

Joint probability for the Pearcey process

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*2000 *Mathematics Subject Classification*. Primary: 60G60, 60G65, 35Q53; secondary: 60G10, 35Q58. *Key words and Phrases*: Dyson's Brownian motion, Pearcey process, random matrices coupled in a chain, random matrices with external potential, infinite-dimensional diffusions.

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The results in this paper form a step in the direction of understanding the behavior of non-intersecting Brownian motions on \mathbb{R} (Dyson's Brownian motions), when the number of particles tends to ∞ . Consider n Brownian particles leaving from points $a_1 < \dots < a_p$ and forced to end up at $b_1 < \dots < b_q$ at time $t = 1$. It is clear that, when $n \rightarrow \infty$, the equilibrium measure for $t \sim 0$ has its support on p intervals and for $t \sim 1$ on q intervals. It is also clear that, when t evolves, intervals must merge, must disappear and be created, leading to various *phase transitions*, depending on the respective fraction of particles leaving from the points a_i and arriving at the points b_j . Therefore the region \mathcal{R} in the space-time strip (x, t) formed by the support ($\subset \mathbb{R}$) of the equilibrium measure as a function of time $0 \leq t \leq 1$ will typically present singularities of different types.

Near the moments, where a phase transition takes place, one expects to find in the limit $n \nearrow \infty$ an infinite-dimensional diffusion, a *Markov cloud*, having some universality properties. Universality here means that the infinite-dimensional diffusion is to depend on the type of singularity only. These Markov clouds are infinite-dimensional diffusions, which 'in principle' could be described by an infinite-dimensional Laplacian with a drift term. We conjecture that each of the Markov clouds obtained in this fashion is related to some *integrable system*, which enables one to derive a non-linear (finite-dimensional) PDE, satisfied by the joint probabilities. The purpose of this paper is to show, for a simple model leading to a cusp, that the joint probabilities at different times, do satisfy such a non-linear PDE. The interrelation between all such equations and the "initial" and "final" ($t \rightarrow \pm\infty$) conditions are interesting and challenging open problems. Moreover, special cases have shown an intimate connection between the integrable system and the Riemann-Hilbert problem associated with the singularity.

The **first question** is the study of the finite Brownian motion model, which, as will be explained in section 2, hinges on the study of the coupled Gaussian Hermitian random matrix ensemble \mathcal{H}_n with external source A ,

define the following differential operators:

$$\begin{aligned}\bar{A}_1^\pm &:= -\frac{1}{2} \left(\sum_{j=1}^m J_{mj} \mathcal{D}_{-1}(E_j) \pm \frac{\partial}{\partial \alpha} \right) \\ \bar{C}_1 &:= \sum_{j=1}^m J_{1j} \mathcal{D}_{-1}(E_j) \\ \bar{A}_2 &:= \frac{1}{2} \left(\mathcal{D}_0(E_m) - \alpha \frac{\partial}{\partial \alpha} - c_{m-1} \frac{\partial}{\partial c_{m-1}} \right) \\ \bar{C}_2 &:= -\mathcal{D}_0(E_1) + c_1 \frac{\partial}{\partial c_1}.\end{aligned}$$

Theorem 0.1 *The log of the probability $\mathbb{P}_n(\alpha; c_1, \dots, c_{m-1}; E_1, \dots, E_m)$ satisfies a fourth-order PDE in $\alpha, c_1, \dots, c_{m-1}$ and in the endpoints $b_1^{(\ell)}, \dots, b_{2r}^{(\ell)}$ of the sets E_ℓ , with quartic non-linearity¹:*

$$\begin{aligned}0 &= \left(F^+ \bar{C}_1 G^- + F^- \bar{C}_1 G^+ \right) \{F^-, F^+\}_{\bar{C}_1} - \left(F^+ G^- + F^- G^+ \right) \bar{C}_1 \{F^-, F^+\}_{\bar{C}_1} \\ &= \det \begin{pmatrix} -G^+ & \bar{C}_1 F^+ & -F^+ & 0 \\ G^- & \bar{C}_1 F^- & -F^- & 0 \\ -\bar{C}_1 G^+ & \bar{C}_1^2 F^+ & 0 & -F^+ \\ \bar{C}_1 G^- & \bar{C}_1^2 F^- & 0 & -F^- \end{pmatrix} \end{aligned} \quad (0.3)$$

where

$$\begin{aligned}F^\pm &:= \bar{A}_1^\pm \bar{C}_1 \log \mathbb{P}_n + k_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}} J_{1m}, \\ G^\pm &:= \left\{ \left(\bar{A}_2 \bar{C}_1 \pm J_{1m} \frac{\partial}{\partial \alpha} \right) \log \mathbb{P}_n \mp K_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}}, F^\pm \right\}_{\bar{C}_1} \\ &\quad + \left\{ (\bar{C}_2 \pm 2\alpha J_{1m} \bar{C}_1) \bar{A}_1^\pm \log \mathbb{P}_n, F^\pm \right\}_{\bar{A}_1^\pm}\end{aligned}$$

with $K_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}}$ a constant, depending on α, c_i, k_1 and k_2 ,

$$K_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}} := J_{1m} \left(2k_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}} \alpha J_{mm} - \frac{k_1 k_2}{\alpha} \right).$$

Note the *robustness* of these equations: the equations always have the same form (0.3), regardless of the length of the chain of matrices; only the quantities F^\pm, G^\pm and H_i^\pm change, via some minors of the matrix J .

¹in terms of the Wronskians $\{f, g\}_X = gXf - fXg$.

The **second question** concerns a simple model of non-intersecting Brownian motions on \mathbb{R} and their behavior, when the number of particles tends to ∞ .

Consider $n = 2k$ non-intersecting Brownian motions on \mathbb{R} , all starting at the origin, such that the k left paths end up at $-a$ and the k right paths end up at $+a$ at time $t = 1$; see [7, 21, 18, 5]. Inspired by [15, 16, 7], the Karlin-McGregor formula [17] enables one to express the transition probability $\mathbb{P}_0^{\pm a}$ in terms of the Gaussian Hermitian random matrices in a chain $\mathbb{P}_n(a; E)$ with external source, explained above; this will be done in section 2.

Let now the number $n = 2k$ of particles go to infinity, and let the points a and $-a$ go to $\pm\infty$. This forces the left k particles to $-\infty$ at $t = 1$ and the right k particles to $+\infty$ at $t = 1$. Since the particles all leave from the origin at $t = 0$, it is natural to believe that for small times the equilibrium measure (mean density of particles) is supported by *one interval*, and for times close to 1, the equilibrium measure is supported by *two intervals*. With a precise scaling, $t = 1/2$ is critical in the sense that for $t < 1/2$, the equilibrium measure for the particles is indeed supported by one, and for $t > 1/2$, by two intervals. The heart-shaped region \mathcal{R} formed by the support of the equilibrium measure as a function of time $0 \leq t \leq 1$ has thus a cusp at $t = 1/2$. The *Pearcey process* $\mathcal{P}(s)$ is now defined as the motion of an infinite number of non-intersecting Brownian paths, just around time $t = 1/2$, with a precise scaling; see [7, 21, 18, 5]. The joint probability that the Pearcey process avoids the windows E_1, \dots, E_m at times s_1, \dots, s_m is defined by

$$\mathbb{P} \left(\begin{array}{c} \mathcal{P}(s_1) \cap E_1 = \emptyset \\ \vdots \\ \mathcal{P}(s_m) \cap E_m = \emptyset \end{array} \right) \\ := \lim_{z \rightarrow 0} \mathbb{P}_0^{\pm 1/z^2} \left(\begin{array}{c} \text{all } x_j \left(\frac{1+s_1 z^2}{2} \right) \notin zE_1 \\ \vdots \\ \text{all } x_j \left(\frac{1+s_m z^2}{2} \right) \notin zE_m \end{array} ; 1 \leq j \leq n \right) \Bigg|_{n=\frac{2}{z^4}},$$

where $\mathbb{P}_0^{\pm a}$ was defined above. The main result of this paper is to show that the infinite-dimensional diffusion equation for the Pearcey process can be replaced by a finite-dimensional non-linear PDE, which is intimately related to the 3-component KP hierarchy and which we now describe.

Given $E_\ell := \bigcup_{i=1}^{r_\ell} [x_{2i-1}^{(\ell)}, x_{2i}^{(\ell)}] \subset \mathbb{R}$, define the space and time gradients

$$\mathcal{X}_{-1} := \sum_{\ell=1}^m \sum_{i=1}^{2r_\ell} \frac{\partial}{\partial x_i^{(\ell)}}, \quad \mathcal{T}_{-1} = \sum_{\ell=1}^m \frac{\partial}{\partial s_\ell},$$

space and time Euler operators $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{T}}_0$ and a mixed space-time operator $\tilde{\mathcal{X}}_{-1}$,

$$\mathcal{X}_0 := \sum_{\ell=1}^m \sum_{i=1}^{2r_\ell} x_i^{(\ell)} \frac{\partial}{\partial x_i^{(\ell)}}, \quad \mathcal{T}_0 = \sum_{\ell=1}^m s_\ell \frac{\partial}{\partial s_\ell}, \quad \tilde{\mathcal{X}}_{-1} = \sum_{\ell=1}^m s_\ell \sum_{i=1}^{2r_\ell} \frac{\partial}{\partial x_i^{(\ell)}}.$$

Theorem 0.2 *Then*

$$\mathbb{Q}(s_1, \dots, s_m; E_1, \dots, E_m) := \log \mathbb{P} \left(\begin{array}{c} \mathcal{P}(s_1) \cap E_1 = \emptyset \\ \vdots \\ \mathcal{P}(s_m) \cap E_m = \emptyset \end{array} \right)$$

satisfies a 4th order and 3rd degree PDE, which can be written as a single Wronskian in the gradient \mathcal{X}_{-1} :

$$\left\{ \begin{array}{l} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}, \frac{1}{8} \{ \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}, \mathcal{X}_{-1}^2 \mathbb{Q} \}_{\mathcal{X}_{-1}} \\ + (\mathcal{X}_0 + 2\mathcal{T}_0 - 2) \mathcal{X}_{-1}^2 \mathbb{Q} - 4(\tilde{\mathcal{X}}_{-1} \mathcal{X}_{-1} - \mathcal{T}_{-1}^2) \mathcal{T}_{-1} \mathbb{Q} \end{array} \right\}_{\mathcal{X}_{-1}} = 0. \quad (0.4)$$

In particular $\mathbb{Q}(s; E) = \log \mathbb{P}(\mathcal{P}(s) \cap E = \emptyset)$ satisfies

$$\left\{ \mathcal{X}_{-1}^2 \frac{\partial \mathbb{Q}}{\partial s}, \frac{1}{8} \left\{ \mathcal{X}_{-1} \frac{\partial \mathbb{Q}}{\partial s}, \mathcal{X}_{-1}^2 \mathbb{Q} \right\}_{\mathcal{X}_{-1}} + (\mathcal{X}_0 - 2) \mathcal{X}_{-1}^2 \mathbb{Q} + 4 \frac{\partial^3 \mathbb{Q}}{\partial s^3} \right\}_{\mathcal{X}_{-1}} = 0.$$

Notice here as well the robustness of the equations. The shape of the equation (0.4) is the same, regardless of the number of times one considers. Moreover the equations are “commutative”: the times and windows can be permuted simultaneously. Notice that the term containing $\tilde{\mathcal{X}}_{-1}$ is the only one which ties up the time s_i with the precise set E_i . We expect that this equation can be used to derive large-time asymptotics, when $t \rightarrow \pm\infty$. Also one expects that the PDE’s for the sine and Airy processes [4] can be obtained from this equation by an appropriate scaling limit. These questions remain challenging open problems.

