

Open boundary Quantum
Knizhnik-Zamolodchikov equation
and the weighted enumeration
of Plane Partitions with symmetries

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We propose new conjectures relating sum rules for the polynomial solution of the qKZ equation with open (reflecting) boundaries as a function of the quantum parameter q and the τ -enumeration of Plane Partitions with specific symmetries, with $\tau = -(q + q^{-1})$. We also find a conjectural relation à la Razumov-Stroganov between the $\tau \rightarrow 0$ limit of the qKZ solution and refined numbers of Totally Symmetric Self Complementary Plane Partitions.

AMS Subject Classification (2000): Primary 05A19; Secondary 82B20

1. Introduction

Integrable lattice models seem to be a constant source of combinatorial wonders. Any statistical lattice model is combinatorial by essence, as it is based on the (weighted) enumeration of configurations. Integrability appears then as the “cherry on the cake” that gives access to exact solutions and, from a purely combinatorial point of view, to exact and/or asymptotic enumeration, involving the analytical computation of critical configuration exponents.

This note is devoted to an extension of the so-called Razumov-Stroganov (RS) conjecture [1], identifying the properly normalized entries of the groundstate (Perron-Frobenius) eigenvector Ψ of the $O(1)$ dense loop model on a cylinder of perimeter $2n$ in the basis of link patterns with the numbers of configurations of the Fully Packed Loop (FPL) model on an $n \times n$ square grid, corresponding to the *same* link patterns. A weaker “sum rule” version [2] of this conjecture simply states that the sum of the components of Ψ equals the total number of FPLs, itself equal to that of Alternating Sign Matrices (ASMs) of the same size. The latter was first proved in [3], by making extensive use of the integrability of a more general inhomogeneous version of the $O(1)$ model. There, it is shown that Ψ may be entirely determined by translating any permutation of the inhomogeneity parameters (spectral parameters) in terms of the local action of the Temperley-Lieb algebra generators, resulting into divided difference equations obeyed by the components of Ψ , that are homogeneous polynomials of the spectral parameters, tending to the above integers in the “homogeneous” limit where all spectral parameters tend to 1. In [3], the sum of components of Ψ is actually computed and identified with a particular case of the Izergin-Korepin determinant [4] [5], reducing to a simple Schur function in that case.

This remarkable link between the $O(1)$ integrable model and ASMs adds up yet another piece to the long lasting puzzle of the Alternating Sign Matrices (see Bressoud’s book [6] for a thrilling tale). ASMs indeed seem to be mysteriously related to other combinatorial objects such as Descending Plane Partitions, and even more interestingly to Plane Partitions with specific symmetries. The latter may all be viewed as rhombus tilings of various domains of the triangular lattice, by means of elementary rhombi made of two adjacent triangles. Particularly interesting are the Totally Symmetric Self Complementary Plane Partitions (TSSCPPs), which may be viewed as rhombus tilings of a regular hexagon of size $(2n) \times (2n) \times (2n)$, and which moreover enjoy all possible symmetries of the hexagon. A Plane Partition is indeed a pile of elementary cubes inside a cube of size $(2n) \times (2n) \times (2n)$.

When viewed in perspective from the $(1, 1, 1)$ direction, the visible individual cube tops and sides form rhombi which tile the large cube's projection, a regular hexagon of size $2n$. The desired symmetries of the pile of cubes, namely that under rotations of $2\pi/3$ around the axis $(1, 1, 1)$ and the self-complementation meaning that the complement of the pile within the large cube is itself a pile with the same structure, translate into a maximal symmetry of the hexagon's rhombus tilings. Although no canonical bijection is known to this day between TSSCPPs and ASMs, their numbers are identical.

Razumov and Stroganov also considered the $O(1)$ loop model on a strip of width L rather than on a cylinder [7], thus trading *periodic* for *open* boundary conditions, and identified again the properly normalized components of the corresponding groundstate vector with the numbers of Vertically Symmetric FPLs (VSFPLs), that is FPLs that are reflection-symmetric with respect to a vertical axis, themselves identified with Vertically Symmetric ASMs (VSASMs) for even size L . Similarly, for odd size L , the sum of components of the properly normalized groundstate vector was conjectured in [8] to be given by the number of Cyclically Symmetric Transpose Complement Plane Partitions (CSTCPPs). The latter Plane Partitions enjoy cyclic rotational symmetry under rotations of $2\pi/3$, and are moreover equal to the complement of their reflection. The sum rules for the open boundaries were computed in [9], along the same lines as [3], resulting in simple determinant and Pfaffian expressions in terms of spectral parameters. From the result of [9], it is a simple exercise to compute the homogeneous limit of the sum rule, which reduces for $L = 2n$ to $A_V(2n + 1)$, the total number of VSASMs of size $2n + 1$, and for $L = 2n - 1$ to $N_8(2n)$, the total number of CSTCPPs of a regular hexagon of size $(2n) \times (2n) \times (2n)$. The duplicity of this result makes one think that the language of Plane Partitions might also be useful to approach the RS conjectures.

An alternative subsequent proof of the periodic boundary RS sum rule may be found in [10], where the integrability of the model is put into perspective within the framework of the Affine Temperley-Lieb algebra and its representation theory. A further reformulation in terms of the $U_q(sl_2)$ quantum Knizhnik-Zamolodchikov (qKZ) equation has led to a host of generalizations, both to higher rank algebras [11] and to different boundary conditions [12] (indexed by root systems of classical Lie algebras). All these extensions involve an extra (quantum group) parameter q , equal to $-e^{i\pi/3}$ in the RS case, via the quantity

$$\tau = -q - \frac{1}{q} \tag{1.1}$$

This is nothing but the weight per loop one would assign within the Temperley-Lieb algebra framework, when dealing with the more general $O(n = \tau)$ model, however when $\tau \neq 1$ (i.e. except at the RS point), no nice cylinder partition function interpretation holds: indeed, for generic q , the boundary conditions are not *periodic*, but only *cyclic* up to a multiplicative shift of q^6 on the spectral parameters, $z_{i+L} \rightarrow q^6 z_i$.

Nevertheless, after taking the homogeneous limit, the solution to the cyclic qKZ equation now produces a vector $\Psi(\tau)$ whose properly normalized components are polynomials of τ , with apparently *non-negative integer* coefficients. In Ref.[13], we have identified the sum rule for the components of $\Psi(\tau)$ with the weighted τ -enumeration of TSSCPPs, carrying a weight τ per vertical step in their Non-Intersecting Lattice Path (NILP) formulation. The latter is a reexpression of the TSSCPPs in a fundamental domain of the hexagon (1/12-th of it) in terms of lattice paths drawn on the rhombi, that do not intersect. The counting of such paths is now a standard exercise. Ref.[13] therefore provides a conjectural combinatorial interpretation for the non-negative integer coefficients of $\Psi(\tau)$ (these non-negative integers were also spotted in [14], but without combinatorial interpretation).

The aim of this paper was to extend the conjecture of [13] to the case of open boundary conditions. On the way, we have found a remarkable coincidence between the leading coefficient of the entries $\Psi_\pi(\tau)$ of the open qKZ solution $\Psi(\tau)$ at small τ and the refined TSSCPP numbers, that count TSSCPPs grouped according to the positions of their endpoints in the NILP formulation. Our next task, to restore symmetry between the cases of even and odd strip width, was to find some Plane Partition interpretation of the VSASM numbers, and we found out that the latter also count rhombus tilings of a hexagon with the same symmetries as for the CSTCPPs, but now with a central triangular hole of size $2 \times 2 \times 2$. Note that this hexagon is no longer regular, but with shape $(2n) \times (2n+2) \times (2n)$. Note also that this allows for a unified NILP interpretation of both VSASM and CSTCPP numbers.

This led us to the main conjectures of this paper:

- (i) *the leading terms in the components of the qKZ solution $\Psi(\tau)$ when $\tau \rightarrow 0$ are the refined TSSCPP numbers arranged according to the endpoints of their associated NILP, with a simple bijection relating these to link patterns.*
- (ii) *the sum rule for the properly normalized solution $\Psi(\tau)$ of the $U_q(sl_2)$ qKZ equation with open boundaries produces a polynomial of τ with non-negative integer coefficients, identical to the generating polynomials for VSASMs (of size $2n + 1$ for $L = 2n$) and*

CSTCPPs (of size $2n$ for $L = 2n - 1$) with a weight τ per vertical step in their respective NILP formulations, except for the steps in one particular central row.

- (iii) *the “maximal” components of $\Psi(\tau)$ corresponding to the link pattern that connects all points $2i - 1$ to $2i$, leaving the point L unmatched in the odd case, are nothing but the generating polynomials for CSSTPPs (of size $2n - 2$ for $L = 2n$) and VSASMs (of size $2n - 1$ for $L = 2n - 1$) with a weight τ per vertical step in their respective NILP formulations, and without any further restriction.*

While the second conjecture is only a sum rule, the first one, like the full RS conjecture, involves separately each component of $\Psi(\tau)$ and gives a combinatorial interpretation for the leading term when $\tau \rightarrow 0$ in terms of TSSCPPs rather than ASMs or FPLs involved in the RS conjecture. This change of point of view, trading ASMs or FPLs for TSSCPPs should be very fruitful, and suggests that the $O(1)$ loop model or the qKZ solutions may be the right place where to look for some ASM/FPL - TSSCPP correspondence.

