe_0} \tilde{\lambda}_L^{b\,Q} \tilde{\lambda}_M^{be_0} = \frac{1}{b\,Qe_0} \lambda_L^{\frac{Q}{b}} \lambda_M^{-\frac{e_0}{b}} \,, \\ Z(\tilde{\lambda}_L, \tilde{\lambda}_M, \tfrac{1}{b}) = \tilde{Z}(\lambda_L, \lambda_M, b) &= -\tfrac{1}{b^2}\, Z(\lambda_L, \lambda_M, b). \end{aligned} \tag{3.21}$$

We can then introduce normalized functions

$$\begin{aligned} \frac{G_3^{\varepsilon_1\varepsilon_2\varepsilon_3}(\alpha_1, \alpha_2, \alpha_3; \lambda_L, \lambda_M, b)}{Z(\lambda_L, \lambda_M, b)} &= -b^{-\sum_i \varepsilon_i - 1}\, e_0\, Q\, \lambda_L^{-\frac{\sum_i \alpha_i}{b}} \lambda_M^{\frac{\sum_i e_i}{b}} \,, \\ \frac{G_3^{\varepsilon_1\varepsilon_2\varepsilon_3}(\alpha_1, \alpha_2, \alpha_3; \lambda_L, \lambda_M, b)}{\tilde{Z}(\lambda_L, \lambda_M, b)} &= b^{-\sum_i \varepsilon_i + 1}\, e_0\, Q\, \lambda_L^{-\frac{\sum_i \alpha_i}{b}} \lambda_M^{\frac{\sum_i e_i}{b}} \,. \end{aligned} \tag{3.22}$$

These correlators are interchanged by the duality transformations (2.16), (2.17), which become equivalent since (3.22) is expressed only through the variables [7]

$$\{b^2, \lambda_L, \lambda_M, bP_i, \varepsilon_i\} \to \{\tfrac{1}{b^2}, \tilde{\lambda}_L, \tilde{\lambda}_M, -\tfrac{P_i}{b}, -\varepsilon_i\} \,. \tag{3.23}$$

---

[7] The first duality relation is satisfied by the constants (3.18), but the second holds up to a sign, *i.e.* $\mathcal{G}^{\{\varepsilon_i\}}(\{P_i\}; b, \lambda_L, \lambda_M) = \mathcal{G}^{\{-\varepsilon_i\}}(\{-P_i\}; \frac{1}{b}, \tilde{\lambda}_L, \tilde{\lambda}_M) = -\mathcal{G}^{\{-\varepsilon_i\}}(\{P_i\}; -\frac{1}{b}, \tilde{\lambda}_L, \tilde{\lambda}_M)$. Nevertheless to simplify notation we shall work with the unnormalized correlators or with other normalizations. Another possible though less intuitive definition is to relate the partition function to the correlator $\langle W_b^- W_b^+ W_b^+ \rangle$; according to (3.19) this removes a factor $b^{-2}\lambda_M^{-e_0/b}$ from Z.

Furthermore each of the simple constants (3.22) is invariant under analytic continuation $b \to \pm ib$, transforming the charges and the coupling constants as

$$\{ b, \lambda_{_L}, \lambda_{_M}, e_i, \alpha_i\} \;\to\; \{\pm ib, \lambda_{_M}, \lambda_{_L}, \mp i\alpha_i, \mp ie_i\} \tag{3.24}$$

or, in terms of target space momenta, $P_i \to \mp i\varepsilon_i P_i$, $\varepsilon_i \to \varepsilon_i$. This matter-Liouville duality transformation reflects the invariance of the actions (2.1), (2.4) and (2.5) under the respective interchange of the matter and Liouville fields $\{\chi, \phi\} \to \{\pm\phi, \mp\chi\}$.

- The final simple expression for the 3-point functions (3.18) satisfies the identities

$$\lambda_{_M}\, G_3^{-++}(\alpha_1, \alpha_2 - \tfrac{b}{2}, \alpha_3) = G_3^{-++}(\alpha_1 - \tfrac{b}{2}, \alpha_2, \alpha_3) = \lambda_{_L}\, G_3^{-++}(\alpha_1, \alpha_2 + \tfrac{b}{2}, \alpha_3)\,, \tag{3.25}$$

and hence,

$$G_3^{-++}(\alpha_1, \alpha_2 - b, \alpha_3) = \frac{\lambda_{_L}}{\lambda_{_M}}\, G_3^{-++}(\alpha_1, \alpha_2, \alpha_3)\,. \tag{3.26}$$

The last identity is also a direct consequence of the matter functional relations (3.7) and the corresponding Liouville ones [3], and can be used itself as a relation determining the 3-point tachyon correlator.

However the r.h.s. of the matter (3.7) or the corresponding Liouville functional relations may become singular, so both the functional relation (3.26) and the simple solution (3.18) are valid for generic momenta. For momenta such that some of the factors in (3.1) becomes singular, there is a $0 \times \infty$ indeterminacy. This ambiguity leads us to reconsider the problem of determining the tachyon 3-point function and not rely on factorization. Then the arbitrary "multiplicity" factors N_{P_1,P_2,P_3} in the first line in (3.18) must satisfy a pair of difference equations, to be derived below as part of the set of functional identities for the n-point tachyon correlators. These equations are weaker than (3.25), (3.26):

$$N_{P_1+b^\varepsilon,P_2,P_3} + N_{P_1-b^\varepsilon,P_2,P_3} = N_{P_1,P_2+b^\varepsilon,P_3} + N_{P_1,P_2-b^\varepsilon,P_3} \qquad \varepsilon = \pm 1\,. \tag{3.27}$$

The expression (3.18), *i.e.* $N_{P_1,P_2,P_3} = 1$, is only the simplest of their solutions. We shall deal with basically two deviations from this generic solution. One is the case when the factor N_{P_1,P_2,P_3} has the meaning of a fusion multiplicity and can take values 1 or 0. In the second, this factor will be rather a distribution. Thus in the simplest example of a gaussian matter $N_{P_1,P_2,P_3} = \delta(P_1 + P_2 + P_3 - e_0)$ replaces the normalization condition (3.9). The duality relations for the 3-point correlators are preserved if $N_{P_1,P_2,P_3}(b) = N_{P_1,P_2,P_3}(-\frac{1}{b}) = N_{-P_1,-P_2,-P_3}(\frac{1}{b})$.

4. The ground ring

The ground ring operators are BRST invariant fields obtained by applying raising operators of level $rs-1$ to the product of two degenerate matter and Liouville fields with Kac labels r,s. The resulting operators have conformal weights $(0,0)$, see [46] for an explicit construction of some of the corresponding states. The ring is generated by the lowest two operators $a_\pm = a_\pm(z)\,\bar a_\pm(\bar z)$ [14],

$$\begin{aligned} a_-(z) &= :\Big(\mathbf{b}(z)\mathbf{c}(z) - \tfrac{1}{b}\partial_z\big[\phi(z)+i\chi(z)\big]\Big)e^{-b(\phi(z)-i\chi(z))}: \\ a_+(z) &= :\Big(\mathbf{b}(z)\mathbf{c}(z) - b\,\partial_z\big[\phi(z)-i\chi(z)\big]\Big)e^{-\frac{1}{b}(\phi(z)+i\chi(z))}: \end{aligned} \tag{4.1}$$

The derivatives $\partial_z a_\pm$ and $\partial_{\bar z} a_\pm$ are $Q_{\rm BRST}$-exact, and therefore any amplitude that involves $a_\pm$ and other BRST invariant operators does not depend on the position of $a_\pm$. This property allows to write recurrence equations for the correlation functions from the OPE of $a_\pm$ and the tachyons W^ε_α [16,17,18,25], which will be generalized below.

4.1. The action of the ring generators on the tachyons

The recurrence equations were initially derived for the free fields with no interaction, or at most accounting for the perturbative first order contribution of the Liouville interaction. The momenta were therefore assumed to satisfy the charge conservation, or "neutrality" condition

$$\tfrac{1}{2}\textstyle\sum_i (e_0 - P_i) \equiv \sum_i e_i = e_0. \tag{4.2}$$

More generally, treating the Liouville and matter screening charges in (2.4) and (2.5) as perturbations amounts to modifying the original ring generators as

$$\begin{aligned} a_- &\to a_-\left(1-\lambda_{_L}\, T^+_b + \ldots\right) = \hat a_-\,, \\ a_+ &\to a_+\left(1-\tilde\lambda_{_L}\, T^-_{1/b} + \ldots\right) = \hat a_+\,. \end{aligned} \tag{4.3}$$

Summarizing, for generic momenta, *i.e.*, taking any complex values excluding the lattice

$$\mathcal{L} := \mathbb{Z}b + \mathbb{Z}\tfrac{1}{b}, \tag{4.4}$$

one finds that the action of the ring generators on the tachyons $W^\varepsilon_\alpha = \mathbf{c}\bar{\mathbf{c}}\; V^\varepsilon_\alpha$ of given chirality contains two terms, up to $Q_{\rm BRST}$ commutators:

$$\begin{aligned} \hat a_-\, W^+_\alpha &= -\lambda_{_L}\, W^+_{\alpha+\frac{b}{2}} - \lambda_{_M} W^+_{\alpha-\frac{b}{2}} \\ \hat a_-\, W^-_\alpha &= -\lambda_{_L}\,\lambda_{_M}\, W^-_{\alpha+\frac{b}{2}} - W^-_{\alpha-\frac{b}{2}} \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
\hat{a}_+ W^-_\alpha &= -\tilde{\lambda}_{\scriptscriptstyle L}\ W^-_{\alpha+\frac{1}{2b}} - \tilde{\lambda}_{\scriptscriptstyle M} W^-_{\alpha-\frac{1}{2b}} \\
\hat{a}_+ W^+_\alpha &= -\tilde{\lambda}_{\scriptscriptstyle L}\, \tilde{\lambda}_{\scriptscriptstyle M}\, W^+_{\alpha+\frac{1}{2b}} - W^+_{\alpha-\frac{1}{2b}} \,,
\end{aligned}
\tag{4.6}
$$

or, in the alternative notation,

$$
\begin{aligned}
\hat{a}_- \mathcal{W}^{(+)}_P &= -\lambda_{\scriptscriptstyle L}\, \mathcal{W}^{(+)}_{P-b} - \lambda_{\scriptscriptstyle M} W^{(+)}_{P+b} \\
\hat{a}_- \mathcal{W}^{(-)}_P &= -\mathcal{W}^{(-)}_{P-b} - \lambda_{\scriptscriptstyle L}\, \lambda_{\scriptscriptstyle M}\, \mathcal{W}^{(-)}_{P+b}
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\hat{a}_+ \mathcal{W}^{(-)}_P &= -\tilde{\lambda}_{\scriptscriptstyle L}\ \mathcal{W}^{(-)}_{P+\frac{1}{b}} - \tilde{\lambda}_{\scriptscriptstyle M} \mathcal{W}^{(-)}_{P-\frac{1}{b}} \\
\hat{a}_+ \mathcal{W}^{(+)}_P &= -\mathcal{W}^{(+)}_{P+\frac{1}{b}} - \tilde{\lambda}_{\scriptscriptstyle L}\, \tilde{\lambda}_{\scriptscriptstyle M}\, \mathcal{W}^{(+)}_{P-\frac{1}{b}} \,.
\end{aligned}
\tag{4.8}
$$

The duality transformations interchange the two ring generators and the two pairs of module relations.

The OPE coefficients in (4.5), (4.6) are found either by direct evaluation of the 3-point function of the ring generator and two tachyons in the presence of a number of screening charges, or by exploiting the factorization to the related known $c < 1$ and $c > 25$ Coulomb gas 3-point constants, see Appendix A for more details. The coefficients, say in (4.5), $C^{(\varepsilon,\varepsilon)\ \alpha'}_{-\frac{b}{2}\ \alpha}$, are expressed as products of the corresponding matter and Liouville constants

$$
\begin{aligned}
C^{(\varepsilon,\varepsilon)\ \alpha-\eta\frac{b}{2}}_{-\frac{b}{2}\ \alpha} = -\lambda_{\scriptscriptstyle M}^{\frac{1+\varepsilon\eta}{2}} \lambda_{\scriptscriptstyle L}^{\frac{1-\eta}{2}} \ &= \ \frac{\gamma(b^\varepsilon(Q-2\alpha))}{\gamma(b^\varepsilon(Q-2\alpha+\eta b))}\, \frac{(Q-2\alpha)^2}{b^2} \\
\times C^{\rm Matt}(\tfrac{b}{2}, e, e_0 - e + \varepsilon\eta\, \tfrac{b}{2})\ &\hat{C}^{\rm Liou}(-\tfrac{b}{2}, \alpha, Q-\alpha+\eta\, \tfrac{b}{2})\,, \qquad \eta = \pm 1\,.
\end{aligned}
\tag{4.9}
$$

Similar formula holds for (4.6). The matter OPE constants in (4.9) are either 1, as in (3.9), or given by the first example in (3.14). The constants $\hat{C}^{\rm Liou}(\alpha_1, \alpha_2, \alpha_3)$ with $\sum_i \alpha_i - Q = (\eta - 1)\,\frac{b}{2}$ are the analogous $c > 25$ Coulomb gas expressions, which are alternatively obtained as residue of the Liouville constant $C^{\rm Liou}$ in (3.2).

- *The case of momenta in the lattice $\mathcal{L}$*

For some momenta $P \in \mathcal{L}$ on the lattice (4.4), the free field 3-point function determining the OPE coefficients is nonvanishing for more than two values, leading to additional terms in (4.5) and (4.6). This typically requires an integer power of one of the screening charges in the dual interaction action, while the generic OPE (4.5) and (4.6) correspond to deformations with the respective actions (2.4), (2.5); see Appendix A.2 for details. The additional OPE terms correspond to reflections with respect to the matter or Liouville, or both, charges of the terms in (4.5) and (4.6). In the first two cases the chirality is inverted.

Let us restrict the consideration to momenta labelled by degenerate matter representations

$$
\mathcal{L}^\pm_M := \Big\{ P \equiv e_0 - 2e = \pm(\frac{n}{b} - mb) \Big\}_{m,n\in\mathbb{N}} \subset \mathcal{L}. \tag{4.10}
$$

For one of the signs in (4.10), $P = P_{m,n} = n/b - mb$, the r.h.s. of (4.5) and (4.6) contains two more terms, [8] while for the other sign, $P = mb - n/b$, the generic formulae (4.5), (4.6) hold. For example,

$$W^{+}_{\alpha\pm\frac{b}{2}} \;\;\to\;\; W^{+}_{\alpha\pm\frac{b}{2}} + \frac{\lambda_M^{\frac{1}{b}(2e\pm b-e_0)}}{b^2}\;\; W^{-}_{\alpha\pm\frac{b}{2}} \qquad 2\alpha - Q = \tfrac{(1\pm1)b}{2} + mb - \tfrac{n}{b}\,, \tag{4.11}$$

$$W^{-}_{\alpha\pm\frac{b}{2}} \to W^{-}_{\alpha\pm\frac{b}{2}} + b^2\,\lambda_M^{\frac{1}{b}(2e\mp b-e_0)}\,W^{+}_{\alpha\pm\frac{b}{2}}\,, \qquad 2\alpha - Q = \tfrac{(1\mp1)b}{2} - \tfrac{mb}{2} + \tfrac{n}{2b}. \tag{4.12}$$

In (4.11) and (4.12) appear the combinations invariant under matter reflection,

$$\begin{aligned}\tilde W^{+}_{\alpha} &:= b\lambda_M^{\frac{1}{2b}(e_0-2e)}\,W^{+}_{\alpha} + \tfrac{1}{b}\,\lambda_M^{\frac{1}{2b}(2e-e_0)}\,W^{-}_{\alpha}\\ &= b\lambda_M^{\frac{1}{2b}(e_0-2e)}\,\gamma[b(Q-2\alpha)]\Big(e^{2ie\chi} + C^{\rm Matt}(e,e,0)\,e^{2i(e_0-e)\chi}\Big)e^{2\alpha\phi}\end{aligned} \tag{4.13}$$

or, in terms of momenta,

$$\tilde{\mathcal W}^{(\pm)}_{P_{m,n}} = b^{\pm1}\,\lambda_M^{\frac{1}{2b}P_{m,n}}\,\mathcal W^{(\pm)}_{P_{m,n}} + \tfrac{1}{b^{\pm1}}\,\lambda_M^{-\frac{1}{2b}P_{m,n}}\,\mathcal W^{(\mp)}_{-P_{m,n}}\,, \qquad P_{m,n} = n/b - mb\,.$$

The relative constant for the unnormalized vertex operators in this linear combination is given by the 2-point function (3.17).

A similar argument can be carried out for the momenta labelled by degenerate Liouville representations

$$\mathcal L^{\pm}_L := \Big\{\varepsilon P \equiv Q - 2\alpha = \pm(mb + \frac{n}{b})\Big\}_{m,n\in\mathbb N} \subset \mathcal L\,. \tag{4.14}$$

In this case Liouville reflected terms appear in the OPEs which correspond to the plus sign in (4.14).

In fact the appearance of the Liouville or matter reflected points is universal and is a consequence of the properties (3.15) and (3.16) of the 3-point functions (3.2) and (3.11); the only peculiarity of the degenerate cases discussed here is that both OPE coefficients are given by a Coulomb gas 3-point correlator, while in general the reflection images correspond to functions satisfying a relation obtained by a reflection from the Coulomb gas charge conservation condition. Taking this into account in particular removes the above asymmetry of the OPEs for the tachyons of momenta $P_{m,n}$ and $-P_{m,n}$, related by a matter charge reflection. To make sense of these relations one should be able to identify in the correlators the matter reflected tachyons in (4.13) (or the Liouville reflected ones in the case of degenerate Liouville case (4.14)). At this stage we shall merely assume that the action of the ring generators is given again by the generic formulae (4.5) and (4.6). This is analogous to what is done in field theory, where only one of the two charges of the same dimension is included in the block decomposition of the 4-point functions.

[8] This happens as well for the border lines $m = 0$, $n = 0$ outside of (4.10).

4.2. The ground ring at non-rational b^2

Assuming that the tachyons at the border lines of the degenerate set (4.10) vanish (at least in the averages), one gets a semi-infinite set, in one to one correspondence with the irreps of $sl(2)\times sl(2)$. The modules of given chirality are generated from the corresponding tachyon of momentum $P = e_0$ serving as an identity. After absorbing the constant $\lambda_{_L}$ in the normalization of the vertex operators, the relations (4.5) and (4.6) are equivalent to the multiplication rule of the characters of $sl(2)$ irreps of dimensions respectively m and n, with the character of the fundamental representation of dimension 2. It allows to represent any character as a polynomial of the fundamental one - the above rule is the functional identity defining the Chebyshev polynomials U_{m-1} of second kind. Analogously (4.5) and (4.6) imply (setting $\lambda_{_M} = 1$)

$$\mathcal{W}^{(\varepsilon)}_{P_{mn}} = \lambda_{_L}^{\frac{1}{2b}\varepsilon(P_{mn}-e_0)}\, U_{m-1}(\tfrac{1}{2}\mathcal{O}_{21}))\, U_{n-1}(\tfrac{1}{2}\mathcal{O}_{12})\, \mathcal{W}^{(\varepsilon)}_{e_0}, \tag{4.15}$$

with $\mathcal{O}_{21} = -\lambda_{_L}^{-\frac{1}{2}}\hat{a}_-$, $\mathcal{O}_{12} = -\tilde{\lambda}_{_L}^{-\frac{1}{2}}\hat{a}_+$. The polynomial acting on the tachyon $\mathcal{W}^{(\varepsilon)}_{e_0}$ represents the ground ring element $\mathcal{O}_{mn}$. The formula (4.15) derived from (4.5) and (4.6) confirms the Ansatz in [24] used in the context of the minimal string theory, see also [37]. The equalities (4.5), (4.6), and hence (4.15), all hold true up to Q-exact terms, which in general disappear only in the 3-point tachyon functions.

4.3. Further OPE channels

The two-term relations (4.5) and (4.6) describe the OPEs of the ring generators $a_\pm$ perturbed by the screening charges in the action. In presence of one or more tachyons, given by an integrated vertex operators, as happens in any n-point function with $n > 3$, the OPE will contain more terms. Indeed now any integrated tachyon serves as a "screening charge". Instead of computing explicitly in an operator form the Q-exact terms, appearing in the product of free exponential fields, and then moving them to the right or left, one can compute all possible OPE relations which send a tachyon W^{ε}_{α} to a tachyon $W^{\varepsilon'}_{\alpha'}$

$$a_\pm W^{\varepsilon}_{\alpha} T^{\varepsilon_1}_{\alpha_1}\dots T^{\varepsilon_t}_{\alpha_t} \;\to\; W^{\varepsilon'}_{\alpha'}\,. \tag{4.16}$$

Thus the effect of the skipped Q - exact terms (if any) is already accounted for in the added new OPE channels, which in turn are valid again up to such terms.

The mass-shell condition implies a relation on the possible combinations of chiralities and momenta in (4.16), see Appendix A.3 for a summary of the consequences of these constraints. The coefficients of these OPEs are computed from free field correlators, see Appendix A.4, here we summarize these results.

The simplest example consistent with the mass-shell condition is given by the OPE relations [18,17]

$$a_- W^+_\alpha T^+_{\alpha_1} = W^+_{\alpha+\alpha_1-\frac{b}{2}}\,, \qquad a_+ W^-_\alpha T^-_{\alpha_1} = W^-_{\alpha+\alpha_1-\frac{1}{2b}}\,. \tag{4.17}$$

The relations (4.17) have been already used for particular values of α_1 and the chiralities in the derivation of the linear terms in the first lines of (4.5), (4.6). They are generalized for generic values of α to a whole series, with $k = 0, 1, 2, \ldots,$ [9]

$$a_- W^+_\alpha T^+_{\alpha_1} \frac{(T^-_0 T^-_{1/b})^k}{k!^2} = W^+_{\alpha+\alpha_1-\frac{b}{2}+\frac{k}{b}}\,, \tag{4.18}$$

$$a_+ W^-_\alpha T^-_{\alpha_1} \frac{(T^+_0 T^+_b)^k}{(k!)^2} = W^-_{\alpha+\alpha_1-\frac{1}{2b}+kb}\,. \tag{4.19}$$

Taken for different k the relations demonstrate the effect of the Q-exact terms,

$$\begin{aligned} a_+ W^-_\alpha T^-_{\alpha_1}\, (T^+_0 T^+_b)^k &= (0+\ldots) T^-_{\alpha_1}\, (T^+_0 T^+_b)^k = (W^-_{\alpha+\alpha_1-\frac{1}{2b}} + \ldots)\, (T^+_0 T^+_b)^k \\ &= (p!^2 \binom{k}{p}^2 W^-_{\alpha+\alpha_1-\frac{1}{2b}+pb} + \ldots)\, (T^+_0 T^+_b)^{k-p} = k!^2\, W^-_{\alpha+\alpha_1-\frac{1}{2b}+kb} + \ldots \end{aligned} \tag{4.20}$$

For the product of interacting fields we obtain combining with (4.5), (4.6),

$$\begin{aligned} \hat{a}_-\, W^+_\alpha T^+_{\alpha_1} &= \sum_{\alpha'} C^{(++)\alpha'}_{-\frac{b}{2}\,\alpha}\; W^+_{\alpha'}\, T^+_{\alpha_1} + \sum_{\alpha'} C^{(+++)\ \alpha'}_{-\frac{b}{2}\,\alpha\,\alpha_1}\; W^+_{\alpha'} \\ &= (-\lambda_{\scriptscriptstyle L}\, W^+_{\alpha+\frac{b}{2}} - \lambda_{\scriptscriptstyle M} W^+_{\alpha-\frac{b}{2}})\, T^+_{\alpha_1} + \sum_{k=0} (\tilde{\lambda}_{\scriptscriptstyle L}\, \tilde{\lambda}_{\scriptscriptstyle M})^k\, \mathcal{W}^+_{P+P_1-\frac{2k+1}{b}} \\ \hat{a}_+\, W^-_\alpha T^-_{\alpha_1} &= \sum_{\alpha'} C^{(--)\ \alpha'}_{-\frac{1}{2b}\,\alpha}\; W^-_{\alpha'}\, T^-_{\alpha_1} + \sum_{\alpha'} C^{(---)\ \ \alpha'}_{-\frac{1}{2b}\,\alpha\,\alpha_1}\; W^-_{\alpha'} \\ &= (-\tilde{\lambda}_{\scriptscriptstyle L}\ \mathcal{W}^{(-)}_{P+\frac{1}{b}} - \tilde{\lambda}_{\scriptscriptstyle M} \mathcal{W}^{(-)}_{P-\frac{1}{b}})\, T^+_{\alpha_1} + \sum_{k=0} (\lambda_{\scriptscriptstyle L}\, \lambda_{\scriptscriptstyle M})^k\, \mathcal{W}^-_{P+P_1+(2k+1)b}\,. \end{aligned} \tag{4.21}$$

The 4-point OPE coefficients in (4.21) are expressed in terms of products of matter and Liouville Coulomb gas 3-point constants, see formula (A.30) below. The relations hold for values for which each of those constants is well defined. In particular (4.18) extends to degenerate values of α (with shifted compared with (4.10), (4.14) notation),

$$P = \tfrac{n+1}{b} \pm (m+1)b\,,\ m,n \in \mathbb{Z}_{\geq 0}\,, \tag{4.22}$$

[9] The first nontrivial example $k = 1$ of these OPE coefficients has been computed by P. Furlan.

for any $k \leq n$. Similarly (4.19) extends for $k \leq m$ if

$$P = \mp \tfrac{n+1}{b} - (m+1)b\,,\ m, n \in \mathbb{Z}_{\geq 0}\,. \tag{4.23}$$

For these values the infinite sums in (4.21) truncate to the first $n+1$ (respectively $m+1$) terms, see Appendix A.4. We can interpret the set $\{\mathcal{W}^{+}_{P_1 - \frac{1}{b} + P - \frac{2k}{b}}\,, k = 0, 1, 2, \ldots\}$ as the states of a $sl(2)$ Verma module of h.w. $bP = n+1$. The state $k = n+1$ of weight $-bP$ corresponds to the singular vector and it is set to zero for the $n+1$- dimensional irrep.