Proposition 1.1 . *The following holds²:*

$$\begin{aligned}
& \mathbb{P}_n(\alpha; c_1, \dots, c_{m-1}; E_1, \dots, E_m) \\
&= \frac{1}{Z_n} \int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(y^{(1)}) \\
& \quad \Delta_{k_1}(y^{(m)'}) \prod_{i=1}^{k_1} e^{-\frac{1}{2} \sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} + \alpha y_i^{(m)}} \prod_{\ell=1}^m dy_i^{(\ell)} \\
& \quad \Delta_{k_2}(y^{(m)''}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2} \sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} - \alpha y_i^{(m)}} \prod_{\ell=1}^m dy_i^{(\ell)},
\end{aligned}$$

where

$$y^{(m)'} := (y_1^{(m)}, \dots, y_{k_1}^{(m)}) \quad \text{and} \quad y^{(m)''} := (y_{k_1+1}^{(m)}, \dots, y_{k_1+k_2}^{(m)}).$$

The *proof* of this statement is a standard application of the Harish-Chandra-Bessis-Itzykson-Zuber formula, combined with the techniques explained in the next section.

2 Non-intersecting Brownian motions

Consider $n = k_1 + k_2$ non-intersecting Brownian motions on \mathbb{R} (Dyson's Brownian motions), all starting at the origin, such that the k_2 left paths end up at $-a$ and the k_1 right paths end up at $+a$ at time $t = 1$:

$$\begin{aligned}
& \mathbb{P}_0^{\pm a}(\text{all } x_j(t) \in E) \\
& := \mathbb{P} \left(\text{all } x_j(t) \in E \left| \begin{array}{l} \text{all } x_j(0) = 0 \\ k_2 \text{ left paths end up at } -a \text{ at time } t = 1, \\ k_1 \text{ right paths end up at } +a \text{ at time } t = 1 \end{array} \right. \right)
\end{aligned} \tag{2.1}$$

In the Proposition below we shall be using the Karlin-McGregor formula for non-intersecting Brownian motions $x_j(t)$ for $0 < t < 1$:

²Throughout this paper, $\Delta_n(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the Vandermonde.

$$\begin{aligned}
& \mathbb{P} \left(\text{all } x_i(t) \in E, 1 \leq i \leq n \mid \begin{array}{l} \text{given } x_i(0) = \gamma_i \\ \text{given } x_i(1) = \delta_i \end{array} \right) \\
&= \int_{E^n} \frac{1}{Z_n} \det(p(t; \gamma_i, x_j))_{1 \leq i, j \leq n} \det(p(1-t; x_{i'}, \delta_{j'}))_{1 \leq i', j' \leq n} \prod_1^n dx_i
\end{aligned}$$

for

$$p(t, x, y) := \frac{1}{\sqrt{\pi t}} e^{-\frac{(y-x)^2}{t}}. \quad (2.2)$$

Consider now the Brownian motions at different times

$$0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m < t_{m+1} = 1$$

and set

$$\tau_i = t_{i+1} - t_i \quad \text{and} \quad \frac{1}{\sigma_j} = \frac{1}{t_j - t_{j-1}} + \frac{1}{t_{j+1} - t_j}, \quad \text{for } 0 \leq j \leq m.$$

Considering m disjoint unions of intervals $E_\ell := \bigcup_{i=1}^r [\tilde{b}_{2i-1}^{(\ell)}, \tilde{b}_{2i}^{(\ell)}] \subset \mathbb{R}$ for $1 \leq \ell \leq m$, we show that the two probabilities $\mathbb{P}_0^{\pm a}$ and \mathbb{P}_n , as in (1.4) and (2.1), are related by a mere change of variables:

Proposition 2.1 *For $0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m < t_{m+1} = 1$,*

$$\mathbb{P}_0^{\pm a}(\text{all } x_i(t_1) \in \tilde{E}_1, \dots, \text{all } x_i(t_m) \in \tilde{E}_m) = \mathbb{P}_n(\alpha; c_1, \dots, c_{m-1}; E_1, \dots, E_m)$$

upon setting

$$E_\ell = \tilde{E}_\ell \sqrt{\frac{2(t_{\ell+1} - t_{\ell-1})}{(t_{\ell+1} - t_\ell)(t_\ell - t_{\ell-1})}}, \quad c_j = \sqrt{\frac{(t_{j+2} - t_{j+1})(t_j - t_{j-1})}{(t_{j+2} - t_j)(t_{j+1} - t_{j-1})}} \quad (2.3)$$

and

$$\alpha = a \sqrt{\frac{2(t_m - t_{m-1})}{(1 - t_m)(1 - t_{m-1})}}. \quad (2.4)$$

Proof: In the following computation, we shall be using the notation

$$x^{(m)'} = (x_1^{(m)}, \dots, x_{k_1}^{(m)}), \quad x^{(m)''} = (x_{k_1+1}^{(m)}, \dots, x_{k_1+k_2}^{(m)}).$$

Remembering $p(t, x, y)$ is the Brownian transition probability (2.2), one computes:

$$\begin{aligned}
& \mathbb{P}_0^{\pm a}(\text{all } x_i(t_1) \in \tilde{E}_1, \dots, \text{all } x_i(t_m) \in \tilde{E}_m) \\
&= \lim_{\substack{x_i^{(0)} \rightarrow 0 \\ a_1, \dots, a_{k_1} \rightarrow a \\ a_{k_1+1}, \dots, a_n \rightarrow -a}} \frac{1}{Z_n} \int_{\tilde{E}_1^n \times \dots \times \tilde{E}_m^n} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} dx_i^{(j)} \\
& \det \left(p(t_1, x_i^{(0)}, x_j^{(1)}) \right)_{1 \leq i, j \leq n} \det \left(p(t_2 - t_1, x_i^{(1)}, x_j^{(2)}) \right)_{1 \leq i, j \leq n} \\
& \dots \det \left(p(t_m - t_{m-1}, x_i^{(m-1)}, x_j^{(m)}) \right)_{1 \leq i, j \leq n} \det \left(p(1 - t_m, x_i^{(m)}, a_j) \right)_{1 \leq i, j \leq n} \\
&= \lim_{\substack{a_1, \dots, a_{k_1} \rightarrow a \\ a_{k_1+1}, \dots, a_n \rightarrow -a}} \frac{1}{Z'_n} \int_{\tilde{E}_1^n \times \dots \times \tilde{E}_m^n} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} dx_i^{(j)} \Delta_n(x^{(1)}) \prod_{j=1}^m e^{-\sum_{i=1}^n x_i^{(j)2} \left(\frac{1}{t_j - t_{j-1}} + \frac{1}{t_{j+1} - t_j} \right)} \\
& \det \left(e^{\frac{2x_i^{(1)} x_j^{(2)}}{t_2 - t_1}} \right)_{1 \leq i, j \leq n} \dots \det \left(e^{\frac{2x_i^{(m-1)} x_j^{(m)}}{t_m - t_{m-1}}} \right)_{1 \leq i, j \leq n} \det \left(e^{\frac{2x_i^{(m)} a_j}{1 - t_m}} \right)_{1 \leq i, j \leq n} \\
&\stackrel{*}{=} \frac{(n!)^m}{Z''_n} \int_{\tilde{E}_1^n \times \dots \times \tilde{E}_m^n} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} dx_i^{(j)} \Delta_n(x^{(1)}) \\
& \prod_{i=1}^{k_1} (x_i^{(m)})^{i-1} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)} x_i^{(j+1)}}{t_{j+1} - t_j} + \frac{2ax_i^{(m)}}{1-t_m}} \\
& \prod_{i=k_1+1}^{k_1+k_2} (x_i^{(m)})^{i-k_1-1} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)} x_i^{(j+1)}}{t_{j+1} - t_j} - \frac{2ax_i^{(m)}}{1-t_m}} \\
&\stackrel{**}{=} \frac{(n!)^m}{(k_1!)(k_2!)Z''_n} \int_{\tilde{E}_1^n \times \dots \times \tilde{E}_m^n} \Delta_n(x^{(1)}) \\
& \Delta_{k_1}(x^{(m)'}) \prod_{i=1}^{k_1} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)} x_i^{(j+1)}}{t_{j+1} - t_j} + \frac{2ax_i^{(m)}}{1-t_m}} \prod_{j=1}^m dx_i^{(j)} \\
& \Delta_{k_2}(x^{(m)''}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\sum_{j=1}^m \frac{x_i^{(j)2}}{\sigma_j} + \sum_{j=1}^{m-1} \frac{2x_i^{(j)} x_i^{(j+1)}}{t_{j+1} - t_j} - \frac{2ax_i^{(m)}}{1-t_m}} \prod_{j=1}^m dx_i^{(j)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(n!)^m}{Z_n'' k_1! k_2!} \int_{\prod_1^m E_i^n} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} dy_i^{(j)} \Delta_n(y^{(1)}) \\
&\quad \Delta_{k_1}(y^{(m)'}) \prod_{i=1}^{k_1} e^{-\frac{1}{2} \sum_{j=1}^m y_i^{(j)2} + \sum_{j=1}^{m-1} c_j y_i^{(j)} y_i^{(j+1)} + \alpha y_i^{(m)}} \\
&\quad \Delta_{k_2}(y^{(m)''}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2} \sum_{j=1}^m y_i^{(j)2} + \sum_{j=1}^{m-1} c_j y_i^{(j)} y_i^{(j+1)} - \alpha y_i^{(m)}}
\end{aligned}$$

by setting, for $1 \leq i \leq n$, $1 \leq j \leq m-1$,

$$\frac{x_i^{(\ell)}}{\sqrt{\sigma_\ell}} = \frac{y_i^{(\ell)}}{\sqrt{2}}, \quad c_j = \frac{\sqrt{\sigma_j \sigma_{j+1}}}{t_{j+1} - t_j} \text{ and } \alpha = \frac{\sqrt{2\sigma_m}}{1 - t_m} a;$$

the change of variables $x^{(\ell)} \mapsto y^{(\ell)}$ induces a change of variables for the boundary terms of the integrals

$$E_\ell = \tilde{E}_\ell \sqrt{\frac{2}{\sigma_\ell}},$$

thus confirming (2.3) and (2.4).

Identity $\stackrel{*}{=}$ in the previous set of identities is established by means of the following argument, which we explain for the indices $j = 1, 2$:

$$\begin{aligned}
&\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq 2}} dx_i^{(j)} \Delta_n(x^{(1)}) \det \left(e^{\frac{2x_i^{(1)} x_j^{(2)}}{\tau_1}} \right)_{1 \leq i, j \leq n} \det \left(e^{\frac{2x_i^{(2)} x_j^{(3)}}{\tau_2}} \right)_{1 \leq i, j \leq n} \\
&= \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq 2}} dx_i^{(j)} \Delta_n(x^{(1)}) \sum_{\pi} (-1)^\pi \prod_1^n e^{\frac{2x_{\pi(i)}^{(1)} x_i^{(2)}}{\tau_1}} \sum_{\pi'} (-1)^{\pi'} \prod_1^n e^{\frac{2x_{\pi'(j)}^{(2)} x_j^{(3)}}{\tau_2}}
\end{aligned}$$

Upon setting

$$\begin{aligned}
x_{\pi'(j)}^{(2)} &\longmapsto x_j^{(2)}, \\
x_{\pi'(i)}^{(1)} &\longmapsto x_i^{(1)},
\end{aligned}$$

this expression turns into

$$\begin{aligned}
&\mapsto \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq 2}} dx_i^{(j)} \sum_{\pi, \pi'} (-1)^{\pi + \pi'} \Delta_n(x_{(\pi \circ \pi')^{-1}(1)}^{(1)}, \dots, x_{(\pi \circ \pi')^{-1}(n)}^{(1)}) \prod_1^n e^{\frac{2x_i^{(1)} x_i^{(2)}}{\tau_1}} \prod_1^n e^{\frac{2x_j^{(2)} x_j^{(3)}}{\tau_2}} \\
&= (n!)^2 \Delta_n(x^{(1)}) \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq 2}} dx_i^{(j)} \prod_1^n e^{\frac{2x_i^{(1)} x_i^{(2)}}{\tau_1} + \frac{2x_i^{(2)} x_i^{(3)}}{\tau_2}}.
\end{aligned}$$

Then one uses the symmetry of the integration ranges vis-à-vis i . In general, one makes not 2 but m synchronized changes of variables.