The paper is organized as follows. In Section 2, we recall some known facts on the qKZ equation with open boundaries and its minimal polynomial solution, which we list for sizes up to $L = 8$ in Appendix A, in their homogeneous form, that keeps only the τ dependence. Section 3 reviews Plane Partitions with various symmetries, namely TSSCPPs and CSTCPPs, and introduces a rhombus tiling problem whose count matches the number of VSASMs, and provides a natural generalization of CSTCPPs in the case of even size L . In Section 4, we introduce the τ -enumeration of CSTCPPs and their even counterparts, which eventually match the sum rules for the homogeneous solutions of the open qKZ equation in odd and even size, listed in Appendix A. Section 5 gathers the various conjectures of the paper, as well as some concluding remarks.

2. qKZ equation with open boundaries

2.1. The equation

We refer to [9] for a detailed presentation. Let us introduce link patterns of size L as configurations of L regularly spaced points on a line, labelled 1 to L from left to right, and connected by pairs through non-intersecting semi-circles with centers on the line, all contained in the upper-half plane delimited by the line. For even L , all points are matched, while for odd L , one of them remains unmatched, and should be thought of as connected to the infinity on the strip via an infinite half-line not crossing any semi-circle (we'll drop this half-line for simplicity in pictorial representations). There are $c_n = (2n)!/(n!(n+1)!)$

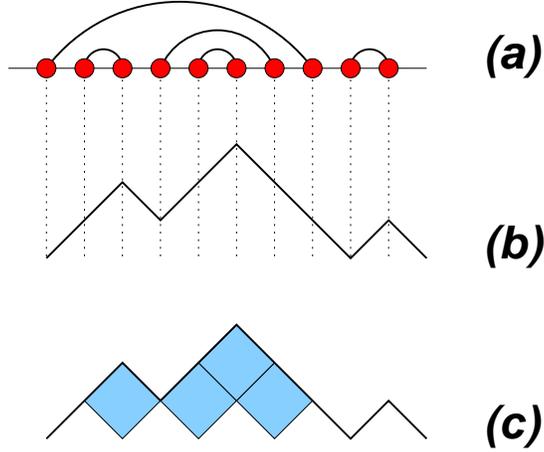


Fig. 1: A sample link pattern π in size $L = 10$ (a) and the associated Dyck path $p(\pi)$ of length 10 (b). We have also indicated the box decomposition of the Dyck path, having $\beta(\pi) = 4$ boxes here.

such link patterns for $L = 2n$ and $L = 2n - 1$, the $L = 2n - 1$ link patterns being in bijection with that at $L = 2n$ upon sending the point labelled L to infinity. A standard bijection replaces link patterns by Dyck paths (see Fig.1), namely lattice paths of L steps, starting from the origin of the integer plane, making steps $(1, 1)$ or $(1, -1)$ only, visiting only points (x, y) with $y \geq 0$ and ending at point $(L, 0)$ if L is even, and $(L, 1)$ if L is odd. Visiting the link pattern π from left to right, we define the Dyck path $p(\pi)$ as follows: it takes an i -th step $(1, 1)$ (resp. $(1, -1)$) if a semi-circle originates (resp. terminates) at point i on π , and an i -th step $(1, 1)$ if the point i is unmatched in π . A useful notion is that of box decomposition of the path, namely expressing it as the hull of the pile of tilted squares of size $\sqrt{2}$ on top of a zig-zag line between the lines $y = 0$ and $y = 1$ (see Fig.1 for an example). We denote by $\beta(\pi)$ the number of boxes in the decomposition of $p(\pi)$.

There is a natural action of the Temperley-Lieb algebra $TL(\tau)$ on link patterns. The generators e_i , $i = 1, 2, \dots, L - 1$ act by inserting a small semi-circle connecting points i and $i + 1$, while gluing the former arcs issued from i and $i + 1$ into a single arch connecting their respective other ends. If i or $i + 1$ is unmatched, e_i simply switches i and $i + 1$ (i.e. interchanges the positions of the unmatched point and of the matched one). In the case where i and $i + 1$ are already connected, the link pattern is left unchanged, but receives a multiplicative factor τ . This leads to the celebrated Temperley-Lieb algebra relations; $e_i^2 = \tau e_i$ and $e_i e_{i \pm 1} e_i = e_i$.

The dense loop model with weight $\tau = -(q + q^{-1})$ per loop is defined via the R -matrix

$$\check{R}_{i,i+1}(z, w) = \frac{q^{-1}z - qw}{q^{-1}w - qz} I + \frac{z - w}{q^{-1}w - qz} e_i \quad (2.1)$$

which we may view as an operator acting on link patterns, I acting as the identity. Let us denote by τ_i the operator that interchanges $z_i \leftrightarrow z_{i+1}$ in any function of the parameters z_1, z_2, \dots, z_L . The level one $U_q(\mathfrak{sl}_2)$ qKZ equation with open boundaries reduces to the system:

$$\begin{aligned} \tau_i \Psi(z_1, \dots, z_L) &= \check{R}_{i,i+1}(z_{i+1}, z_i) \Psi(z_1, \dots, z_L), \quad 1 \leq i \leq L-1 \\ \Psi\left(\frac{r}{z_1}, z_2, \dots, z_{L-1}, z_L\right) &= c_1(z_1) \Psi(z_1, \dots, z_L) \\ \Psi\left(z_1, z_2, \dots, z_{L-1}, \frac{rs}{z_L}\right) &= c_L(z_L) \Psi(z_1, \dots, z_L) \end{aligned} \quad (2.2)$$

where $s = q^6$ and c_1 and c_L two functions to be determined, and Ψ a vector in the link pattern basis. In the following we will restrict ourselves to the values $r = 1$, $rs = q^6$, of the boundary terms¹.

Using the expression for $\check{R}_{i,i+1}$ (2.1), we may rewrite the first equation of (2.2) above in components (indexed by link patterns π) as:

$$\Delta_i \Psi_\pi = \sum_{\substack{\pi' \neq \pi \\ e_i \pi' = \pi}} \Psi_{\pi'}, \quad 1 \leq i \leq L-1 \quad (2.3)$$

where the operator Δ_i reads

$$\Delta_i = \frac{q^{-1} z_{i+1} - q z_i}{z_i - z_{i+1}} (\tau_i - 1) \quad (2.4)$$

In [9], it was shown that it is sufficient to solve these equations in the case of even size $L = 2n$, as the solution for size $L-1$ may then be obtained by taking the limit $z_L \rightarrow 0$, namely

$$\Psi_{f(\pi)}(z_1, \dots, z_{L-1}) \propto \Psi_\pi(z_1, \dots, z_{L-1}, 0) \quad (2.5)$$

while the link patterns are mapped bijectively $\pi \rightarrow f(\pi)$ by removing the point L and leaving unmatched the point formerly connected to it. Unless otherwise stated, we restrict ourselves to $L = 2n$ in the following.

¹ It seems that only the cases $r = 1$ and $r = 1/q^6$ produce nice polynomials of τ with integer coefficients for the components of Ψ , the two being interchanged under the reflection of link patterns with respect to a vertical line.

2.2. Minimal polynomial solution Ψ

In Ref.[9] it was argued that the minimal polynomial solution to the qKZ equation has the following basic component corresponding to the fully nested link pattern π_0 that connects points i to $2n + 1 - i$, $i = 1, 2, \dots, n$:

$$\Psi_{\pi_0} = \prod_{1 \leq i < j \leq n} (qz_i - q^{-1}z_j)(q - q^{-1}z_i z_j) \prod_{n+1 \leq i < j \leq 2n} (qz_i - q^{-1}z_j)(q^{-2}z_i z_j - q^2) \quad (2.6)$$

which clearly satisfies the boundary conditions of Eq. (2.2) with $c_1(x) = 1/x^{2n-2}$ and $c_L(x) = (q^3/x)^{2n-2}$. Then, as explained in Ref.[9], using Eq. (2.3), all other components of Ψ are expressed in a triangular way as linear combinations of products of operators Δ acting on Ψ_{π_0} .

The first few solutions for $L = 1, 2, \dots, 8$ are given in Appendix A below for completeness, in the homogeneous limit where all $z_i \rightarrow 1$ (for even size L), except for $z_L \rightarrow 0$ (for odd size $L - 1$), and upon dividing out by a global factor $(q - q^{-1})^{2n(n-1)}$ for even size L and $(-q)^{3n-3}(q - q^{-1})^{2(n-1)^2}$ for odd size $L - 1$, and using the variable τ of Eq. (1.1). We define the sum rule $\Pi_L(\tau)$ to be simply the sum of components of $\Psi(\tau)$ normalized in this way.

2.3. Miscellaneous conjectures

Inspecting the examples of Appendix A, we have come up with a few conjectures that we list below.

Degree and valuation: We note the following pattern for the degree and valuation (highest and lowest powers of τ) of $\Psi_\pi(\tau)$ as a polynomial of τ . Expressing the link patterns as Dyck paths, recall that $\beta(\pi)$ denotes the number of boxes in the decomposition of $p(\pi)$. In the odd case $L = 2n - 1$, let us also record the position $u(\pi)$ of the unmatched point in π , $u(\pi) = 1, 3, 5, \dots, 2n - 1$. Then we have:

$$\begin{aligned} \deg(\Psi_\pi) &= n(n-1) - \beta(\pi) && \text{for } L = 2n \\ \deg(\Psi_\pi) &= (n-1)^2 - \beta(\pi) && \text{for } L = 2n - 1 \\ \text{val}(\Psi_\pi) &= \beta(\pi) && \text{for } L = 2n \\ \text{val}(\Psi_\pi) &= \beta(\pi) + u(\pi) - n && \text{for } L = 2n - 1 \end{aligned} \quad (2.7)$$

Parity: Like in the cyclic case of Ref.[13], the components of Ψ have a definite parity as polynomials of τ . As this parity is reversed by each action of Δ_i (i.e. each action of e_i on

the link patterns) we may define unambiguously a sign $\epsilon(\pi)$ for each link pattern π , with the boundary condition that the “maximal component” with link pattern π_{max} connecting points $2i - 1$ to $2i$ (with the last point unmatched in odd size) has sign $\epsilon(\pi_{max}) = 1$. Then $\Psi_\pi(-\tau) = \epsilon(\pi)\Psi_\pi(\tau)$ for all π . Note that with this definition we also have

$$\epsilon(\pi) = (-1)^{\beta(\pi)} \tag{2.8}$$

for all L , as a consequence of (2.7).