• The kinematical mass-shell constraints imply that for generic momenta $P_\alpha, P_{\alpha_1} \notin \mathcal{L}$ and $P_\alpha + P_{\alpha_1} \notin \mathcal{L}$ the identities (4.21) exhaust all OPEs in presence of one integrated tachyon. Furthermore under the same type of restrictions there is no contribution of two or more such integrated tachyons to the OPE of the interacting fields. Therefore for a product of p tachyons $T^{\varepsilon}_{\alpha_i}$ there are p terms in the OPE of the type in (4.21).

However for values in $\mathcal{L}$ there are more solutions already for the case of one integrated tachyon. In particular its momentum P_1 can be given any $c < 1$ or $c > 25$ degenerate value. The OPE coefficients in this case are computed for generic values of α and any integers $n, m \geq 0$:

$$\begin{aligned}
\hat{a}_{-}\, W^{-}_{\alpha}\, T^{+}_{\alpha_1} - \sum_{\alpha'} C^{(--)\ \ \alpha'}_{-\frac{b}{2}\,\alpha\,\alpha_1}\, W^{-}_{\alpha'}\, T^{+}_{\alpha_1} &= \sum_{\alpha'} C^{(-+-)\ \alpha'}_{-\frac{b}{2}\,\alpha\,\alpha_1}\, W^{-}_{\alpha'} \\
&= \sum_{s=0}^{n} \tilde{\lambda}^{n-s}_{M}\, \tilde{\lambda}^{s}_{L}\, W^{-}_{\alpha + \frac{(m+1)b}{2} + \frac{2s-n}{2b}} \begin{cases} \lambda^{m+1}_{M} & \text{if } P_1 = \frac{n+1}{b} - (m+1)b \\ \lambda^{m+1}_{L} & \text{if } P_1 = \frac{n+1}{b} + (m+1)b \end{cases} \\
\hat{a}_{+}\, W^{+}_{\alpha}\, T^{-}_{\alpha_1} - \sum_{\alpha'} C^{(++)\ \ \alpha'}_{-\frac{1}{2b}\,\alpha}\, W^{+}_{\alpha'}\, T^{-}_{\alpha_1} &= \sum_{\alpha'} C^{(+-+)\ \ \alpha'}_{-\frac{1}{2b}\,\alpha\,\alpha_1}\, W^{+}_{\alpha'} \\
&= \sum_{k=0}^{m} \lambda^{k}_{M}\, \lambda^{m-k}_{L}\, W^{-}_{\alpha + \frac{(m-2k)b}{2} + \frac{n+1}{2b}} \begin{cases} \tilde{\lambda}^{n+1}_{M} & \text{if } P_1 = \frac{n+1}{b} - (m+1)b \\ \tilde{\lambda}^{n+1}_{L} & \text{if } P_1 = -\frac{n+1}{b} - (m+1)b \end{cases}
\end{aligned} \tag{4.24}$$

More generally, one can have a product of any number of arbitrary tachyons with partial sums of momenta in $\mathcal{L}$, depending on the chiralities. The simplest computable examples with two integrated tachyons are given by

$$\begin{aligned}
a_{-}\, W^{-}_{\alpha}\, T^{+}_{\alpha_1}\, T^{+}_{\alpha_2} &= -W^{-}_{\alpha + \frac{b}{2}}\,, \qquad \alpha_1 + \alpha_2 = b\,, \\
a_{+}\, W^{+}_{\alpha}\, T^{-}_{\alpha_1}\, T^{-}_{\alpha_2} &= -W^{+}_{\alpha + \frac{1}{2b}}\,, \qquad \alpha_1 + \alpha_2 = \tfrac{1}{b}\,,
\end{aligned} \tag{4.25}$$

or $P_1 + P_2 = 2/b, 2b$ respectively. These identities were used for $\alpha_1 = 0$ in the derivation of the last terms in (4.5) and (4.6); for this value they reduce to (4.24). There are also cases in which the chirality of the tachyon in the OPE is inverted. In our consideration below we shall restrict to combinations of momenta which allow at most the basic series in (4.21), (4.24).

4.4. Diagonal ground ring

As we mentioned in the Introduction, it is possible to construct a discrete model of non-rational 2D quantum gravity in which the order operators, *i.e.* the degenerate fields labelled by the diagonal ($m = n$) of the infinite Kac table, have a simple realization as observables.

It happens that in this theory the 4-point function of order fields contains only order fields in the intermediate channels. Therefore in the corresponding CFT the order field tachyons must form a closed algebra under OPE. This is not possible in the matter CFT on a rigid surface, where the OPE of the diagonal fields generates the whole spectrum of degenerate fields. The question arises, is it possible, after switching on the Liouville field, that the order fields form a closed algebra? We shall argue that such a theory exists.

First we notice that the ground ring element $\mathcal{O}_{2,2}$ obtained by combining (4.5) and (4.6), has four term OPE with the tachyons of given chirality. They involve shifts of the momenta with $\pm Q$ and $\pm e_0$. To preserve the diagonal $m = n$ of (4.10) we need rather a projection to the two terms with shifts by $\pm e_0$. Indeed, such a projection exists but it requires a different deformation of the free field ring elements. This new theory is defined by an interaction which contains the two Liouville screening charges, as well as the two possible Liouville dressings of the non-trivial vertex operator with zero dimension: $T^{+}_{1/b} = \mathcal{T}^{(+)}_{-e_0}$ and $T^{-}_{b} = \mathcal{T}^{(-)}_{-e_0}$:

$$\begin{aligned}\mathcal{A}^{\rm int} &= \int \Big(\mu_{_L} e^{2b\phi} + \tilde{\mu}_{_L} e^{2/b\phi} - \tfrac{\pi^2}{e_0^2}\, \mu_{_M} \tilde{\mu}_{_M}\, e^{2ie_0\chi} \Big(b^2 \mu_L e^{2b\phi} + b^{-2} \tilde{\mu}_L e^{2/b\phi}\Big)\Big). \\ &= \lambda_{_L} T^{+}_{b} + \lambda'_{_L}\, T^{+}_{1/b} + \tilde{\lambda}_{_L} T^{-}_{1/b} + \tilde{\lambda}'_{_L}\, T^{-}_{b}\,, \qquad \lambda'_{_L} = \tilde{\lambda}_{_L} \lambda_{_M} \tilde{\lambda}_{_M}\,.\end{aligned} \tag{4.26}$$

The matter charges in the correlators computed with this action can be screened only by multiples of $e_0 = \frac{1}{b} - b$, whence the name "diagonal", by which we refer to it.[10] The duality transformations (2.14), or (2.15), with a simultaneous change of sign of one of the two fields, as discussed above, exchange the Liouville screening charges as well as the two new terms, so that the action is invariant. On the other hand, the matter-Liouville transformations $\{b, \lambda_{_L}, \lambda_{_M}, \phi, \chi\} \;\to\; \{\pm ib, \lambda_{_M}, \lambda_{_L}, \mp\chi, \pm\phi\}$ map it to the action (4.27).

The deformation (4.26) leads to an operator $a_{-}a_{+} \to A$ with the following OPEs

$$\begin{aligned} A\, W^{+}_{\alpha} &= \lambda_{_L}\, W^{+}_{\alpha - \frac{e_0}{2}} + \tilde{\lambda}_{_L} \lambda_{_M}\, \tilde{\lambda}_{_M}\, W^{+}_{\alpha + \frac{e_0}{2}}\,, \\ A\, W^{-}_{\alpha} &= \tilde{\lambda}_{_L}\, W^{-}_{\alpha + \frac{e_0}{2}} + \lambda_{_L} \lambda_{_M}\, \tilde{\lambda}_{_M}\, W^{-}_{\alpha - \frac{e_0}{2}}\,. \end{aligned} \tag{4.28}$$

---

[10] In the same way one can consider an interaction theory described by the two matter screening charges and their Liouville reflected counterparts $T^{\pm}_{Q}$.

$$\mathcal{A}^{\rm dg}_{\rm int} = \int \Big(\mu_{_M} e^{-2ib\chi} + \tilde{\mu}_{_M} e^{\frac{2i}{b}\chi} - \frac{\pi^2\, \mu_L \tilde{\mu}_L}{Q^2}\, e^{2Q\phi}\big(b^2 \mu_{_M} e^{-2ib\chi} + \frac{\tilde{\mu}_M}{b^2} e^{\frac{2i}{b}\chi}\big)\Big) \tag{4.27}$$

These relations are obtained combining the free field formulae used in the derivation of (4.5), (4.6) and (4.17); see Appendix A.1. Comparing with the composition of (4.5) and (4.6), the product $T^{-}_{1/b}\, T^{-}_{0}\, T^{+}_{0}$ of two matter and one Liouville screening charges, which leads to the shift $\alpha \to \alpha + e_0/2$, is now traded for the tachyon $T^{+}_{1/b} = \mathcal{T}^{(+)}_{-e_0}$. This explains the expression $\lambda' = \tilde{\lambda}_{_L}\, \tilde{\lambda}_{_M}\, \lambda_{_M}$ for the coupling constant in (4.26), (4.28).

Note that we now need all the four terms in the interaction action (4.26) in order to determine the OPEs of the ring generator with tachyons of both chiralities, in contrast with any of the relations (4.5), or (4.6). Let us stress that at this stage we will consider the diagonal action (4.26) as a formal tool, which provides us in a systematic way with certain rules. In particular, we will not discuss its possible semiclassical limits.

It appears that in this theory the mass-shell condition applied to the potential OPE channels is much more restrictive. Thus there are no additional terms in the OPE as far as we consider either generic momenta, or the set of diagonal momenta $P = ke_0$. Similarly for the momenta of interest (generic, or diagonal degenerate) there are no more OPE terms in the presence of integrated tachyons besides (4.17). The operator a_+a_-, perturbed by the diagonal action (4.26), generates a $sl(2)$ type ring, as does each of the operators $a_{\mp}$ perturbed by the actions (2.4) and (2.5). Applying (4.28) to the set of order parameter fields we get a formula analogous to (4.15), representing the diagonal ring elements as Chebyshev polynomial of the generator A.

5. Functional relations for the closed string tachyon amplitudes

5.1. 3-point solutions of the ring identities

In this section we shall apply the free field computed OPEs of the ring generators assuming that they hold in a general tachyon correlator.

The general 4-point function with one of the ring generators and arbitrary three tachyons W_{α_i} can be computed in two ways exploiting the operator product expansions (4.5), (4.6). This leads to the finite difference identities (3.27) for the tachyon 3-point correlators. Similarly in the diagonal theory (4.28) implies the relation

$$N_{P_1+e_0,P_2,P_3} + N_{P_1-e_0,P_2,P_3} = N_{P_1,P_2+e_0,P_3} + N_{P_1,P_2-e_0,P_3}\,. \tag{5.1}$$

The simplest solution of (5.1), as that for the identities (3.27), is $N_{P_1,P_2,P_3} = 1$. To fix here the overall normalization constant we assume that the correlators with zero overall matter charge are the same in both theories, *i.e.*, they are given by the normalized with the leg factors Liouville 3-point constant (3.5).

Non-trivial solutions exist whenever the momenta take values corresponding to degenerate Virasoro representations. Let us first consider the correlators in the diagonal theory for tachyons with momenta $P = \mathbb{Z}e_0$. We require that the 3-point function vanishes whenever one of the momenta is zero, that is, outside of the set of degenerate values. We choose for definiteness the sign in (4.10), taken for $m = n$, to coincide with the chirality ε_i. Then the diagonal ring relation (5.1) for the 3-point function

$$N_{P_1,P_2,P_3} = N_{m_1,m_2,m_3}\,, \qquad P_i = \varepsilon_i m_i e_0 \leftrightarrow \alpha_i = \tfrac{Q}{2} - m_i \tfrac{e_0}{2}\,, \tag{5.2}$$

turns into the standard recurrence relation for the tensor-product decomposition multiplicities of the irreps of $sl(2)$ of finite dimensions m_k:

$$N_{m_1,m_2,m_3} = \begin{cases} 1 \quad \text{if} & \begin{matrix} |m_1 - m_2| + 1 \le m_3 \le m_1 + m_2 - 1 \\ \text{and} \;\; m_1 + m_2 + m_3 = \text{odd} \end{matrix} \\ 0 \quad \text{otherwise}\,. \end{cases} ; \tag{5.3}$$

Any of the two sides in (5.1) is equal to the 4-point multiplicity $N_{m_1,m_2,m_3,2}$, where,

$$\begin{aligned} N_{P_1,P_2,P_3,P_4} &= \sum_{m=1} N_{m_1,m_2,m}\, N_{m,m_3,m_4} = N_{m_1,m_2,m_3,m_4} \\ &= \tfrac{1}{2}\left(\min(m_1+m_2, m_3+m_4) - \max(|m_1-m_2|, |m_3-m_4|)\right)\,. \end{aligned} \tag{5.4}$$

Similarly for general degenerate momenta (4.10) the identities (3.27) are solved by the product

$$N_{P_1,P_2,P_3} = N_{m_1,m_2,m_3} N_{n_1,n_2,n_3}\,, \qquad P_s = \pm(n_s/b - m_s b)\,, \tag{5.5}$$

assuming the vanishing of the tachyons on the border lines $m = 0$, or $n = 0$ of (4.10); see also [24] for the rational case. The solution is symmetric with respect to matter charge reflections $P_s \to -P_s$ and thus can be identified as a correlator of the invariant combinations (4.13). These $sl(2) \times sl(2)$ decomposition multiplicities are the fusion multiplicities in the quasi-rational matter theory at generic values of b^2, described by the infinite set of fields of momenta $P_{m,n}$. The same solution of (3.27) is found if the tachyon momenta take the Liouville degenerate values $\varepsilon_s P_s = \pm(n_s/b + m_s b)$ as in (4.14).

The fusion multiplicities (5.5) coincide, when restricted to the diagonal $m_s = n_s$, with those obtained in the diagonal theory, (5.2) and (5.3). However the 4-point fusion multiplicities

$$N_{P_1,P_2,P_3,P_4} = \sum_{m,n=1} N_{P_1,P_2,P_{m,n}}\, N_{P_{m,n},P_3,P_4} = N_{m_1,m_2,m_3,m_4}\, N_{n_1,n_2,n_3,n_4} \tag{5.6}$$

taken for such values differ from their counterparts (5.4) in the diagonal theory.

For other 3-point solutions see also Appendix B.

5.2. Recurrence relations

We next apply the OPE relations inserting a ring generator in the 4-point function

$$G_4^{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left\langle\, W^{\varepsilon_1}_{\alpha_1}(0)\;\; W^{\varepsilon_2}_{\alpha_2}(1)\;\, T^{\varepsilon_3}_{\alpha_3}\;\;\, W^{\varepsilon_4}_{\alpha_4}(\infty) \,\right\rangle\, . \tag{5.7}$$

The relation one gets for a_- approaching the first or the second tachyons, reads

$$\begin{aligned}\sum_\alpha C^{(\varepsilon)}{}_{-\frac{b}{2}\,\alpha_1}{}^{\alpha}\; G_4^{(\varepsilon)}(\alpha,\alpha_2,\alpha_3,\alpha_4) + \sum_\alpha C^{(\varepsilon)}{}_{-\frac{b}{2}\,\alpha_1\alpha_3}{}^{\alpha}\; G_3^{(\varepsilon)}(\alpha,\alpha_2,\alpha_4) \\ = \sum_\alpha C^{(\varepsilon)}{}_{-\frac{b}{2}\,\alpha_2}{}^{\alpha}\; G_4^{(\varepsilon)}(\alpha_1,\alpha,\alpha_3,\alpha_4) + \sum_\alpha C^{(\varepsilon)}{}_{-\frac{b}{2}\,\alpha_2\alpha_3}{}^{\alpha}\; G_3^{(\varepsilon)}(\alpha_1,\alpha,\alpha_4)\end{aligned} \tag{5.8}$$

where we have omitted for simplicity the explicit dependence of the tachyon correlators $G^{(\varepsilon)}$ and the OPE coefficients $C^{(\varepsilon)}$ on each of the chiralities. The OPE relations (4.5) determine the first terms in both sides of (5.8). On the other hand (4.21), and for special values of the momenta, (4.24), give the explicit expressions for the 4-point OPE coefficients, which determine the inhomogeneous, 3-point "contact", terms $\sim G_3^{(\varepsilon)}$ in (5.8). For generic momenta the ring relation (5.8) generalizes straightforwardly to a correlator with an arbitrary number $p-3$ of integrated tachyons $T^+_{\alpha_i}$ with summations over $(p-1)$-point contact terms. If however some partial sums of momenta "degenerate", i.e., lie on the lattice $\mathcal{L}$, there are other possible solutions of the mass-shell conditions, as explained in Appendix A.3, and hence potentially new m-point contact terms, $3 \le m \le p-1$. The OPE coefficients would require the computation of higher $p-m+3$-point free field functions matrix elements, generalizing the $p = 4 = m+1$ case of Appendix A.4.

The inhomogeneous associativity identities (5.8) can be interpreted as string analogs of the duality equations for the local 4-point correlators of the $c < 1$ or $c > 25$ Virasoro theory. Given the OPE coefficients $C^{(\varepsilon)}$ and a choice of the 3-point terms, the set of these relations determines the 4-point tachyon correlators. What has also to be added to this set of recursive difference equations is a choice of some boundary conditions, *i.e.* particular known values of the 4-point tachyon correlators.

• We shall now specialize the contact terms in (5.8) for two basic classes of 4-point tachyon correlators. For the 3-point tachyon correlators in (5.8) we shall take the generic solution (3.18). Let us choose $\varepsilon_2 = \varepsilon_3 = 1 = -\varepsilon_1$. Then the first series in (4.21) contributes to the r.h.s. of (5.8) (with $W^+_\alpha T^+_{\alpha_1}$ now denoted $W^+_{\alpha_2} T^+_{\alpha_3}$). As we have mentioned above, if the tachyon $W^+_{\alpha_2}$ is labelled by the degenerate momentum (4.22) of matter or Liouville type, the r.h.s. of (4.21) terminates. Hence only the first $n+1$ contact terms have to be taken into account in the r.h.s. of (5.8). Using the relation (3.25) satisfied by the generic solution (3.18) one obtains $(n+1)$ times a power of the coupling constants $\lambda_{_L}, \lambda_{_M}$. For generic values $P_1\,, P_3 \notin \mathcal{L}$, and $P_2+P_3 \notin \mathcal{L}$ these are the only contact terms in (5.8) and we obtain the equation

$$\begin{aligned}
&G_4^{-+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) + \lambda_{_L}\lambda_{_M}\, G_4^{-+++}(\alpha_1+b,\alpha_2,\alpha_3,\alpha_4)\\
&-\lambda_{_L}\, G_4^{-+++}(\alpha_1+\tfrac{b}{2},\alpha_2+\tfrac{b}{2},\alpha_3,\alpha_4) - \lambda_{_M}\, G_4^{-+++}(\alpha_1+\tfrac{b}{2},\alpha_2-\tfrac{b}{2},\alpha_3,\alpha_4)\\
&=\sum_{k=0}^{n}(\tilde\lambda_{_L}\tilde\lambda_{_M})^k\, G_3^{-++}(\alpha_1+\tfrac{b}{2},\alpha_2+\alpha_3-\tfrac{b}{2}+\tfrac{k}{b},\alpha_4)\\
&=-(n+1)\, G_3^{-++}(\alpha_1+\tfrac{b}{2},\alpha_2+\alpha_3-\tfrac{b}{2},\alpha_4)\,.
\end{aligned} \tag{5.9}$$

In the last line we have used the relation (3.25) satisfied by the generic solution (3.18); in general one should keep the r.h.s. of the first equality.

There is another class of correlators in which the OPE relations (4.21) produce finite number of contact terms in (5.8). These are the correlators with all generic momenta, but restricted by an overall charge conservation condition. It can be a relation involving the two matter charges

$$\sum_{i=1}^{4} e_i - e_0 = mb - \tfrac{n}{b} \;\leftrightarrow\; \sum_{i=1}^{4} P_i = 2e_0 - 2mb + \tfrac{2n}{b}\,, \quad m,n\in\mathbb{Z}_{\geq 0}\,, \tag{5.10}$$

or it can be $c>25$ charge conservation condition

$$\sum_i \alpha_i - Q = -mb - n/b \;\leftrightarrow\; \sum_{i=1}^{4}\varepsilon_i P_i = 2Q + 2mb + \tfrac{2n}{b}\,, \quad m,n\in\mathbb{Z}_{\geq 0}\,. \tag{5.11}$$

For a fixed n in (5.10) (or (5.11)) k of the charges T_0^- (or $T_{1/b}^-$), $k=0,1,\dots,n$, can be assigned to the OPE in (4.21). The general identity (5.8) takes again the form of (5.9).

The equation (5.9) is recursive with m, e.g., in the case of the matter type charge conservation condition (5.10), the $\lambda_{_M}$-independent and $\lambda_{_M}$-dependent terms in the l.h.s. correspond to m and $m-1$ respectively. In the case of degenerate P_2 (4.22), terms with the three values $m, m\pm 1$ appear.

If on the other hand the degenerate field appears as an integrated tachyon, *i.e.* in our notation P_3 is degenerate as in the r.h.s. of (4.24), then (changing the notation $n\to n_3$ in (4.24)), this OPE relation leads to new (n_3+1) contact terms. This implies, using once again the simple solution (3.18), that the coefficient in the r.h.s. of (5.9) is modified to $(n-n_3)$. For $n_3=n$ the contact term in the r.h.s. disappears and the relation (5.9) becomes homogeneous.