Identity $\stackrel{**}{=}$ follows by simultaneously setting

$$\begin{aligned}
x_i^{(\ell)} &\mapsto x_{\pi'(i)}^{(\ell)}, \quad 1 \leq i \leq k_1, \quad 1 \leq \ell \leq m, \quad \pi' \in S_{k_1} \\
x_{k_1+j}^{(\ell)} &\mapsto x_{k_1+\pi''(j)}^{(\ell)}, \quad 1 \leq j \leq k_2, \quad 1 \leq \ell \leq m, \quad \pi'' \in S_{k_2},
\end{aligned}$$

subsequently summing over $\pi' \in S_{k_1}$, $\pi'' \in S_{k_2}$, giving rise to $\Delta_{n_1}(x^{(m)'})$ and $\Delta_{n_2}(x^{(m)''})$, because of the presence of $\Delta_n(x^{(1)})$, and then dividing by $k_1!k_2!$, thus ending the proof of Proposition 2.1. \blacksquare

When taking the limit for $n \rightarrow \infty$, we shall need the following scaling (see [7]), assuming $k = k_1 = k_2$:

$$n = 2k = \frac{2}{z^4}, \quad \pm a = \pm \frac{1}{z^2}, \quad x_i \mapsto x_i z, \quad t_j = \frac{1}{2}(1 + s_j z^2), \quad \text{for } z \rightarrow 0. \tag{2.5}$$

Proposition 2.2 . Given $\tilde{E}_\ell = \bigcup_{i=1}^r [\tilde{b}_{2i-1}^{(\ell)}, \tilde{b}_{2i}^{(\ell)}] \subset \mathbb{R}$, the following holds:

$$\begin{aligned}
&\mathbb{P}_0^{\pm a}(\text{all } x_i(t_1) \in \tilde{E}_1, \dots, \text{all } x_i(t_m) \in \tilde{E}_m) \Bigg| \begin{array}{l} t_j = \frac{1}{2}(1 + s_j z^2) \\ \tilde{b}_j^{(\ell)} = u_j^{(\ell)} z \\ a = \frac{1}{z^2} \\ n = \frac{2}{z^4} \end{array} \\
&= \mathbb{P}_n(\alpha; c_1, \dots, c_{m-1}; b^{(1)}, \dots, b^{(m)}) \Bigg|_{n=\frac{2}{z^4}}, \tag{2.6}
\end{aligned}$$

with

$$\begin{aligned}
c_j &= \frac{\sqrt{\sigma_j \sigma_{j+1}}}{t_{j+1} - t_j} = \sqrt{\frac{(t_j - t_{j-1})(t_{j+2} - t_{j+1})}{(t_{j+1} - t_{j-1})(t_{j+2} - t_j)}} \\
&= \begin{cases} \sqrt{\frac{(1+s_1 z^2)(s_3 - s_2)}{(1+s_2 z^2)(s_3 - s_1)}} & \text{for } j = 1 \\ \sqrt{\frac{(s_j - s_{j-1})(s_{j+2} - s_{j+1})}{(s_{j+1} - s_{j-1})(s_{j+2} - s_j)}} & \text{for } 2 \leq j \leq m-2 \\ \sqrt{\frac{(s_{m-1} - s_{m-2})(1 - s_m z^2)}{(s_m - s_{m-2})(1 - s_{m-1} z^2)}} & \text{for } j = m-1 \end{cases} \\
\alpha &= \frac{a\sqrt{2\sigma_m}}{1 - t_m} = \frac{\sqrt{2}}{z^2} \frac{\sqrt{t_m - t_{m-1}}}{\sqrt{(1 - t_m)(1 - t_{m-1})}} = \frac{2}{z} \frac{\sqrt{s_m - s_{m-1}}}{\sqrt{(1 - s_m z^2)(1 - s_{m-1} z^2)}}
\end{aligned}$$

$$\begin{aligned}
b_i^{(\ell)} &= \tilde{b}_i^{(\ell)} \sqrt{\frac{2}{\sigma_\ell}} = \tilde{b}_i^{(\ell)} \sqrt{\frac{2(t_{\ell+1} - t_{\ell-1})}{(t_\ell - t_{\ell-1})(t_{\ell+1} - t_\ell)}} \\
&= \begin{cases} 2u_i^{(1)} \sqrt{\frac{1+s_2 z^2}{(1+s_1 z^2)(s_2 - s_1)}}, & \text{for } \ell = 1 \\ 2u_i^{(\ell)} \sqrt{\frac{s_{\ell+1} - s_{\ell-1}}{(s_\ell - s_{\ell-1})(s_{\ell+1} - s_\ell)}}, & \text{for } 2 \leq \ell \leq m-1 \\ 2u_i^{(m)} \sqrt{\frac{1 - s_{m-1} z^2}{(s_m - s_{m-1})(1 - s_m z^2)}}, & \text{for } \ell = m \end{cases}
\end{aligned}$$

Proof: Straightforward from Proposition 2.1, combined with the scaling, appearing in (2.6). ■

3 The inverse of a tridiagonal matrix and its derivatives

Consider the $(k+1) \times (k+1)$ tridiagonal matrix, with non-diagonal entries c_1, \dots, c_k :

$$J^{-1}(c_1, \dots, c_k) = \begin{pmatrix} -1 & c_1 & & & \mathbf{0} \\ & & \ddots & & \\ c_1 & -1 & & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & -1 & c_k \\ \mathbf{0} & & & \ddots & \\ & & & c_k & -1 \end{pmatrix}$$

with

$$D(c_1, \dots, c_k) := \begin{cases} \det J^{-1}(c_1, \dots, c_k), & \text{for } k \geq 1 \\ -1 & \text{for } k = 0 \\ 1 & \text{for } k = -1. \end{cases} \quad (3.1)$$

Then one checks, that for $1 \leq j \leq m$,

$$J_{1j}(c_1, \dots, c_{m-1}) = (-1)^{j-1} c_1 \dots c_{j-1} \frac{D(c_{j+1}, \dots, c_{m-1})}{D(c_1, \dots, c_{m-1})}$$

and

$$J_{mj}(c_1, \dots, c_{m-1}) = (-1)^{m-j} c_j \dots c_{m-1} \frac{D(c_1, \dots, c_{j-2})}{D(c_1, \dots, c_{m-1})}. \quad (3.2)$$

Define

$$\begin{aligned} r_j &:= \frac{J_{mj}}{J_{1j}} \quad \text{for } 1 \leq j \leq m \\ &= 0 \quad \text{for } j = 0. \end{aligned} \quad (3.3)$$

For later use, we shall need the following identities:

Lemma 3.1

$$\begin{aligned}
c_1 \frac{\partial}{\partial c_1} \frac{J_{m1} J_{mi}}{J_{1i}} &= -2J_{m1}^2 \quad , \quad c_1 \frac{\partial}{\partial c_1} \log \frac{J_{1m}}{J_{1i}} = \delta_{1i} \quad , \quad c_1 \frac{\partial}{\partial c_1} J_{mm} = -2J_{m1}^2 \\
c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \frac{J_{m1} J_{mi}}{J_{1i}} &= 2 \left(-1 - 2J_{mm} + \frac{J_{m1} J_{mi}}{J_{1i}} \right) \\
c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \frac{J_{m1}}{J_{1i}} &= -1 - \frac{2D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} = -1 - 2 \frac{J_{mm} J_{1i} - J_{m1} J_{mi}}{J_{1i}} \\
c_{m-1} \frac{\partial}{\partial c_{m-1}} \log J_{mi} &= 1 - 2c_{m-1} J_{m,m-1} - \delta_{im} = -2J_{mm} - 1 - \delta_{im}
\end{aligned}$$

Proof: At first notice that

$$\begin{aligned}
c_1 \frac{\partial}{\partial c_1} D(c_1, \dots, c_j) &= -2c_1^2 D(c_3, \dots, c_j) \\
&= 2(D(c_1, \dots, c_j) + D(c_2, \dots, c_j))
\end{aligned}$$

$$\begin{aligned}
c_{m-1} \frac{\partial}{\partial c_{m-1}} D(c_j, \dots, c_{m-1}) &= -2c_{m-1}^2 D(c_j, \dots, c_{m-3}) \\
&= 2(D(c_j, \dots, c_{m-1}) + D(c_j, \dots, c_{m-2}))
\end{aligned} \tag{3.4}$$

and

$$D(c_2, \dots, c_{m-1})D(c_1, \dots, c_{i-2}) - D(c_1, \dots, c_{m-1})D(c_2, \dots, c_{i-2}) = D(c_{i+1}, \dots, c_{m-1}) \prod_1^{i-1} c_k^2.$$

$$D(c_1, \dots, c_{m-2})D(c_{i+1}, \dots, c_{m-1}) - D(c_1, \dots, c_{m-1})D(c_{i+1}, \dots, c_{m-2}) = D(c_1, \dots, c_{i-2}) \prod_i^{m-1} c_k^2. \tag{3.5}$$

Then one checks, by (3.2) and (3.5), that

$$\begin{aligned}
\frac{J_{m1} J_{mi}}{J_{1i}} &= \frac{c_i^2 \dots c_{m-1}^2 D(c_1, \dots, c_{i-2})}{D(c_{i+1}, \dots, c_{m-1}) D(c_1, \dots, c_{m-1})} \\
&= \frac{D(c_1, \dots, c_{m-2})}{D(c_1, \dots, c_{m-1})} - \frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} \\
&= J_{mm} - \frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})}.
\end{aligned} \tag{3.6}$$

Hence

$$\frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} = \frac{J_{mm}J_{1i} - J_{m1}J_{mi}}{J_{1i}},$$

and by (3.2),

$$\frac{J_{m1}}{J_{1i}} = \frac{(-1)^{m-i}c_i \dots c_{m-1}}{D(c_{i+1}, \dots, c_{m-1})}. \quad (3.7)$$

Moreover, explicit differentiation of (3.6) and using the identities (3.5), (3.2), one is led to

$$\begin{aligned} & c_1 \frac{\partial}{\partial c_1} \frac{J_{m1}J_{mi}}{J_{1i}} \\ &= \frac{-2c_i^2 \dots c_{m-1}^2 (D(c_2, \dots, c_{m-1})D(c_1, \dots, c_{i-2}) - D(c_2, \dots, c_{i-2})D(c_1, \dots, c_{m-1}))}{D(c_{i+1}, \dots, c_{m-1})D^2(c_1, \dots, c_{m-1})} \\ &= \frac{-2 \prod_1^{m-1} c_i^2}{D^2(c_1, \dots, c_{m-1})} \\ &= -2J_{m1}^2, \end{aligned}$$

and, setting $i = m$, yields at once the last identity on the first line of the statement of Lemma 3.1. Also by (3.6) and (3.5)

$$\begin{aligned} & \frac{1}{2}c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \frac{J_{m1}J_{mi}}{J_{1i}} \\ &= -1 - 2J_{mm} + \frac{D(c_1, \dots, c_{m-2})D(c_{i+1}, \dots, c_{m-1}) - D(c_1, \dots, c_{m-1})D(c_{i+1}, \dots, c_{m-2})}{D(c_1, \dots, c_{m-1})D(c_{i+1}, \dots, c_{m-1})} \\ &= -1 - 2J_{mm} + \frac{J_{m1}J_{mi}}{J_{1i}} \end{aligned}$$

From (3.7) compute at once

$$c_1 \frac{\partial}{\partial c_1} \log \frac{J_{1m}}{J_{1i}} = \delta_{1i}$$

and from (3.7), (3.4) and (3.6),

$$\begin{aligned} c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \frac{J_{1m}}{J_{1i}} &= 1 - 2 \frac{D(c_{i+1}, \dots, c_{m-1}) + D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} \\ &= -1 - 2 \frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} \\ &= -1 - 2 \frac{J_{mm}J_{1i} - J_{m1}J_{mi}}{J_{1i}}. \end{aligned}$$