Integrality, symmetry: All entries of $\Psi(\tau)$ are polynomials with non-negative integer coefficients, and so are the sum rules $\Pi_L(\tau)$. We note that the entries of Ψ are not in general symmetric under reflection $\pi \rightarrow \pi^t$ of link patterns with respect to a vertical axis, namely $\Psi_\pi(\tau) \neq \Psi_{\pi^t}(\tau)$ in general, although this symmetry is restored at the RS point, where $\Psi_\pi(1) = \Psi_{\pi^t}(1)$. This is because the boundary conditions on the left and right are *not* the same in general: Ψ is indeed symmetric under $z_1 \rightarrow 1/z_1$ on the left and $z_L \rightarrow q^6/z_L$ on the right (up to multiplicative factors c_1 or c_L , see Eq. (2.2)), and the left-right symmetry is restored only when $q^6 = 1$ (thus including the RS point $q = -e^{i\pi/3}$, $\tau = 1$ and its “conjugate” $q = e^{i\pi/3}$, $\tau = -1$).

3. Plane Partitions with symmetries

In this section, we recall a few know facts on Plane Partitions with symmetries, related in particular to their expression as NILP and to their explicit enumeration. We also introduce a generalization of CSTCPPs to reproduce the number of VSASMs.

3.1. TSSCPPs and a first conjecture

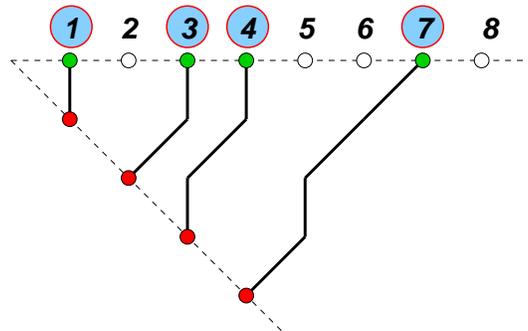


Fig. 2: A sample NILP in bijection with a TSSCPP of size 10. The corresponding endpoints are circled, and read $r_1 = 1, r_2 = 3, r_3 = 4, r_4 = 7$.

We recall the expression for the number $N_{10}(2n)$ of TSSCPPs in a box of size $2n \times 2n$. We refer for instance to [6] for details and further references. As TSSCPPs are maximally symmetric rhombus tilings of a regular hexagon of the triangular lattice of size $2n$, they are entirely determined by the tiling of a fundamental domain of area $\frac{1}{12}$ th of the hexagon, with the shape of a triangle. Following the sequences of two of the three types of rhombi used, one easily ends up with an equivalent configuration of $n - 1$ NILP drawn on the integer plane (see Fig.2 for an illustration for $n = 5$), starting at points $(i, -i)$, $i = 1, 2, \dots, n - 1$ and ending on the line $y = 0$, making only vertical $(0, 1)$ or diagonal steps $(1, 1)$. We record by an increasing sequence $1 \leq r_1 < r_2 < \dots < r_{n-1}$ the endpoints $(r_i, 0)$. Note that $r_i \leq 2i$, as the largest r 's are attained by using only diagonal steps.

The total number of TSSCPPs of size $2n$ equals

$$N_{10}(2n) = \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\binom{i}{r_j - i} \right) = 1, 2, 7, 42, 429, \dots \quad (3.1)$$

for $n = 1, 2, 3, 4, 5 \dots$. The latter is expressed as the sum of minors of size $n - 1$ of the $(n - 1) \times (2n - 2)$ matrix Q with entries $Q_{i,r} = \binom{i}{r-i}$, $1 \leq i \leq n - 1$, $1 \leq r \leq 2n - 2$. This is in fact a particular case of the Lindström Gessel Viennot (LGV) formula [15], expressing the number of lattice paths with fixed origins and endpoints as a (fermionic Slater) determinant.

In Ref.[16], it was noted that the TSSCPPs may be regrouped (refined) according to their common endpoints $\{r_1, r_2, \dots, r_{n-1}\}$ into exactly c_n sets, corresponding to the conditions that $1 \leq r_1 < \dots < r_{n-1}$ and $r_i \leq 2i$ for all i , where the latter conditions ensure that the binomial coefficient $\binom{i}{r_i - i} = \binom{i}{2i - r_i}$ does not vanish. Listing the endpoints in lexicographic order, we may form vectors $\Theta^{(2n)}$ with c_n components, with entries equal to $\det_{1 \leq i, j \leq n-1} \left(\binom{i}{r_j - i} \right)$, that sum to $N_{10}(2n)$. The first few such vectors read

$$\begin{aligned} \Theta^{(2)} &= \{1\} \\ \Theta^{(4)} &= \{1, 1\} \\ \Theta^{(6)} &= \{1, 2, 1, 2, 1\} \\ \Theta^{(8)} &= \{1, 3, 3, 1, 5, 6, 2, 3, 1, 5, 6, 2, 3, 1\} \end{aligned} \quad (3.2)$$

summing respectively to 1, 2, 7, 42. In Ref.[16], a simple bijection between the set of admissible endpoints and the link patterns was proposed. Here we use a slight modification thereof, as we compose it with a reflection with respect to a vertical axis. This is

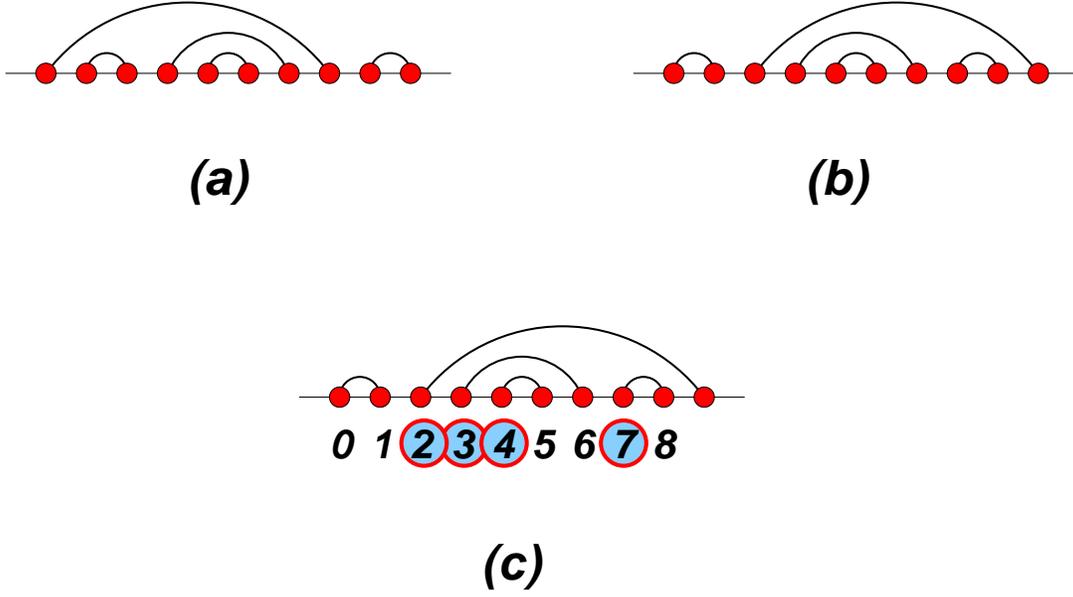


Fig. 3: The bijection between link patterns of size $L = 2n$ and sequences of integers $1 \leq r_1 < r_2 < \dots < r_{n-1}$ with $r_i \leq 2i$ for all i is illustrated on an example for $n = 5$. Starting from the link pattern (a), we first reflect it with respect to a vertical axis (b), and then record the positions (c) of all origins of semi-circles by the corresponding point label minus one, omitting the first (at position 0). These read $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 7$ here.

summarized in Fig.3. Starting from a link pattern π of size $L = 2n$, we first reflect it with respect to a vertical axis, and then record the positions of all origins of semi-circles forming it, by the point label minus one, omitting the first one. This gives a bijective mapping $\pi \rightarrow \{r_i(\pi)\}_{1 \leq i \leq n-1}$. Conversely, given the r_i 's, there is a unique link pattern $\{r_1, \dots, r_{n-1}\} \rightarrow \pi(r_1, \dots, r_{n-1})$. In Ref.[13], the TSSCPPs were enumerated with a weight τ per vertical step, resulting in generating polynomials

$$N_{10}(2n; \tau) = \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j} \binom{i}{r_j - i} \right) \quad (3.3)$$