In particular in the class of correlators with $n = 0 = n_3$ the r.h.s of (5.9) simplifies to one or zero contact terms, which we write as

$$
\begin{aligned}
&G_4^{-+++}(\alpha_1 - \tfrac{b}{2}, \alpha_2, \alpha_3, \alpha_4) + \lambda_{_L}\,\lambda_{_M}\, G_4^{-+++}(\alpha_1 + \tfrac{b}{2}, \alpha_2, \alpha_3, \alpha_4) \\
&-\lambda_{_L}\, G_4^{-+++}(\alpha_1, \alpha_2 + \tfrac{b}{2}, \alpha_3, \alpha_4) - \lambda_{_M}\, G_4^{-+++}(\alpha_1, \alpha_2 - \tfrac{b}{2}, \alpha_3, \alpha_4) \\
&= -G_3^{-++}(\alpha_1, \alpha_2 + \alpha_3 - \tfrac{b}{2}, \alpha_4) \\
&+ \sum_{m_3=0} (\lambda_{_M}^{m_3+1}\, \delta_{\alpha_3, b+\frac{m_3 b}{2}} + \lambda_{_L}^{m_3+1}\, \delta_{\alpha_3, -\frac{m_3 b}{2}})\, G_3^{-++}(\alpha_1 + \tfrac{(m_3+1)b}{2}, \alpha_2, \alpha_4)\,.
\end{aligned}
\tag{5.12}
$$

The relation dual to (5.12) reads

$$
\begin{aligned}
&G_4^{+---}(\alpha_1 - \tfrac{1}{2b}, \alpha_2, \alpha_3, \alpha_4) + \tilde\lambda_{_L}\,\tilde\lambda_{_M}\, G_4^{+---}(\alpha_2 + \tfrac{1}{2b}, \alpha_2, \alpha_3, \alpha_4) \\
&-\tilde\lambda_{_L}\, G_4^{+---}(\alpha_1, \alpha_2 + \tfrac{1}{2b}, \alpha_3, \alpha_4) - \tilde\lambda_{_M}\, G_4^{+---}(\alpha_1, \alpha_2 - \tfrac{1}{2b}, \alpha_3, \alpha_4) \\
&= -G_3^{+--}(\alpha_1, \alpha_2 + \alpha_3 - \tfrac{1}{2b}, \alpha_4) \\
&+ \sum_{n_3=0} (\tilde\lambda_{_M}^{n_3+1}\, \delta_{\alpha_3, \frac{1}{b}+\frac{n_3}{2b}} + \tilde\lambda_{_L}^{n_3+1}\, \delta_{\alpha_3, -\frac{n_3}{2b}})\, G_3^{+--}(\alpha_1 + \tfrac{(n_3+1)}{2b}, \alpha_2, \alpha_4)\,.
\end{aligned}
\tag{5.13}
$$

In these simplified equations only one of the matter and one of the Liouville charges is effectively contributing, namely the pairs in the action (2.4), or (2.5) respectively. The equations (5.12), (5.13) become homogeneous if, in particular, $T^{\varepsilon}_{\alpha_3}$ coincides with one of the four screening charges. For example, if $\alpha_3 = 0$ ($e_3 = -b$), the simplest solution of the homogeneous relation (5.12) is

$$
\begin{aligned}
G_4^{(-+++)}(\alpha_1, \alpha_2, 0, \alpha_4) &= -\frac{1}{b^2}\big(\sum_i e_i - e_0 + b\big)\, \lambda_{_L}^{\frac{1}{b}(Q - \sum_i \alpha_i)}\, \lambda_{_M}^{-\frac{1}{b}(e_0 - \sum_i e_i)} \\
&= -\frac{\partial}{\partial \lambda_{_M}} G_3(\alpha_1, \alpha_2, \alpha_4)\,,
\end{aligned}
\tag{5.14}
$$

while for $a_3 = b$ ($e_3 = 0$) it is

$$
\begin{aligned}
G^{(-+++)}(\alpha_1, \alpha_2, b, \alpha_4) &= \frac{1}{b^2}\,\big(\sum_i \alpha_i - Q - b\big)\, \lambda_{_L}^{\frac{1}{b}(Q - \sum_i \alpha_i)}\, {\lambda_{_M}}^{-\frac{1}{b}(e_0 - \sum_i e_i)} \\
&= -\frac{\partial}{\partial \lambda_{_L}} G(\alpha_1, \alpha_2, \alpha_4)\,.
\end{aligned}
\tag{5.15}
$$

Setting $\lambda_{_M} = 0$ in (5.13) (or $\tilde\lambda_{_M} = 0$ in (5.12)) reduces furthermore (5.12), (5.13), so that they apply to correlators with momenta satisfying trivial matter condition $m = 0 = n$ in (5.10). Note that besides the main contact terms, each of these equations with two terms in the l.h.s. still contains an "accidental" contact term, e.g., $m_3 = 0$ in (5.12), missed in the old considerations. Taking into account such terms in the (4+m)-point generalisations of the reduced two term equations, derived with only the Liouville interaction included, leads to an alternative derivation of the four term identities (5.12), (5.13).

In a similar way, one derives generically homogeneous relations fusing the ring generators with two tachyons of the same chirality. E.g., for $\varepsilon_s = 1\,, s = 2,3,4$

$$\begin{aligned} &\lambda_{_L} G_4(\alpha_1,\alpha_2,\alpha_3+\tfrac{b}{2},\alpha_4) + \lambda_{_M}\ G_4(\alpha_1,\alpha_2,\alpha_3-\tfrac{b}{2},\alpha_4) \\ &= \lambda_{_L} G_4(\alpha_1,\alpha_2+\tfrac{b}{2},\alpha_3,\alpha_4) + \lambda_{_M} G_4(\alpha_1,\alpha_2-\tfrac{b}{2},\alpha_3,\alpha_4), \end{aligned} \tag{5.16}$$

cancelling the difference of the two contact terms. We stress that this and the above discussed cancellations occur when the simplest constant 3-point solution (3.18) is used; in general we should keep the full linear combinations of 3-point contact terms.

- The functional equations take a more compact form after rescaling

$$G_n^{(\varepsilon)} \equiv b^{-\sum_i \varepsilon_i}\ \lambda_{_L}^{\frac{1}{b}(Q-\alpha)}\,\lambda_{_M}^{\frac{1}{b}(e-e_0)}\ \hat{G}_n^{(\varepsilon)}, \qquad \alpha = \sum_{i=1}^n \alpha_i \quad e = \sum_{i=1}^n e_i. \tag{5.17}$$

The normalized correlators $\hat{G}_n$ do not depend on the constants $\lambda_{_L}, \lambda_{_M}$ as is standardly checked by shifting $\phi \to \phi - \frac{\log\lambda_L}{2b}\,, \chi \to \chi + \frac{\log\lambda_M}{2bi}$. The rescaling by the power of b in (5.17) is equivalent to a change of the leg factor normalization

$$V_\alpha^\varepsilon \to \hat{V}_\alpha^\varepsilon = b^\varepsilon\, V_\alpha^\varepsilon \tag{5.18}$$

which removes the chirality-dependent power of b in the 3-point function (3.18). This normalisation does not change the OPE ring identities (4.5), (4.6), but changes the coefficients in front of the contact terms. With a slight abuse of notation, in what follows we shall write G_n for the corresponding correlators with just the powers of b removed, *i.e.*, the ones differing from $\hat{G}_n$ only by the powers of $\lambda_{_L}$ and $\lambda_{_M}$. For further reference we write the ring relations (5.12) and (5.13) also in terms of the target space momenta. For the rescaled functions

$$\hat{\mathcal{G}}_4^{(\pm)}(P_1|P_2,P_3,P_4) \equiv \hat{G}^{\mp\pm\pm\pm}(\alpha_1,\alpha_2,\alpha_3,\alpha_4), \qquad \varepsilon_i P_i = Q - 2\alpha_i,$$

the equations take the form

$$\begin{aligned} &2(\cosh b\partial_{P_1} - \cosh b\partial_{P_2})\ \hat{\mathcal{G}}_4^{(+)}(P_1|P_2,P_3,P_4) = -b(n+1) N_{P_1,P_2+P_3-b^{-1},P_4}\,, \\ &2(\cosh \tfrac{1}{b}\partial_{P_1} - \cosh \tfrac{1}{b}\partial_{P_2})\ \hat{\mathcal{G}}_4^{(-)}(P_1|P_2,P_3,P_4) = -\tfrac{1}{b}(m+1) N_{P_1,P_2+P_3+b,P_4} \end{aligned} \tag{5.19}$$

- In the diagonal theory the ring relations read

$$\begin{aligned} &\tilde{\lambda}_{_L} G_4^{-,+,\varepsilon,\varepsilon}(\alpha_1+\tfrac{e_0}{2},\alpha_2,\alpha_3,\alpha_4) + \lambda_{_L}\lambda_{_M}^{-\frac{e_0}{b}}\ G_4^{-,+,\varepsilon,\varepsilon}(\alpha_1-\tfrac{e_0}{2},\alpha_2,\alpha_3,\alpha_4) \\ &- \lambda_{_L} G_4^{-,+,\varepsilon,\varepsilon}(\alpha_1,\alpha_2-\tfrac{e_0}{2},\alpha_3,\alpha_4) - \tilde{\lambda}_{_L}\lambda_{_M}^{-\frac{e_0}{b}}\ G_4^{-,+,\varepsilon,\varepsilon}(\alpha_1,\alpha_2+\tfrac{e_0}{2},\alpha_3,\alpha_4) \\ &= \begin{cases} -G_3^{-,+,\varepsilon}(\alpha_1,\alpha_2+\alpha_3-\frac{Q}{2},\alpha_4)\,, & \text{if } \varepsilon = + \\ \ \ G_3^{-,+,\varepsilon}(\alpha_1+\alpha_3-\frac{Q}{2},\alpha_2,\alpha_4)\,, & \text{if } \varepsilon = - \end{cases}. \end{aligned} \tag{5.20}$$

Here we shall normalize the correlators (taking also into account the rescaling (5.18)), as

$$G_4^{(+)} = -\tfrac{b}{e_0}\,\lambda_{L}^{\frac{1}{b}(Q-\alpha)}\,\lambda_{M}^{\frac{1}{b}(e-e_0)}\,\hat{G}_4^{(+)}, \quad G_4^{(-)} = \tfrac{1}{be_0}\,\lambda_{L}^{\frac{1}{b}(Q-\alpha)}\,\lambda_{M}^{\frac{1}{b}(e-e_0)}\,\hat{G}_4^{(-)}. \tag{5.21}$$

With this normalization (5.20) becomes

$$2\,(\cosh e_0\partial_{P_1} - \cosh e_0\partial_{P_2})\,\hat{\mathcal{G}}_4^{(\varepsilon)}(P_1|P_2,P_3,P_4) = \varepsilon e_0\,\hat{\mathcal{G}}_3^{(\varepsilon)}(P_1|P_2+P_3,P_4)\,. \tag{5.22}$$

The solutions of the mass-shell condition, restricted to diagonal momenta $P_3 \in \mathbb{Z}e_0 \subset \mathcal{L}$ allow only $P_3 = 0$ as a possible momentum leading to an "accidental" contact term. This is the tachyon $T_{Q/2}$ of no definite chirality.

In the same way one derives relations with α_3 exchanged with α_2 or α_1 respectively. The derivation can be repeated also with the third field taken at infinity. The collection of these identities for generic momenta imply a set of symmetry relations for the contact terms in the r.h.s., e.g. for $\varepsilon = 1$,

$$\mathcal{G}_3(P_1, P_2+P_3, P_4) = \mathcal{G}_3(P_1, P_2+P_4, P_3) = \mathcal{G}_3(P_1, P_4+P_3, P_2). \tag{5.23}$$

6. Solutions of the ring relations in the absense of matter (or Liouville) screening charges

In this section we will describe solutions of the ring generated functional equations in the simplest case of only Liouville or only matter perturbation.

6.1. Solutions with matter charge conservation

In the case of gaussian matter field (formally $\lambda_{M} = 0 = \tilde{\lambda}_{M}$) the neutrality condition (4.2) holds. The l.h.s. of the functional relations (5.12), (5.13) reduces to a difference of two terms. For generic momenta the equations extend [25] straighforwardly to p-point correlators satisfying the "chirality rule", *i.e.*, one of the tachyons has the opposite chirality $-\varepsilon$ to the chirality ε of the other $p-1$ ones; these are in fact the only correlators comparable with the microscopic approach. If we restrict to the resonant correlators, satisfying also the Liouville type conservation condition $\sum_{s=1}^{p}\alpha_s = Q$, the equations simplify with only the λ_{L} - independent term surviving in the l.h.s. The r.h.s. is recursively reduced to a 3-point function and the solution is a constant independent of the momenta. In general the functions are symmetric with respect to $p-1$ of the charges and since they have to reproduce as a special case the resonant amplitudes they depend only on the sum $\alpha = \sum_{s=1}^{p}\alpha_s$ of the Liouville charges. Taking into account the new normalization in (5.18) the equations read,

$$\hat{G}_p^{(\varepsilon)}(\alpha) -\ \lambda_{L}\ \hat{G}_p^{(\varepsilon)}(\alpha+b^{\varepsilon}) = -(p-3)b^{\varepsilon}\,\hat{G}_{p-1}^{(\varepsilon)}(\alpha)\,, \qquad \alpha = \sum_{s=1}^{p}\alpha_s\,. \tag{6.1}$$

Starting with the 4-point case, the solution involves an arbitrary solution of the homogeneous equation, *i.e.*, a periodic in b or $1/b$ function. We shall use as a boundary condition the known expressions (5.15) and its dual, in which one of the Liouville charges is b or $1/b$. This leaves us with a linear function of α

$$G^{(\varepsilon)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \lambda_L^{\frac{1}{b}(Q-\alpha)}(\alpha - Q - b^{\varepsilon}) = \lambda_L^{\frac{1}{2b}(-2Q+\sum_i \varepsilon_i P_i)}(b^{-\varepsilon} - \tfrac{1}{2}\textstyle\sum_i \varepsilon_i P_i)\,. \tag{6.2}$$

The two choices of boundary conditions are correlated with the two effective actions, (2.4) and (2.5). The $p \geq 4$ relation (6.1) is solved by

$$\begin{aligned} G_p^{(+)}(\alpha) &= \Big(-b\,\partial_{\lambda_L}\Big)^{p-3}\,\lambda_L^{\frac{1}{b}(Q-\alpha)+p-3}\,,\\ G_p^{(-)}(\alpha) &= \Big(-\tfrac{1}{b}\,\partial_{\tilde{\lambda}_L}\Big)^{p-3}\,\tilde{\lambda}_L^{b(Q-\alpha)+p-3}\,, \end{aligned} \tag{6.3}$$

recovering formula (2.53) of [34]. The solutions of the two equations are interchanged by the duality transformations (2.16) and (2.17), up to a sign in the second case; it disappears for the properly normalized correlators as in (3.22).

The formulae (6.2) are valid also for the 4-point functions with $\sum_i \varepsilon_i = \pm 4$, which are constants, because of the matter charge conservation condition. These constants and the solutions with $\sum_i \varepsilon_i = \pm 2$ in (6.2) are related with an inhomogeneous analog of the Liouville reflection relation, in contrast with what we had for the 3-point functions in (3.19). For example,

$$\begin{aligned} &G^{++++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \\ &= \lambda_L^{\frac{1}{b}(Q-2\alpha_1)} G^{-+++}(Q-\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (Q-2\alpha_1)\,\lambda_L^{\frac{1}{b}(Q-\sum_{i=1}^{4}\alpha_i)}\,. \end{aligned} \tag{6.4}$$

The second term in the last equality compensates the contact term in the ring relation (5.12) for the 4-point function of type $^{(-+++)}$ in the r.h.s. of (6.4), so that the l.h.s. satisfies a homogeneous equation without contact terms, as it should. So far we have excluded from the discussion the correlators with two equal chiralities, *i.e.*, of the type $^{(++--)}$. The reason is that the contact terms depend on the choice for the integrated tachyon and one obtains an inconsistent set of relations. These correlators have been neglected in the earlier considerations, *e.g.* [34] and [9], basically because of the vanishing of the unnormalized perturbative expressions, as discussed above in the comment after (3.6). Furthermore the correlator of type $^{(+---)}$ is also trivial constant when determined by the action (2.4), and similarly for the correlator of type $^{(-+++)}$ computed with (2.5), since the corresponding functional equations are homogeneous.

• The chirality rule satisfying solutions of the two type equations are related by pairs of inhomogeneous Liouville reflections

$$\begin{aligned}\hat{\mathcal{G}}^{-+++}(P_1,P_2,P_3,P_4) &= P_1+b = -\hat{\mathcal{G}}^{+---}(P_1,P_2,P_3,P_4)+Q\\ &= \hat{\mathcal{G}}^{+-++}(P_1,P_2,P_3,P_4)+(P_1-P_2)\\ &= \hat{\mathcal{G}}^{---+}(P_1,P_2,P_3,P_4)+(P_1+P_4-e_0)\\ &= \hat{\mathcal{G}}^{+---}(P_1,P_2,P_3,P_4)+2P_1-e_0\,.\end{aligned} \tag{6.5}$$

The matter reflections do not make sense since they violate the charge conservation condition (4.2). Using this condition $\sum_i P_i = 2e_0$, we can rewrite (6.2) as

$$\mathcal{G}^{(\varepsilon=-\varepsilon_1)}(P_1|P_2,P_3,P_4) = \lambda_L^{\frac{1}{2b}(-2Q+\sum_{i=1}^4 \varepsilon_i P_i)}\big(Q-\varepsilon_1\sum_{s\neq 1}(e_0-P_1-P_s)\big) \tag{6.6}$$

i.e., in the form of (1.2), with $N_{P_1,P_2,P_3}=1$, if $\sum_i P_i = e_0$ and $N_{P_1,P_2,P_3}=0$ otherwise.

If we restrict the momenta to the range $\varepsilon_s(P_s-\frac{e_0}{2})>0\,,\ s\neq 1$ (physical for $\varepsilon\, e_0>0$ and implying $\varepsilon_1(P_1-\frac{e_0}{2})>0$ as well), the correlator (6.6) reproduces the three channel expansion formula of [34],

$$\begin{aligned}\mathcal{G}(P_1,P_2,P_3,P_4) &= \lambda_L^{\frac{1}{2b}(\sum_{i=1}^4|P_i|-2Q)}\,\hat{\mathcal{G}}(P_1,P_2,P_3,P_4)\,,\\ \hat{\mathcal{G}}(P_1,P_2,P_3,P_4) &= \tfrac{1}{2}\,\left(Q-|P_1+P_2-e_0|-|P_1+P_3-e_0|-|P_1+P_4-e_0|\right)\,.\end{aligned} \tag{6.7}$$

This formula holds irrespectively of which of the four momenta is chosen with opposite sign since, unlike (6.6), it is symmetric in them, but at the price that it is not analytic. Vice versa, for any choice of the signs of the combinations

$$\{P_{st}:=P_s+P_t-e_0\}_{s,t=1,\dots,4}$$

compatible with the conservation condition $\sum_i P_i = 2e_0$, the formula (6.7) recovers one of the correlators with $\sum_i\varepsilon_i=\pm 2$. In other words (6.7) is a symmetrization over the chiralities,

$$\hat{\mathcal{G}}(P_1,P_2,P_3,P_4) = \sum_{\varepsilon_t}\prod_{s\neq t}\theta(\varepsilon_t(P_t+P_s-e_0))\,\hat{\mathcal{G}}^{(-\varepsilon_t)}(P_1,P_2,P_3,P_4) \tag{6.8}$$

where $\theta(x)=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}$. Here $\sum_{\varepsilon_t}=\sum_t\sum_{\varepsilon_t=\pm}$ and ε_t in $\hat{\mathcal{G}}^{(-\varepsilon_t)}$ indicates as before the chirality opposite to the remaining three.[11]

[11] We could restrict to the subdomain in which the signs of $P_i-e_0/2$, instead of the chiralities, satisfy the chirality rule. Then replacing the step function factor in the symmetrization formula (6.8) with $\prod_t\theta(\varepsilon_t(P_t-\frac{e_0}{2}))$ makes the correspondence of the two types of correlators "local", i.e., depending only on the individual momenta. However the subdomain is not preserved by the shifts, combinations with two positive and two negative signs of $\{P_t-\frac{e_0}{2}\}$ may appear.

The permutation symmetry with respect to the four matter charges e_1, e_2, e_3, e_4 is an analog of the locality of the 4-point euclidean correlation functions, so we shall refer to formulae of this type as "local" or "physical". The other symmetric combination with a plus relative sign corresponds to generically unphysical momenta $\varepsilon_i(P_i - e_0) < 0$. The local correlators do not depend on the chiralities and so they are invariant under Liouville reflections. Formula (6.7) is reproduced in the discrete model framework. For one of the momenta coinciding with e_0 (6.7) reduces to a derivative of the "physical" 3-point function

$$\mathcal{G}(P_1, P_2, P_3, e_0) = -b\partial_{\lambda_L} \mathcal{G}_3(P_1, P_2, P_3) = -b\partial_{\lambda_L} \lambda_L^{\frac{1}{2b}(-Q+\sum_s |P_s|)} \tag{6.9}$$

if $e_0 > 0$ and analogously a derivative with respect to $\tilde{\lambda}_L$ for $e_0 < 0$. On the level of 3-point functions the physical tachyons are identified by the fixed chirality fields $\mathcal{W}_P^+$ for $P > 0$ or $\mathcal{W}_P^-$ for $P < 0$. The functional identities rewritten for the correlators of these physical representatives contain in general P-dependent powers of λ_L, coming from a Liouville reflection as in (3.19), whenever unphysical value $\varepsilon P < 0$ is reached. As it is clear from (6.8) this local representation of the physical fields is not possible on the level of the 4-point function. Note that there are other symmetric combinations locally reproduced by the eight solutions of the equations. They are obtained by replacing $\sum_i \varepsilon_i P_i$ with $\sum_i |P_i|$ in (6.2): one gets two combinations which are interchanged under the transformation (2.16). What distinguishes the correlator (6.7) is that it preserves the simple fusion rule of the underlying local matter theory in each of the s, t, u channels so that the notion of "locality" of the 4-point tachyon "correlation numbers" matches that of standard locality. Furthermore with the chosen normalisation (5.18) this correlator is self-dual with respect to the Liouville type transformation (2.16); in the initial normalization the two analogs of (6.7) differ only by an overall power of b^2. If we further normalize with the partition functions (3.21) we can define two correlators, depending only on $\{b^2, bP_i\}$, which are exchanged by (3.23)

$$\begin{aligned} {}^{(n)}\mathcal{G}_4(P_1, P_2, P_3, P_4; \lambda_L, \lambda_M, b) &= -\frac{e_0 Q}{2b^2} \lambda_L^{\frac{1}{2b}(\sum_{i=1}^4 |P_i| - 4Q)} \lambda_M^{\frac{e_0}{b}} \left(\frac{Q}{2b} - \sum_{s \neq 1} |\tfrac{1}{2b} P_{1s}| \right) \\ = {}^{(n)}\tilde{\mathcal{G}}_4(-P_1, -P_2, -P_3, -P_4; \tilde{\lambda}_L, \tilde{\lambda}_M, \tfrac{1}{b}) &= {}^{(n)}\tilde{\mathcal{G}}_4(P_1, P_2, P_3, P_4; \tilde{\lambda}_L, \tilde{\lambda}_M, -\tfrac{1}{b}) \,. \end{aligned} \tag{6.10}$$

• The difference identities for the correlators with definite chiralities like (6.6) do not preserve the physical regions, neither the region determined by the set of inequalities above. Accordingly (6.7) does not satisfy globally the equations which apply by definition only to the partially symmetric, fixed chirality correlators. One can compute directly the shift relations for the local correlators from the explicit expression (6.7), or derive them from the initial identities.

It is instructive to compare the two types of equations. If the shift crosses the boundary of the momenta region in which a given fixed chirality correlaror represents the local one, the shifted correlator can be replaced via pairs of Liouville reflections (6.5) by the proper local representative in the new region. In this replacement there appear linear in the momenta terms which can be moved to the r.h.s. and interpreted as a modification of the contact terms. By the same mechanism any of the homogeneous relations acquires a non-trivial r.h.s., if the shift crosses the boundary of the corresponding region. Since the coefficients in the linear relation (6.8) project to different regions of momenta, this effectively implies that the (4-point) OPE coefficients in the analog of (5.8) for the local correlators will be no more constants but will depend themselves on the momenta. The rule which is extracted from the explicit expression (6.7) is that whenever the boundary is crossed, a Liouville reflected tachyon in the OPE appears, "dressed" with the inverse propagator (3.20). Namely (taking as usual $b > 0$) we compute from (6.7)

$$\begin{aligned} & - \hat{\mathcal{G}}_4(P_1, P_2, P_3, P_4) + \hat{\mathcal{G}}_4(P_1 + b, P_2 - b, P_3, P_4) \\ & = \sum_{s=3,4} \left((\theta(P_{2s}) - \theta(-P_{2s}))\, \tfrac{b}{2} + \theta(-P_{2s} + b)\theta(P_{2s})\,(P_{2s} - b) \right) \end{aligned} \tag{6.11}$$

We shall take $P_{2s} > 0\,, s = 3, 4\,,\ P_{34} > 0$ so that the first correlator in the l.h.s. of (6.11) is identified with $\mathcal{G}^{-+++}$. Then the r.h.s. of (6.11) reduces to

$$b + \sum_{s=3,4} \theta(-(P_{2s} - b))\,(P_{2s} - b) = b - \sum_{s=3,4} \theta(-P_{2s} + b)\, \hat{\mathcal{G}}_2(P_{23} - b, -P_{23} + b)^{-1}\,. \tag{6.12}$$

If both shifted momenta change sign, $P_{2s} - b < 0\,, s = 3, 4$ the shifted correlator in the l.h.s. is identified with $\mathcal{G}^{+-++}$. If only $P_{23} - b < 0$, while $P_{24} - b > 0$ (i.e., $P_{13} + b < 0$), this shifted correlator is of type $\mathcal{G}^{---+}$. Irrespectively of the signs of $P_{2s} - b$ the first term in (6.12) corresponds to the standard constant contact term. For negative $P_{23} - b$ the physical tachyon in the 3-point function $\hat{\mathcal{G}}_3(P_1 + b, P_{23} - b, P_4)$ has to be identified with the Liouville reflection of the tachyon $W^+_{\alpha_2+\alpha_3-b/2} = \mathcal{W}^+_{P_{23}-b}$. However the new contact term cannot be identified simply with the product of this 3-point correlator and the Liouville reflected OPE constant $C^{++-}_{-\frac{b}{2}\,\alpha_2\alpha_3}{}^{Q-\alpha_2-\alpha_3+\frac{b}{2}} = b^{-2}\lambda_L^{\frac{P_{23}-b}{b}}$ (see (A.32) below); rather it is related to the derivative of this constant with respect to λ_L.