Also, using

$$D(c_1, \dots, c_{m-1}) = -D(c_1, \dots, c_{m-2}) - c_{m-1}^2 D(c_1, \dots, c_{m-3})$$

and the explicit formula (3.2) for J_{mi} , one computes, using (3.4),

$$\begin{aligned} c_{m-1} \frac{\partial}{\partial c_{m-1}} \log J_{mi} &= c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \left((-1)^{m-i} c_i \dots c_{m-1} \frac{D(c_1, \dots, c_{i-2})}{D(c_1, \dots, c_{m-1})} \right) \\ &= 1 + 2c_{m-1}^2 \frac{D(c_1, \dots, c_{m-3})}{D(c_1, \dots, c_{m-1})} - \delta_{im} \\ &= 1 - 2c_{m-1} J_{m,m-1} - \delta_{im} \\ &= 2(1 - c_{m-1} J_{m,m-1}) - 1 - \delta_{im} \\ &= 2 \left(\frac{D(c_1, \dots, c_{m-1}) + c_{m-1}^2 D(c_1, \dots, c_{m-3})}{D(c_1, \dots, c_{m-1})} \right) - 1 - \delta_{im} \\ &= -2 \frac{D(c_1, \dots, c_{m-2})}{D(c_1, \dots, c_{m-1})} - 1 - \delta_{im} \\ &= -2J_{mm} - 1 - \delta_{im}, \end{aligned}$$

confirming the last formula in the statement of Lemma 3.1. ■

Proposition 3.2 *For arbitrary $z \in \mathbb{C}$, consider the map of Proposition 2.2, namely*

$$((s_1, \dots, s_m), (u^{(1)}, \dots, u^{(m)})) \longmapsto ((c_1, \dots, c_{m-1}, \alpha), (b^{(1)}, \dots, b^{(m)}))$$

where

$$\begin{aligned} c_j &= \begin{cases} \sqrt{\frac{(1+s_1 z^2)(s_3-s_2)}{(1+s_2 z^2)(s_3-s_1)}} & \text{for } j = 1 \\ \sqrt{\frac{(s_j-s_{j-1})(s_{j+2}-s_{j+1})}{(s_{j+1}-s_{j-1})(s_{j+2}-s_j)}} & \text{for } 2 \leq j \leq m-2 \\ \sqrt{\frac{(s_{m-1}-s_{m-2})(1-s_m z^2)}{(s_m-s_{m-2})(1-s_{m-1} z^2)}} & \text{for } j = m-1 \end{cases} \\ \alpha &= \frac{2}{z} \sqrt{\frac{s_m - s_{m-1}}{(1-s_{m-1} z^2)(1-s_m z^2)}} \\ b^{(i)} &= \frac{u^{(i)}}{U_i(s, z)} \quad 1 \leq i \leq m, \end{aligned} \tag{3.8}$$

with

$$U_i(s; z) := \begin{cases} \frac{1}{2} \sqrt{\frac{(1 + s_1 z^2)(s_2 - s_1)}{1 + s_2 z^2}}, & \text{for } i = 1, \\ \frac{1}{2} \sqrt{\frac{(s_i - s_{i-1})(s_{i+1} - s_i)}{s_{i+1} - s_{i-1}}}, & \text{for } 2 \leq i \leq m - 1, \\ \frac{1}{2} \sqrt{\frac{(1 - s_m z^2)(s_m - s_{m-1})}{1 - s_{m-1} z^2}}, & \text{for } i = m, \end{cases}$$

The inverse map involves the tridiagonal matrix J^{-1} and can be expressed as a fractional linear map in $\frac{J_{m1} J_{mi}}{J_{1i}}$ for $1 \leq i \leq m$

$$\begin{aligned} s_i &= s_i(\alpha, c; z) = \frac{1}{z^2} \frac{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} + 2}{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} - 2} \\ u^{(i)} &= b^{(i)} U_i(s(\alpha, c; z); z) \end{aligned} \quad (3.9)$$

with

$$U_i(s(\alpha, c; z); z) = \frac{-\alpha z J_{m1}}{\alpha^2 z^4 J_{m1} J_{mi} - 2 J_{1i}} = \frac{-1}{J_{mi}} \left(\frac{\alpha z \frac{J_{m1} J_{mi}}{J_{1i}}}{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} - 2} \right)$$

Note that also the entries of J can be expressed as functions of s_i :

$$\begin{aligned} J_{1i} &= \frac{-(1 - z^2 s_i)}{4z^2 U_i(s, z)} \sqrt{\frac{(1 + s_1 z^2)(1 + s_2 z^2)}{s_2 - s_1}} \\ J_{mi} &= \frac{-(1 + z^2 s_i)}{4z^2 U_i(s, z)} \sqrt{\frac{(1 - s_{m-1} z^2)(1 - s_m z^2)}{s_m - s_{m-1}}} \end{aligned} \quad (3.10)$$

with $U_i(s, z)$ as in (3.8).

Proof: Step 1: Upon expanding $D(c_1, \dots, c_{m-1})$ along the j -th row, one

checks by (3.2) and (3.3), that for $1 \leq j \leq m-1$,

$$\begin{aligned}
r_{j+1} - r_j &= \frac{\prod_{i=1}^{m-1} (-c_i)}{c_1^2 \dots c_j^2} \\
&\quad \frac{D(c_1, \dots, c_{j-1})D(c_{j+1}, \dots, c_{m-1}) - c_j^2 D(c_{j+2}, \dots, c_{m-1})D(c_1, \dots, c_{j-2})}{D(c_{j+2}, \dots, c_{m-1})D(c_{j+1}, \dots, c_{m-1})} \\
&= \frac{\prod_{i=1}^{m-1} (-c_i)}{\prod_{i=1}^j c_i^2} \cdot \frac{D(c_1, \dots, c_{m-1})}{D(c_{j+1}, \dots, c_{m-1})D(c_{j+2}, \dots, c_{m-1})}, \tag{3.11}
\end{aligned}$$

and similarly, for $1 \leq j \leq m-2$,

$$\begin{aligned}
r_{j+2} - r_j &= \frac{\prod_{i=1}^{m-1} (-c_j)}{\prod_{i=1}^{j+1} c_i^2} \\
&\quad \frac{D(c_1, \dots, c_j)D(c_{j+1}, \dots, c_{m-1}) - c_j^2 c_{j+1}^2 D(c_{j+3}, \dots, c_{m-1})D(c_1, \dots, c_{j-2})}{D(c_{j+3}, \dots, c_{m-1})D(c_{j+1}, \dots, c_{m-1})} \\
&= \frac{\prod_{i=1}^{m-1} (-c_i)}{\prod_{i=1}^{j+1} c_i^2} \cdot \frac{D(c_1, \dots, c_{m-1})}{D(c_{j+1}, \dots, c_{m-1})D(c_{j+3}, \dots, c_{m-1})}.
\end{aligned}$$

These identities then lead to:

$$\begin{aligned}
\frac{(r_i - r_{i-1})(r_{i+2} - r_{i+1})}{(r_{i+1} - r_{i-1})(r_{i+2} - r_i)} &= c_i^2, \text{ for } 1 \leq i \leq m-2 \\
\frac{(r_i - r_{i-1})(r_{i+1} - r_i)}{(r_{i+1} - r_{i-1})} &= -\frac{J_{m1}}{J_{1i}^2}, \text{ for } 1 \leq i \leq m-1
\end{aligned}$$

and

$$\begin{aligned} \frac{r_{m-1} - r_{m-2}}{r_m - r_{m-2}} &= c_{m-1}^2, \\ r_m - r_{m-1} &= -\frac{1}{J_{1m}}. \end{aligned} \quad (3.12)$$

Step 2: It is easier to show that the inverse map of (3.9) is given by (3.8). So, from inverting the fractional linear map, appearing in (3.9), one computes

$$J_{m1}r_i = \frac{J_{m1}J_{mi}}{J_{1i}} = -\frac{2}{\alpha^2 z^4} \left(\frac{1 + z^2 s_i}{1 - z^2 s_i} \right) \quad (3.13)$$

and so, inverting this map, one computes, in cascade,

$$\begin{aligned} 1 - z^2 s_i &= -\frac{4}{\alpha^2 z^4 r_i J_{m1} - 2}, & 1 + z^2 s_i &= \frac{2\alpha^2 z^4 r_i J_{m1}}{\alpha^2 z^4 r_i J_{m1} - 2} \\ s_i - s_{i-1} &= \frac{4\alpha^2 z^4 (r_{i-1} - r_i) J_{m1}}{(\alpha^2 z^4 r_i J_{m1} - 2)(\alpha^2 z^4 r_{i-1} J_{m1} - 2)} \\ s_{i+1} - s_{i-1} &= \frac{4\alpha^2 z^2 (r_{i-1} - r_{i+1}) J_{m1}}{(\alpha^2 z^4 r_{i-1} J_{m1} - 2)(\alpha^2 z^4 r_{i+1} J_{m1} + 2)}. \end{aligned}$$

Therefore, using (3.12), one checks

$$\begin{aligned} \frac{(s_i - s_{i-1})(s_{i+2} - s_{i+1})}{(s_{i+1} - s_{i-1})(s_{i+2} - s_i)} &= \frac{(r_i - r_{i-1})(r_{i+2} - r_{i+1})}{(r_{i+1} - r_{i-1})(r_{i+2} - r_i)} = c_i^2 \text{ for } 2 \leq i \leq m-2 \\ \frac{(1 + s_1 z^2)(s_3 - s_2)}{(1 + s_2 z^2)(s_3 - s_1)} &= \frac{r_1(r_3 - r_2)}{r_2(r_3 - r_1)} = c_1^2 \\ \frac{(1 - s_m z^2)(s_{m-1} - s_{m-2})}{(1 - s_{m-1} z^2)(1 - s_m z^2)} &= \frac{r_{m-1} - r_{m-2}}{r_m - r_{m-2}} = c_{m-1}^2 \\ \frac{4}{z^2} \frac{s_m - s_{m-1}}{(1 - s_{m-1} z^2)(1 - s_m z^2)} &= -J_{m1}(r_m - r_{m-1})\alpha^2 = \alpha^2 \end{aligned}$$

and similarly for the expressions U_i in (3.8). The signs are all specified from $\sqrt{s_{i+1} - s_i}$ and $\sqrt{s_{i+2} - s_i}$.