The latter were then conjectured to match the sum rules for the suitably normalized cyclic boundary qKZ solutions. This τ -enumeration leads naturally to the vectors $\Theta^{(2n)}(\tau)$, the entries of which count the TSSCPPs with fixed endpoints (still listed in lexicographic order) and with a weight τ per vertical step, summing to $N_{10}(2n; \tau)$. The first few of them

read

$$\begin{aligned}
\Theta^{(2)}(\tau) &= \{1\} \\
\Theta^{(4)}(\tau) &= \{\tau, 1\} \\
\Theta^{(6)}(\tau) &= \{\tau^3, 2\tau^2, \tau, 2\tau, 1\} \\
\Theta^{(8)}(\tau) &= \{\tau^6, 3\tau^5, 3\tau^4, \tau^3, 5\tau^4, 6\tau^3, 2\tau^2, 3\tau^2, \tau, 5\tau^3, 6\tau^2, 2\tau, 3\tau, 1\}
\end{aligned} \tag{3.4}$$

Let us now look at the qKZ solutions of Appendix A, corresponding to even sizes $L = 2, 4, 6, 8$. We note that the terms of smallest degree in τ , namely the valuation terms, coefficients of $\tau^{\beta(\pi)}$ in $\Psi_\pi(\tau)$, produce exactly the entries of the vectors (3.2). More precisely, we have

$$\Theta_{r_1, \dots, r_{n-1}}^{(2n)}(\tau) = \Psi_{\pi(r_1, \dots, r_{n-1})}(\tau) \Big|_{min} \tag{3.5}$$

where the subscript *min* stands for the lowest order term in τ , (term $\tau^{\beta(\pi)}$ in $\Psi_\pi(\tau)$), and this holds for $n = 1, 2, 3, 4$. Note that indeed

$$\beta(\pi) = \sum_{i=1}^{n-1} 2i - r_i(\pi) \tag{3.6}$$

produces the same power of τ both in $\Psi_\pi(\tau)|_{min}$ and in $\Theta_{r_1(\pi), \dots, r_{n-1}(\pi)}^{(2n)}(\tau)$. The same phenomenon is observed for the qKZ solution of odd size $L = 2n - 1$, with the obvious extension of the bijection between sets of admissible endpoints and link patterns of odd size. We are led to conjecture that the relation (3.5) holds for arbitrary n , namely that

$$\Psi_\pi(\tau) \sim \det_{1 \leq i, j \leq n-1} \left(\binom{i}{r_j(\pi) - i} \right) \times \tau^{\beta(\pi)} \times \begin{cases} 1 & \text{if } L = 2n \\ \tau^{u(\pi) - n} & \text{if } L = 2n - 1 \end{cases} \tag{3.7}$$

when $\tau \rightarrow 0$.

As a concluding remark, recall the above observation that, in general, the entries of $\Psi(\tau)$ are not symmetric under reflection of link patterns $\pi \rightarrow \pi^t$. This is clearly the case for the conjectured leading $\tau \rightarrow 0$ term in $\Psi_\pi(\tau)$ (3.7), say for even $L = 2n$. Indeed, the set $\{2n - 1 - r_j(\pi^t)\}_{j=1}^{n-1}$ is the complement of the set $\{r_i(\pi)\}_{i=1}^{n-1}$ within $\{1, 2, \dots, 2n - 2\}$, hence the coefficients (3.7) for π and π^t are in general distinct (although $\beta(\pi) = \beta(\pi^t)$). This lack of symmetry in refined TSSCPPs is a puzzle when we compare the entries (3.2) to those of the cyclic qKZ solution at the RS point (counting FPLs), that are indeed reflection-symmetric. It suggests the existence of a non-symmetric change of basis relating the vector of FPL numbers to that of refined TSSCPPs.

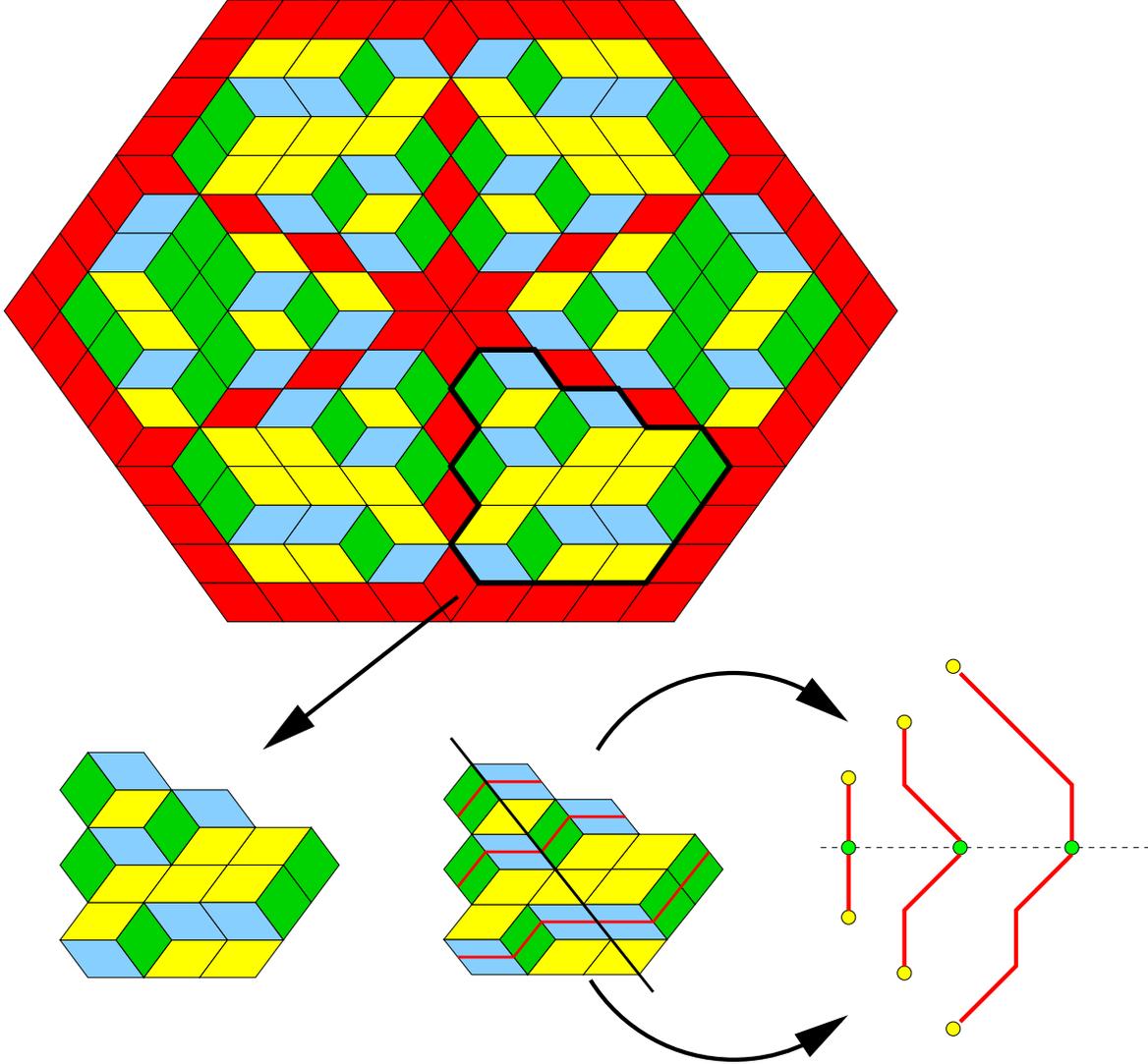


Fig. 4: The bijection between CSTCPPs and pairs of TSSCPPs. The CSTCPP at hand is a tiling of a regular hexagon of size $2n = 8$ here, symmetric with respect to all axes passing through the center of the hexagon and the middle of each edge, resulting in fixed rhombi (represented in red). We have delimited a fundamental domain (thick broken black line), which is further mapped onto a NILP configuration. The latter is cut into two halves, each of which is identified with the NILP formulation of a TSSCPP of same size.

3.2. CSTCPPs and the case $L = 2n - 1$

The number of CSTCPPs in a regular hexagon of size $(2n) \times (2n) \times (2n)$, denoted by $N_8(2n)$, was first obtained [17] by mapping the corresponding rhombus tiling configurations to NILP, easily enumerated via a LGV-type determinant:

$$N_8(2n) = \det_{1 \leq i, j \leq n-1} \left(\binom{i+j}{2i-j} \right) \quad (3.8)$$

The determinant was then evaluated as the product (see [18] for a simpler, illuminating proof):

$$N_8(2n) = \prod_{i=0}^{n-1} (3i+1) \frac{(6i)!(2i)!}{(4i+1)!(4i)!} = 1, 2, 11, 170, 7429, \dots \quad (3.9)$$

for $n = 1, 2, 3, 4, 5, \dots$

We may regard each CSTCPP as a *pair* of TSSCPPs. The bijection between CSTCPPs and pairs of TSSCPPs is illustrated in Fig.4. It is obtained by simply cutting each CSTCPP into two halves, after rewriting it in terms of NILP.

This bijection results in the following identity, counting the total number of pairs of TSSCPPs with common arrival points $r_1 < r_2 < \dots < r_{n-1}$:

$$N_8(2n) = \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \left(\det_{1 \leq i, j \leq n-1} \left(\binom{i}{r_j - i} \right) \right)^2 \quad (3.10)$$

One may derive this formula directly from (3.8), by noting the following matrix identity: denoting by A and B the matrices with entries $A_{i,r} = \binom{i}{r-i} = \binom{i}{2i-r}$ and $B_{i,j} = \binom{i+j}{2i-j}$, with $i, j = 1, 2, \dots, n-1$ and $r = 1, 2, \dots, 2n-2$, we have indeed that $B = AA^t$, as a consequence of the binomial identity $\binom{i+j}{2i-j} = \sum_{r=\text{Max}(i,j)}^{\text{Min}(2i,2j)} \binom{i}{2i-r} \binom{j}{r-j}$. Eq.(3.10) is nothing but a rewriting of the determinant of B in terms of the minors of A .