We stress that (6.11) is just an alternative rewriting of the initial shift relation as a relation for the local correlators; otherwise the new terms $P_{23} + P_{24} - 2b = (P_2 - b) - (P_1 + b)$ (or $P_{23} - b = (P_1 + b) + P_4 - e_0$) in (6.12) are precisely the inhomogeneous terms of the Liouville reflections in (6.5), needed to represent the shifted correlator in the l.h.s. by a function of type $\mathcal{G}^{+-++}$ (or of type $\mathcal{G}^{---+}$) respectively.

There are several remarks in order concerning the identity (6.11), (6.12):

i) The appearance of P_{24}-dependent terms, besides P_{23}, serves as a symmetrization as in the simple relation (5.23) (in which the OPE coefficients are set to 1). These terms correspond to correlators in which the fourth tachyon is represented by an integral T_{α_4}; the shift equation for the symmetrised correlator does not distinguish the two situations. Or, alternatively, the relation (6.11) represents the "splitting" of the local 5-point function with a ring generator into various products of 3- times 4-point, times the inverse of a 2-point, correlators. Effectively the shift equation rewritten for the symmetrized correlator manifests the short distance expansion around all the three points $0, 1, \infty$.[12]

ii) On the other hand, once extracted from (6.12), these modified contact terms can be used to extend the r.h.s. of the general 4-term ring relation (5.9) for $n = 0$. Namely for $P_{st} > 0\,, s, t = 2, 3, 4$ (ensured by the physical values $\varepsilon_s P_s > e_0/2$) we have

$$\begin{aligned} &\hat{G}(P_1 + b, P_2 - b, P_3, P_4) - \hat{G}^{-+++}(P_1, P_2, P_3, P_4) \\ &+ \hat{G}^{-+++}(P_1 + b, P_2 + b, P_3, P_4) - \hat{G}^{-+++}(P_1 + 2b, P_2, P_3, P_4) \\ &= b + \sum_{s=3,4} \theta(-P_{2s} + b)(P_{2s} - b)\,, \qquad P_{2s} > 0\,, s = 3, 4\,, P_{34} > 0\,. \end{aligned} \tag{6.13}$$

This extended relation can be taken as a definition of the local correlator in a range of momenta, larger than the physical range in which it is represented by the fixed chirality correlators. Combining (6.13) with the initial fixed chirality relation (5.9) we obtain alternatively, relabeling the momenta,

$$\begin{gathered} \hat{G}(P_1, P_2, P_3, P_4) = \hat{G}^{-+++}(P_1, P_2, P_3, P_4) + \sum_{s=3,4} \theta(-P_{2s})\, P_{2s}\,, \\ \text{for } P_{2s} + b > 0\,, s = 3, 4\,,\ P_{34} > 0\,. \end{gathered} \tag{6.14}$$

The momenta of the three fixed chirality correlators in (6.13) are in the range, in which these correlators coincide with the local correlators so that (6.13) can be also interpreted as a shift relation for local correlators. Similarly one can derive shift relations and their duals in other regions of momenta, "neighbouring" the physical range. In particular, the dual of (6.13), extending the second equation in (5.19) for the case $m = 0$, implies

$$\begin{gathered} \hat{G}(P_1, P_2, P_3, P_4) = \hat{G}^{+---}(P_1, P_2, P_3, P_4) - \sum_{s=3,4} \theta(P_{2s})\, P_{2s}\,, \\ \text{for } P_{2s} - \tfrac{1}{b} < 0\,, s = 3, 4\,,\ P_{34} < 0\,. \end{gathered} \tag{6.15}$$

We shall exploit all these relations in section 7 below.

[12] This is analogous to the general discussion in [19] (see also [20-22]), where functional relations for the tachyon correlators in the $c = 1$ theory without interactions are derived starting from Ward identities of non-scalar currents. The resonant amplitudes described in these works are too simple to actually make a distinction between the two types of equations but an extension of the method might be appropriate for the problem under consideration.

6.2. Distribution type solution of the two term ring relations.

We can also interpret the solution of the ring relation in distribution sense, accounting for the charge conservation condition (4.2) by a δ-function. The correlators are expressed in terms of the p-point "multiplicities" for gaussian matter

$$N_{P_1,P_2,\dots,P_p} = N_p(P_1 + P_2 + \ldots + P_p), \tag{6.16}$$

$$N_p(P) = \delta(P - (p-2)e_0)\,, \qquad p \geq 3\,. \tag{6.17}$$

We interpret the 3-point multiplicity N_{P_1,P_2,P_3} as the factor modifying the generic 3-point constant (3.18), i.e. as the matter part of the 3-point correlator instead of (3.9). It satisfies the second relation in (3.25) and its dual, which are the 3-point ring relations in the absence of matter screening charges. Now (6.6) is replaced by the integral representation

$$\hat{\mathcal{G}}^{(\varepsilon)}(P_1|P_2,P_3,P_4) = \tfrac{1}{2}\Big[N_{P_1,P_2,P_3,P_4}\,Q - \varepsilon\,\langle\, P\,\rangle_{P_1;P_2,P_3,P_4}\Big]\,, \tag{6.18}$$

where

$$\langle\, P\,\rangle_{P_1;P_2,P_3,P_4} := \int_{-\infty}^{\infty} dP\,\left(N_{P_1,P_2,P}\,P\,N_{-P,P_3,P_4} + \left\{ \begin{matrix} \text{permutations} \\ P_2 \to P_3, P_4 \end{matrix} \right\}\right)\,. \tag{6.19}$$

The analog of the local 4-point function (6.7) is obtained replacing $\varepsilon P \to |P| = |Q - 2\alpha|$ in the three channels in (6.19).

- The ring relation (5.9) remains a relation with two terms in the l.h.s. for the correlators satisfying (5.10) with fixed $m = 0$ and nontrivial $n \neq 0$. A second boundary condition is provided by the correlator with $\alpha_1 = 0, e_1 = 1/b$, *i.e.*, the negative chirality field is given by the dual matter charge T_0^-

$$G_4^{(+)}(0,\alpha_2,\alpha_3,\alpha_4)_{\big|e_1=1/b} = (-\tfrac{1}{b}\partial_{\tilde{\lambda}_M})G_3(\alpha_2,\alpha_3,\alpha_4) = -\frac{1}{b}(n+1)\,\lambda_L^{\frac{1}{b}(Q-\alpha)}\,\tilde{\lambda}_M^n \tag{6.20}$$

and similarly

$$G_4^{(-)}(0,\alpha_2,\alpha_3,\alpha_4)_{\big|e_1=1/b} = (-b\partial_{\lambda_M})G_3(\alpha_2,\alpha_3,\alpha_4) = -(m+1)b\,\lambda_L^{\frac{1}{b}(Q-\alpha)}\,\lambda_M^m\,. \tag{6.21}$$

The solutions (to be discussed in more detail below in section 7. generalize (6.2)

$$\begin{aligned} G_4^{(+)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \lambda_L^{\frac{1}{b}(Q-\alpha)}\tilde{\lambda}_M^n\,(n+1)(\alpha - Q - b + \tfrac{n}{b}) \\ G_4^{(-)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \lambda_L^{\frac{1}{b}(Q-\alpha)}\lambda_M^m\,(m+1)(\alpha - Q - \tfrac{1}{b} + mb)\,. \end{aligned} \tag{6.22}$$

6.3. Solution with Liouville charge conservation

One can obtain similarly the solutions of the ring relations in the absence of Liouville screening charges, so that $\sum_{s=1}^{p} \alpha_s - Q = 0$, or $\sum_{s=1}^{p} \varepsilon_s P_s = (p-2)Q$. Such a constraint again goes beyond the normalization assumptions which led us to (3.18) since (3.2) is singular, i.e., it rather corresponds to the Coulomb gas constant $\hat{C}^{\rm Liou}$ obtained as a residuum of (3.2). Nevertheless the final expression in (3.18) satisfies the ($\lambda_{_L} = 0$) equations for the tachyon 3-point functions and can be taken as the solution in this case. Then the analog of (6.3) is given by a derivative with respect to the matter constant $\lambda_{_M}$

$$G_p^{(+)}(\alpha_1, \alpha_2, \cdots, \alpha_p) = \Big(-b\,\partial_{\lambda_{_M}}\Big)^{p-3} \lambda_{_M}^{-\frac{1}{b}(e_0 - \sum_{i=1}^{p} e_i)+p-3} \qquad (6.23)$$

and a similar formula for the opposite chiralities. The normalized with the partition functions $Z_M(\lambda_{_L}, \lambda_{_M}, b) = -Z(\lambda_{_L}, \lambda_{_M}, b)$ and $\tilde{Z}_M(\lambda_{_L}, \lambda_{_M}, b) = -\tilde{Z}(\lambda_{_L}, \lambda_{_M}, b)$ solution (6.23) and its dual are related to the normalized correlators (6.3), by the matter-Liouville duality (3.24), now equivalent to $\{b^2, \lambda_{_L}, b\alpha, be\} \to \{-b^2, \lambda_{_M}, be, b\alpha\}$. The analog of the formula (6.6) reads

$$\hat{G}_4^{(\varepsilon)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \varepsilon\big(e_0 - e - \varepsilon b^{\varepsilon}\big) = -\varepsilon \left(\frac{e_0}{2} - \varepsilon \sum_{s=2}^{4} (\alpha_1 + \alpha_s - \frac{Q}{2}) \right) . \qquad (6.24)$$

One can introduce also different analogs of the "local" correlator (6.7), now symmetric with respect to the four Liouville momenta $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. This correlator can be used to define another "local" extension of the general ring relations, analogously to (6.13). This case is however more speculative since we lack a selection rule of the type of Seiberg inequality and moreover we have no independent information on the generic $c < 1$ n-point correlators.

7. The 4-point function for fixed number of screening charges

7.1. The fixed chirality solutions

In this section we analyse the difference equations (5.19) in the case when the total sum of momenta is restricted by integer numbers of matter screening charges as in (5.10). Reducing recursively with m to the two term identities for $m = 0$ discussed above one obtains

$$\begin{aligned} &G_4^{(\varepsilon)}(\alpha; m, n) - \lambda_{_L}^{b^{\varepsilon-1}}\, G_4^{(\varepsilon)}(\alpha + b^{\varepsilon}; m, n) \\ &\quad = \sum_{l=0}^{m} \sum_{k=0}^{n} \lambda_{_M}^{l}\, (\tilde{\lambda}_{_L} \tilde{\lambda}_{_M})^k G_3^{(\varepsilon)}(\alpha_1 + \frac{l-1}{2} b, \alpha_2 + \alpha_3 - \frac{l-1}{2} b + \frac{k}{b}, \alpha_4; m-l, k) \\ &\quad = -b^{\varepsilon}\,(m+1)(n+1)\,\lambda_{_L}^{\frac{1}{b}(Q-\alpha)} \lambda_{_M}^{m} \tilde{\lambda}_{_M}^{n} , \end{aligned} \qquad (7.1)$$

where $\alpha = \sum_{s=1}^{4} \alpha_s$ is as before the total Liouville charge. The recursive equations for the 3-point functions are similarly reduced to the second identity in (3.25) and its dual. In the last line of (7.1) we have inserted again the simplest solution (3.18) of these identities.

The equation (7.1) admits a solution generalizing (6.22), and which can be cast into a "three channel" expansion form, consistent with a $sl(2) \times sl(2)$ type decomposition rule:

$$\begin{aligned}&\hat{G}^{(\varepsilon)}(\alpha; m, n) = (m+1)(n+1)(\alpha - Q - b^{\varepsilon} + \tfrac{n}{b}\,\tfrac{1+\varepsilon}{2} + mb\,\tfrac{1-\varepsilon}{2}) = \\ &\tfrac{1}{2}(m+1)(n+1)(Q + mb + \tfrac{n}{b}) + \tfrac{\varepsilon}{2}\sum_{s\neq 1}\sum_{k=0}^{m}\sum_{l=0}^{n}\left(P_1 + P_s - e_0 + 2kb - \tfrac{2l}{b}\right).\end{aligned} \tag{7.2}$$

Let us take for definiteness $\varepsilon = +1$. The solution (7.2) reduces to (6.2) for $m = 0 = n$, and to (6.22) for $m = 0$. For $n = 0$ the correlator is compared with another solution of (5.12), namely the solution (6.23), (6.24) with Liouville charge conservation $\sum_i \alpha_i - Q = \alpha - Q = 0$, *i.e.*, it is consistent with a third boundary condition given by the matter charge T_0^{+},[13]

$$G^{(+)}(\alpha; m, 0)\Big|_{\alpha=Q} = -b(m+1)\lambda_{M}^{m} = (e_0 - e - b)\lambda_{M}^{\frac{1}{b}(e-e_0)}\,. \tag{7.3}$$

The boundary conditions described do not fix uniquely the arbitrariness in the solution of the homogeneous difference equation - a term of the type $mn(m+1)(n+1)P(m,n)$ with an arbitrary polynomial $P(m,n)$ is still allowed. As we shall see below, (7.2) is the solution "smoothly" related to another class of solutions of (7.1), the ones with one degenerate field, which are constructed recursively starting from a boundary value.

Let us summarize. We have imposed three boundary conditions corresponding to derivatives of 3-point correlators with respect to $\lambda_L, \lambda_M, \tilde{\lambda}_M$ - for $\varepsilon = 1$ (or $\tilde{\lambda}_L, \lambda_M, \tilde{\lambda}_M$ - for $\varepsilon = -1$). These are the coupling constants in a three-term interaction which includes one of the Liouville and both matter screening charges. The "doubled" matter interaction contributes perturbatively, i.e, with integer powers of the screening charges.

The duality transformation (2.16) exchanges the two solutions (7.2) $\varepsilon = \pm 1$. Pairs of inhomogeneous Liouville reflections interchange the solutions of different chiralities, generalising the relations (6.5).

• We note that there is a special case involving a non-integer number of screening charges. For $n = 0, \varepsilon = 1$ or for $m = 0, \varepsilon = -1$, the fixed chirality solutions (7.2) of the functional equations (5.12) and (5.13) can be written in a form which allows to extend them to arbitrary (non-integer) values of m or n respectively,

$$\begin{aligned}G_4^{(+)} &= \lambda_L^{\frac{1}{b}(Q-\alpha)}\lambda_M^{\frac{1}{b}(e-e_0)}\,\frac{1}{b}(e - e_0 + b)(\alpha - Q - b) = -b\,\partial_{\lambda_L}\,\partial_{\lambda_M}\,\lambda_L^{\frac{1}{b}(Q-\alpha)+1}\lambda_M^{\frac{1}{b}(e-e_0)+1}\\ G_4^{(-)} &= \tilde{\lambda}_L^{b(Q-\alpha)}\tilde{\lambda}_M^{b(e_0-e)}\,b(e_0 - e + \tfrac{1}{b})(\alpha - Q - \tfrac{1}{b}) = -\tfrac{1}{b}\,\partial_{\tilde{\lambda}_L}\,\partial_{\tilde{\lambda}_M}\,\tilde{\lambda}_L^{b(Q-\alpha)+1}\tilde{\lambda}_M^{b(e_0-e)+1}\,.\end{aligned} \tag{7.4}$$

[13] The two simultaneous restrictions on the matter and Liouville charges lead to a value $P_1 \in \mathcal{L}$, which implies a new accidental contact term in (4.5), see section A.3. The solution here is consistent with (4.5) becoming a homogeneous relation.

7.2. Correlators satisfying the locality requirement

Let us now look for a "local" 4-point function, symmetric in the four matter charges, which reproduces for certain range of their values the fixed chirality correlators (7.2). Now the symmetry constraint has many solutions. We observe that the representation in the last line of (7.2) takes the form of (1.2) with fusion multiplicities determined by the charge conservation condition, *i.e.* $N(P_1, P_2, P_3) = 1$ if P_i satisfy $\sum_i P_i = 2e_0 - 2kb + 2\frac{l}{b}$ with some integers k, l between 0 and m or n, and $N(P_1, P_2, P_3) = 0$ otherwise. These are the fusion rules of the underlying local matter correlators of this type and it is natural to solve the symmetry requirement, i.e., to determine the correlator of "local" tachyons so that to preserve these fusion rules. Of the two possible such combinations we choose the one which reduces to (6.7) for $m = 0 = n$,

$$\begin{aligned}
&\hat{\mathcal{G}}(P_1, P_2, P_3, P_4; m, n) = \\
&= \tfrac{1}{2}(m+1)(n+1)(Q + mb + \tfrac{n}{b}) - \tfrac{1}{2}\textstyle\sum_{s=2}^{4}\sum_{k=0}^{m}\sum_{l=0}^{n} |P_1 + P_s - e_0 + 2kb - \tfrac{2l}{b}| \\
&= \tfrac{1}{2}(m+1)(n+1)(Q + mb + \tfrac{n}{b}) \\
&\qquad - \sum_{s=2}^{4}\sum_{\substack{r=-m \\ \text{mod } 2}}^{m}\sum_{\substack{t=-n \\ \text{mod } 2}}^{n} |e_1 + e_s - \tfrac{e_0}{2} - \tfrac{mb}{2} + \tfrac{n}{2b} - \tfrac{rb}{2} - \tfrac{t}{2b}|\,.
\end{aligned} \tag{7.5}$$

The symmetry under permutations of the momenta is ensured by the charge conservation condition (5.10). The duality properties of the properly normalised correlators (7.5) are analogous to those in (6.10); to ensure that the transformation (2.16) and (2.17) become identical, we should include a power of b under the modulus in (7.5). The shift equations satisfied by the local correlators are derived from the explicit expression (7.5).

We shall now give another argument in support of the formula (7.5).

Clearly unlike the simplest example (6.7) discussed in section 4, we now lack a complete "atlas" of fixed chirality solutions to match locally (7.5) in all regions of the momenta. Consider the case $n = 0$. In the physical region $P_{st} = P_s + P_t - e_0 > 0,\ s,t = 2,3,4$ (or equivalently $P_{1i} + 2mb < 0\,, i = 1,2,3$) the local correlator is represented by the solution $G^{-+++}(P_1, P_2, P_3, P_4; m, 0)$ in (7.2). On the other hand we can use the extended identities as (6.14) to find a representation of the local correlator in the "vicinity" of any region described by the eight fixed chirality correlators. The identities (6.13), (6.14) imply that in the extended range of momenta the shift relation (7.1) is replaced by a relation for the local correlators, namely

$$\begin{aligned}
&-\hat{G}(P_1, P_2, P_3, P_4; m, 0) + \hat{G}(P_1 + b, P_2 - b, P_3, P_4; m, 0) \\
&\qquad = 2b + \sum_{s=3,4} \theta(-P_{2s} + b)\,(P_{2s} - b)\,, \qquad \text{for } P_{st} > 0\,, s,t = 2,3,4\,.
\end{aligned} \tag{7.6}$$

The proposed correlators (7.5) do indeed satisfy (6.14) and the shift equation (7.6), as well as all other similar identities. In fact these equations determine completely the local correlators for $m = 1\,, n = 0$ (or the correlators for $m = 0\,, n = 1$), taking also into account the inhomogeneous Liouville reflection relations generalizing (6.5). The solution is identical to the one prescribed by (7.5). Then in the next step we can use this solution, as we did in the case $m = 0 = n$, in order to extend further the general ring relations as identities for the local correlators, generalising (6.13). This in particular determines the local correlator in the "next to the nearest" range, extending (6.14),

$$\hat{G}(P_1, P_2, P_3, P_4; m, 0) = \hat{G}^{-+++}(P_1, P_2, P_3, P_4; m, 0) + \sum_{s=3,4} \left(\theta(-P_{2s})\, P_{2s} + \theta(-P_{2s} - 2b)\,(P_{2s} + 2b)\right)\,, \quad \text{for } P_{2s} + 3b > 0\,, s = 3, 4\,,\ P_{34} > 0\,. \tag{7.7}$$

In principle one can reproduce in this way recursively the correlators (7.5) for the two thermal cases $n = 0$, or $m = 0$. Furthermore we can combine the two types of shift relations. Thus starting again from the local correlator $\hat{G}(P_1, P_2, P_3, P_4; 1, 0)$, which is represented by $\hat{G}^{+---}(P_1, P_2, P_3, P_4; 1, 0)$ in the range $\{P_{st} + 2b < 0\,, s = 2, 3, 4\}$, we can compute the r.h.s. of the second identity in (5.19) and use the new contact terms to extend this identity for arbitrary n and the fixed $m = 1$,

$$\begin{aligned}
&\hat{G}(P_1 - \tfrac{1}{b}, P_2 + \tfrac{1}{b}, P_3, P_4; 1, n) - \hat{G}^{+---}(P_1, P_2, P_3, P_4; 1, n) \\
&+ \hat{G}^{+---}(P_1 - \tfrac{1}{b}, P_2 - \tfrac{1}{b}, P_3, P_4; 1, n-1) - \hat{G}^{+---}(P_1 - \tfrac{2}{b}, P_2, P_3, P_4; 1, n-1) \\
&= \tfrac{2}{b} - \sum_{s=3,4} \left(\theta(P_{2s} + \tfrac{1}{b})(P_{2s} + \tfrac{1}{b}) + \theta(P_{2s} + 2b + \tfrac{1}{b})(P_{2s} + 2b + \tfrac{1}{b})\right)\,,\ P_{st} + 2b < 0\,, s = 2, 3, 4\,;
\end{aligned}$$

or,

$$\hat{G}(P_1, P_2, P_3, P_4; 1, n) = \hat{G}^{+---}(P_1, P_2, P_3, P_4; 1, n) + \sum_{s=3,4} \left(\theta(P_{2s})\, P_{2s} + \theta(P_{2s} + 2b)\,(P_{2s} + 2b)\right)\,,\ \text{for } P_{2s} + 2b - \tfrac{1}{b} < 0\,, s = 3, 4\,, P_{34} + 2b < 0\,, \tag{7.8}$$

etc., confirming (7.5).

One finds also symmetric with respect to the Liouville labels α_i tachyon correlators - they preserve the fusion rules of the Coulomb gas $c > 25$ theory.

7.3. Distribution type solutions

Furthermore a distribution type solution generalizing (6.18) is obtained by multiplying (7.2) with $\delta(P - 2e_0 + 2mb - 2n/b)$ and summing over nonnegative m, n. The n-point "multiplicities" are again distributions, depending only on the total momentum P, but instead of (6.17) they are given by semi-infinite double sums of δ-functions. They are expressed in terms of the 3-point "multiplicity"

$$N_3(P) = \sum_{m,n=0}^{\infty} \delta(P - e_0 + 2mb - 2n/b). \tag{7.9}$$

It satisfies the two relations (3.27), while (3.26) is replaced by the difference identities

$$\begin{aligned} N_{P_1-b,P_2,P_3} - N_{P_1+b,P_2,P_3} &= \sum_{n=0}^{\infty} \delta(\sum_{i=1}^{3} P_i - (2n+1)/b) \\ N_{P_1+1/b,P_2,P_3} - N_{P_1-1/b,P_2,P_3} &= \sum_{m=0}^{\infty} \delta(\sum_{i=1}^{3} P_i + (2m+1)b) \,. \end{aligned} \tag{7.10}$$

We define the quantity $\langle P \rangle$ as in (6.19), but with the new 3-point multiplicity (7.10). Using the properties of (7.9) one reproduces the functional relation

$$\begin{aligned} \langle P \rangle_{P_1;P_2,P_3,P_4} + \langle P \rangle_{P_1+2b;P_2,P_3,P_4} - \langle P \rangle_{P_1+b;P_2-\,b,P_3,P_4)} - \langle P \rangle_{P_1+b;P_2+\,b,P_3,P_4)} \\ = 2b \sum_{m,n=0}^{\infty} (n+1)\delta(\sum_i P_i - 2e_0 + 2mb - 2n/b) \,. \end{aligned} \tag{7.11}$$

The identity is equivalent to (5.9), when projected to a fixed sum of momenta, since the irreducible part of the 4-point function satisfies the homogeneous equation. A local correlator with $\langle P \rangle$ replaced by $\langle |P| \rangle$ is also obtained.