Identity (3.10) is obtained by solving

$$s_i = s_i(\alpha, c; z) = \frac{1}{z^2} \frac{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} + 2}{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} - 2}$$

for $\frac{J_{m1}J_{mi}}{J_{1i}}$, by substituting the result in (3.9), i.e.,

$$U_i(s(\alpha, c; z); z) = -\frac{\alpha z \frac{J_{m1}}{J_{1i}}}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} - 2},$$

and then solving for $\frac{J_{m1}}{J_{1i}}$. Finally expressing $J_{mi} = \frac{J_{m1}J_{mi}}{\frac{J_{1i}}{J_{m1}}}$ leads to the second relation (3.10). The first relation (3.10) is obtained from the ratio $J_{1i} = \frac{J_{m1}}{J_{m1}/J_{1i}}$ and using the result previously obtained for J_{mi} at $i = 1$. ■

Lemma 3.3 *The following identities hold*

$$\begin{aligned} \alpha \frac{\partial}{\partial \alpha} s_i(\alpha, c; z) &= -8U_i^2(s, z) \frac{J_{mi}J_{1i}}{J_{m1}} = \frac{1}{z^2} - s_i^2 z^2 \\ \alpha \frac{\partial}{\partial \alpha} \log U_i(s(\alpha, c; z), z) &= -z^2 s_i \\ c_1 \frac{\partial}{\partial c_1} s_i(\alpha, c; z) &= 8U_i^2(s, z) J_{1i}^2 = \frac{(1 - s_i z^2)^2 (1 + s_1 z^2) (1 + s_2 z^2)}{2z^4 (s_2 - s_1)} \\ c_1 \frac{\partial}{\partial c_1} \log U_i(s(\alpha, c; z), z) &= \delta_{i1} - 2\alpha z^3 J_{m1} J_{1i} U_i(s, z) \\ &= \delta_{i1} - \frac{1}{2z^2 (s_2 - s_1)} (1 + s_1 z^2) (1 + s_2 z^2) (1 - s_i z^2) \\ c_{m-1} \frac{\partial}{\partial c_{m-1}} s_i(\alpha, c; z) &= 8U_i^2(s, z) \frac{J_{mi}J_{1i}}{J_{m1}} \left(1 + 2J_{mm} - \frac{J_{mi}J_{m1}}{J_{1i}} \right) \\ &= -\frac{1 + s_i z^2}{2z^4 (s_m - s_{m-1})} \\ &\quad \left((1 - s_m z^2) (1 - s_{m-1} z^2) (1 + s_i z^2) - 2 (1 - s_m s_{m-1} z^4) (1 - s_i z^2) \right) \\ c_{m-1} \frac{\partial}{\partial c_{m-1}} \log U_i(s_i(\alpha, c; z), z) &= -1 - 2 \frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} - 2\alpha z^3 U_i(s, z) J_{mi} \left(1 + 2J_{mm} - \frac{J_{mi}J_{m1}}{J_{1i}} \right) \end{aligned}$$

where

$$\begin{aligned}
& 1 + 2J_{mm} - \frac{J_{mi}J_{m1}}{J_{1i}} \\
&= \frac{1}{2z^2(s_m - s_{m-1})(1 - s_i z^2)} \\
&\quad \times ((1 - s_m z^2)(1 - s_{m-1} z^2)(1 + s_i z^2) - 2(1 - s_m s_{m-1} z^4)(1 - s_i z^2))
\end{aligned}$$

$$\begin{aligned}
\frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} &= \frac{J_{mm}J_{1i} - J_{m1}J_{mi}}{J_{1i}} \\
&= -\frac{(s_m - s_i)(1 - z^2 s_{m-1})}{(s_m - s_{m-1})(1 - z^2 s_i)} \\
J_{mm} &= -\frac{(1 - s_{m-1} z^2)(1 + s_m z^2)}{2z^2(s_m - s_{m-1})} \\
J_{m1} &= -\frac{1}{2z^2} \sqrt{\frac{(1 + s_1 z^2)(1 + s_2 z^2)(1 - s_{m-1} z^2)(1 - s_m z^2)}{(s_2 - s_1)(s_m - s_{m-1})}} \\
\frac{J_{mi}J_{m1}}{J_{1i}} &= -\frac{(1 - s_m z^2)(1 - s_{m-1} z^2)(1 + s_i z^2)}{2z^2(s_m - s_{m-1})(1 - s_i z^2)} \\
J_{mi}U_i \alpha &= \frac{-(1 + s_i z^2)}{2z^3} \\
J_{1i}U_i &= -\frac{1 - s_i z^2}{4z^2} \sqrt{\frac{(1 + s_1 z^2)(1 + s_2 z^2)}{s_2 - s_1}}.
\end{aligned}$$

Proof: Differentiating the first identity (3.9) with regard to a and using the second expression (3.9) yield

$$a \frac{\partial}{\partial a} s_i(a, c; z) = -\frac{8a^2 z^2 \frac{J_{m1}J_{mi}}{J_{1i}}}{\left(a^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} - 2\right)^2} = -8U_i^2(s, z) \frac{J_{mi}J_{1i}}{J_{m1}}.$$

Differentiating U_i as in (3.9), and using the expression (3.9) for s_i , yield:

$$\alpha \frac{\partial}{\partial \alpha} \log U_i(\alpha, c; z) = -\frac{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} + 2}{\alpha^2 z^4 \frac{J_{m1}J_{mi}}{J_{1i}} - 2} = -z^2 s_i,$$

while (3.9) and Lemma 3.1 yield

$$\begin{aligned}
c_1 \frac{\partial}{\partial c_1} s_i(\alpha, c; z) &= \frac{-4\alpha^2 z^2}{\left(\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2\right)^2} c_1 \frac{\partial}{\partial c_1} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= \frac{8\alpha^2 z^2 J_{m1}^2}{\left(\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2\right)^2} \\
&= 8U_i(s, z)^2 J_{1i}^2
\end{aligned}$$

$$\begin{aligned}
c_{m-1} \frac{\partial}{\partial c_{m-1}} s_i(\alpha, c; z) &= \frac{-4\alpha^2 z^2}{\left(\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2\right)^2} c_{m-1} \frac{\partial}{\partial c_{m-1}} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= 8U_i(s, z)^2 \frac{J_{mi} J_{1i}}{J_{m1}} \left(1 + 2J_{mm} - \frac{J_{m1} J_{mi}}{J_{1i}}\right)
\end{aligned}$$

$$\begin{aligned}
c_1 \frac{\partial}{\partial c_1} \log U_i(s; z) &= c_1 \frac{\partial}{\partial c_1} \log \frac{J_{m1}}{J_{1i}} - \frac{\alpha^2 z^4}{\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2} c_1 \frac{\partial}{\partial c_1} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= \delta_{1i} - 2\alpha z^3 J_{m1} J_{1i} U_i(s, z)
\end{aligned}$$

$$\begin{aligned}
&c_{m-1} \frac{\partial}{\partial c_{m-1}} \log U_i(s; z) \\
&= c_{m-1} \frac{\partial}{\partial c_{m-1}} \log \frac{J_{m1}}{J_{1i}} - \frac{\alpha^2 z^4}{\alpha^2 \frac{J_{m1} J_{mi}}{J_{1i}} z^4 - 2} c_{m-1} \frac{\partial}{\partial c_{m-1}} \frac{J_{m1} J_{mi}}{J_{1i}} \\
&= -1 - 2 \frac{D(c_{i+1}, \dots, c_{m-2})}{D(c_{i+1}, \dots, c_{m-1})} \\
&\quad - 2\alpha z^3 U_i(s, z) J_{mi} \left(1 + 2J_{mm} - \frac{J_{mi} J_{m1}}{J_{1i}}\right)
\end{aligned}$$

yielding most of the differential identities of Lemma 3.3. The remaining relations are a consequence of (3.8), (3.9), (3.10) and (3.13). \blacksquare

4 Integrable deformations and the Virasoro constraints

In order to compute the differential equation for

$$\begin{aligned}
& \mathbb{P}_n(\alpha; c_1, \dots, c_{m-1}; E_1 \times \dots \times E_m) \\
&= \frac{1}{Z_n} \int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^{k_1+k_2} \prod_{\ell=1}^m dx_i^{(\ell)} \\
& \quad \Delta_{k_1}(x^{(m)'}) \prod_{i=1}^{k_1} e^{-\frac{1}{2} \sum_{\ell=1}^m x_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell x_i^{(\ell)} x_i^{(\ell+1)} + \alpha x_i^{(m)}} \\
& \quad \Delta_{k_2}(x^{(m)''}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2} \sum_{\ell=1}^m x_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell x_i^{(\ell)} x_i^{(\ell+1)} - \alpha x_i^{(m)}}, \tag{4.1}
\end{aligned}$$

we need to add to the numerator of \mathbb{P}_n many auxiliary variables

$$\left\{ \begin{array}{l} \bar{t} := (\bar{t}_1, \bar{t}_2, \dots), \bar{s} := (\bar{s}_1, \bar{s}_2, \dots), \bar{u} := (\bar{u}_1, \bar{u}_2, \dots) \text{ and } \beta, \\ \gamma^{(\ell)} := (\gamma_1^{(\ell)}, \gamma_2^{(\ell)}, \dots) \text{ for } 2 \leq \ell \leq m-1 \\ c^{(\ell)} := (c_{p,q}^{(\ell)})_{p,q \geq 1} \text{ for } 1 \leq \ell \leq m-1 \end{array} \right. .$$

Note that the time variables \bar{t} , \bar{s} , \bar{u} are totally different from the t -variables appearing in the Brownian motion. yielding the following integral³

$$\begin{aligned}
& \tau_{k_1 k_2}(\bar{t}, \bar{s}, \bar{u}; \beta, \gamma^{(2)}, \dots, \gamma^{(m-1)}, c^{(1)}, \dots, c^{(m-1)}, \alpha, E_1 \times \dots \times E_m) \\
&= \frac{1}{k_1! k_2!} \int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^{k_1+k_2} \left(e^{\sum_{j=1}^{\infty} \bar{t}_j x_i^{(1)j}} \prod_{\ell=1}^m dx_i^{(\ell)} \right) \\
& \quad \Delta_{k_1}(x_1^{(m)}, \dots, x_{k_1}^{(m)}) \\
& \quad \prod_{i=1}^{k_1} e^{-\frac{1}{2} \sum_{\ell=1}^m x_i^{(\ell)2} + \alpha x_i^{(m)} + \beta x_i^{(m)2} - \sum_{j=1}^{\infty} \bar{s}_j (x_i^{(m)})^j + \sum_{p,q \geq 1} \sum_{\ell=1}^{m-1} c_{p,q}^{(\ell)} (x_i^{(\ell)})^p (x_i^{(\ell+1)})^q + \sum_{\ell=2}^{m-1} \sum_{r=1}^{\infty} \gamma_r^{(\ell)} (x_i^{(\ell)})^r} \\
& \quad \Delta_{k_2}(x_{k_1+1}^{(m)}, \dots, x_{k_1+k_2}^{(m)}) \\
& \quad \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2} \sum_{\ell=1}^m x_i^{(\ell)2} - \alpha x_i^{(m)} - \beta x_i^{(m)2} - \sum_{j=1}^{\infty} \bar{u}_j (x_i^{(m)})^j + \sum_{p,q \geq 1} \sum_{\ell=1}^{m-1} c_{p,q}^{(\ell)} (x_i^{(\ell)})^p (x_i^{(\ell+1)})^q + \sum_{\ell=2}^{m-1} \sum_{r=1}^{\infty} \gamma_r^{(\ell)} (x_i^{(\ell)})^r} \\
&= \det \left(\begin{array}{c} (\mu_{ij}^+)_{1 \leq i \leq k_1, 1 \leq j \leq k_1+k_2} \\ (\mu_{ij}^-)_{1 \leq i \leq k_2, 1 \leq j \leq k_1+k_2} \end{array} \right) \tag{4.2}
\end{aligned}$$

where

$$\begin{aligned}
\mu_{ij}^{\pm} &= \int_{\prod_1^m E_i} \left(\prod_{\ell=1}^m dx^{(\ell)} \right) e^{\sum_1^{\infty} (\bar{t}_k x^{(1)k} - (\bar{s}_k / \bar{u}_k) x^{(m)k})} (x^{(1)})^{j-1} (x^{(m)})^{i-1} \\
& \quad e^{-\frac{1}{2} \sum_{\ell=1}^m x^{(\ell)2} \pm \alpha x^{(m)} \pm \beta x^{(m)2} + \sum_{p,q \geq 1} \sum_{\ell=1}^{m-1} c_{p,q}^{(\ell)} (x^{(\ell)})^p (x^{(\ell+1)})^q + \sum_{\ell=2}^{m-1} \sum_{r=1}^{\infty} \gamma_r^{(\ell)} (x^{(\ell)})^r}
\end{aligned} \tag{4.3}$$

This is to say, the integral (4.2), along the locus

$$\mathcal{L} := \left\{ \begin{array}{l} \bar{t}_i = 0, \bar{s}_i = 0, \bar{u}_i = 0, \beta = 0, \gamma_r^{(\ell)} = 0, \\ c_{11}^{(\ell)} = c_{\ell} \text{ and } c_{ij}^{(\ell)} = 0 \text{ for } i, j \geq 1 \text{ with } (i, j) \neq (1, 1) \end{array} \right\} \tag{4.4}$$