3.3. VSASMs and new Plane Partitions for the case $L = 2n$

The total number $A_V(2n+1)$ of VSASMs of size $(2n+1) \times (2n+1)$ equals:

$$A_V(2n+1) = \prod_{i=1}^n (3i-1) \frac{(6i-3)!(2i-1)!}{(4i-1)!(4i-2)!} = 1, 3, 26, 646, 45885, \dots \quad (3.11)$$

for $n = 1, 2, 3, 4, 5, \dots$ It may also be expressed as a number of NILP, and actually reinterpreted as the number of rhombus tilings of a hexagon of shape $(2n) \times (2n+2) \times (2n)$ with a central triangular hole² of size $2 \times 2 \times 2$ (see Fig. 5 for an illustration). The counting of NILP yields a determinant formula for the above numbers

$$A_V(2n+1) = \det_{1 \leq i, j \leq n-1} \left(\binom{i+j+1}{2i-j} \right) \quad (3.12)$$

² We refer to [18] and [19], for the weighted enumeration of very similar objects, also in relation with Descending Plane Partitions.

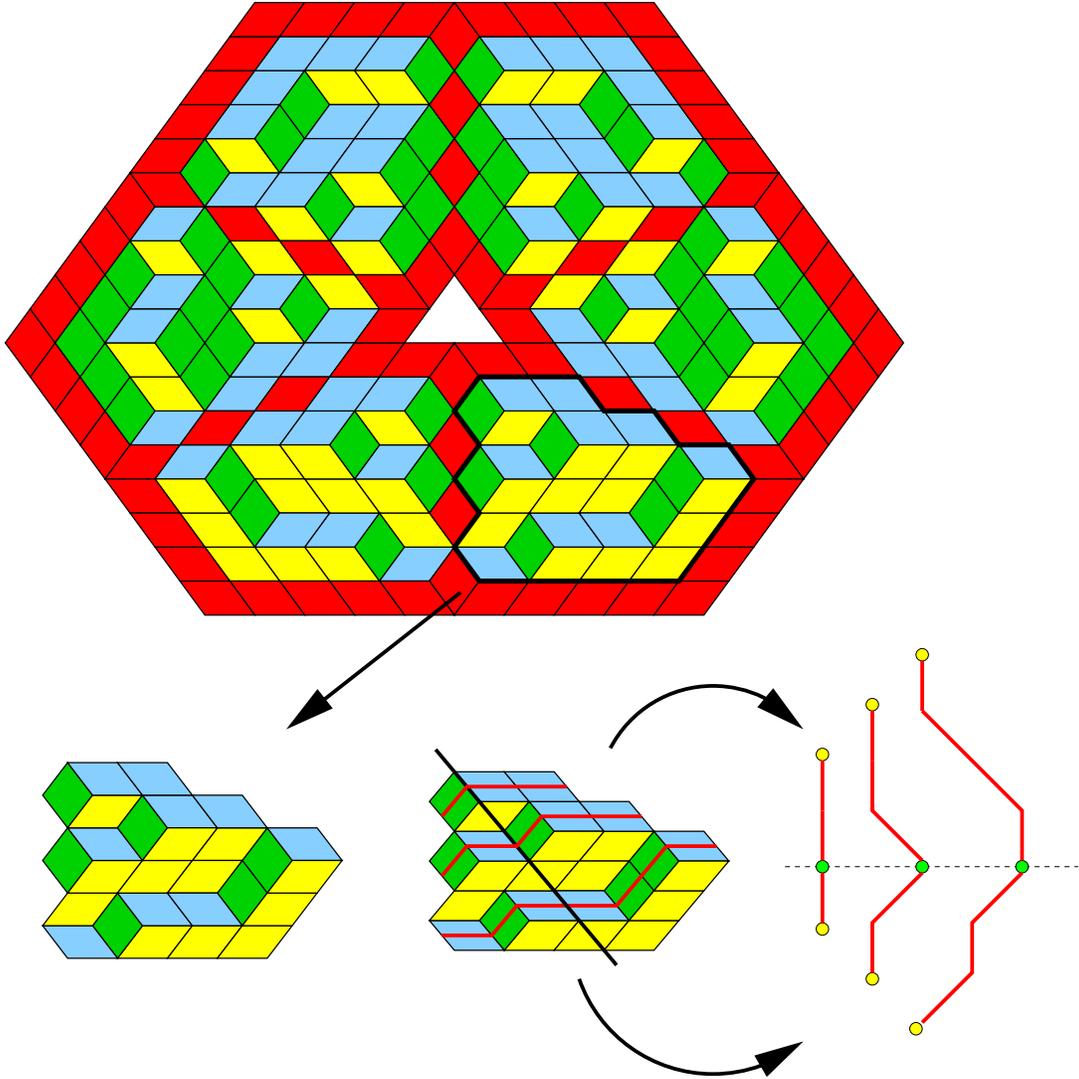


Fig. 5: A typical rhombus tiling of an hexagon of size $(2n) \times (2n + 2) \times (2n)$ (with $n = 4$ here) with a central triangular hole of size $2 \times 2 \times 2$, symmetric with respect to all bisecting lines of its edges. We have extracted a fundamental domain (thick black broken line) and transformed it into a configuration of NILP, by following successions of tiles of two of the three types used. The latter are cut again into two different halves, one of which is a TSSCPP, the other with paths of length one more.

The latter determinant can be evaluated, using a more general result [17], leading to the product formula (3.11). As before, the NILP may be cut into two halves, one of which is identified with a TSSCPP, and the other with a set of NILP of length one more (see Fig.5).

This bijection results in the following representation for the VSASM numbers:

$$A_V(2n+1) = \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\binom{i}{r_j - i} \right) \times \det_{1 \leq i, j \leq n-1} \left(\binom{i+1}{r_j - i} \right) \quad (3.13)$$

which may also be derived from (3.12) via the binomial identity $\binom{i+j+1}{2i-j} = \sum_{r=\text{Max}(i-1, j)}^{\text{Min}(2i, 2j)} \binom{i+1}{2i-r} \binom{j}{r-j}$.

4. τ -enumeration of Plane Partitions with symmetries

In this section we introduce polynomials $N_8(2n; \tau)$ and $A_V(2n+1; \tau)$ that generate the rhombus tilings of Sects. 3.2 and 3.3 with a specific weighting by the parameter τ . These turn out to match the sum rule $\Pi_L(\tau)$ for the open qKZ solution, respectively for $L = 2n - 1$ and $L = 2n$ for all the examples of Appendix A, and we conjecture that they do in general.

4.1. Odd case $L = 2n - 1$

We introduce the polynomials $N_8(2n; \tau)$ which enumerate the pairs of TSSCPPs involved in the CSTCPPs, with a weight τ per vertical step, except in the last step of say the second TSSCP of the pair. The latter reads:

$$\begin{aligned} N_8(2n; \tau) = & \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j} \binom{i}{r_j - i} \right) \\ & \times \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j} \binom{i-1}{r_j - i - 1} + \tau^{2i-r_j-1} \binom{i-1}{r_j - i} \right) \end{aligned} \quad (4.1)$$

in which we have performed a decomposition of the paths of the second TSSCPP according to their last step (receiving no weight τ). Alternatively, the result (4.1) may be put in the form of a single determinant, namely

$$\begin{aligned} N_8(2n; \tau) = & \det_{1 \leq i, j \leq n-1} \left(\sum_{r=1}^{2n-2} \tau^{2j-r} \binom{j}{2j-r} \right. \\ & \left. \times \left\{ \tau^{2i-r} \binom{i-1}{2i-r} + \tau^{2i-r-1} \binom{i-1}{2i-r-1} \right\} \right) \end{aligned} \quad (4.2)$$

The first few polynomials $N_8(2n; \tau)$ take the values

$$\begin{aligned} N_8(2; \tau) &= 1 \\ N_8(4; \tau) &= 1 + \tau \\ N_8(6; \tau) &= 1 + 3\tau + 4\tau^2 + 2\tau^3 + \tau^4 \\ N_8(8; \tau) &= 1 + 6\tau + 19\tau^2 + 32\tau^3 + 41\tau^4 + 35\tau^5 + 21\tau^6 + 11\tau^7 + 3\tau^8 + \tau^9 \end{aligned} \quad (4.3)$$

Comparing with the data of Appendix A, we note that $\Pi_{2n-1}(\tau) = N_8(2n; \tau)$ for $n = 1, 2, 3, 4$. We also note the first few values of $N_8(2n; \tau)$ for $\tau = 1, 2, -1$:

$$\begin{aligned} N_8(2n; 1) &= 1, 2, 11, 170, 7429, 920460, 323801820, \dots \\ N_8(2n; 2) &= 1, 3, 55, 6153, 4196961, 17446527483, 441865841817751, \dots \\ N_8(2n; -1) &= 1, 0, 1, 0, 81, 0, 456976, \dots \end{aligned} \quad (4.4)$$

for $n = 1, 2, 3, 4, 5, 6, 7, \dots$. Apart from the obvious $N_8(2n; 1) = N_8(2n)$, we have found the following identifications:

$$\begin{aligned} N_8(2n; 2) &= \det_{1 \leq i, j \leq n-1} \left(\begin{pmatrix} 2i + 2j - 1 \\ 2i - 1 \end{pmatrix} \right) \\ N_8(2n; -1) &= \begin{cases} A_V(2p+1)^4 & \text{if } n = 2p+1 \\ 0 & \text{if } n = 2p \end{cases} \end{aligned} \quad (4.5)$$