• Now let us turn to the diagonal theory defined by the action (4.26). We shall look for solutions for the 4-point function assuming a "diagonal" ($m = n$) charge conservation condition (5.10). This leads to a single sum of δ functions representing the 3-point multiplicity

$$N_{P_1,P_2,P_3} \equiv N_3(P) = \sum_{k=0}^{\infty} \delta(P - (2k+1)e_0). \tag{7.12}$$

The 4-point multiplicity is accordingly

$$N_{P_1,P_2,P_3,P_4} \equiv \int_{-\infty}^{\infty} dP \, N_{P_1,P_2,-P} N_{P,P_3,P_4} = \sum_{m=0}^{\infty} (m+1)\, \delta(\sum_{i=1}^{4} P_i - 2(m+1)e_0) \,. \tag{7.13}$$

Instead of (7.11) one obtains

$$\sum_{\sigma=\pm 1} \langle P \rangle_{P_1+\sigma e_0;P_2,P_3,P_4} - \sum_{\sigma=\pm} \langle P \rangle_{P_1;P_2+\sigma e_0,P_3,P_4} = -2e_0 \; N_3(P_1+P_2+P_3+P_4) \,. \tag{7.14}$$

The the r.h.s. of (6.18), now with the multiplicities defined in (7.12), (7.13), provides a solution of (5.22)[14],

$$
\begin{aligned}
\hat{\mathcal{G}}_4^{(\varepsilon)}(P_1|P_2,P_3,P_4) &= \frac{1}{2}\,\Big(Q\,N_{P_1,P_2,P_3,P_4} - \varepsilon\,\langle\,P\,\rangle_{P_1;P_2,P_3,P_4}\Big)\\
&= \frac{1}{2}\sum_{m=0}^{\infty}\left[Q(m+1) + \varepsilon\sum_{s=2}^{4}\sum_{k=0}^{m}(P_1+P_s-(2k+1)e_0)\right]\delta(\sum_{i=1}^{4}P_i - 2(m+1)e_0) \qquad (7.15)\\
&= \sum_{m=0}^{\infty}(m+1)\big(\sum_i \alpha_i - Q - b^{\varepsilon} + \varepsilon\, m\,\tfrac{e_0}{2}\big)\delta(\sum_{i=1}^{4}P_i - 2(m+1)e_0)\,.
\end{aligned}
$$

Note that in contrast with (7.15), in the non-diagonal theory the solution cannot be expressed entirely in terms of the 3-point "multiplicities" and the inverse 2-point correlator due to the more complicated form of the 1pi part.

From (7.15) one extrapolates the symmetric correlator

$$
\hat{\mathcal{G}}(P_1,P_2,P_3,P_4;m) = \tfrac{Q}{2}(m+1) - \tfrac{1}{2}\textstyle\sum_{s=2}^{4}\sum_{\substack{r=-m\\ \text{mod } 2}}^{m}\ |P_1+P_s-(m+1)e_0 - re_0|\,. \qquad (7.16)
$$

In this case there is no underlying local matter theory to compare with, rather we preserve the fusion rules (7.12). A formula of this type is reproduced in the microscopic approach in [29], with the delta-functions replaced by periodic delta's.

• The "multiplicities" introduced in this section are considered for real momenta only, but they can be expressed in terms of meromorphic functions defined in the whole complex plane. Thus the 3-point multiplicity (7.10) is given by the discontinuity on the real axis of a meromorphic function,

$$
N_3(P) = \frac{1}{4\pi i}\Big(f(\tfrac{e_0-P}{2}+i0) - f(\tfrac{e_0-P}{2}-i0)\Big), \qquad (7.17)
$$

namely the logarithmic derivative of the double Γ-function

$$
f(z) \equiv -\partial_z\log\Gamma_b(z+b) = -\int_0^{\infty}dt\left(\frac{e^{-zt}}{(1-e^{bt})(1-e^{-t/b})} - \frac{z-\frac{e_0}{2}}{t}e^{-t} + \frac{1}{t^2}\right). \qquad (7.18)
$$

The diagonal multiplicity (7.12) is expressed as the discontinuity of $\psi(z) = \partial_z\log\Gamma(z)$.

[14] The arbitrariness in the diagonal case is fixed comparing the first term, $m=0$, with (6.18), and furthermore, with the solutions with one degenerate field, to be discussed in the next section.

8. Degenerate fields in the diagonal theory

The most interesting correlation functions, especially from the point of view of comparing with the microscopic theory, are those involving four degenerate fields. In this section we solve the difference equations for the spectrum of momenta corresponding to the degenerate matter fields (order operators) in the diagonal theory. This spectrum is given by the diagonal $e_0\mathbb{Z}$ of the grid (4.4), with the point $P = 0$ excluded. We will assume that there is no tachyon with $P = 0$, *i.e.*, the tachyon correlation functions vanish if one of the momenta is zero.

• To begin with, we will find the solutions of the diagonal ring relations with one degenerate field $P_2 = (m+1)e_0$, $\alpha_2 = b - \frac{me_0}{2}$, and three generic. As initial condition we take

$$\hat{\mathcal{G}}_4^{(+)}(P_1, e_0, P_3, P_4) = N_{m_1,m_3,m_4}(\sum_{s\neq 2}\alpha_s - Q) = (\sum_{s\neq 2}\alpha_s - Q)\,, \tag{8.1}$$

where the generic solution with $N_{P_1,P_2,P_3} = 1$ for the 3-point correlator is inserted. We solve (5.22) recursively, under the assumption that at the point $P_2 = 0$ the correlator vanishes. This is achieved automatically if the fields are interpreted as linear combinations of vertex operators antisymmetric under a composition of matter and Liouville reflections:

$$\mathcal{V}_P^{(\varepsilon,A)} = \mathcal{V}_P^{\varepsilon} - \lambda_L^{\frac{\varepsilon P}{b}}\lambda_M^{-\frac{P}{b}}\,\mathcal{V}_{-P}^{\varepsilon} = -\lambda_L^{\frac{\varepsilon P}{b}}\lambda_M^{-\frac{P}{b}}\,\mathcal{V}_{-P}^{(\varepsilon,A)}\,. \tag{8.2}$$

Since the degenerate field is assumed anti-symmetric, the contact terms cancel, as the generic solution (3.18) satisfies the reflection identity (3.19). We get

$$\begin{aligned}\hat{G}_4^{(+)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= (m+1)(\alpha - Q - b + \frac{e_0}{2}m)\\ &= \tfrac{Q}{2}(m+1) - \sum_{s\neq 2}\sum_{\substack{r=-m\\ \text{mod } 2}}^{m}\tfrac{1}{2}(\varepsilon_s P_s - r\,e_0)\,.\end{aligned} \tag{8.3}$$

This expression has the form (1.2), with trivial multiplicities $N_{P,P_s,P_r} = 1$ for $s, r \neq 2$ and a non-trivial multiplicity $N_{P_t,P_2=(m+1)e_0,P}$, representing a continuation of the $sl(2)$ decomposition rule (5.3) to non-integer isospins; the shifts by re_0 in (8.3) correspond to the weight diagram of the irrep of dimension $m+1$. The solution (8.3) also justifies the choice of the linear combination in (7.15). The meaning of the nonnegative integer m in the two types of solutions is different, but in both cases $m+1$ counts the number of intermediate contributions in each channel. Projecting (7.15) to a fixed charge m and inserting the value $P_2 = (m+1)e_0$ reproduces (8.3).

• Now let us consider correlators in which all tachyons correspond to degenerate fields, $P_i = \varepsilon_i m_i e_0$, $m_i \in \mathbb{N}$, *i.e.*, $\alpha_i = \frac{Q}{2} - \frac{m_i e_0}{2}$. These tachyons satisfy fusion rules given by the $sl(2)$ decomposition multiplicity (5.3), which is also expressed by an integral, in general

$$N_{m_1,\dots,m_p} = \tfrac{1}{\pi}\int_0^{2\pi} d\theta \sin^2\theta\ \prod_{i=1}^{p}\tfrac{\sin(m_i\theta)}{\sin\theta}\,, \tag{8.4}$$

in terms of the characters $\chi_{m_i}(\theta) = \frac{\sin(m_i\theta)}{\sin\theta}$. These multiplicities preserve the homogeneous identity, implied by (5.1), with respect to any pair of variables:

$$N_{m_1+1,m_2,\dots,m_p} + N_{m_1-1,\dots,m_p} = N_{m_1,m_2+1,\dots,m_p} + N_{m_1,m_2-1,\dots,m_p}\,. \qquad (8.5)$$

They are symmetric under permutations and extend to arbitrary integer values of the weights m_i by the (shifted) Weyl reflection property

$$N_{-m_1,m_2,\dots,m_p} = -N_{m_1,m_2,\dots,m_p},$$

so that they vanish if some $m_i = 0 = P_i$.

We start with a simple example in which $P_2 = 2e_0$, illustrating the recursive determination of the 4-point correlators. As an "initial" condition we take again the correlator in the first equality in (8.1), but instead of the generic solution $N_{m_1,m_3,m_4} = 1$ we take the $sl(2)$ 3-point multiplicity (5.2). Up to the contact terms, which we will neglect at this stage, eqn. (5.22) gives for $\hat{\mathcal{G}}_4^{(+)}(P_1|2e_0,P_3,P_4) = \hat{G}_4(\alpha_1, b-\frac{1}{2}e_0,\alpha_3,\alpha_4)$

$$\begin{aligned}&\hat{G}_4(\alpha_1, b-\tfrac{1}{2}e_0,\alpha_3,\alpha_4) = \textstyle\sum_{\sigma=\pm1} N_{m_1+\sigma,m_3,m_4}(\alpha - Q - b + e_0\frac{1-\sigma}{2}) + \dots \\ &= N_{m_1,2,m_3,m_4}(\alpha-Q-b) + (e_0 N_{m_1-1,m_3,m_4} + \dots)\,.\end{aligned} \qquad (8.6)$$

To obtain the first term in the second line we have used the homogeneous relation (8.5) for the 4-point multiplicity, applied for $m_2 = 1$ and using that $N_{m_1,m_3,m_4} = N_{m_1,1,m_3,m_4}$. The result should be symmetric with respect to m_1, m_3, m_4, so instead of the incomplete second term in the last line of (8.6) we should have a symmetric expression, which vanishes if some $m_s = 1$, recovering (8.1). A solution to these conditions is given by

$$\hat{G}_4(\alpha_1, b-\tfrac{1}{2}e_0,\alpha_3,\alpha_4) = \ N_{m_1,2,m_3,m_4}\left(\alpha - Q - b + \tfrac{e_0}{2}(N_{m_1,2,m_3,m_4}-1)\right). \qquad (8.7)$$

The normalization of the added term is fixed to $+\frac{e_0}{2}$, since generically $N_{m_1,2,m_3,m_4} = 2$ and this is in agreement with our previous solution (8.3) taken for $m+1 = m_2 = 2$. In the next step of the recursion we take $P_2 = 3e_0$ and use the result in (8.7). Once again we recover the first term $N_{m_1,3,m_3,m_4}(\alpha-Q-b)$ uniquely, while we get an expression for the second term which is not symmetric, and generically should be equal to $2e_0 = e_0(N_{m_1,3,m_3,m_4}-1)$, if compared with (8.3). The end result is a formula in which $N_{m_1,2,m_3,m_4}$ in (8.7) is replaced by N_{m_1,m_2,m_3,m_4}. This formula can be cast in the form

$$\begin{aligned}&\hat{G}_4^{(+)}(m_1,m_2,m_3,m_4) = \\ &\frac{1}{2}\left(Q\,N_{m_1,m_2,m_3,m_4} - \sum_{m=1}(N_{m_1,m_2,m}\,(me_0)\,N_{m,m_3,m_4} + \text{permutations})\right) \\ &= N_{m_1,m_2,m_3,m_4}\left(\sum_i \alpha_i - Q - b + \frac{e_0}{2}(N_{m_1,m_2,m_3,m_4}-1)\right), \quad \alpha_i = \tfrac{Q}{2} - m_i\tfrac{e_0}{2}\,.\end{aligned} \qquad (8.8)$$

To connect the two expressions in the second and the third lines we have used the relation

$$\sum_{m=0}(N_{m_1,m_2,m}\, m\, N_{m,m_3,m_4} + \text{permutations}) = N_{m_1,m_2,m_3,m_4}\Big(\sum_{i=1}^{4} m_i - N_{m_1,m_2,m_3,m_4}\Big)\,. \tag{8.9}$$

This identity has a purely group theoretical formulation being expressed in terms of the $sl(2)$ tensor product decomposition multiplicities (5.3) and the dimensions m_i of the irreps. It is derived using the definition (5.3). We stress that by construction the diagonal degenerate fields satisfy closed fusion algebra, in contrast with the standard $c < 1$ matter quasi-rational theory.

• We obtained recursively the solution (8.8) from the difference equations (5.22) without referring to the exact form of the contact terms. Instead, we strongly used the expected symmetries of the solution and the requirement that whenever $N_{m_1m_2m_3m_4} = m_2$, the solution coincides with (8.3), derived for one degenerate and three generic momenta. In the particular case $P_2 = (m+1)e_0 = P_2 = P_4 = -P_1$ in which $\sum_i P_i = 2(m+1)e_0$ all the three formulae (8.8), (8.3) and (7.15) coincide.

We shall now show that (8.8) satisfies the difference relations (5.22), but with a contact term proportional to the difference of two $sl(2)$ multiplicities,

$$[N]_{m_1,m_2+m_3,m_4} := N_{m_1,m_2+m_3,m_4} - N_{m_1,|m_2-m_3|,m_4}\,. \tag{8.10}$$

The quantity $[N]$ can take values $0, \pm 1$. The second term reflects the interpretation (8.2) of the fields. Indeed, if we represent the fields as in (8.2) and assume that the 3-point functions of the initial fields are given by (5.3), we have finally to retain two of the four resulting contact terms – namely the ones with positive labels, as they appear in (8.10).

To prove the above statement we need some identities for the $sl(2)$ multiplicities. In particular we shall exploit

$$N_{m_1+1,m_2,m_3} - N_{m_1-1,m_2,m_3} = N_{m_1,|m_2-m_3|,1} - N_{m_1,m_2+m_3,1}\,. \tag{8.11}$$

This identity – the r.h.s. of which represents the "deviation" from the simpler relation in (3.25), is derived using the general integral representation (8.4); two of the initially four terms in the r.h.s. survive, as in (8.11), when the equality is restricted to positive indices, *i.e.*, when the multiplicities of the l.h.s. are given by (5.3), as we assume throughout this section. Applying (8.11) to both sides of the following equality

$$\begin{aligned}&\sum_{m=0} N_{m_1,m_3,m}(N_{m+1,m_2,m_4} - N_{m-1,m_2,m_4}) = \\ &-\sum_{m=0}(N_{m+1,m_1,m_3} - N_{m-1,m_1,m_3})N_{m,m_2,m_4}\end{aligned} \tag{8.12}$$

we obtain

$$\sum_m N_{m_1,m_3,m}(N_{m+1,m_2,m_4} - N_{m-1,m_2,m_4}) = -[N]_{m_2+m_4,m_1,m_3} = [N]_{m_1+m_3,m_2,m_4} \,. \tag{8.13}$$

If $m_1 \geq m_s$, $s = 2, 3, 4$, the linear combination in (8.10), (8.13) is symmetric with respect to the three variables m_2, m_3, m_4. Indeed in this case

$$\begin{aligned}[N]_{m_1+m_3,m_2,m_4} &= \tfrac{1}{\pi} \int_0^{2\pi} d\theta \, \tfrac{\cos m_1\theta}{\sin\theta} \prod_{s=2,3,4} \sin m_s\theta \\ &= -[N]_{m_1,m_2+m_4,m_3} = -[N]_{m_1,m_2+m_3,m_4} = -[N]_{m_1,m_4+m_3,m_2} \,.\end{aligned} \tag{8.14}$$

Thus choosing the largest of the labels m_i, say, m_1, as the one corresponding to the negative chirality $\varepsilon_1 = -1$, we arrive at the symmetry relation (8.14) of the type of (5.23).

We shall now check that (8.8) satisfies the ring relation with the contact term given by the linear combination (8.10). Indeed if we compute the shifts of the function (8.8) – interpreted as $\hat{\mathcal{G}}^{(+)}(P_1|P_2, P_3, P_4)$ – we get, using (8.5), (8.13), (8.14),

$$\begin{aligned}&\sum_{\sigma=\pm1} \hat{G}_4^{(+)}(m_1+\sigma, m_2, m_3, m_4) - \sum_{\sigma=\pm1} \hat{G}_4^{(+)}(m_1, m_2+\sigma, m_3, m_4) \\ &\quad = \tfrac{e_0}{2} \, ([N]_{m_2+m_3,m_1,m_4} + [N]_{m_2+m_4,m_1,m_3}) = e_0[N]_{m_2+m_3,m_1,m_4} \,, \\ &\sum_{\sigma=\pm1} \hat{G}_4^{(+)}(m_1, m_2+\sigma, m_3, m_4) - \sum_{\sigma=\pm1} \hat{G}_4^{(+)}(m_1, m_2, m_3+\sigma, m_4); \\ &\quad = \tfrac{e_0}{2} \, ([N]_{m_1,m_3+m_4,m_2} - [N]_{m_1,m_2+m_4,m_3}) = 0.\end{aligned} \tag{8.15}$$

The r.h.s of the second relation in (8.15) vanishes due to (8.14), so that it takes the form of the diagonal version of the homogeneous relation (5.16). The two terms in the r.h.s of the first relation are identical and sum up to one term (which can now take the values $0, 1$). We stress that these identities hold in the region of validity of (8.14), *i.e.*, when the field of negative chirality is chosen to coincide with the largest of the integers m_i. Otherwise the formula (8.8) is symmetric with respect to the four labels. Eq. (8.15) is an analog of the formula (6.7) in the sense that, similarly to (6.7), it reproduces solutions of the ring relations with $\sum_i \varepsilon_i = 2$ in certain regions of momenta (*i.e.*, it does not distinguish the negative chirality sign unless we specify which m_i is bigger.) What simplifies here the correlator and the shift equations is that the various local regions are determined by the individial momenta and furthermore the intermediate momenta all have an identical sign.

- The first line in (8.8) extends to negative m_s, so that $G_4^{(+)}(m_1, m_2, m_3, -m_4) = -(\frac{\lambda_M}{\lambda_L})^{\frac{e_0}{b}} \, G_4^{(+)}(m_1, m_2, m_3, m_4)$. The values $-m_4 = -1, m_4 = 1$ correspond to $T^+_{1/b}$, T^+_b

respectively. Restoring the prefactor $-b/e_0$ in (5.21) we can write

$$\begin{aligned}(-b\partial_{\lambda_L})\, \lambda_L^{\frac{1}{b}(Q-\sum_{s=1}^{3}\alpha_s)}\, N_{m_1,m_2,m_3} &= -\tfrac{e_0}{b}\, G_4^{(+)}(m_1,m_2,m_3,1) \\ &= G_4^{(+)}(m_1,m_2,m_3,1) + \tfrac{1}{b^2}\, (\tfrac{\lambda_L}{\lambda_M})^{\frac{e_0}{b}}\, G_4^{(+)}(m_1,m_2,m_3,-1)\,.\end{aligned} \tag{8.16}$$

We can interpret (8.16) as a "boundary" condition obtained from the first two of the four terms in the diagonal action (4.26); the differentiation with respect to λ_L gives the linear combination in the second line in (8.16) (taken with a prefactor $1/b$ due to the rescaling in (5.18)). In that sense the action defining our correlators is given by the two positive chirality terms of the diagonal perturbation (4.26).

- The 4-point correlator with $\sum_i \varepsilon_i = -2$ is constructed in a similar way, parametrizing the momenta as $P_i = -\varepsilon_i m_i e_0$ (so that they are physical for $e_0 < 0$),

$$\begin{aligned}&\hat{G}_4^{(-)}(m_1,m_2,m_3,m_4) := \\ &\frac{1}{2}\left(Q\, N_{m_1,m_2,m_3,m_4} - \sum_{m=0}(N_{m_1,m_2,m}\,(-me_0)\, N_{m,m_3,m_4} + \text{permutations})\right) \\ &= N_{m_1,m_2,m_3,m_4}\left(\sum_i \alpha_i - Q - \tfrac{1}{b} - \frac{e_0}{2}(N_{m_1,m_2,m_3,m_4} - 1)\right), \quad \alpha_i = \tfrac{Q}{2} + m_i \tfrac{e_0}{2}\,.\end{aligned} \tag{8.17}$$

Then

$$\begin{aligned}(-\frac{1}{b}\partial_{\tilde\lambda_L})\tilde\lambda_L^{b(Q-\sum_{s=1}^{3}\alpha_s)}\, N_{m_1,m_2,m_3} &= e_0 b\, G_4^{(-)}(m_1,m_2,m_3,1) \\ &= G_4^{(-)}(m_1,m_2,m_3,1) + b^2\, (\tfrac{\tilde\lambda_L}{\tilde\lambda_M})^{-e_0\, b}\, G_4^{(-)}(m_1,m_2,m_3,-1)\end{aligned} \tag{8.18}$$

can be interpreted as a boundary value related to the two negative chirality terms in (4.26).

The solution (8.8) with $be_0 > 0$ (or (8.17) with $be_0 < 0$) reproduces the 4-point correlation function of the microscopic model [29]. Duality interchanges the normalised with (3.21) correlators; effectively both (8.8) and (8.17) get multiplied by Q times the standard powers of λ_L, λ_M. On the other hand the transformations (3.24) lead to correlators of the same type, in which the positive integers $m_i = 2j_i + 1$ parametrize the diagonal degenerate Liouville points. (*i.e.*, $Q - 2\alpha = (2j+1)Q$, so that now the Liouville scaling dimension takes a "Sugawara" form $\triangle_L = -j(j+1)Q^2$). These correlators are solutions of a ring relation computed with the dual diagonal action (4.27).

9. Degenerate fields in the conventional theory

As we have discussed, in order to extend all ring relations to the whole lattice $\mathcal{L}$, one needs to know all possible additional contact terms. On the other hand when only one of the tachyons in the correlator is degenerate, solving some of the ring equations already determines the unique solution.

- *One degenerate, three generic fields*

It will be convenient to shift the notation compared with (4.10), so that the matter degenerate momenta $P = e_0 - mb + n/b$ are parametrized by nonnegative integers $m, n \in \mathbb{Z}_{\geq 0}$. We take $\mathcal{W}^{(\varepsilon)}_{P_2}$ as the degenerate tachyon, while the momenta of the remaining three operators are assumed generic. According to the analysis in Appendix A.3. there are no additional unknown contact terms in this case. We shall solve recursively the equations, assuming that the tachyons at the border lines $n = -1$ and $m = -1$ have vanishing correlators.

Let us start with the "thermal" cases $n = 0$, or $m = 0$. As before we take as initial conditions

$$\hat{G}^{(\varepsilon)}_4(\alpha_1, b^\varepsilon, \alpha_3, \alpha_4) = \sum_{s\neq 2} \alpha_s - Q = \alpha - Q - b^\varepsilon \, . \tag{9.1}$$

Solving recursively (5.12) we obtain

$$\hat{G}^{(+)}_4(\alpha_1, \alpha_2 = b + \tfrac{mb}{2}, \alpha_3, \alpha_4) = \sum_{\substack{r=-m \\ \text{mod } 2}}^{m} \hat{G}_4(\alpha_1 - r\tfrac{b}{2}, \alpha_2 = b, \alpha_3, \alpha_4) + (m+1)\tfrac{mb}{2} \tag{9.2}$$

which can be also rewritten as

$$\begin{aligned}\hat{G}^{(+)}_4(\alpha_1, \alpha_2 = b + \tfrac{mb}{2}, \alpha_3, \alpha_4) &= (m+1)(\sum_{s\neq 2} \alpha_s - Q + \tfrac{mb}{2}) = (m+1)(\alpha - Q - b) \\ &= (m+1)(\tfrac{Q}{2} + \tfrac{mb}{2}) - \sum_{s\neq 2}\sum_{k=0}^{m}(\tfrac{Q}{2} - \alpha_s - \tfrac{mb}{2} + kb) \, .\end{aligned} \tag{9.3}$$

Similarly (5.13) gives

$$\begin{aligned}\hat{G}^{(-)}_4(\alpha_1, \alpha_2 = \tfrac{1}{b} + \tfrac{n}{2b}, \alpha_3, \alpha_4) &= (n+1)(\sum_{s\neq 2} \alpha_s - Q + \tfrac{n}{2b}) = (n+1)(\alpha - Q - \tfrac{1}{b}) \\ &= (n+1)(\tfrac{Q}{2} + \tfrac{n}{2b}) - \sum_{s\neq 2}\sum_{k=0}^{m}(\tfrac{Q}{2} - \alpha_s - \tfrac{n}{2b} + kb) \, .\end{aligned} \tag{9.4}$$

The $m+1$ or $n+1$ terms in each of the three channels of the above expansions correspond to the weight diagram of the $sl(2)$ irreps of dimension $m+1$, or $n+1$.