³If $m = 1$ or 2 , the formulae below must be reinterpreted; e.g., for $m = 1$, the c_i 's are absent and for $m = 2$, γ is not present.

yields the integral (4.1); also for the sake of brevity, set $\gamma_\ell := \gamma_1^{(\ell)}$. The following locus

$$\mathcal{L}_\beta := \left\{ \begin{array}{l} \bar{t}_i = 0, \bar{s}_i = 0, \bar{u}_i = 0, \gamma_r^{(\ell)} = 0, \\ c_{11}^{(\ell)} = c_\ell \text{ and } c_{ij}^{(\ell)} = 0 \text{ for } i, j \geq 1 \text{ with } (i, j) \neq (1, 1) \end{array} \right\} \quad (4.5)$$

will also be used. Then the following statement holds:

Proposition 4.1 *Given a disjoint union of intervals and the associated algebra of differential operators*

$$E_\ell := \bigcup_{i=1}^r [b_{2i-1}^{(\ell)}, b_{2i}^{(\ell)}] \subset \mathbb{R} \quad \text{and} \quad \mathcal{D}_k(E_\ell) = \sum_{i=1}^{2r} (b_i^{(\ell)})^{k+1} \frac{\partial}{\partial b_i^{(\ell)}}, \quad 1 \leq \ell \leq m$$

the integrals (4.2) satisfy, besides the trivial relations,

$$-\frac{\partial}{\partial \bar{s}_1} + \frac{\partial}{\partial \bar{u}_1} = \frac{\partial}{\partial \alpha}, \quad -\frac{\partial}{\partial \bar{s}_2} + \frac{\partial}{\partial \bar{u}_2} = \frac{\partial}{\partial \beta}, \quad (4.6)$$

and, upon setting

$$\gamma_1 := \bar{t}_1, \quad \text{and} \quad \frac{\partial}{\partial \gamma_m} := -\frac{\partial}{\partial \bar{s}_1} - \frac{\partial}{\partial \bar{u}_1},$$

the following Virasoro relations ($2 \leq \ell \leq m-1$):

$$\mathcal{D}_{-1}(E_1)_{\tau_{k_1 k_2}} = \left\{ \begin{array}{l} \overbrace{-\frac{\partial}{\partial \gamma_1}} \\ -\frac{\partial}{\partial \bar{t}_1} + c_1 \frac{\partial}{\partial \gamma_2} + (k_1 + k_2) \bar{t}_1 \\ + \sum_{i \geq 2} i \bar{t}_i \frac{\partial}{\partial \bar{t}_{i-1}} + \sum_{\substack{i \geq 2 \\ j \geq 1}} i c_{ij}^{(1)} \frac{\partial}{\partial c_{i-1, j}^{(1)}} + \sum_{j \geq 2} c_{1j}^{(1)} \frac{\partial}{\partial \gamma_j^{(2)}} \end{array} \right\}_{\tau_{k_1 k_2}}$$

⋮

$$\begin{aligned}
\mathcal{D}_{-1}(E_\ell)\tau_{k_1 k_2} &= \left\{ \begin{aligned} &c_{\ell-1} \frac{\partial}{\partial \gamma_{\ell-1}} - \frac{\partial}{\partial \gamma_\ell} + c_\ell \frac{\partial}{\partial \gamma_{\ell+1}} + (k_1 + k_2) \gamma_\ell \\ &+ \sum_{\substack{i \geq 1 \\ j \geq 2}} j c_{ij}^{(\ell-1)} \frac{\partial}{\partial c_{i,j-1}^{(\ell-1)}} + \sum_{\substack{i \geq 2 \\ j \geq 1}} i c_{ij}^{(\ell)} \frac{\partial}{\partial c_{i-1,j}^{(\ell)}} \\ &+ \sum_{i \geq 2} c_{i1}^{(\ell-1)} \frac{\partial}{\partial \gamma_i^{(\ell-1)}} + \sum_{j \geq 2} c_{1j}^{(\ell)} \frac{\partial}{\partial \gamma_j^{(\ell+1)}} \\ &+ \sum_{r \geq 2} r \gamma_r^{(\ell)} \frac{\partial}{\partial \gamma_{r-1}^{(\ell)}} \end{aligned} \right\} \tau_{k_1 k_2} \\
&\vdots \\
\mathcal{D}_{-1}(E_m)\tau_{k_1 k_2} &= \left\{ \begin{aligned} &c_{m-1} \frac{\partial}{\partial \gamma_{m-1}} + \overbrace{\left(\frac{\partial}{\partial \bar{s}_1} + \frac{\partial}{\partial \bar{u}_1} \right)}^{-\frac{\partial}{\partial \gamma_m}} + 2\beta \overbrace{\left(-\frac{\partial}{\partial \bar{s}_1} + \frac{\partial}{\partial \bar{u}_1} \right)}^{\frac{\partial}{\partial \alpha}} \\ &-k_1(\bar{s}_1 - \alpha) - k_2(\bar{u}_1 + \alpha) \\ &+ \sum_{i \geq 2} i \left(\bar{s}_i \frac{\partial}{\partial \bar{s}_{i-1}} + \bar{u}_i \frac{\partial}{\partial \bar{u}_{i-1}} \right) \\ &+ \sum_{i \geq 2} c_{i1}^{(m-1)} \frac{\partial}{\partial \gamma_i^{(m-1)}} + \sum_{\substack{i \geq 1 \\ j \geq 2}} j c_{ij}^{(m-1)} \frac{\partial}{\partial c_{i,j-1}^{(m-1)}} \end{aligned} \right\} \tau_{k_1 k_2} \tag{4.7}
\end{aligned}$$

and (only needed for $\ell = 1, m$)

$$\mathcal{D}_0(E_1)\tau_{k_1 k_2} = \left(\begin{aligned} &-\frac{\partial}{\partial \bar{t}_2} + c_1 \frac{\partial}{\partial c_1} + \frac{(k_1 + k_2)(k_1 + k_2 + 1)}{2} \\ &\sum_{i \geq 1} i \bar{t}_i \frac{\partial}{\partial \bar{t}_i} + \sum_{\substack{i, j \geq 1 \\ (i, j) \neq (1, 1)}} i c_{ij}^{(1)} \frac{\partial}{\partial c_{i,j}^{(1)}} \end{aligned} \right) \tau_{k_1 k_2}$$

$$\mathcal{D}_0(E_m)\tau_{k_1 k_2} = \left(\begin{array}{c} \overbrace{\left(-\frac{\partial}{\partial \bar{s}_1} + \frac{\partial}{\partial \bar{u}_1}\right)}^{\frac{\partial}{\partial \alpha}} + c_{m-1} \frac{\partial}{\partial c_{m-1}} + \left(\frac{\partial}{\partial \bar{s}_2} + \frac{\partial}{\partial \bar{u}_2}\right) \\ + 2\beta \overbrace{\left(-\frac{\partial}{\partial \bar{s}_2} + \frac{\partial}{\partial \bar{u}_2}\right)}^{\frac{\partial}{\partial \beta}} + \frac{k_1(k_1+1)}{2} + \frac{k_2(k_2+1)}{2} \\ \sum_{i \geq 1} \left(i \bar{s}_i \frac{\partial}{\partial \bar{s}_i} + i \bar{u}_i \frac{\partial}{\partial \bar{u}_i} \right) + \sum_{\substack{i, j \geq 1 \\ (i, j) \neq (1, 1)}} j c_{ij}^{(m-1)} \frac{\partial}{\partial c_{i, j}^{(m-1)}} \end{array} \right) \tau_{k_1 k_2} \quad (4.8)$$

Before giving the proof of Proposition 4.1, we need the following lemma, concerning the expressions:

$$dI_n := \Delta_n(x) \prod_{k=1}^n dx_k e^{\sum_{i=1}^{\infty} \bar{t}_i x_k^i}$$

$$\mathcal{E}(x, y) := \prod_{k=1}^n e^{\sum_{i=1}^{\infty} \bar{t}_i x_k^i + \sum_{i, j \geq 1} c_{ij} x_k^i y_k^j + \sum_{i=1}^{\infty} \bar{s}_i y_k^i}$$

Lemma 4.2

$$\frac{\partial}{\partial \varepsilon} \prod_{i=1}^n d(x_i + \varepsilon x_i^{k+1}) \Big|_{\varepsilon=0} = \begin{cases} 0, & k = -1 \\ n \prod_{i=1}^n dx_i, & k = 0 \end{cases}$$

$$\frac{\partial}{\partial \varepsilon} dI_n(x + \varepsilon x^{k+1}) \Big|_{\varepsilon=0} = \begin{cases} \left(\sum_{i \geq 2} i \bar{t}_i \frac{\partial}{\partial \bar{t}_{i-1}} + n \bar{t}_1 \right) dI_n(x), & k = -1 \\ \left(\sum_{i \geq 1} i \bar{t}_i \frac{\partial}{\partial \bar{t}_i} + \frac{n(n+1)}{2} \right) dI_n(x), & k = 0 \end{cases}$$

$$\begin{aligned}
& \left. \frac{\partial}{\partial \varepsilon} \mathcal{E}(x + \varepsilon x^{k+1}, y) \right|_{\varepsilon=0} \\
&= \begin{cases} \left(\sum_{i \geq 2} i \bar{t}_i \frac{\partial}{\partial \bar{t}_{i-1}} + n \bar{t}_1 + \sum_{i \geq 2, j \geq 1} i c_{ij} \frac{\partial}{\partial c_{i-1, j}} + \sum_{j \geq 1} c_{1j} \frac{\partial}{\partial \bar{s}_j} \right) \mathcal{E}(x, y), & k = -1 \\ \left(\sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} + \sum_{i, j \geq 1} i c_{ij} \frac{\partial}{\partial c_{ij}} \right) \mathcal{E}(x, y), & k = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \left. \frac{\partial}{\partial \varepsilon} \mathcal{E}(x, y + \varepsilon y^{k+1}) \right|_{\varepsilon=0} \\
&= \begin{cases} \left(\sum_{i \geq 2} i \bar{s}_i \frac{\partial}{\partial \bar{s}_{i-1}} + n \bar{s}_1 + \sum_{i \geq 1, j \geq 2} j c_{ij} \frac{\partial}{\partial c_{i, j-1}} + \sum_{i \geq 1} c_{i1} \frac{\partial}{\partial \bar{t}_i} \right) \mathcal{E}(x, y), & k = -1 \\ \left(\sum_{i \geq 1} i \bar{s}_i \frac{\partial}{\partial \bar{s}_i} + \sum_{i, j \geq 1} j c_{ij} \frac{\partial}{\partial c_{ij}} \right) \mathcal{E}(x, y), & k = 0 \end{cases}
\end{aligned}$$

Proof: This is obtained by setting $x_i \mapsto x_i + \varepsilon x_i^{k+1}$ and then $y_i \mapsto y_i + \varepsilon y_i^{k+1}$; then take the derivative with regard to ε and set $\varepsilon = 0$.

Proof of Proposition 4.1:

Case 1: Performing the infinitesimal change of variables, for all $1 \leq i \leq n$,

$$x_i^{(1)} \mapsto x_i^{(1)} + \varepsilon x_i^{(1)k+1}$$

in the integral (4.2) involves the following integral only

$$\prod_{i=1}^{k_1+k_2} dx_i^{(1)} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^{k_1+k_2} e^{\sum_{j=1}^{\infty} \bar{t}_j (x_i^{(1)})^j} \prod_{i=1}^{k_1} e^{-\frac{1}{2} x_i^{(1)2} + \sum_{p, q \geq 1} c_{p, q}^{(1)} (x_i^{(1)})^p (x_i^{(2)})^q + \sum_{r=1}^{\infty} \gamma_r^{(2)} (x_i^{(2)})^r}$$

and leads to the first Virasoro constraints in (4.7), upon differentiation with regard to ε , setting $\varepsilon = 0$, applying Lemma 4.2 and taking into account the variation of the boundary term in the integral, $b_j^{(1)} \mapsto b_j^{(1)} - \varepsilon (b_j^{(1)})^{k+1} + O(\varepsilon^2)$. The shifts in the time parameter produces the terms in the first line of the Virasoro constraint.