The $\tau = 2$ identification is proved as follows. In Ref.[13], it was shown that the two rectangular $(n-1) \times (2n-2)$ matrices B and A with respective entries $B_{i,r} = \binom{2i}{r}$ and $A_{i,r} = 2^{2i-r} \binom{i}{r-i}$, $i = 1, 2, \dots, n-1$ and $r = 1, 2, \dots, 2n-2$, actually share the same minors of size $n-1$, as one has $B = QA$, Q the square matrix with entries $Q_{i,k} = \binom{k}{i}$. Here we will use also a slight modification of this identity. Introducing the matrices C, D with entries $C_{i,r} = \binom{2i-1}{r-1}$ and $D_{i,r} = 2^{2i-r} \binom{i-1}{2i-r} + 2^{2i-r-1} \binom{i-1}{2i-r-1}$, $i = 1, 2, \dots, n-1$ and $r = 1, 2, \dots, 2n-2$, we have that $C = RD$ for a matrix R with entries $R_{i,k} = \binom{i-1}{k-1}$, $i, k = 1, 2, \dots, n-1$. The determinant in (4.2) at $\tau = 2$ simply reads $\det(AD^t)$, now reexpressed as $\det(QAD^tR^t) = \det(BC^t)$ as both square matrices Q and R^t have determinant 1. But BC^t has entries

$$(BC^t)_{i,j} = \sum_{r=1}^{2n-2} \binom{2i}{r} \binom{2j-1}{r-1} = \begin{pmatrix} 2i + 2j - 1 \\ 2i - 1 \end{pmatrix} \quad (4.6)$$

and the first line of (4.5) follows.

As to the $\tau = -1$ identification, we may recast (4.2) at $\tau = -1$ as the determinant $N_8(2n; -1) = \det(E)$ of a skew-symmetric matrix E , with entries $E_{i,j} = \binom{i+j-1}{2i-j} - \binom{i+j-1}{2i-j-1}$, thanks to standard binomial sum identities. In Ref.[13], as a consequence of a $\tau = -1$ -enumeration of TSSCPPs, a Pfaffian identity was conjectured for the square of the number of VSASMs, in the form

$$A_V(2n+1)^2 = \text{Pf}_{1 \leq i < j \leq 2n} \left(\sum_{i \leq r < s \leq 2j} (-1)^{r+s-1} \left\{ \binom{i}{r-i} \binom{j}{s-j} - \binom{i}{s-i} \binom{j}{r-j} \right\} \right) \quad (4.7)$$

We simply note that

$$\begin{aligned} \sum_{i \leq r < s \leq 2j} (-1)^{r+s-1} \binom{i}{r-i} \binom{j}{s-j} &= \binom{i+j-1}{2i-j} \\ \sum_{i \leq r < s \leq 2j} (-1)^{r+s-1} \binom{i}{s-i} \binom{j}{r-j} &= \binom{i+j-1}{2i-j-1} = \binom{i+j-1}{2j-i} \end{aligned} \quad (4.8)$$

hence $\det(E)$ is nothing but the square of the Pfaffian (4.7), and the second line of (4.5) boils down to the conjectured formula (4.7).

4.2. Even case $L = 2n$

Like in the odd case, we introduce the polynomial $A_V(2n+1; \tau)$ that enumerates the rhombus tilings of the holed hexagon of Sect. 3.3, expressed as pairs of NILP, with a weight τ per vertical step, except for the last steps of the second NILP of the pair. It reads:

$$\begin{aligned} A_V(2n+1; \tau) &= \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j} \binom{i}{r_j-i} \right) \\ &\quad \times \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j+1} \binom{i}{r_j-i-1} + \tau^{2i-r_j} \binom{i}{r_j-i} \right) \end{aligned} \quad (4.9)$$

Alternatively, this may be recast into a single determinant

$$\begin{aligned} A_V(2n+1; \tau) &= \det_{1 \leq i, j \leq n-1} \left(\sum_{r=1}^{2n-2} \tau^{2i-r} \binom{i}{2i-r} \right. \\ &\quad \left. \times \left\{ \tau^{2j-r+1} \binom{j}{2j-r+1} + \tau^{2j-r} \binom{j}{2j-r} \right\} \right) \end{aligned} \quad (4.10)$$

The first few polynomials $A_V(2n+1; \tau)$ take the values:

$$\begin{aligned} A_V(3; \tau) &= 1 \\ A_V(5; \tau) &= 1 + \tau + \tau^2 \\ A_V(7; \tau) &= 1 + 3\tau + 7\tau^2 + 6\tau^3 + 6\tau^4 + 2\tau^5 + \tau^6 \\ A_V(9; \tau) &= 1 + 6\tau + 25\tau^2 + 54\tau^3 + 102\tau^4 + 119\tau^5 + 131\tau^6 + 94\tau^7 + 67\tau^8 + 29\tau^9 \\ &\quad + 14\tau^{10} + 3\tau^{11} + \tau^{12} \end{aligned} \quad (4.11)$$

Comparing with the data of Appendix A, we note that $\Pi_{2n}(\tau) = A_V(2n+1; \tau)$ for $n = 1, 2, 3, 4$. We also note the first few values of $A_V(2n+1; \tau)$ for $\tau = 1, 2, -1$:

$$\begin{aligned} A_V(2n+1; 1) &= 1, 3, 26, 646, 45885, 9304650, \dots \\ A_V(2n+1; 2) &= 1, 7, 307, 82977, 137460201, 1392263902567, \dots \\ A_V(2n+1; -1) &= 1, 1, 4, 36, 1089, 81796, \dots \end{aligned} \quad (4.12)$$

for $n = 1, 2, 3, 4, 5, 6, \dots$. Apart from the obvious $A_V(2n + 1; 1) = A_V(2n + 1)$, we have found the following identifications:

$$\begin{aligned} A_V(2n + 1; 2) &= \det_{1 \leq i, j \leq n} \left(\binom{2i + 2j - 3}{2i - 1} \right) \\ A_V(2n + 1; -1) &= \left(N_8 \left(2 \left[\frac{n+1}{2} \right] \right) A_V \left(2 \left[\frac{n}{2} \right] + 1 \right) \right)^2 \end{aligned} \quad (4.13)$$

where $[x]$ stands for the integer part of x .

The first line of (4.13) may be proved exactly by the same argument as before. As to the second line, we note that at $\tau = -1$ (4.10) boils down to the determinant of a matrix F with entries $F_{i,j} = \binom{i+j}{2i-j} - \binom{i+j}{2i-j-1}$, $i, j = 1, 2, \dots, n-1$, thanks to standard binomial summation formulae. By simple row manipulations, we may slightly transform F as follows: let us introduce the matrix P with entries $P_{i,j} = \delta_{i,j} + \delta_{i+1,j}$, $i, j = 1, 2, \dots, n-1$. Then FP has the entries $(FP)_{i,j} = \delta_{i,1} \delta_{j,1} + \binom{i+j-1}{2i-j-2} - \binom{i+j-1}{2j-i-2}$, $i, j = 1, 2, \dots, n-1$. We note that when n is odd, removing the first term $\delta_{i,1} \delta_{j,1}$ does not change the value of the determinant, as the corresponding minor is that of a skew-symmetric matrix of odd size $(n-2)$, hence vanishes. Hence for $n = 2p + 1$, $\det(F) = \det(G)$, where G is the skew-symmetric matrix with entries $G_{i,j} = \binom{i+j-1}{2j-i-2} - \binom{i+j-1}{2i-j-2}$, $i, j = 1, 2, \dots, 2p$. Its determinant is therefore the square of its Pfaffian, which we conjecture to be given by $\text{Pf}(G) = N_8(2p+2)A_V(2p+1)$. When n is even, let us multiply the term $\delta_{i,1} \delta_{j,1}$ by some arbitrary real number x . Then the corresponding determinant takes the form $ax + b$, as is readily seen by expanding it, say with respect to the first column. We have $b = 0$ as it is nothing but the determinant at $x = 0$, in which case it is the determinant of a skew-symmetric matrix of odd size. Finally a is the 1,1 minor. So at $x = 1$, we get that for $n = 2p$, $\det(F) = \det(H)$, where H is the matrix with entries $H_{i,j} = \binom{i+j+1}{2j-i-1} - \binom{i+j+1}{2i-j-1}$, $i, j = 1, 2, \dots, 2p-2$. The determinant of this skew-symmetric matrix is the square of its Pfaffian, which we conjecture to be equal to $\text{Pf}(H) = N_8(2p)A_V(2p+1)$.

5. Conjectures and conclusion

5.1. Conjectures

We list and comment the main conjectures of this paper, and add up a few.

Conjecture 1. The coefficients of smallest degree in τ of the homogeneous open boundary qKZ solution $\Psi^{(L)}(\tau)$ form a vector identical to that of TSSCPP numbers arranged

according to their (admissible) endpoints, via the bijection with link patterns described in Sect. 3.1 (see Eq. (3.7)).