In deriving these formulae we have used only one of the ring relations. In the other channels one has to take into account the additional contact terms. If the degenerate field is represented by an integrated tachyon, then the accidental contact term due to (4.24), taken for $n=0$, precisely compensates the generic one. Indeed (9.3), (9.4) satisfy homogeneous relations, e.g. [15]

$$\sum_{\pm} \hat{G}_4^{(+)}(\alpha_1 \pm \tfrac{b}{2}, \alpha_2, \alpha_3, \alpha_4) = \sum_{\pm} \hat{G}_4^{(+)}(\alpha_1, \alpha_2, \alpha_3 \pm \tfrac{b}{2}, \alpha_4)\,.$$

We can compare these solutions with the ones in (7.2), extending the latter to the values $P_2 = e_0 - mb$ or $P_2 = e_0 + n/b$ for $\varepsilon = \pm 1$ respectively. For these special values (7.2) coincides with (9.3), or (9.4), and this justifies the choice in the 1pi-term in (7.2), obtained by a different argument.

Now let us consider an arbitrary degenerate momentum $P_2 = e_0 - mb + n/b$. Solving (5.9) recursively with m we get instead of (9.2)

$$\begin{aligned} &\hat{G}_4^{(+)}(\alpha_1, \alpha_2 = b + \tfrac{mb}{2} - \tfrac{n}{2b}, \alpha_3, \alpha_4) = \\ &\sum_{\substack{r=-m \\ \mathrm{mod}\ 2}}^{m} \hat{G}_4^{(+)}(\alpha_1 - r\tfrac{b}{2}, \alpha_2 = b - \tfrac{n}{2b}, \alpha_3, \alpha_4) + (n+1)(m+1)\tfrac{mb}{2}. \end{aligned} \tag{9.5}$$

To proceed further we need to identify $\hat{G}_4^{(+)}$ with a negative chirality correlator (9.4) with the same value of the degenerate momentum P_2. We choose

$$\begin{aligned} \hat{G}_4^{(+)}(\alpha_1, b - \tfrac{n}{2b}, \alpha_3, \alpha_4) &\equiv \hat{G}_4^{(-)}(\alpha_1, \tfrac{1}{b} + \tfrac{n}{2b}, \alpha_3, \alpha_4) \\ \hat{G}_4^{(-)}(\alpha_1, \tfrac{1}{b} - \tfrac{mb}{2}, \alpha_3, \alpha_4) &\equiv \hat{G}_4^{(+)}(\alpha_1, b + \tfrac{m}{2}, \alpha_3, \alpha_4) \end{aligned} \tag{9.6}$$

so that in particular the initial condition (9.1) for $n = 0 = m$ is preserved. Inserting (9.6) in (9.5) and using the first equalities in (9.4), (9.3), we obtain

$$\begin{aligned} &\hat{G}_4^{(\varepsilon)}(\alpha_1, \alpha_2 = b^{\varepsilon} + \varepsilon(\tfrac{mb}{2} - \tfrac{n}{2b}), \alpha_3, \alpha_4) = (n+1)(m+1)\big(\alpha - Q - b^{\varepsilon} + \tfrac{n}{b}\tfrac{1+\varepsilon}{2} + mb\tfrac{1-\varepsilon}{2}\big) \\ &= (n+1)(m+1)\big(\tfrac{Q}{2} + \tfrac{mb}{2} + \tfrac{n}{2b}\big) - \sum_{s\neq 2} \sum_{\substack{r=-m \\ \mathrm{mod}\ 2}}^{m} \sum_{\substack{t=-n \\ \mathrm{mod}\ 2}}^{n} \big(\tfrac{Q}{2} - \alpha_s + \tfrac{rb}{2} + \tfrac{t}{2b}\big)\,. \end{aligned} \tag{9.7}$$

The identification (9.6) is suggested by the comparison with (7.2) - the latter coincides with (9.7) if $P_2 = e_0 + n/b - mb$.

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[15] On the other hand the homogeneous equation for the correlator of type $^{++++}$ is solved by $G^{++++}(\alpha_1, \alpha_2 = b + \tfrac{mb}{2}, \alpha_3, \alpha_4) = (m+1)(\sum_{\neq 2} \alpha_s - Q)$.

The second line of (9.7) illustrates the general form (1.2). The first fusion multiplicity corresponds to the shift of P_s with the weight diagram of the degenerate field, *i.e.* $P = P_s + P_2 - e_0 + 2kb - 2l/b\,, k = 0, \dots m\,, l = 0, \dots n$, while the multiplicity depending on three generic momenta corresponds to the trivial solution (3.18). As in section 7 we shall choose a solution of the symmetry requirement preserving these fusion rules since once again they correspond to the fusion rules of the underlying local matter correlator. We obtain a symmetric in the three generic momenta P_s formula

$$\begin{aligned}&\hat{\mathcal{G}}_4(P_1, P_2 = e_0 + n/b - mb, P_3, P_4) = \\ &\quad \tfrac{1}{2}\Big((n+1)(m+1)(Q + mb + \tfrac{n}{b}) - \textstyle\sum_{s\neq 2}\sum_{\substack{r=-m \\ \mathrm{mod}\ 2}}^{m}\ \sum_{\substack{t=-n \\ \mathrm{mod}\ 2}}^{n}\ |P_s + rb + \tfrac{t}{b}|\Big)\,.\end{aligned} \tag{9.8}$$

To check this result let us analyse directly the equation (6.13) for the local correlators similarly as we did in section 7. We rewrite (6.13) as

$$\begin{aligned}&\hat{G}(P_1, P_2 - b = e_0 - mb, P_3, P_4) = \sum_{\pm}\hat{G}(P_1 \pm b, P_2, P_3, P_4) - \hat{G}(P_1, P_2 + b, P_3, P_4) \\ &+ b + \sum_{s=3,4}\theta(-P_{2s} + b)(P_{2s} - b)\,, \qquad P_{2s} = P_s - (m-1)b > 0\,, s = 3, 4\,.\end{aligned} \tag{9.9}$$

We shall illustrate this identity for $m = 1$ in which case the last correlator corresponding to the border momentum $P_2 = e_0 + b$ drops. We start with the local counterpart of (9.1) as an initial condition

$$\hat{\mathcal{G}}(P_1, P_2 = e_0, P_3, P_4) = \tfrac{Q}{2} - \tfrac{1}{2}\textstyle\sum_{s\neq 2}|P_s|\,.$$

Then

$$\begin{aligned}&\hat{\mathcal{G}}(P_1, e_0 - b, P_3, P_4) = \sum_{\pm}\hat{\mathcal{G}}(P_1 \pm b, e_0, P_3, P_4) + b - \sum_{s=3,4}\theta(-P_{2s} + b)|P_{2s} - b| \\ &= Q + b - \tfrac{1}{2}|P_1 - b| - \tfrac{1}{2}|P_1 + b| - \tfrac{1}{2}\textstyle\sum_{s=3,4}\left((P_s + b) + |P_s - b|(\theta(P_s - b) + \theta(-P_s + b))\right) \\ &= Q + b - \tfrac{1}{2}\textstyle\sum_{s\neq 2}(|P_s + b| + |P_s - b|)\,, \qquad \text{for } P_3, P_4 > 0\end{aligned} \tag{9.10}$$

thus confirming formula (9.8) for this particular example.

The fixed chirality formulae (9.3), (9.4) were presented in [35]. The physical correlator (9.8) reproduces the expression found by a different method in [37], in which the locality of the underlying correlators is automatically taken into account.

Similarly one solves the equations in the case when one of the tachyons is Liouville degenerate.

- *Four degenerate fields – a conjecture*

When all four fields are labelled by degenerate matter representations the 3-point function $\hat{G}_3 = 1$ is to be replaced by the fusion multiplicity in (5.3), (5.5). Accordingly the initial value (9.1) gets multiplied by this multiplicity. The equations themselves get more complicated due to many additional contact terms and the possible cancellations between them. We conjecture that the effect will be, like in the diagonal case, an expression in which the 3-point $sl(2)$ fusion multiplicities (5.5) determine the expansion range, while the factors $n+1$ and $m+1$ in (9.7) are replaced by the 4-point sl(2) multiplicities in (5.6), symmetric under the change of sign of any of the momenta. For $\alpha_i = \frac{Q}{2} - \varepsilon(\frac{n_i}{2b} - \frac{m_i b}{2})\,, i = 1,2,3,4$ and $P_{m,n} = n/b - mb$, $n_i, m_i, n, m \in \mathbb{N}$, this leads to

$$
\begin{aligned}
\hat{G}_4^{(\varepsilon)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \tfrac{1}{2}[N_{P_1,P_2,P_3,P_4}\left(bN_{m_1,m_2,m_3,m_4} + \tfrac{1}{b}\,N_{n_1,n_2,n_3,n_4}\right) \\
&\quad - \varepsilon \sum_{m,n=1} \left(N_{P_1,P_2,P_{m,n}}\,(\tfrac{n}{b} - mb)\,N_{P_{m,n},P_3,P_4} + \text{permutations}\{2,3,4\}\right)] \\
&= N_{P_1,P_2,P_3,P_4}\left(\alpha - Q - b^{\varepsilon} + \tfrac{1-\varepsilon}{2} b(N_{m_1,m_2,m_3,m_4} - 1) + \tfrac{1+\varepsilon}{2b}\,(N_{n_1,n_2,n_3,n_4} - 1)\right)\,.
\end{aligned}
\tag{9.11}
$$

In the last equality we used (8.9). The conjectured local correlator is given by a formula as in the first line of (9.11) with intermediate momenta $\varepsilon(\frac{n}{b} - mb)$ replaced by $|\frac{n}{b} - mb|$.

10. Summary and discussion

In this paper we reported the results of our study of 2d quantum gravity, or non-critical bosonic string theory, with generic non-rational values of the matter central charge (1.1).

The main point of our investigation is the systematic study of the effects of including matter interactions in the 2d string. Conventionally one adds to the gaussian action the two matter screening charges, which together with the Liouville ones serve as interaction terms. Motivated by the comparison with a discrete, microscopic approach, to be discussed in a subsequent paper [29], we introduced and studied also another deformation of the Liouville theory, defined by the interaction action (4.26). While in the first, "conventional" theory, the $c < 1$ (matter) and $c > 25$ (Liouville) parts factorize before moduli integration, there is no such factorization in the second theory, which we called "diagonal".

To construct the tachyon correlators we have adopted and extended the ground ring approach introduced long ago [14-18]. In this approach the matter-Liouville factorization of the integrand of the 4-point tachyon correlators (in the conventional theory) is not directly exploited, and so the precise realization of any of these $c < 1$ and $c > 25$ correlators is not apriori required. In particular no assumption on the existence of a fully consistent non-rational matter theory is made. Indeed such a theory has not been rigorously established in the conventional theory, and does not exist in the second, diagonal theory. For our purposes it was sufficient to derive a 3-point generic matter OPE constant, formula (3.8), a $c < 1$

analog of the Liouville DOZZ formula, (3.2) which extends the Dotsenko-Fateev Coulomb gas constant. The ground ring method is based on the derivation of functional equations for the tachyon correlators, using the module action (operator product) of the fundamental ground ring elements on a (0,0)-form tachyon W^{ε}_{α} in the presence of integrated tachyons. The OPE coefficients of the ground ring action are determined by well defined free field correlators, computed either by using the matter-Liouville Coulomb gas representation, or exploiting the factorization into known $c < 1$ and $c > 25$ Coulomb gas correlators.

The explicit 3-point OPE coefficients in (4.5), (4.6) confirm the ground ring structure conjectured in [24]. The functional relations for the 3-point functions are closely related to a standard identity for the tensor product decomposition multiplicities of sl(2) finite dimensional irreps, which are reproduced as a particular case.[16] Besides those in the non-rational case one has more 3-point solutions and some were used as a building block in the construction of the 4-point solutions we have described. The diagonal theory admits an action of the ground ring generated by the new deformations of the product of ring generators $a_- a_+$. The result (4.28) is an effective projection of the ring action to a diagonal $sl(2)$ type identities.

What complicates the case of n-point functions, $n > 3$, are the additional contact terms in the functional relations due to the fact that the fourth, *etc.* field, given by an integrated tachyon T^{ε}_{α}, serves as a new "screening charge". Thus, besides the two operator terms in (4.5), (4.6) which correspond to perturbations by the screening charges in the interaction actions (2.4), (2.5), there are other channels in the OPE of a ring generator and a tachyon W^{ε}_{α}. These OPE terms account for the effect of the $Q_{\rm BRST}$-exact terms, skipped in the r.h.s. of (4.5), (4.6). We have computed two series of 4-point OPE coefficients, (4.21), (4.24), sufficient for the class of tachyon correlators we consider. The diagonal model is more restrictive on the content of the operator products and in particular leaves less room for contact terms.

We have found basically two types of 4-point solutions of the functional equations (5.8). Apart from a particular example, both involve an integer number of some of the screening charges. We have presented in more detail the solutions with matter screening charges, however, because of the symmetry of the ring identities, in the conventional non-diagonal theory some of these solutions have Liouville analogs as well.

The first class of solutions, (7.2), appears for generic values of the four tachyon momenta, such that their sum is restricted by a matter charge conservation, thus generalizing the tachyon correlators for gaussian matter of [34]. The arbitrariness in the solutions of the homogeneous equations, or, effectively, in the determination of the 1pi part in (1.2), is partially fixed by comparison with 4-point functions in which one of the tachyons is a

16 We stress that these multiplicities would not be allowed if the formal "matter $\times$ Liouville factorisation" was taken too literary.

screening charge. The choice of these “boundary” conditions corresponds to the type of interaction action, which otherwise enters the definition of the correlators rather formally. To fix the remaining arbitrariness we have required that different classes of solutions are related to each other, whenever their partial wave expansions, as in (1.2), are comparable.

The second class of solutions found in sections 8 and 9 represents 4-point functions in which one field (9.7), or all fields (formulae (8.8), (8.17) in the diagonal theory) correspond to a degenerate Virasoro representation. For the correlators with four degenerate fields in the conventional theory we only give a conjecture. The problem is complicated by the unknown additional contact terms at degenerate values.

The equations we have derived and studied apply by definition to the correlators satisfying the chirality rule. Besides these fixed chirality, and hence partially symmetric, 4-point functions we have described also correlators symmetric with respect to the four (or the three generic) matter charges. We interpreted this symmetry as tachyon ”locality”. Until this point locality of the underlying $c < 1$ and $c > 25$ correlators is only partially exploted in the computation of the OPE coefficients. In the simplest example in section 6 the set of fixed chirality solutions serves as a local basis for the local correlators. Then the original equations are rewritten equivalently as equations for these symmetrised correlators. To fix in general the arbitrariness in the solution of the symmetry requirement we have exploited the fact that all our solutions admit the channel decomposition form (1.2). Our universal choice was to preserve the fusion multiplicities in the symmetric counterpart of this expression - the formal rule is to replace the inverse propagator εP with $|P|$. This choice indirectly takes into account the locality of the underlying $c < 1$ theory, since these fusion rules correspond to the ones manifested by the local correlators of that theory. Furthermore the local correlators (6.7), (7.5), (9.8) are invariant under Liouville reflections.

The symmetric correlators (as well as their analogs, symmetrised with respect to the Liouville labels) do not satisfy globally the original ring relations, rather satisfy shift equations with modified and momenta dependent inhomogeneous terms. We have proposed an alternative recursive derivation of these equations for the local correlators starting with the simplest case of sect. 6.1. It also yields a full set of local representatives of the symmetric tachyon correlators, extending the set of fixed chirality correlators obtained as solutions of the initial equations.

Our treatment of the local correlators remains however rather “phenomenological” and the direct derivation of their equations is still an open problem which requires an extension of the Coulomb gas based technique we had mostly exploited. Conceptually this is important since it is natural to interpret the local correlators as the true ”physical” ones, while the sets of partially symmetric, fixed chirality correlators, though basic in our construction, should be considered rather as auxiliary objects. This is confirmed by the matrix model approach [29] (formula (7.16)) and also by the comparison with the recent paper [37], in which the underlying matter and Liouville theories are explicitly exploited in the computation of the 4-point function with one matter degenerate and three generic fields: the local correlator (9.8) coincides with the expression in [37] computed by this more constructive method. A notable exceptional case, avoiding these problems and confirmed by the discrete model in [29], are the 4-point functions (8.8) of four degenerate fields in the diagonal theory.

Our analysis has been restricted so far to the bulk quantities. However, as it is well known from the studies in the rational matter and the generic Liouville BCFT, the bulk 3-point correlators, *i.e.*, the (properly normalized) OPE coefficients, give information about the boundaries, since the matrices diagonalizing them are closely related to the disc 1-point functions, as briefly discussed in Appendix B. In Appendix A.5. we have also computed some chiral OPE coefficients in the presence of matter charges, including the four OPE coefficients of the boundary ground ring, which has a similar to (4.5), (4.6) two-term structure. The functional relations for the boundary tachyon correlators, generalizing the trivial matter case [18], [25], will be discussed elsewhere, see also the paper [36] which appeared meanwhile, which deals with this problem too, but in the minimal string theory.

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Appendix A. Coulomb gas computations

In this appendix we shall compute some matrix elements of the type

$$
\begin{aligned}
&\langle\alpha'|\mathbf{c}_{-1}\mathbf{c}_0 \int_{C_n} dz_n\, V^{\varepsilon_n}_{\alpha_n}(z_n)\cdots\int_{C_2} dz_2\, V^{\varepsilon_2}_{\alpha_2}(z_2)\, a_-(z)\,\mathbf{c}V^{\varepsilon_1}_{\alpha_1}(z_1)|0\rangle_{\rm free} = \\
&\int\cdots\int\left((\frac{\alpha_1-e_1}{b}-1)\frac{1}{z-z_1} - \sum_{i=2}^{n}\frac{\alpha_i-e_i}{b}\frac{1}{z_i-z}\right)\times \\
&\langle\alpha'|V^{\varepsilon_n}_{\alpha_n}(z_n)\cdots V_{(\frac{b}{2},-\frac{b}{2})}(z)\,V^{\varepsilon_1}_{\alpha_1}(z_1)|0\rangle_{\rm free} = \\
&\int\cdots\int\left(\frac{1}{b^2}\,\langle...\rangle_M\partial_z\langle...\rangle_L - \frac{1}{b^2}\,\langle...\rangle_L\partial_z\langle...\rangle_M - \frac{1}{z-z_1}\langle...\rangle_M\,\langle...\rangle_L\right)
\end{aligned}
\tag{A.1}
$$

and their volume integral counterparts, which determine the OPE coefficients of the ring generator a_- with the tachyon fields. Everywhere here V^{ε}_{α} , or $V_{(e,\alpha)}$ denote *unnormalized* products of vertex operators, with no relation necessarily of the type in (2.7) on the pair of matter and Liouville charges (e,α). There is a similar formula for the other generator.

Conventions:

$$\langle\phi(x_1)\phi(x_2)\rangle = -\frac{1}{2}\log x_{12}^2 = \langle\chi(x_1)\chi(x_2)\rangle$$

$$\phi(z) = \phi^{(+)}(z)+\phi^{(-)}(z) := \frac{i}{\sqrt{2}}\big(-a_0\log z+\sum_{n>0}\frac{a_n}{n}z^{-n}\big)+\frac{i}{\sqrt{2}}\big(-iq-\sum_{n>0}\frac{a_{-n}}{n}z^{n}\big)$$

$$[a_n,a_{-m}] = n\delta_{n,m}\,,\qquad [a_0,q]=-i$$

$$V^{(+)}_{(e_1,\alpha_1)}(z_1)\,V^{(-)}_{(e_2,\alpha_2)}(z_2) = z_{12}^{2(e_1e_2-\alpha_1\alpha_2)}\,V^{(-)}_{(e_2,\alpha_2)}(z_2)\,V^{(+)}_{(e_1,\alpha_1)}(z_1)\,,\qquad |z_1|>|z_2|$$

Let us also recall some ghost field correlation functions. The correlators of the ghost $\mathbf{b},\mathbf{c}$ fields decouple as the full correlators factorize. The 2-point function is computed in the vacua $\langle 0|\mathbf{c}_{-1}\mathbf{c}_0\mathbf{c}_1$ and $|0\rangle$, normalizing $\langle 0|\mathbf{c}_{-1}\,\mathbf{c}_0\,\mathbf{c}_1|0\rangle=1$,

$$\begin{aligned}\langle\mathbf{b}(z_1)\ \mathbf{c}(z_2)\rangle &= \langle 0|\mathbf{c}_{-1}\mathbf{c}_0\,\mathbf{c}_1\sum_{k=-1}\mathbf{b}_k\,z_1^{-k-2}\sum_{m=-1}\mathbf{c}_{-m}z_2^{m+1}\,|0\rangle = \\ &\langle 0|\mathbf{c}_{-1}\mathbf{c}_0\mathbf{c}_1|0\rangle\frac{1}{z_1}\sum_{p=0}\left(\frac{z_2}{z_1}\right)^p = \frac{1}{z_{12}}\,,\quad |z_1|>|z_2|\,.\end{aligned} \tag{A.2}$$

The 3-point ghost $\mathbf{c}$ function is

$$\langle 0|\mathbf{c}(z_1)\,\mathbf{c}(z_2)\,\mathbf{c}(z_3)|0\rangle = \langle 0|\prod_{i=1}^{3}\sum_{k_i=-1}^{1}\mathbf{c}_{k_i}\,z_i^{-k_i+1}\,|0\rangle = z_{12}\,z_{13}\,z_{23}\,. \tag{A.3}$$

The 3-point function with the insertion of one field $:\mathbf{bc}:$ reads

$$\begin{aligned}&\langle 0|\mathbf{c}_{-1}\,\mathbf{c}(z_2)\ :\mathbf{bc}(z):\ \mathbf{c}(z_1)|0\rangle = \\ &\langle 0|\mathbf{c}_{-1}\sum_{s=0}\mathbf{c}_s\,z_2^{-s+1}\sum_{m}z^{-m-1}\Big(\sum_{k=2}\mathbf{b}_{-k}\mathbf{c}_{m+k}-\sum_{k=-1}\mathbf{c}_{m-k}\mathbf{b}_k\Big)\sum_{p=-1}\mathbf{c}_{-p}z_1^{p+1}|0\rangle \\ &= \langle 0|\mathbf{c}_{-1}\mathbf{c}(z_2)\,\mathbf{c}(z_1)|0\rangle\ \big(\frac{1}{z_2-z}-\frac{1}{z-z_1}\big)\,,\quad |z_2|>|z|>|z_1|\,,\end{aligned} \tag{A.4}$$

while

$$\langle 0|\,\mathbf{c}_{-1}\,\mathbf{c}_0\ :\mathbf{bc}(z):\ \mathbf{c}(z_1)|0\rangle = -\frac{1}{z-z_1}\,,\quad |z|>|z_1|\,. \tag{A.5}$$

The last formula is used in (A.1) producing the shifts by -1. In particular it leads to the last term in the matter-Liouville factorized expression in the last line, where one is using the representation of the ring operator in terms of derivatives,

$$a_-(z) =: e^{bi\chi}\big(\frac{1}{b^2}\overleftrightarrow{\partial}_z+\mathbf{bc}(z)\big)\,e^{-b\phi}:\,, \tag{A.6}$$

meaning action of the derivative to the right minus action to the left.

A.1. 3-point volume integral matrix elements

We start with some bulk correlators, most of which have been already computed [16,17,18]. In these examples we shall use the first representation in (A.1), while in the next subsection we will exploit the matter-Liouville factorized expression in the second line.