Case 2: Performing the infinitesimal change of variables, for all $1 \leq i \leq n$,

$$x_i^{(\ell)} \longmapsto x_i^{(\ell)} + \varepsilon x_i^{(\ell)k+1}$$

in the integral (4.2) involves the following integral only

$$\prod_{i=1}^{k_1+k_2} dx_i^{(\ell)} \prod_{i=1}^{k_1} e^{\sum_{r=1}^{\infty} \gamma_r^{(\ell)} (x_i^{(\ell)})^r - \frac{1}{2} x_i^{(\ell)2} + \sum_{r=1}^{\infty} \gamma_r^{(\ell-1)} (x_i^{(\ell-1)})^r + \sum_{p,q \geq 1} c_{p,q}^{(\ell-1)} (x_i^{(\ell-1)})^p (x_i^{(\ell-1)})^q}$$

$$e^{\sum_{r=1}^{\infty} \gamma_r^{(\ell+1)} (x_i^{(\ell+1)})^r + \sum_{p,q \geq 1} c_{p,q}^{(\ell)} (x_i^{(\ell)})^p (x_i^{(\ell+1)})^q}$$

and leads to the Virasoro constraints for $2 \leq \ell \leq m-1$ in (4.7), upon differentiation with regard to ε , setting $\varepsilon = 0$, applying Lemma 4.2 and taking into account the variation of the boundary term in the integral, $b_j^{(\ell)} \mapsto b_j^{(\ell)} - \varepsilon (b_j^{(\ell)})^{k+1} + O(\varepsilon^2)$. Also here one must take into account the shifts.

Case 3: Performing the infinitesimal change of variables, for all $1 \leq i \leq n$,

$$x_i^{(m)} \longmapsto x_i^{(m)} + \varepsilon x_i^{(m)k+1}$$

in the integral (4.2) involves the following integral only

$$\prod_{i=1}^{k_1+k_2} dx_i^{(m)} \Delta_{k_1}(x_1^{(m)}, \dots, x_{k_1}^{(m)}) \Delta_{k_2}(x_{k_1+1}^{(m)}, \dots, x_{k_1+k_2}^{(m)})$$

$$\prod_{i=1}^{k_1} e^{-\sum_{j=1}^{\infty} [\bar{s}_j - \delta_{j1} \alpha + \delta_{j2} (\frac{1}{2} - \beta)] (x_i^{(m)})^j + \sum_{r=1}^{\infty} \gamma_r^{(m-1)} (x_i^{(m-1)})^r + \sum_{p,q \geq 1} c_{p,q}^{(m-1)} (x_i^{(m-1)})^p (x_i^{(m-1)})^q}$$

$$\prod_{i=k_1+1}^{k_1+k_2} e^{-\sum_{j=1}^{\infty} [\bar{u}_j + \delta_{j1} \alpha + \delta_{j2} (\frac{1}{2} + \beta)] (x_i^{(m)})^j + \sum_{r=1}^{\infty} \gamma_r^{(m-1)} (x_i^{(m-1)})^r + \sum_{p,q \geq 1} c_{p,q}^{(m-1)} (x_i^{(m-1)})^p (x_i^{(m-1)})^q},$$

and leads to the last Virasoro constraint for $\ell = m$ with a similar argument; this ends the proof of Proposition 4.1. The relations (4.6) follow at once by inspection of the integral (4.2). \blacksquare

Setting

$$J^{-1} := J^{-1}(c_1, \dots, c_{m-1}) := \begin{pmatrix} -1 & c_1 & & & & \\ & & \ddots & & \mathbf{0} & \\ c_1 & -1 & & \ddots & & \\ & \ddots & & \ddots & & \ddots \\ & & \ddots & & -1 & c_{m-1} \\ \mathbf{0} & & & \ddots & & \\ & & & & c_{m-1} & -1 \end{pmatrix}, \quad (4.9)$$

define the differential operators $\mathcal{A}_i^\pm, \mathcal{C}_i$ and $\bar{\mathcal{A}}_i, \bar{\mathcal{C}}_i$:

$$\mathcal{A}_1^\pm := -\frac{1}{2} \sum_{j=1}^m J_{mj} \left(\mathcal{D}_{-1}(b^{(j)}) - 2\delta_{jm}\beta \frac{\partial}{\partial \alpha} \right) \mp \frac{1}{2} \frac{\partial}{\partial \alpha} =: \bar{\mathcal{A}}_1^\pm + \beta J_{mm} \frac{\partial}{\partial \alpha}$$

$$\mathcal{C}_1 := \sum_{j=1}^m J_{1j} \left(\mathcal{D}_{-1}(b^{(j)}) - 2\delta_{jm}\beta \frac{\partial}{\partial \alpha} \right) =: \bar{\mathcal{C}}_1 - 2\beta J_{1m} \frac{\partial}{\partial \alpha}$$

$$\mathcal{A}_2^\pm := \frac{1}{2} \left(\mathcal{D}_0(b^{(m)}) - \alpha \frac{\partial}{\partial \alpha} - c_{m-1} \frac{\partial}{\partial c_{m-1}} \right) - (\beta \pm \frac{1}{2}) \frac{\partial}{\partial \beta} =: \bar{\mathcal{A}}_2 \mp \frac{1}{2} \frac{\partial}{\partial \beta} - \beta \frac{\partial}{\partial \beta}$$

$$\mathcal{C}_2 := -\mathcal{D}_0(b^{(1)}) + c_1 \frac{\partial}{\partial c_1} =: \bar{\mathcal{C}}_2$$

We now state:

Proposition 4.3 *Along the locus \mathcal{L} , the partials and second partials of $f := \log \tau_{k_1 k_2}$ with regard to $\bar{t}_i, \bar{s}_i, \bar{u}_i$ can be expressed in terms of the operators $\bar{\mathcal{A}}_i, \bar{\mathcal{C}}_i$ and $\partial/\partial \beta$:*

$$\begin{aligned}
\bar{\mathcal{A}}_1^+ f &= \frac{\partial f}{\partial \bar{s}_1} + \frac{\alpha(k_2 - k_1)}{2} J_{mm} \\
\bar{\mathcal{A}}_1^- f &= \frac{\partial f}{\partial \bar{u}_1} + \frac{\alpha(k_2 - k_1)}{2} J_{mm} \\
\bar{\mathcal{C}}_1 f &= \frac{\partial f}{\partial \bar{t}_1} - \alpha(k_2 - k_1) J_{m1} \\
\left(\bar{\mathcal{A}}_2 - \frac{1}{2} \frac{\partial}{\partial \beta} \right) f &= \frac{\partial f}{\partial \bar{s}_2} + \frac{1}{4} (k_2^2 + k_2 + k_1^2 + k_1) \\
\left(\bar{\mathcal{A}}_2 + \frac{1}{2} \frac{\partial}{\partial \beta} \right) f &= \frac{\partial f}{\partial \bar{u}_2} + \frac{1}{4} (k_2^2 + k_2 + k_1^2 + k_1) \\
\bar{\mathcal{C}}_2 f &= \frac{\partial f}{\partial \bar{t}_2} - \frac{1}{2} (k_2 + k_1) (k_2 + k_1 + 1) \\
\bar{\mathcal{A}}_1^+ \bar{\mathcal{C}}_1 f &= \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_1} - k_1 J_{1m} \\
\left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m} \frac{\partial}{\partial \alpha} - \frac{1}{2} \bar{\mathcal{C}}_1 \frac{\partial}{\partial \beta} \right) f &= \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_2} - \alpha(k_2 - k_1) J_{1m} J_{mm} \\
\bar{\mathcal{C}}_2 \bar{\mathcal{A}}_1^+ f &= \frac{\partial^2 f}{\partial \bar{t}_2 \partial \bar{s}_1} - \alpha(k_2 - k_1) J_{1m}^2 \quad (4.10)
\end{aligned}$$

Proof: The Virasoro relations (4.7) can be written as (remembering $t_1 = \gamma_1$)

$$\begin{pmatrix} \mathcal{D}_{-1}(b^{(1)}) \\ \vdots \\ \mathcal{D}_{-1}(b^{(m-1)}) \\ \mathcal{D}_{-1}(b^{(m)}) - 2\beta \frac{\partial}{\partial \alpha} - (k_1 - k_2)\alpha \end{pmatrix} f = J^{-1} \begin{pmatrix} \frac{\partial}{\partial \gamma_1} \\ \vdots \\ \frac{\partial}{\partial \gamma_m} \end{pmatrix} f + O(\mathcal{L}_\beta),$$

remembering the locus \mathcal{L}_β as in (4.5), and the tridiagonal matrix J^{-1} defined in (4.9). The symbol $O(\mathcal{L}_\beta)$ means a term which vanishes along \mathcal{L}_β .

Therefore, using the trivial relations (4.6), one finds

$$J \begin{pmatrix} \mathcal{D}_{-1}(b^{(1)}) \\ \mathcal{D}_{-1}(b^{(2)}) \\ \vdots \\ \mathcal{D}_{-1}(b^{(m-1)}) \\ \mathcal{D}_{-1}(b^{(m)}) - 2\beta \frac{\partial}{\partial \alpha} - (k_1 - k_2)\alpha \end{pmatrix} f = \begin{pmatrix} \frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial \gamma_2} \\ \vdots \\ \frac{\partial}{\partial \gamma_{m-1}} \\ -\frac{\partial}{\partial \bar{s}_1} - \frac{\partial}{\partial \bar{u}_1} \end{pmatrix} f + O(\mathcal{L}_\beta)$$

which combined with

$$-\frac{\partial}{\partial \bar{s}_1} + \frac{\partial}{\partial \bar{u}_1} = \frac{\partial}{\partial \alpha},$$

leads to the following equations for $f = \log \tau_{k_1 k_2}$ along the locus \mathcal{L}_β :

$$\mathcal{A}_1^+ f = \frac{\partial f}{\partial \bar{s}_1} + \frac{\alpha}{2}(k_2 - k_1)J_{mm}$$

$$\mathcal{A}_1^- f = \frac{\partial f}{\partial \bar{u}_1} + \frac{\alpha}{2}(k_2 - k_1)J_{mm}$$

$$\mathcal{C}_1 f = \frac{\partial f}{\partial \bar{t}_1} - \alpha(k_2 - k_1)J_{m1}$$

The second set of Virasoro relations, combined with

$$-\frac{\partial}{\partial \bar{s}_2} + \frac{\partial}{\partial \bar{u}_2} = \frac{\partial}{\partial \beta},$$

leads to the following equations, also along the locus \mathcal{L}_β ,

$$\mathcal{A}_2^+ f = \frac{\partial f}{\partial \bar{s}_2} + \frac{1}{4}(k_2^2 + k_2 + k_1^2 + k_1)$$

$$\mathcal{A}_2^- f = \frac{\partial f}{\partial \bar{u}_2} + \frac{1}{4}(k_2^2 + k_2 + k_1^2 + k_1)$$

$$\mathcal{C}_2 f = \frac{\partial f}{\partial \bar{t}_2} - \frac{1}{2}(k_2 + k_1)(k_2 + k_1 + 1)$$

Moreover

$$\mathcal{A}_1^+ \mathcal{C}_1 f = \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_1} - k_1 J_{1m} - \beta J_{mm} J_{m1} (k_2 - k_1)$$

$$\mathcal{A}_2^+ \mathcal{C}_1 f = \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_2} - \alpha (k_2 - k_1) J_{1m} J_{mm}$$

$$\mathcal{C}_2 \mathcal{A}_1^+ f = \frac{\partial^2 f}{\partial \bar{t}_2 \partial \bar{s}_1} - \alpha (k_2 - k_1) J_{1m}^2.$$