This conjecture looks very promising, as it relates for the first time in a way similar to the full RS conjecture two different objects, one of them purely combinatorial, here the TSSCPPs arranged according to their endpoints, and the other purely algebraic, in the form of the leading coefficients of the qKZ solution $\Psi_\pi(\tau)$ when $\tau \rightarrow 0$. This remarkable coincidence suggests that the correspondingly refined TSSCPP numbers might be directly obtainable from the qKZ equation. Moreover, if we were able to relate directly the cyclic and open boundary solutions of qKZ, we would have a natural way of going from the components $\Psi(\tau)$ of the cyclic case, equal presumably to some τ -enumeration of ASMs or FPLs with fixed connectivities, to the TSSCPP numbers sorted according to their endpoints, which would provide us with a new refinement in a possible TSSCPP-ASM correspondence.

Conjecture 2. The sum rule for homogeneous open boundary qKZ solution $\Psi^{(L)}(\tau)$ equates the generating polynomial for the corresponding Plane Partitions or rhombus tilings with the suitable reflection and cyclic symmetries, namely $N_8(L+1; \tau)$ if $L = 2n - 1$ and $A_V(L+1; \tau)$ if $L = 2n$.

This produces a refinement of the sum rules proved in [9], that incorporates the “quantum” deformation parameter q explicitly.

Conjecture 3. The ”maximal” component $\Psi_{max}(L)$ of the qKZ solution in size L with link pattern π_{max} connecting points $2i - 1$ to $2i$, and the last point unmatched for odd size, reads respectively for even and odd sizes:

$$\Psi_{\pi_{max}}(2n) = \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \left(\det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j} \binom{i}{r_j - i} \right) \right)^2 \quad (5.1)$$

with values $1, 1 + \tau^2, 1 + 5\tau^2 + 4\tau^4 + \tau^6, 1 + 14\tau^2 + 49\tau^4 + 62\tau^6 + 34\tau^8 + 9\tau^{10} + \tau^{12}$ for $n = 1, 2, 3, 4$.

$$\begin{aligned} \Psi_{\pi_{max}}(2n-1) = & \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j} \binom{i}{2i-r_j} \right) \\ & \times \det_{1 \leq i, j \leq n-1} \left(\tau^{2i-r_j-1} \binom{i-1}{2i-r_j-1} \right) \end{aligned} \quad (5.2)$$

with values $1, \tau, 2\tau^2 + \tau^4, 6\tau^3 + 13\tau^5 + 6\tau^7 + \tau^9$ for $n = 1, 2, 3, 4$. The component of its reflected link pattern π_{max}^t , that leaves point 1 un matched and connects all other points $2i$ to $2i + 1$, $i \geq 1$, reads:

$$\Psi_{\pi_{max}^t}(2n - 1) = \sum_{1 \leq r_1 < r_2 < \dots < r_{n-1}} \det_{1 \leq i, j \leq n-1} \left(\tau^{r_j - i} \binom{i}{r_j - i} \right) \det_{1 \leq i, j \leq n-1} \left(\tau^{r_j - i} \binom{i-1}{r_j - i} \right) \quad (5.3)$$

with values $1, 1, 1 + 2\tau^2, 1 + 6\tau^2 + 13\tau^4 + 6\tau^6$ for $n = 1, 2, 3, 4$.

These are simply the complete τ -enumeration of both CSTCPPs and VSASMs in the form of pairs of NILP, with a weight τ per vertical step in the two first cases and per diagonal step in the last one.

Conjecture 4. In the open boundary case, the Razumov-Stroganov conjecture identifies the components $\Psi_\pi(2n)$ at $\tau = 1$ with the numbers $\text{VSFPL}_{2n+1}(\pi)$ of Vertically Symmetric Fully-Packed Loop configurations on a square grid of size $2n + 1$ reproducing the same connectivity pattern π . Combining this with our observation of Sect. 2.3 on the parity of the components of Ψ as polynomials of τ and with our -1 -enumeration of VSASMs (4.13), we deduce a new alternating sum rule for the numbers $\text{VSFPL}_{2n+1}(\pi)$:

$$\sum_{\pi} \epsilon(\pi) \text{VSFPL}_{2n+1}(\pi) = \left(N_8 \left(2 \left\lfloor \frac{n+1}{2} \right\rfloor \right) A_V \left(2 \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \right)^2 \quad (5.4)$$

with $\epsilon(\pi)$ given by (2.8).

This is the open boundary version of the alternating sum rule (5.1) of Ref.[13].

Conjecture 5. We have the two following identities for Pfaffians:

$$\begin{aligned} \text{Pf}_{1 \leq i < j \leq 2p} \left(\binom{i+j-1}{2j-i-2} - \binom{i+j-1}{2i-j-2} \right) &= N_8(2p+2) A_V(2p+1) \\ \text{Pf}_{1 \leq i < j \leq 2p-2} \left(\binom{i+j+1}{2j-i-1} - \binom{i+j+1}{2i-j-1} \right) &= N_8(2p) A_V(2p+1) \end{aligned} \quad (5.5)$$

These have arisen from the $\tau = -1$ -enumeration of VSASMs in the NILP form, and should be compared with the determinantal expressions (3.8) and (3.12) respectively for the numbers $N_8(2n)$ and $A_V(2n + 1)$. We suspect this is by far the easiest to prove in the list of conjectures above, as both sequences N_8 and A_V are known explicitly and take simple product forms. This should presumably be done using techniques developed in [20].

5.2. Conclusion

In this paper, we have found new conjectures giving a combinatorial interpretation of the level one $U_q(sl_2)$ open qKZ minimal polynomial solution in terms of Plane Partitions with various symmetries. The main interest is to have kept the dependence in the quantum parameter q via the combination $\tau = -(q + q^{-1})$, and to have related the qKZ solution to generating polynomials for the τ -enumeration of Plane Partitions. In particular, we have found a unifying framework for VSASMs and CSTCPPs, allowing to view both as rhombus tilings of (possibly holed) hexagons with the same symmetries, and to τ -enumerate them in a similar way.

The main and most promising conjecture regards the $\tau \rightarrow 0$ behavior of the components of the solution, whose coefficients are interpreted as the numbers of TSSCPPs with fixed endpoints in their NILP formulation. This points to a possible approach of the ASM-TSSCPP correspondence, by trying to relate the cyclic and open qKZ solutions. Note that no such nice coincidence seems to happen in the cyclic case of [13], but note however that the coefficients of top degree in τ coincide in the cyclic and open cases. This should not come as a surprise, as the $\tau \rightarrow \infty$ limit amounts to taking $q \rightarrow \infty$, and therefore leaves us with the *same* renormalized operator $\Delta_i \rightarrow -z_i/(z_i - z_{i+1})(\tau_i - 1)$ and the *same* renormalized fundamental component $\Psi_{\pi_0} = z_1^{n-1} z_2^{n-2} \cdots z_{n-1} z_{n+1}^{n-1} z_{n+2}^{n-2} \cdots z_{2n-1}$, for the open and cyclic cases.

As it clearly appears from the studies of Ref.[13] and the present paper, there should exist some sort of unifying interpretation of the minimal polynomial solutions of the qKZ equation in terms of Plane Partitions or rhombus tilings. This should include also the other boundary conditions considered in [12] as well as those with a point at infinity along the cylinder [21]. The sum rules found in those cases so far do not have all plane partition counterparts, but we believe such interpretations should always exist.

Another interesting question concerns the generalization to higher rank groups [11], where sum rules again have produced nice integer sequences, without combinatorial interpretation yet. Maybe one should hunt for some higher dimensional generalizations of Plane Partitions, presumably with many symmetries.

Finally, let us comment on the specialization $\tau = 2$, corresponding to the rational limit $q \rightarrow -1$, known to produce for the components of Ψ the multidegrees of some variety of upper triangular nilpotent matrices with additional symmetries [12]. Remarkably, we have obtained for the total degree of these varieties (first lines of eqs.(4.5) and (4.13)) the *same*

total degree as that of the Brauer scheme of Refs.[22] and [23], based on a completely different loop model with crossings, and moreover with periodic boundary conditions. This striking coincidence awaits a good geometrical explanation. On the other hand, the polynomials $N_8(2n; 2)$ and $A_V(2n + 1; 2)$ provide a nice reexpression of the total degree of the Brauer scheme as a sum over CSTCPPs or their even counterparts of powers of 2, suggesting that, like in the cyclic case for the variety $M^2 = 0$, these Plane Partitions play the role of “pipe dreams” [24] for the Brauer scheme, that would be decomposable into complete intersections of linear and quadratic varieties.

Acknowledgments: This work was partially supported by the ENRAGE European network MRTN-CT-2004-5616, the ANR program GIMP ANR-05-BLAN-0029-01, the ACI GEOCOMP and the ENIGMA European network MRT-CT-2004-5652.