The factor $\mathbf{c}_{-1}\,\bar{\mathbf{c}}_{-1}\,\mathbf{c}_0\bar{\mathbf{c}}_0$ is denoted $(\mathbf{c}\bar{\mathbf{c}})_{-1}\,(\mathbf{c}\bar{\mathbf{c}})_0$. Consider first the matrix element

$$\langle\alpha'|(\mathbf{c}\bar{\mathbf{c}})_{-1}\,(\mathbf{c}\bar{\mathbf{c}})_0\;a_-(x_0)(\mathbf{c}\bar{\mathbf{c}}V^{\varepsilon}_{\alpha})(x_1)\rangle = \langle\alpha'|(e+\frac{b}{2},\alpha-\frac{b}{2})\rangle\,(\frac{\alpha-e-b}{b})^2\,(x^2_{01})^{b(\alpha+e-\frac{1}{b})}$$
$$=\begin{cases} 0 & \text{if } \varepsilon=1,\\ (\frac{2\alpha-Q}{b})^2=-\frac{\gamma(\frac{1}{b}(Q-2\alpha+b)}{\gamma(\frac{1}{b}(Q-2\alpha)} & \text{if } \varepsilon=-1\end{cases} \tag{A.7}$$

and $\alpha'=Q-\alpha+\frac{b}{2}$, recovering

$$a_-W^-_{\alpha}=-W^-_{\alpha-\frac{b}{2}}\,,\qquad a_+W^+_{\alpha}=-W^+_{\alpha-\frac{1}{2b}}\,, \tag{A.8}$$

$$a_-W^+_{\alpha}=a_+W^-_{\alpha}=0\,. \tag{A.9}$$

Our next example is

$$\int\frac{d^2x_2}{\pi}\,\langle\alpha'|(\mathbf{c}\bar{\mathbf{c}})_{-1}\,(\mathbf{c}\bar{\mathbf{c}})_0\,a_-(x_0)(\mathbf{c}\bar{\mathbf{c}}V^{\varepsilon}_{\alpha})(x_1)\,V^{\varepsilon_2}_{\alpha_2}(x_2)\rangle_{\text{free}}=\langle\alpha'|(e+e_2+\frac{b}{2},\alpha+\alpha_2-\frac{b}{2})\rangle$$
$$\int\frac{d^2x_2}{\pi}\left|\frac{\alpha-e-b}{b}\,\frac{1}{z_{01}}+\frac{\alpha_2-e_2}{b}\,\frac{1}{z_{02}}\right|^2\,(x^2_{01})^{b(\alpha+e)}\,(x^2_{02})^{b(\alpha_2+e_2)}\,(x^2_{21})^{2ee_2-2\alpha\alpha_2} \tag{A.10}$$

In the three of the four possible cases this integral vanishes for generic momenta, either due to factors $\gamma(1)$, or, because of sign compensation of the various terms. It survives only for $\varepsilon=\varepsilon_2=1$ producing the constant

$$\frac{\gamma(b(2\alpha_2-b))\,\gamma(b(2b-2\alpha-2\alpha_2)+1)}{\gamma(b(b-2\alpha)+1)}=\frac{\gamma(b(Q-2(\alpha+\alpha_2-\frac{b}{2}))}{\gamma(b(Q-2\alpha))\,\gamma(b(Q-2\alpha_2))}$$

which precisely provides the leg factor normalization of the three tachyons, thus recovering the first formula in (4.17).[17] In agreement with the BRST invariance, both in (A.7) and the integrated (A.10) only the combination satisfying the mass-shell condition survives, while all the other terms, possible in the analogous pure matter or Liouville 3-point matrix elements, now cancel out automatically, due to the effect of the raising prefactor in the ring generator. In particular choosing $\alpha_2=b$ or $\alpha_2=0$ one recovers the generic two term action in the first line in (4.5).

[17] In particular the non-generic value $\alpha+\alpha_2-\frac{b}{2}=\frac{Q}{2}$ corresponds to a tachyon of no definite chirality, for which the numerator and equally the compensating leg factor, become singular.

Finally for $\varepsilon_2=\varepsilon_3=1$ and $\alpha_2+\alpha_3=b$ there is a double integral matrix element

$$\int \frac{d^2x_2}{\pi} \int \frac{d^2x_3}{\pi} \langle\alpha'|(\mathbf{c}\bar{\mathbf{c}})_{-1}\,(\mathbf{c}\bar{\mathbf{c}})_0\, a_-(x_0)(c\bar{c}V^-_{\alpha})(x_1)\, V^+_{\alpha_2}(x_2)\, V^+_{b-\alpha_2}(x_3)\rangle_{\rm free} = \\ = -\frac{\gamma(\frac{1}{b}(Q-2\alpha-b))}{\gamma(\frac{1}{b}(Q-2\alpha))\,\gamma(b(Q-2\alpha_2))\,\gamma(b(Q-2(b-\alpha_2)))}\,. \tag{A.11}$$

This constant reproduces again the relevant leg factors and thus we obtain for the normalized fields the first of the relations (4.25).

A.2. The general 3-point constant

In general accounting for all possible matter and Liouville screening charges one computes the 3-point function of the ring generator with two tachyons using the representation in the last line in (A.1). The result for the 2d integral (for the unnormalized tachyons) is proportional to the product of the 3-point $c<1$ Coulomb gas OPE constant computed in [32] and its $c>25$ counterpart given by an analytic continuation of the same formula,

$$\langle 0|(\mathbf{c}\bar{\mathbf{c}}V^{\varepsilon_3}_{(e_0-e_3,Q-\alpha_3)})(\infty)\,(\mathbf{c}\bar{\mathbf{c}})_0\, a_-\,(\mathbf{c}\bar{\mathbf{c}}V^{\varepsilon_2}_{(e_2,\alpha_2)})\,|0\rangle = c(\alpha_2,\alpha_3)\times \\ C^{\rm Matt}(\frac{b}{2},e_2,e_0-(e_2+\frac{b}{2}-k_1b+k_2/b))\hat{C}^{\rm Liou}(-\frac{b}{2},\alpha_2,Q-(\alpha_2-\frac{b}{2}+s_1b+s_2/b))\,, \tag{A.12}$$

where

$$c(\alpha_2,\alpha_3) = \frac{1}{b^4}\,\big((\alpha_3-\alpha_2+\tfrac{b}{2})(\alpha_3+\alpha_2-Q-\tfrac{b}{2})+(\alpha_3-\alpha_2-\tfrac{b}{2})(\alpha_3+\alpha_2-Q+\tfrac{b}{2})\big)^2 \\ = \frac{1}{b^4}\,\big((e_3-e_2+\tfrac{b}{2})(e_3+e_2-e_0-\tfrac{b}{2})+(e_3-e_2-\tfrac{b}{2})(e_3+e_2-e_0+\tfrac{b}{2})\big)^2 \tag{A.13}$$

Here $e_3=e_2+\frac{b}{2}-k_1b+\frac{k_2}{b}$, $\alpha_3=\alpha_2-\frac{b}{2}+s_1b+\frac{s_2}{b}$ and the four integers s_1,s_2,k_1,k_2 - the number of screening charges of type T^+_b, $T^-_{1/b}$, T^+_0, T^-_0, are restricted by (2.7) depending on the combination of chiralities, *i.e.*,

$$(\varepsilon_3-\varepsilon_2)e_2+b\frac{1+\varepsilon_3}{2}+b^{\varepsilon_3}-b^{\varepsilon_2}=(s_1+\varepsilon_3k_1)b+(s_2-\varepsilon_3k_2)\tfrac{1}{b}\,. \tag{A.14}$$

The OPE coefficient for the normalized tachyons is given, by the r.h.s., of (A.12) times the ratio of leg factors, *i.e.*,

$$a_-W^{\varepsilon_2}_{\alpha_2} = \sum_{\alpha_3,\varepsilon_3} c(\alpha_2,\alpha_3)C^{\rm Matt}(\frac{b}{2},e_2,e_0-e_3)\,\hat{C}^{\rm Liou}(-\frac{b}{2},\alpha_2,Q-\alpha_3)\,\frac{\gamma(b^{\varepsilon_2}(Q-2\alpha_2))}{\gamma(b^{\varepsilon_3}(Q-2\alpha_3))}\,W^{\varepsilon_3}_{\alpha_3} \tag{A.15}$$

The coefficient in the r.h.s. is examined either using directly the expressions of the two $c<1$ and $c>25$ Coulomb gas constants, or by exploiting the compact formula (3.8) for the matter constant and the relation of $\hat{C}^{\rm Liou}$ to $C^{\rm Liou}$ in (3.2), regularizing $\alpha_1 = -e_1 = -b/2 + \epsilon$. We get that the overall constant goes to zero like ϵ^2, unless one of the four factors

$$(e_{123} - e_0)^2\, (e^1_{23} - e_0)^2\, (e^3_{12})^2\, (e^2_{13})^2$$

vanishes as well.

The values $e_3 = e_2 \pm b/2$ are equivalent to $k_2 = 0\,, k_1 = 0, 1$. When the chirality is preserved, $\varepsilon_2 = \varepsilon_3$, we obtain taking into account the condition (A.14) two solutions for each sign

$$\begin{aligned} &k_2 = 0 = s_2\,,\ k_1 = 1 - s_1 = 0, 1\,, \quad \text{for } \varepsilon_2 = 1 = \varepsilon_3\,, \\ &k_2 = 0 = s_2\,,\ k_1 = s_1 = 0, 1\,, \quad \text{for } \varepsilon_2 = -1 = \varepsilon_3\,, \end{aligned} \tag{A.16}$$

altogether leading to the generic OPE relations (4.9), (4.5).

For $e_3 = e_2 \pm b/2$ but $\varepsilon_2 = -\varepsilon_3 = 1$ the values of e_2 become restricted by (A.14)

$$\begin{aligned} e_0 - 2e_2 = P_2 = Q - 2\alpha_2 = s_1 b + \tfrac{s_2}{b} - \tfrac{b}{2} \pm \tfrac{b}{2}\,, \\ (e_3, \alpha_3) = (e_2 \pm \tfrac{b}{2}, Q - \alpha_2 \mp \varepsilon_2 \tfrac{b}{2}) \end{aligned} \tag{A.17}$$

The resulting (e_3, α_3) correspond to the Liouville reflected counterparts of the two terms in (4.5); they have to be added whenever the momenta take the special discrete values in (A.17). These values include the Liouville degenerate points (with the plus sign in (4.14)). Similar formula arises for $\varepsilon_2 = -1 = -\varepsilon_3$.

One gets a nonzero expression also for $e_3 = e_0 - e_2 \pm b/2$. The combination of chiralities $\varepsilon_2 = -\varepsilon_3$ reproduces the matter reflected points occurring for

$$\begin{aligned} e_0 - 2e_2 = P = -k_1 b + \tfrac{k_2}{b} + \tfrac{b}{2} \mp \tfrac{b}{2}\,, \\ (e_3, \alpha_3) = (e_0 - e_2 \pm \tfrac{b}{2}, \alpha_2 \mp \varepsilon_2 \tfrac{b}{2}) \end{aligned} \tag{A.18}$$

with $s_2 = 0\,, s_1 = 0, 1\,,$ for $\varepsilon_2 = 1$, and $s_2 = 0, s_1 = 1, 0$ for $\varepsilon_2 = -1$ respectively. These values include the matter degenerate momenta (with the positive sign in (4.10)). Finally for $e_3 = e_0 - e_2 \pm b/2$ and $\varepsilon_2 = \varepsilon_3$ there are two series of solutions, corresponding to both matter and Liouville reflections

$$\begin{aligned} e_0 - 2e_2 = \tfrac{b}{2} \mp \tfrac{b}{2} - k_1 b + \tfrac{k_2}{b}\,, \\ (e_3, \alpha_3) = (e_0 - e_2 \pm \tfrac{b}{2}, Q - \alpha_2 \pm \varepsilon_2 \tfrac{b}{2}), \end{aligned} \tag{A.19}$$

with $s_1 + k_1 = 1\,, s_2 = k_2$ for $\varepsilon_2 = 1$ and $k_2 = 0 = s_2$, $s_1 = k_1$ for $\varepsilon_2 = -1$.

In all cases the constant in (A.13) becomes $c(\alpha_2, \alpha_3) = \frac{(Q-2\alpha_2)^2}{b^2}$ and the final result simplifies to powers of $\lambda_{_L}\,, \lambda_{_M}$ and b, e.g., in the case (A.18), $e_0 - 2e_2 = Q - 2\alpha_2 = k_1 b - k_2/b$ we have

$$C^{(+-)\,\alpha - \frac{b}{2}}_{-\frac{b}{2}, \alpha} = -\frac{\lambda_{_M}^{k_1 - \frac{k_2}{b^2}}}{b^2} = \frac{(Q-2\alpha)^2}{b^2} \frac{\gamma(b(Q-2\alpha))}{\gamma(\frac{1}{b}(Q - 2\alpha + b))}\, C^{\rm Matt}(\tfrac{b}{2}, e_2, e_2 - \tfrac{b}{2}) \tag{A.20}$$

using that $\hat{C}^{\rm Liou}(-\frac{b}{2},\alpha_2,Q-\alpha_2+\frac{b}{2})=1$, while $C^{\rm Matt}(\frac{b}{2},e_2,e_2-\frac{b}{2})$ is determined by matter reflection as in (3.16) from (3.14). This constant is finite for positive integers $k_1,k_2\neq 0$, which are the values of the degenerate representations in (4.10). Similar formulae hold for the other cases in (A.17), (A.18) and the constituent matter or Liouville Coulomb gas constants $C^{\rm Matt}$, $\hat{C}^{\rm Liou}$ are finite for the degenerate values of the momenta. On the other hand in the last case (A.19), which involves values on the boundary of the degenerate regions appear, there may appear singularities in the constants or the leg factors. In all these considerations we have assumed that b^2 is generic, non-rational.

• In contrast to the above result in the diagonal case the 3-point function computed with k_1,k_2,s_1,s_2 tachyons of type $T^-_b\,,T^+_{1/b},T^+_b\,,T^-_{1/b}$ respectively, is more severely restricted. We have $e_3=e_2-\frac{e_0}{2}+(k_1+k_2)e_0\,,\alpha_3=\alpha_2-\frac{Q}{2}+(s_1+k_1)b+\frac{s_2+k_2}{b}$ and the condition (A.14) is replaced by

$$(\varepsilon_3-\varepsilon_2)e_2+\tfrac{Q}{2}-\varepsilon_3\tfrac{e_0}{2}+b^{\varepsilon_3}-b^{\varepsilon_2}=(s_1+k_1+\varepsilon_3(k_1+k_2))b+(s_2+k_2-\varepsilon_3(k_1+k_2))\tfrac{1}{b}\,. \quad \text{(A.21)}$$

Thus in the generic case $\varepsilon_2=\varepsilon_3$ we have $s_1=1\,,s_2=k_1=0=k_2$, or $k_2=1\,,s_2=k_1=0=s_1$ for $\varepsilon_2=1$ and $s_2=1\,,\ s_1=k_2=0=k_1$, or $k_1=1\,,s_1=k_2=0=s_2$, - for $\varepsilon_2=-1$. These solutions all involve one of the interaction terms in (4.26). The action of a_-a_+ can be understood as the composition of the free field formulae (A.8), (A.9), (4.17) for the two generators

$$\begin{aligned}
&a_+\,a_-T^+_{\alpha_1}\,W^+_\alpha=a_+\,W^+_{\alpha+\alpha_1-\frac{b}{2}}=-W^+_{\alpha+\alpha_1-\frac{Q}{2}}\quad\Rightarrow\\
&a_+(-\,a_-T^+_b)\,W^+_\alpha=W^+_{\alpha-\frac{e_0}{2}}\\
&a_+\,(-a_-T^+_{\frac{1}{b}})\,W^+_\alpha=W^+_{\alpha+\frac{e_0}{2}}
\end{aligned}\qquad \text{(A.22)}$$

and

$$\begin{aligned}
&a_+\,T^-_{\alpha_1}\,a_-\,W^-_\alpha=-a_+\,T^-_{\alpha_1}\,W^-_{\alpha-\frac{b}{2}}=-W^-_{\alpha+\alpha_1-\frac{Q}{2}}\quad\Rightarrow\\
&(-a_+\,T^-_{\frac{1}{b}})\,a_-\,W^-_\alpha=W^-_{\alpha+\frac{e_0}{2}}\\
&(-a_+\,T^-_b)\,a_-\,W^-_\alpha=W^-_{\alpha-\frac{e_0}{2}}\,.
\end{aligned}\qquad \text{(A.23)}$$

If the chirality is inversed there are more solutions. However, restricting to diagonal momenta the only soltions are $P=\pm e_0,0$. Since the tachyon of $P=0$ has no definite chirality, all these solutions effectively fit the generic formula (4.28).

A.3. Mass-shell restrictions on the contact terms

The contact terms in the difference equations for the 4-point tachyon correlators are determined by the OPE coefficients computed by the 4-point Coulomb gas functions

$$C^{\;(\varepsilon_2\varepsilon_3\varepsilon_4)\,\alpha_4}_{-\frac{b}{2}\,\alpha_2\,\alpha_3} = \frac{1}{\gamma((Q-2\alpha_4)b^{\varepsilon_4})\gamma((2\alpha_4-Q)b^{\varepsilon_4})}\,\langle 0|W^{\varepsilon_4}_{Q-\alpha_4}(\infty)\,(\mathbf{c}\bar{\mathbf{c}})_0\,a_-\,W^{\varepsilon_2}_{\alpha_2}\,T^{\varepsilon_3}_{\alpha_3}|0\rangle\,,$$

$$\alpha_4=\alpha_2+\alpha_3-\tfrac{b}{2}+s_1 b+\tfrac{s_2}{b}\,,\quad e_4=e_2+e_3+\tfrac{b}{2}-k_1 b+\tfrac{k_2}{b}\,. \tag{A.24}$$

As in the computation of the 3-point OPE coefficients the denominator comes from the leg factors in the trivial 2-point matrix element, cf. (3.20),

$$\langle 0|W^{\varepsilon_4}_{Q-\alpha_4}(\infty)(\mathbf{c}\bar{\mathbf{c}})_0 W^{\varepsilon_4}_{\alpha_4}|0\rangle = \langle 0|\mathbf{c}(z)\partial_z\bar{\mathbf{c}}(\bar z)\partial_{\bar z}W^{\varepsilon_4}_{Q-\alpha_4}(z,\bar z)W^{\varepsilon_4}_{\alpha_4}(z',\bar z')|0\rangle\,.$$

If α_2 coincides with one of the four values of the screening charges (A.24) reduces to a 3-point function. The mass-shell condition (2.7) implies

$$(\varepsilon_4-\varepsilon_2)e_2+(\varepsilon_4-\varepsilon_3)e_3+\frac{1+\varepsilon_4}{2}\,b+b^{\varepsilon_4}-b^{\varepsilon_2}-b^{\varepsilon_3}=(s_1+\varepsilon_4 k_1)b+(s_2-\varepsilon_4 k_2)\tfrac{1}{b}\,. \tag{A.25}$$

For generic momenta $P_2, P_3 \notin \mathcal{L}$ and $P_2+P_3\notin\mathcal{L}$ the only solution of the mass-shell condition occurs for $\varepsilon_2=\varepsilon_3=\varepsilon_4=1$, with $k_2=s_2=k$, $k_1+s_1=0$, whence $k_1=0=s_1$. The case $k=0$, corresponds to (4.17) while $k\geq 1$ to (4.18), (4.19).

The relation (A.25) is generalized to a product $W^{\varepsilon_2}_{\alpha_2}T^{\varepsilon_3}_{\alpha_3}\dots T^{\varepsilon_{N-1}}_{\alpha_{N-1}}$ in the N-point analog of the Coulomb gas correlator (A.24), which contains $N-3\geq 1$ integrated tachyons. If the momenta are generic, i.e, $P_{i_1}+\ldots+P_{i_s}\notin\mathcal{L}$ for any partial sum, the mass shell condition implies $\varepsilon_i=\varepsilon_N\,, i=2,3,\dots,N-1$. Hence for $\varepsilon_N=1$ one gets $s_1+k_1+N-4=0\,, k_2=s_2$, while for $\varepsilon_N=-1$, the constraint is $k_1=s_1$, $k_2+s_2+N-3=0$. In both cases there are no solutions for $N-3>1$. As a consequence, there are no new contact terms coming from two or more integrated tachyons in the equations for the n-point functions, $n\geq 5$ for those generic values of the momenta. If the given chiralities $\varepsilon_i\,, i=2,\dots,N-1$ are all identical, imposing the condition $\sum_{i=2}^{N-1}P_i\neq\mathcal{L}$ forces ε_N to be of the same sign which again implies that there are no solutions of the mass shell condition for $N>4$. Similarly for $\varepsilon_2=-1$ and $\varepsilon_i=1\,, i=3,\dots,N-1$ imposing $P_2\neq\mathcal{L}\,, \sum_{i=3}^{N-1}P_i\neq\mathcal{L}$ excludes any contact terms. These properties are taken into account in writing the equations (6.1).

Apart from the above series of contact terms for $\varepsilon_2=\varepsilon_3=\varepsilon_4$, which occurs for generic momenta, there are various possibilities taking place for particular values of α_2 or α_3, determined by the choice of the signs ε_i in (A.25). E.g.,

$$\begin{aligned}&\varepsilon_2=-1=-\varepsilon_3=-\varepsilon_4\,,\quad e_0-2e_2=P_2=(k_2-s_2)/b-(s_1+k_1)b\,,\\&\varepsilon_2=-1=-\varepsilon_3=\varepsilon_4\,,\quad e_0-2e_3=P_3=(k_2+s_2+1)/b-(k_1-s_1)b\,.\end{aligned} \tag{A.26}$$

In particular these conditions admit solutions for P_2 or P_3 which correspond to the degenerate matter values when the interaction involves the two matter charges. Another example of (A.25) is given by $\varepsilon_2 = \varepsilon_3 = 1 = -\varepsilon_4$ which may occur for generic values of P_2 and P_3 but their sum restricted by $P_2 + P_3 = (s_1 - k_1)b + (s_2 + k_2 + 1)/b$.

For the N-point generalization of (A.24) with $p = N-3 > 1$ integrated tachyons, there are further possibilities. E.g., the second condition in (A.26) is replaced by $\sum_{i=3}^{p+2} P_i = (k_2+s_2+p)/b-(k_1-s_1)b$. etc. The simplest example with $p = 2$ and no screening charges was computed in (A.11).

In the diagonal theory described by the action (4.26) the mass-shell condition for the 4-point function (A.24) is again more restrictive. We have $e_4 = e_2+e_3-\frac{e_0}{2}+(k_1+k_2)e_0\,, \alpha_4 = \alpha_2 + \alpha_3 - \frac{Q}{2} + (s_1 + k_1)b + (s_2 + k_2)/b$ and in the generic case $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \pm 1$ the condition admits the unique solution $k_1 = 0 = k_2 = s_1 = s_2$. It reproduces the OPE in the first lines in (A.22) and (A.23). The analog of the second example in (A.26) is

$$e_0 - 2e_3 = P_3 = (s_2 + 2k_2 + k_1)/b - (k_2 - s_1)b\,.$$

Restricting to diagonal values $P_3 = ke_0$, we obtain $s_1 + k_1 + k_2 + s_2 = 0$, *i.e.*, the only possible value is $P_3 = 0$, which is beyond the degenerate matter range.

The above conditions on the momenta are kinematical. As in the analysis of the general 3-point function (A.12), further restrictions appear from the fusion rules of the degenerate fields dictated by the 3-point constants in the decompositions of the 4-point matter and Liouville functions. In the next section we consider some examples.

A.4. 4-point OPE coefficients

We derive here the 4-point OPE coefficients in (4.18) and (4.24).