Appendix A. Polynomial solution of the qKZ equation for open boundaries in the homogeneous limit

$$L = 1 \quad \varphi_{\text{---}\bullet\text{---}} = 1$$

$$\Pi_1(\tau) = 1 \tag{A.1}$$

$$L = 2 \quad \varphi_{\text{---}\bullet\text{---}\bullet\text{---}} = 1$$

$$\Pi_2(\tau) = 1 \tag{A.2}$$

$$L = 3 \quad \varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 1$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}} = \tau$$

$$\Pi_3(\tau) = 1 + \tau \tag{A.3}$$

$$L = 4 \quad \varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = \tau$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 1 + \tau^2$$

$$\Pi_4(\tau) = 1 + \tau + \tau^2 \tag{A.4}$$

$$L = 5 \quad \varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = \tau$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 1 + 2\tau^2$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 2\tau + \tau^3$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}} = \tau^3$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 2\tau^2 + \tau^4$$

$$\Pi_5(\tau) = 1 + 3\tau + 4\tau^2 + 2\tau^3 + \tau^4 \tag{A.5}$$

$$L = 6 \quad \varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = \tau^3$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 2\tau^2 + 2\tau^4$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = \tau + 3\tau^3 + \tau^5$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 2\tau + 2\tau^3 + \tau^5$$

$$\varphi_{\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---}} = 1 + 5\tau^2 + 4\tau^4 + \tau^6$$

$$\Pi_6(\tau) = 1 + 3\tau + 7\tau^2 + 6\tau^3 + 6\tau^4 + 2\tau^5 + \tau^6 \tag{A.6}$$

$$\begin{aligned}
L=7 \quad \varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= \tau^3 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 3\tau^2 + 5\tau^4 + \tau^6 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 5\tau^3 + 3\tau^5 + \tau^7 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= \tau^6 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 2\tau^2 + 3\tau^4 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 3\tau + 11\tau^3 + 10\tau^5 + 2\tau^7 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 5\tau^4 + 4\tau^6 + \tau^8 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 2\tau + 3\tau^3 + 3\tau^5 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 3\tau^4 + 4\tau^6 + \tau^8 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= \tau + 6\tau^3 + 3\tau^5 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 6\tau^2 + 13\tau^4 + 6\tau^6 + \tau^8 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 3\tau^5 + 2\tau^7 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 1 + 8\tau^2 + 12\tau^4 + 5\tau^6 \\
\varphi \quad \text{---} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{---} &= 6\tau^3 + 13\tau^5 + 6\tau^7 + \tau^9
\end{aligned} \tag{A.7}$$

$$\Pi_7(\tau) = 1 + 6\tau + 19\tau^2 + 32\tau^3 + 41\tau^4 + 35\tau^5 + 21\tau^6 + 11\tau^7 + 3\tau^8 + \tau^9$$

$$\begin{aligned}
L = 8 \quad \varphi \quad \text{[diagram]} &= \tau^6 \\
\varphi \quad \text{[diagram]} &= \tau^3 + 6\tau^5 + 6\tau^7 + \tau^9 \\
\varphi \quad \text{[diagram]} &= 3\tau^2 + 9\tau^4 + 12\tau^6 + 5\tau^8 + \tau^{10} \\
\varphi \quad \text{[diagram]} &= 5\tau^3 + 5\tau^5 + 3\tau^7 + \tau^9 \\
\varphi \quad \text{[diagram]} &= 3\tau^5 + 3\tau^7 \\
\varphi \quad \text{[diagram]} &= 2\tau^2 + 15\tau^4 + 24\tau^6 + 13\tau^8 + 2\tau^{10} \\
\varphi \quad \text{[diagram]} &= 3\tau + 15\tau^3 + 29\tau^5 + 20\tau^7 + 7\tau^9 + \tau^{11} \\
\varphi \quad \text{[diagram]} &= 5\tau^4 + 6\tau^6 + 3\tau^8 \\
\varphi \quad \text{[diagram]} &= 2\tau + 15\tau^3 + 27\tau^5 + 19\tau^7 + 7\tau^9 + \tau^{11} \\
\varphi \quad \text{[diagram]} &= 3\tau^4 + 8\tau^6 + 3\tau^8 \\
\varphi \quad \text{[diagram]} &= \tau + 12\tau^3 + 28\tau^5 + 25\tau^7 + 8\tau^9 + \tau^{11} \\
\varphi \quad \text{[diagram]} &= 6\tau^2 + 21\tau^4 + 18\tau^6 + 9\tau^8 + 2\tau^{10} \\
\varphi \quad \text{[diagram]} &= 6\tau^3 + 21\tau^5 + 18\tau^7 + 5\tau^9 \\
\varphi \quad \text{[diagram]} &= 1 + 14\tau^2 + 49\tau^4 + 62\tau^6 + 34\tau^8 + 9\tau^{10} + \tau^{12}
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
\Pi_8(\tau) &= 1 + 6\tau + 25\tau^2 + 54\tau^3 + 102\tau^4 + 119\tau^5 + 131\tau^6 + 94\tau^7 \\
&\quad + 67\tau^8 + 29\tau^9 + 14\tau^{10} + 3\tau^{11} + \tau^{12}
\end{aligned}$$

References

- [1] A.V. Razumov and Yu.G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, *Theor. Math. Phys.* **138** (2004) 333–337; *Teor. Mat. Fiz.* 138 (2004) 395–400, [math.CO/0104216](#).
- [2] M.T. Batchelor, J. de Gier and B. Nienhuis, *The quantum symmetric XXZ chain at $\Delta = -1/2$, alternating sign matrices and plane partitions*, *J. Phys.* A34 (2001) L265–L270, [cond-mat/0101385](#).
- [3] P. Di Francesco and P. Zinn-Justin, *Around the Razumov–Stroganov conjecture: proof of a multi-parameter sum rule*, *Elec. Jour. Comb.* 12 (1) (2005), R6, [math-ph/0410061](#).
- [4] A. Izergin, *Partition function of the six-vertex model in a finite volume*, *Sov. Phys. Dokl.* **32** (1987) 878–879.
- [5] V. Korepin, *Calculation of norms of Bethe wave functions*, *Comm. Math. Phys.* **86** (1982) 391–418.
- [6] D. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Cambridge Univ. Pr., 1999.
- [7] A.V. Razumov and Yu.G. Stroganov, *$O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices*, *Theor. Math. Phys.* **142** (2005) 237–243; *Teor. Mat. Fiz.* 142 (2005) 284–292, [cond-mat/0108103](#).
- [8] P. Pearce, V. Rittenberg and J. de Gier, *Critical $Q=1$ Potts model and Temperley–Lieb stochastic processes*, [cond-mat/0108051](#); P. Pearce, V. Rittenberg, J. de Gier and B. Nienhuis, *Temperley–Lieb Stochastic Processes*, *J. Phys.* **A35** (2002) L661–L668, [math-ph/0209017](#).
- [9] P. Di Francesco, *Inhomogenous loop models with open boundaries*, *J. Phys. A: Math. Gen.* **38** (2005) 6091–6120, [math-ph/0504032](#)
- [10] V. Pasquier, *Quantum incompressibility and Razumov Stroganov type conjectures*, [cond-mat/0506075](#).
- [11] P. Di Francesco and P. Zinn-Justin, *Quantum Knizhnik–Zamolodchikov equation, generalized Razumov–Stroganov sum rules and extended Joseph polynomials*, *J. Phys. A* **38** (2006) L815–L822, [math-ph/0508059](#).
- [12] P. Di Francesco and P. Zinn-Justin, *From Orbital Varieties to Alternating Sign Matrices*, submitted to FPSAC '06, [math-ph/0512047](#).
- [13] P. Di Francesco, *Totally Symmetric Self Complementary Plane Partitions and the quantum Knizhnik–Zamolodchikov equation: a conjecture*, *J. Stat. Mech.: Theor. Exp.* (2006) P09008, [cond-mat/0607499](#).
- [14] M. Kasatani and V. Pasquier, *On polynomials interpolating between the stationary state of a $O(n)$ model and a $Q.H.E.$ ground state*, [cond-mat/0608160](#).
- [15] B. Lindström, *On the vector representations of induced matroids*, *Bull. London Math. Soc.* **5** (1973) 85–90;

- I. M. Gessel and X. Viennot, *Binomial determinants, paths and hook formulae*, Adv. Math. **58** (1985) 300–321.
- [16] P. Di Francesco, *A refined Razumov–Stroganov conjecture*, J. Stat. Mech. P08009 (2004), [cond-mat/0407477](#).
- [17] W. Mills, D. Robbins and H. Rumsey, *Enumeration of a symmetry class of plane partitions*, Discrete Math. **67** (1987) 43–55.
- [18] M. Ciucu and K. Krattenthaler, *Plane partitions: $5\frac{1}{2}$ symmetry classes*, in "Combinatorial Methods in Representation Theory," (M. Kashiwara, K. Koike, S. Okada, I. Terada, and H. Yamada, Eds.), Advanced Studies in Pure Mathematics **28** (2000) 81–103, [math.CO/9808018](#).
- [19] M. Ciucu, T. Eisenkölbl, C. Krattenthaler and D. Zare, *Enumeration of lozenge tilings of hexagons with a central triangular hole*, J. Combin. Theory Ser. A **95**(2001) 251–334, [math.CO/9912053](#); C. Krattenthaler, *Descending plane partitions and rhombus tilings of a hexagon with triangular hole*, Europ. J. Combin. **27** (2006) 1138–1146, [math.CO/0310188](#).
- [20] C. Krattenthaler, *Advanced determinant calculus*, Séminaire Lotharingien Combin. ("The Andrews Festschrift") **42** (1999) B42q.
- [21] P. Di Francesco, P. Zinn-Justin and J.-B. Zuber, *Sum rules for the ground states of the $O(1)$ loop model on a cylinder*, J. Stat. Mech.: Theor. Exp. (2006) P08011, [math-ph/0603009](#).
- [22] P. Di Francesco and P. Zinn-Justin, *Inhomogeneous model of crossing loops and multidegrees of some algebraic varieties*, to appear in *Commun. Math. Phys.* (2005), [math-ph/0412031](#).
- [23] A. Knutson and P. Zinn-Justin, *A scheme related to the Brauer loop model*, [math.AG/0503224](#).
- [24] A. Knutson and E. Miller, *Gröbner geometry of Schubert polynomials*, *Annals of Mathematics* (2003), [math.AG/0110058](#).