The vertex part of the function (A.24) for

$$e_4 = e_2 + e_3 + \tfrac{b}{2} + \tfrac{n}{b}\,, \quad \alpha_4 = \alpha_2 + \alpha_3 - \tfrac{b}{2} + \tfrac{n}{b}\,, \ \varepsilon_i = 1\,, i = 2, 3, 4 \qquad \text{(A.27)}$$

is realized by a $2n$-multiple integral coming from the power of screening charges $(T_0^- \, T_{1/b}^-)^n$. We shall use instead the matter-Lioville factorization formula as in the last line in (A.1), with an alternative realization of each of the two types of correlators. Since one of the fields is the simplest degenerate field, these $c < 1$ and $c > 25$ correlators are standard, given by sums of products of hypergeometric functions. In particular in our example only one of the OPE channels survives. E.g., the matter matrix element is given for the values

e_i in (A.27) by the product of blocks with intermediate charge $e_2 + \frac{b}{2}$,

$$\begin{aligned}
&\langle e_0 - e_4 | V_{e_3}(x_3)\, V_{\frac{b}{2}}(x_0)\, V_{e_2}(x_2)|0\rangle_M = (x_{32}^2)^{\triangle_M(e_4)-\triangle_M(e_2)-\triangle_M(e_3)-\triangle_M(b/2)} \\
&\times\ {}_2F_1(-n, (2e_2+2e_3+2b)b+n-1; (2e_2+b)b; \frac{z_{02}}{z_{32}})\ \ (\text{same, with } z\to\bar z) \\
&\times C^{\mathrm{Matt}}(\tfrac{b}{2}, e_2, e_0-e_2-\tfrac{b}{2})\, C^{\mathrm{Matt}}(e_2+\tfrac{b}{2}, e_3, e_0-e_2-e_3-\tfrac{b}{2}-\tfrac{n}{b})\ f_M(z) f_M(\bar z)\,, \\
& \\
&f_M(z) = \big(\frac{z_{02}}{z_{32}}\big)^{be_2}\,\big(\frac{z_{30}}{z_{32}}\big)^{be_3}\,.
\end{aligned} \tag{A.28}$$

The expression for the Liouville correlator $\langle Q-\alpha_4|V_{\alpha_3}(x_3)\, V_{-\frac{b}{2}}(x_0)\, V_{\alpha_2}(x_2)|0\rangle_L$ is analogous, with be_i , b^2 replaced by $b\alpha_i$, $-b^2$, etc., while the constants C^{Matt} are replaced by the $c > 25$ Coulomb gas constants $\hat{C}^{\mathrm{Liou}}$. The first of the OPE constants in both cases is trivial, $C^{\mathrm{Matt}}(\frac{b}{2}, e_2, e_0-e_2-\frac{b}{2}) = 1 = \hat{C}^{\mathrm{Liou}}(-\frac{b}{2}, \alpha_2, Q-\alpha_2+\frac{b}{2})$. The hypergeometric function is the same, using that $e_i = \alpha_i - b$, and it reduces to a finite series of $n+1$ terms. We then apply the derivatives with respect to z_0 and $\bar z_0$ term by term. Using that the difference of the powers of $z_{0i}\,, i = 2,3$ from the matter and the Liouville functions is a constant $b(\alpha_i - e_i) = b^2$ one gets simply an overall factor

$$\begin{aligned}
&\int \tfrac{d^2\,x_3}{\pi}\,(\sum_i \frac{1}{z_{0i}} - \frac{1}{z_{02}})\, f_M(z)\, f_L(z)\big({}_2F_1(-n, (2\alpha_2+2\alpha_3-2b)b+n-1; (2\alpha_2-b)b; \frac{z_{02}}{z_{32}})\big)^2 \\
&\quad\times\ (\text{same, with } z\to\bar z) \\
&= (x_{02}^2)^u \int \tfrac{d^2\,x_3}{\pi}\, \big({}_2F_1(-n, -w+n-1; u; \frac{z_{02}}{z_{32}})\big)^2\, \big({}_2F_1(-n, -w+n-1; u; \frac{\bar z_{02}}{\bar z_{32}})\big)^2 (x_{30}^2)^{v-1}\,(x_{32}^2)^w \\
& \\
&= \left(\sum_k \frac{(-n)_k\,(-w+n-1)_k}{k!(-w)_k}\ {}_3F_2(-n, -w+n-1, u+k; -w+k, u; 1)\right)^2\ \frac{\gamma(v)\gamma(w+1)}{\gamma(-u+1)} \\
&= \left(n!\ \frac{(v)_n}{(u)_n\,(-w)_n}\,\frac{w-n+1}{w-2n+1}\right)^2\,\frac{\gamma(v)\gamma(w+1)}{\gamma(-u+1)} =: C(\alpha_2,\alpha_3;n)
\end{aligned} \tag{A.29}$$

where

$$u := (2\alpha_2 - b)b\,,\ v := (2\alpha_3-b)b\,, w := -u-v = (b-2\alpha_4)b + 2n\,.$$

Altogether one obtains for the values in (A.27) the OPE coefficient in the n-th term in (4.21)

$$\begin{aligned}
(\tilde\lambda_{{}_L}\,\tilde\lambda_{{}_M})^n = C_{-\frac{b}{2}\,\alpha_2\,\alpha_3}^{\ (+++)\ \ \alpha_4} = C(\alpha_2,\alpha_3;n)\ \frac{\gamma(b(Q-2\alpha_2))\,\gamma(b(Q-2\alpha_3))}{\gamma(b(Q-2\alpha_4))} \\
\times\ C^{\mathrm{Matt}}(e_2+\tfrac{b}{2}, e_3, e_0-e_4)\,\hat{C}^{\mathrm{Liou}}(\alpha_2-\tfrac{b}{2}, \alpha_3, Q-\alpha_4)\,.
\end{aligned} \tag{A.30}$$

This result has been derived for generic values of the momenta. If however α_2 takes a degenerate value, $\alpha_2 = b \pm \frac{mb}{2} - \frac{p}{2}$, $m, p \in \mathbb{Z}_{\geq 0}$, one of the Coulomb gas constants in (A.30) vanishes, the other becomes singular. The vanishing of the constant reflects a null vector factorization so it is natural to resolve this ambiguity for such fields by restricting the validity of (A.30) to the values of n avoiding the singularity, i.e., $n \leq p$. On the other hand we can still keep (A.30) for any n for fields with degenerate labels but not obeying the factorization conditions.

This derivation generalizes to other cases discussed in the previous section. In particular let us consider the case described by the second line in (A.26)

$$\alpha_2 = -e_2 + \tfrac{1}{b}\,,\ \alpha_3 = e_3 + b = \tfrac{(k_1 - s_1 + 1)b}{2} - \tfrac{k_2 + s_2}{2b}\,,\ \alpha_4 = \alpha_2 + \alpha_3 - \tfrac{b}{2} + s_1 b + \tfrac{s_2}{b} = -e_4 + \tfrac{1}{b}\,.$$

Examining the product $C^{\rm Matt}(e_2 + \eta\frac{b}{2}, e_3, e_0 - e_4)\, C^{\rm Liou}(\alpha_2 + \eta'\frac{b}{2}, e_3, Q - \alpha_4)$ one observes that for generic values of α_2 it becomes singular only if $\eta = \eta' = 1$, $k_1 = 0\,, s_1 \geq 1$, or $\eta = \eta' = -1$, $s_1 = 0\,, k_1 \geq 1$; alternatively for these values the corresponding products $C^{\rm Matt}\hat{C}^{\rm Liou}$ are finite (as well as each of the two constants itself). This implies that in each of these two cases only one of the four possible products of matter and Liouville blocks survives in (A.24). Furthermore the hypergeometric functions corresponding to the matter and Liouville local correlators are identical again and the differences of overall powers of z_{0i} are b^2 as before. E.g., in the case $s_1 = 0$, the chiral factor $f_M(z)\,{}_2F_1$ in (A.28) is replaced by

$$\begin{aligned} &z^{b(e_0 - \varepsilon_2)}(1-z)^{be_3}\,{}_2F_1(-s_2, b(2e_3 - 2e_2) + s_2 + 1; 1 + b(e_0 - 2e_2); z) \\ &= z^{b\alpha_2 - b^2}(1-z)^{b\alpha_3 - b^2}\,{}_2F_1(-s_2, (2\alpha_2 + 2\alpha_3 - 2b)b + s_2 - 1; (2\alpha_2 - b)b; z) \end{aligned}$$

Then all the steps in the derivation of (A.29) are repeated with n replaced by s_2 or k_2 respectively. We summarize these results by relations analogous to (A.30) for $s_1 = 0\,, k_1 \geq 1$, and $k_1 = 0\,, s_1 \geq 1$, respectively:

$$\begin{aligned} \lambda_{M}^{k_1}\,\tilde{\lambda}_{M}^{k_2}\,\tilde{\lambda}_{L}^{s_2} &= C^{\ (-+-)\ \ \alpha_4}_{-\frac{b}{2}\alpha_2\ \alpha_3} = C(\alpha_2, \alpha_3; s_2)\ \frac{\gamma(\frac{1}{b}(Q - 2\alpha_2))\,\gamma(b(Q - 2\alpha_3))}{\pi\,\gamma(\frac{1}{b}(Q - 2\alpha_4))} \\ &\times\ C^{\rm Matt}(e_2 - \tfrac{b}{2}, e_3, e_0 - e_4)\, C^{\rm Matt}(\tfrac{b}{2}, e_2, e_0 - e_2 + \tfrac{b}{2})\, \hat{C}^{\rm Liou}(\alpha_2 - \tfrac{b}{2}, e_3, Q - \alpha_4)\,, \end{aligned}$$

$$\begin{aligned} \tilde{\lambda}_{M}^{k_2}\,\lambda_{L}^{s_1}\,\tilde{\lambda}_{L}^{s_2} &= C^{\ (-+-)\ \ \alpha_4}_{-\frac{b}{2}\alpha_2\ \alpha_3} = C(Q - \alpha_2, \alpha_3; k_2)\ \frac{\gamma(\frac{1}{b}(Q - 2\alpha_2))\,\gamma(b(Q - 2\alpha_3))}{\pi\,\gamma(\frac{1}{b}(Q - 2\alpha_4))} \\ &\times\ C^{\rm Matt}(e_2 + \tfrac{b}{2}, e_3, e_0 - e_4)\, \hat{C}^{\rm Liou}(-\tfrac{b}{2}, \alpha_2, Q - \alpha_2 - \tfrac{b}{2})\, \hat{C}^{\rm Liou}(\alpha_2 + \tfrac{b}{2}, e_3, Q - \alpha_4)\,. \end{aligned} \tag{A.31}$$

Denoting $s_2 + k_2 = n\,, s_2 - k_2 = 2s - n$ and $k_1 = m + 1$, or $s_1 = m + 1$, respectively we arrive at (4.24). The case in the first line of (A.26) can be analysed similarly.

On the other hand the alternative multiple integral representation of any of these 4-point correlators is not of the type in [32]. Comparing with (A.31) one effectively computes these nonstandard integrals.

One can compute also the 4-point correlator with $W^{+}_{Q-\alpha_4}$ replaced by the tachyon $W^{-}_{\alpha_4}$ (or by $W^{-}_{Q-\alpha_4}$) so that the labels α_i (or the labels e_i) satisfy a Liouville (matter) reflected version of the respective charge conservation condition. This does not change the hypergeometric functions and it remains to use (3.15) or (3.16) respectively. The result is

$$b^2 \lambda_{L}^{\frac{2\alpha_4-Q}{b}} C^{\,(++-)\,Q-\alpha_4}_{-\frac{b}{2}\,\alpha_2\,\alpha_3} = C^{\,(+++)\,\alpha_4}_{-\frac{b}{2}\,\alpha_2\,\alpha_3} = b^2 \lambda_{M}^{\frac{2e_4-e_0}{b}} C^{\,(++-)\,\alpha_4}_{-\frac{b}{2}\,\alpha_2\,\alpha_3} = \left(\frac{\lambda_L}{\lambda_M}\right)^{\frac{2\alpha_4-Q}{b}} C^{\,(+++)\,Q-\alpha_4}_{-\frac{b}{2}\,\alpha_2\,\alpha_3} \tag{A.32}$$

Thus the reflection properties of the underlying 4-point Liouville and matter correlators ensure the validity of (3.19) on the level of these particular string 4-point correlators.

A.5. Some chiral OPE coefficients

Now we consider a few chiral matrix elements, some of which have been computed in [18], [25]. The chiral analog of the simplest matrix element (A.7) reads for $|z_0| > |z_1|$

$$\langle \alpha' | \mathbf{c}_{-1} \mathbf{c}_0\, a_-(z_0) (\mathbf{c} V^{\varepsilon}_{\alpha})(z_1) \rangle = \begin{cases} 0 & \text{if } \varepsilon = 1, \\ \frac{2\alpha - Q}{b} = -\frac{\Gamma(\frac{1}{b}(Q-2\alpha+b)}{\Gamma(\frac{1}{b}(Q-2\alpha)} & \text{if } \varepsilon = -1 \end{cases} \tag{A.33}$$

We recognize in the r.h.s. the leg factor normalization exploited in the boundary theory, which is obtained replacing in (2.6) $\gamma(x) \to \Gamma(x)$.

The fields are radially ordered as above, accordingly the bounds on the integrals are given by the arguments of the neighbouring fields, the utmost left one being at $+\infty$, the utmost right one - in $-\infty$. E.g., let us look at the chiral analog of (A.10) for $\varepsilon = 1 = \varepsilon_2$. We choose $|z_0| > |z_1|$ and send these two arguments to 1 and 0 respectively. The coordinate z_2 is 'floating' and we can collect the result for the three possible insertions of the integral by writing the linear combination with coefficients indicating the contours of integration

$$\sum_{(ij)} c_{i,j} \int_{C_{ij}} dz_2 \langle \alpha' | \mathbf{c}_{-1} \mathbf{c}_0\, a_-(z_0) (\mathbf{c} V^{+}_{\alpha})(z_1)\, V^{+}_{\alpha_2}(z_2) \rangle = \frac{\pi\, \Gamma(b(Q-2\alpha-2\alpha_2+b))}{\Gamma(b(Q-2\alpha))\Gamma(b(Q-2\alpha_2))}$$
$$\Big(- c_{\infty,1} \frac{\sin \pi b(Q-2\alpha-2\alpha_2+b)}{\sin \pi b(Q-2\alpha)\, \sin \pi b(Q-2\alpha_2))} + c_{1,0} \frac{1}{\sin \pi b(Q-2\alpha_2)} + c_{0,-\infty} \frac{1}{\sin \pi b(Q-2\alpha)} \Big) \tag{A.34}$$

The overall constant reproduces the chiral leg factor normalization of the three fields. Taking $\alpha_2 = b$ or $\alpha_2 = 0$ the r.h.s. can be identified with a (linear combination of) Liouville or matter matrix elements of three chiral vertex operators (CVO). The intermediate states are described by a proper choice of the coefficients in (A.34), as it has been done in the boundary Liouville case [5]; the boundary fields are linear combinations of CVO[18]. Each of the two constants determine the corresponding OPE of CVO, *i.e.*, has the meaning of a particular Liouville or matter fusing matrix elements in a certain gauge. The parametrization of the Liouville or matter intermediate states (boundary conditions) is taken as $z_L(\sigma) = \lambda_L^{1/2} \cos \pi b(Q - 2\sigma)$ or $z_M(a) = \lambda_M^{1/2} \cos \pi b(e_0 - 2a)$, with

$$\begin{aligned} c_{\infty,1} &= -z_L(\sigma_3)\,,\ c_{1,0}^{\pm} = -z_L(\sigma_3 \mp \tfrac{b}{2})\,,\ c_{0,-\infty} = -z_L(\sigma_1) \\ c'_{\infty,1} &= -z_M(a_3)\,,\ c'^{\pm}_{1,0} = -z_M(a_3 \pm \tfrac{b}{2})\,,\ c'_{0,-\infty} = -z_M(a_1)\,. \end{aligned} \tag{A.35}$$

Up to the overall leg factor normalization this gives for the constants in the r.h.s.

$$C^L_{\sigma_3 \mp \frac{b}{2}\,,\alpha + \frac{b}{2}} \begin{bmatrix} -\frac{b}{2} & \alpha \\ \sigma_3 & \sigma_1 \end{bmatrix} = -\lambda_L^{\frac{1}{2}} \frac{2 \sin \pi b(\alpha - \frac{b}{2} \pm (\sigma_3 + \sigma_1 - Q)) \sin \pi b(\alpha - \frac{b}{2} \pm (\sigma_3 - \sigma_1))}{\sin \pi b(Q - 2\alpha)}\,, \tag{A.36}$$

$$C^M_{a_3 \pm \frac{b}{2}\,,e - \frac{b}{2}} \begin{bmatrix} \frac{b}{2} & e \\ a_3 & a_1 \end{bmatrix} = -\lambda_M^{\frac{1}{2}} \frac{2 \sin \pi b(e + \frac{b}{2} \pm (a_3 + a_1 - e_0)) \sin \pi b(e + \frac{b}{2} \pm (a_3 - a_1))}{\sin \pi b(e_0 - 2e)} \tag{A.37}$$

and the case without screening charges (A.33) corresponds to (a product) of trivial constants

$$C^M_{a_2\,,e + \frac{b}{2}} \begin{bmatrix} \frac{b}{2} & e_2 \\ a_3 & a_1 \end{bmatrix} = 1 = C^L_{\sigma_2, \alpha - \frac{b}{2}} \begin{bmatrix} -\frac{b}{2} & \alpha \\ \sigma_3 & \sigma_1 \end{bmatrix}\,.$$

The first of these expressions (A.36) has been derived in [25] combining the formulae in [18]; it differs by an overall constant from the boundary Liouville constant computed in [5]. The analytic continuation of the latter is similarly related to the matter constant (A.37); vice versa (A.37) is obtained from (A.36) via the analytic continuation formula (3.24). Finally let us look at the chiral analog of the double integral matrix element (A.11) with the various possible positions of the two inserted vertex operators. E.g. for $\alpha_2 = b$ we obtain for $|z_0| > |z_1|$

$$\begin{aligned} \int_{C_2} dz_2 \int_{C_3} dz_3 \langle \alpha' | \mathbf{c}_{-1} \mathbf{c}_0\, a_-(z_0) (\mathbf{c} V_\alpha^-)(z_1)\, V_b^+(z_2)\, V_0^+(z_3) \rangle &= \frac{\pi^2\, \Gamma(\frac{1}{b}(Q - 2\alpha - b))}{\Gamma(\frac{1}{b}(Q - 2\alpha))\, \Gamma(bQ)\, \Gamma(be_0)} \times \\ \frac{\sin \pi b(Q - 2\alpha)}{\sin \pi 2b\alpha}\, C^L_{\sigma_2\,,\alpha + \frac{b}{2}} \begin{bmatrix} -\frac{b}{2} & \alpha \\ \sigma_3 & \sigma_1 \end{bmatrix} C^M_{a_2\,,e - \frac{b}{2}} &\begin{bmatrix} \frac{b}{2} & e \\ a_3 & a_1 \end{bmatrix}\,, \end{aligned} \tag{A.38}$$

where the ratio of leg factors is extracted in the r.h.s. of the first line.

These formulae will be applied to the boundary tachyon operators at generic values of momenta. Whenever the relation is applied to the left, *i.e.*, with the opposite order of the fields, there appears an overall minus sign. The four constants, (A.36), (A.37) for

[18] See [47] for the precise meaning of this statement in the rational case.

plus chirality, and -1 in (A.33), and the constant in the second line in (A.38) for minus chirality, correspond to the four OPE coefficients in the bulk identities (4.5). One can write down a formula analogous to (4.9), expressing any of the four OPE coefficients in terms of a product of the corresponding Liouville and matter boundary OPE constants.

The formula (A.34) for arbitrary α_2, but keeping one of the terms, provides the simplest contact term in the boundary counterpart of the recurrence relation (5.12). It remains to compute the boundary 4-point functions determining the analogs of (4.21), (4.24) and hence of the contact terms in (5.9). The details will be presented elsewhere.

Appendix B. More 3-point solutions and boundary CFT interpretation

We have encountered several examples of solutions of the ring relations for the 3-point functions described by the various "fusion" multiplicities N_{P_1,P_2,P_n}. They generate n-point multiplicities which can be cast formally into the general form

$$
\begin{aligned}
N_{P_1,P_2,\cdots,P_n} &= \int d\mu(a) \prod_i \chi_{P_i}(a) \,, \\
\chi_{P+b^\varepsilon}(a) + \chi_{P-b^\varepsilon}(a) \; (&= 2\cosh(b^\varepsilon \partial_P)\, \chi_P(a)) = f_{b^\varepsilon}(a)\, \chi_P(a) \,, \qquad \varepsilon = \pm 1 \,.
\end{aligned} \tag{B.1}
$$

The relation in (B.1) is a sufficient condition ensuring the validity of the homogeneous relations (3.27); analogous identity holds in the diagonal case.

This formula is specialized by certain range of the variable a, dual to the spectrum of momenta P, and by some choice of the measures in the two spaces, the "characters" $\chi_P(a)$ and the function $f_b(a)$. E.g., the simplest example (6.16) corresponds to $\chi_{P_i}(a) = e^{ia(\varepsilon_i e_0 - P_i)}$ with $\sum_{i=1}^p \varepsilon_i = p-2$, and $f_b(a) = 2\cos ba$. Another explicit solution of the homogeneous relations (3.27) is given by a formula dual to (5.5),

$$
\begin{aligned}
N_{P_1,P_2,P_3} &= \sum_{m=0} \sum_{n=0} \left(4 \sin \pi m e_0 b \, \sin \pi n \tfrac{e_0}{b}\right)^2 \chi_{P_1}(m,n) \chi_{P_2}(m,n)\, \chi_{P_3}(m,n) \,, \\
\chi_{P(m,n)} &= \frac{\sin \pi m P b}{\sin \pi m e_0 b} \frac{\sin \pi n P/b}{\sin \pi n e_0/b} = \chi_{-P}(m,n) \,, \quad f_b(a_{m,n}) = 2\cos(b(\tfrac{n}{b} \pm mb)) \,.
\end{aligned} \tag{B.2}
$$

The "characters" satisfying the relation in (B.1) coincide up to a normalization with the tachyon disk 1-point functions $\langle \mathcal{V}_P^\varepsilon \rangle_a \sim \lambda_L^{\varepsilon \frac{P}{2b}} \chi_P(a)$, with a boundary label a,

$$
\lambda_L \, \langle \mathcal{V}_{P-\varepsilon b}^{(\varepsilon)} \rangle_a + \langle \mathcal{V}_{P+\varepsilon b}^{(\varepsilon)} \rangle_a = -\langle a_- \mathcal{V}_P^{(\varepsilon)} \rangle_a = \sqrt{\lambda_L}\, f_b(a)\, \langle \mathcal{V}_P^{(\varepsilon)} \rangle_a \tag{B.3}
$$

In the first equality we have used (4.5) (with λ_M set to 1). The second is a version of the standard bulk - boundary equations, yet to be established in this context. It implies [24] that the eigenvalue of the operator a_- is identified up to a power of λ_L with the function

$f_b(a)$ in (B.1) for any solution N_{P_1,P_2,P_3} of the ring relations (3.27); in [24] this reasoning has been used in the rational case, assuming the validity of the OPE relations (4.5),(4.6).

The solution (B.2) provides in the boundary CFT interpretation an example of boundaries parametrized by the degenerate matter (or Liouville, as in [6]) representations; the two transformations (3.24) preserve the formula for the characters inverting the sign $(m,n) \to (m,-n)$ or $(m,n) \to (-m,n)$ in agreement with the $c < 1$ versus $c > 25$ parametrizations (4.10) and (4.14). In this case taking $P = e_0$ and $\varepsilon = \pm 1$ the first (or the second) term in (B.3) disappears, respectively, so that we have for $a = a_{m,n}$ that $\lambda_{_L} \langle \mathcal{V}^{(+)}_{e_0-b} \rangle_a = -\langle a_- \mathcal{V}^{(+)}_{e_0} \rangle_a = \partial_{\lambda_{_L}} \langle a_- \rangle_a$. This determines the 1-point function $\langle a_- \rangle_a$.

Finally let us mention another symmetric under the change of sign $P \to -P$ solution of (3.27), represented as in (B.1),

$$
\begin{aligned}
N_{P_1,P_2,P_3} &\sim 4 \int_0^\infty dt \left(\frac{\prod_{i=1}^3 \cosh P_i t}{\sinh bt \sinh \frac{t}{b}} - \frac{1}{t^2} \right) \\
&= -\frac{\partial}{\partial_{\alpha_1}} \log \left(S_b(\alpha_{123} - Q) S_b(Q - \alpha^1_{23}) S_b(\alpha^3_{12}) S_b(\alpha^2_{13}) \right)
\end{aligned}
\tag{B.4}
$$

The formula applies to complex values of the momenta P_i. This 3-point function is similar to the density $\rho(P_1)$ which appears in the disk partition function [2], [48], with the two boundary parameters replaced by the two momenta P_2, P_3. It is interpreted as the derivative of the log of a particular fusion matrix element. The 'diagonalizing' matrix here is a disc bulk 1-point function $\cosh(Q - 2\alpha)t$, analogous to the solution in [5].

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