Indeed, one first needs to check

$$\mathcal{A}_1(\alpha) = J_{mm} \beta - \frac{1}{2},$$

$$\begin{aligned} \mathcal{A}_2^+(\alpha J_{m1}) &= -\frac{1}{2} \left(\alpha \frac{\partial}{\partial \alpha} + c_{m-1} \frac{\partial}{\partial c_{m-1}} \right) \alpha J_{m1} \\ &= -\frac{1}{2} (\alpha J_{m1} + \alpha J_{m1} (-2J_{mm} - 1)) \\ &= \alpha J_{m1} J_{mm} \end{aligned}$$

and

$$\mathcal{C}_2(J_{mm}) = c_1 \frac{\partial}{\partial c_1} J_{mm} = -2J_{m1}^2.$$

Then, using the expressions for the $\mathcal{A}_i^\pm f$ and $\mathcal{C}_i f$, one computes along the locus⁴ \mathcal{L}_β ,

$$\begin{aligned} \mathcal{A}_1^+ \mathcal{C}_1 f &= \mathcal{A}_1^+ \left(\frac{\partial f}{\partial \bar{t}_1} - \alpha (k_2 - k_1) J_{m1} \right) \\ &= \frac{\partial}{\partial \bar{t}_1} (\mathcal{A}_1^+ f) - (k_2 - k_1) J_{m1} (\mathcal{A}_1^+ \alpha) \\ &= \frac{\partial}{\partial \bar{t}_1} \left(\frac{\partial f}{\partial \bar{s}_1} + \frac{\alpha}{2} (k_2 - k_1) J_{mm} - \frac{\bar{t}_1}{2} J_{m1} (k_1 + k_2) \right) - (k_2 - k_1) J_{m1} (J_{mm} \beta - \frac{1}{2}) \\ &= \frac{\partial^2 f}{\partial \bar{t}_1 \partial \bar{s}_1} - J_{m1} k_1 - \beta J_{mm} J_{m1} (k_2 - k_1) \end{aligned}$$

⁴Since one takes the derivative with regard to \bar{t}_1 in the expression below, one must keep track of the term containing \bar{t}_1 in the identity for $\mathcal{A}_1^+ f$ before setting $\bar{t}_1 = 0$.

and

$$\begin{aligned}
\mathcal{A}_2\mathcal{C}_1f &= \mathcal{A}_2\left(\frac{\partial f}{\partial t_1} - \alpha(k_2 - k_1)J_{m1}\right) \\
&= \frac{\partial}{\partial t_1}\mathcal{A}_2f - \mathcal{A}_2(\alpha(k_2 - k_1)J_{m1}) \\
&= \frac{\partial^2}{\partial t_1\partial \bar{s}_2}f - (k_2 - k_1)\alpha J_{m1}J_{mm}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_2\mathcal{A}_1^+f &= \frac{\partial}{\partial \bar{s}_1}\mathcal{C}_2f + \mathcal{C}_2\left(\frac{\alpha}{2}(k_2 - k_1)J_{mm}\right) \\
&= \frac{\partial^2 f}{\partial \bar{s}_1\partial t_2} - \alpha(k_2 - k_1)J_{m1}^2.
\end{aligned}$$

Finally, set $\beta = 0$ in each of these expressions. That means that we can replace all $\mathcal{A}_i, \mathcal{C}_i$ by $\bar{\mathcal{A}}_i, \bar{\mathcal{C}}_i$, except for $\bar{\mathcal{A}}_2$ and $\bar{\mathcal{B}}_2$, which gives an extra $\partial/\partial\beta$ and the composition

$$\begin{aligned}
\mathcal{A}_2^+\mathcal{C}_1f|_{\beta=0} &= \left(\bar{\mathcal{A}}_2 - \frac{1}{2}\frac{\partial}{\partial\beta}\right)\left(\bar{\mathcal{C}}_1 - 2J_{1m}\beta\frac{\partial}{\partial\alpha}\right)f|_{\beta=0} \\
&= \left(\bar{\mathcal{A}}_2\bar{\mathcal{C}}_1 + J_{1m}\frac{\partial}{\partial\alpha}\right)f - \frac{1}{2}\bar{\mathcal{C}}_1\frac{\partial f}{\partial\beta}|_{\beta=0}.
\end{aligned}$$

This establishes Proposition 4.3. ■

Remark: The five operators $\bar{\mathcal{A}}_1^\pm, \bar{\mathcal{A}}_2, \bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2$ form a Lie algebra (upon using Lemma 3.1):

$$\begin{aligned}
[\bar{\mathcal{A}}_1^\pm, \bar{\mathcal{C}}_1] &= 0, \quad [\bar{\mathcal{A}}_1^+, \bar{\mathcal{A}}_1^-] = 0, \quad [\bar{\mathcal{A}}_2, \bar{\mathcal{C}}_2] = 0 \\
[\bar{\mathcal{A}}_1^\pm, \bar{\mathcal{A}}_2] &= -\left(\frac{1}{2} + J_{mm}\right)\bar{\mathcal{A}}_1^\pm, \quad [\bar{\mathcal{A}}_1^\pm, \bar{\mathcal{C}}_2] = -J_{1m}\bar{\mathcal{C}}_1 \\
[\bar{\mathcal{A}}_2, \bar{\mathcal{C}}_1] &= -J_{1m}(\bar{\mathcal{A}}_1^+ + \bar{\mathcal{A}}_1^-), \quad [\bar{\mathcal{C}}_2, \bar{\mathcal{C}}_1] = -(1 + 2J_{11})\bar{\mathcal{C}}_1
\end{aligned}$$

5 Integrable deformations and 3-component KP

The following integral

$$\begin{aligned} \tau_{k_1 k_2}(\bar{t}, \bar{s}, \bar{u}; \alpha, \beta, \gamma^{(2)}, \dots, \gamma^{(m-1)}, c^{(1)}, \dots, c^{(m-1)}, E_1 \times \dots \times E_m) \\ = \det \begin{pmatrix} (\mu_{ij}^+)_{1 \leq i \leq k_1, 1 \leq j \leq k_1+k_2} \\ (\mu_{ij}^-)_{1 \leq i \leq k_2, 1 \leq j \leq k_1+k_2} \end{pmatrix} \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} \mu_{ij}^\pm(\bar{t}, \bar{s}, \bar{u}, c, \gamma) &= \int_{\prod_1^m E_i} (x^{(1)})^{j-1} (x^{(m)})^{i-1} e^{\sum_1^\infty (\bar{t}_k x^{(1)k} - \left(\frac{\bar{s}_k}{\bar{u}_k}\right) x^{(m)k})} e^{\pm \alpha x_m \pm \beta x_m^2} \\ &\quad F_m(x^{(1)}, \dots, x^{(m)}) \prod_{\ell=1}^m dx^{(\ell)} \\ &= \left\langle x^{i-1} e^{-\sum_1^\infty \left(\frac{\bar{s}_k}{\bar{u}_k}\right) x^k} e^{\pm \alpha x \pm \beta x^2} \mid x^{j-1} e^{\sum_1^\infty \bar{t}_k x^k} \right\rangle_m \end{aligned} \quad (5.2)$$

with regard to the inner-product ($m \geq 2$)

$$\langle f \mid g \rangle_m = \int_{\prod_1^m E_i} f(x_m) g(x_1) F_m(x_1, \dots, x_m) dx_1 \dots dx_m,$$

for

$$F_m(x_1, \dots, x_m) := \left(\prod_1^m e^{-\frac{x_i^2}{2}} \right) e^{\sum_{p,q \geq 1} \sum_{\ell=1}^{m-1} c_{pq}^{(\ell)} x_\ell^p x_{\ell+1}^q + \sum_{\ell=2}^{m-1} \sum_{r=1}^\infty \gamma_r^{(\ell)} x_\ell^r}.$$

From ([4]), it follows that the function $\tau_{k_1 k_2}$ above, which is expressed as the determinant of a moment matrix with regard to two different weights, satisfies the 3-component KP and thus it satisfies in particular, the following PDE's:

$$\begin{aligned} \frac{\partial^2}{\partial \bar{s}_1 \partial \bar{t}_1} \log \tau_{k_1 k_2} &= -\frac{\tau_{k_1+1, k_2} \tau_{k_1-1, k_2}}{\tau_{k_1, k_2}^2} \\ \frac{\partial^2}{\partial \bar{u}_1 \partial \bar{t}_1} \log \tau_{k_1 k_2} &= -\frac{\tau_{k_1, k_2+1} \tau_{k_1, k_2-1}}{\tau_{k_1, k_2}^2} \\ \frac{\partial^2}{\partial \bar{s}_1 \partial \bar{u}_1} \log \tau_{k_1 k_2} &= \frac{\tau_{k_1+1, k_2-1} \tau_{k_1-1, k_2+1}}{\tau_{k_1, k_2}^2} \end{aligned} \quad (5.3)$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial \bar{s}_1 \partial \bar{t}_2} \log \tau_{k_1 k_2} &= -\frac{1}{\tau_{k_1 k_2}^2} \left[\left(\frac{\partial}{\partial \bar{t}_1} \tau_{k_1+1, k_2} \right) \tau_{k_1-1, k_2} - \tau_{k_1+1, k_2} \left(\frac{\partial}{\partial \bar{t}_1} \tau_{k_1-1, k_2} \right) \right] \\
\frac{\partial^2}{\partial \bar{s}_2 \partial \bar{t}_1} \log \tau_{k_1 k_2} &= -\frac{1}{\tau_{k_1 k_2}^2} \left[\left(\frac{\partial}{\partial \bar{s}_1} \tau_{k_1-1, k_2} \right) \tau_{k_1+1, k_2} - \tau_{k_1-1, k_2} \left(\frac{\partial}{\partial \bar{s}_1} \tau_{k_1+1, k_2} \right) \right] \\
\frac{\partial^2}{\partial \bar{s}_2 \partial \bar{u}_1} \log \tau_{k_1 k_2} &= \frac{1}{\tau_{k_1 k_2}^2} \left[\left(\frac{\partial}{\partial \bar{s}_1} \tau_{k_1-1, k_2+1} \right) \tau_{k_1+1, k_2-1} - \tau_{k_1-1, k_2+1} \left(\frac{\partial}{\partial \bar{s}_1} \tau_{k_1+1, k_2-1} \right) \right].
\end{aligned} \tag{5.4}$$

Thus, upon taking the ratio of the first, second and third equations of (5.4) and (5.3), one finds

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}_1} \log \frac{\tau_{k_1+1, k_2}}{\tau_{k_1-1, k_2}} &= \frac{\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} \log \tau_{k_1, k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2}} \\
\frac{\partial}{\partial \bar{s}_1} \log \frac{\tau_{k_1+1, k_2}}{\tau_{k_1-1, k_2}} &= -\frac{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_2} \log \tau_{k_1, k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2}}. \\
\frac{\partial}{\partial \bar{s}_1} \log \frac{\tau_{k_1+1, k_2-1}}{\tau_{k_1-1, k_2+1}} &= -\frac{\frac{\partial^2}{\partial \bar{u}_1 \partial \bar{s}_2} \log \tau_{k_1, k_2}}{\frac{\partial^2}{\partial \bar{u}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2}}.
\end{aligned} \tag{5.5}$$

6 A PDE for the Gaussian matrices coupled in a chain, with external potential

The purpose of this section is to prove Theorem 0.2. Notice the probability

$$\begin{aligned}
&\mathbb{P}_n(\alpha; c_1, \dots, c_{m-1}; E_1 \times \dots \times E_m) \\
&= \frac{1}{Z_n} \int_{\prod_1^m E_i^{k_1+k_2}} \Delta_{k_1+k_2}(x^{(1)}) \prod_{i=1}^m dy_i^{(\ell)} \\
&\quad \Delta_{k_1}(x^{(m)'}) \prod_{i=1}^{k_1} e^{-\frac{1}{2} \sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} + \alpha y_i^{(m)}} \\
&\quad \Delta_{k_2}(x^{(m)'}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2} \sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} - \alpha y_i^{(m)}}, \tag{6.1}
\end{aligned}$$

is invariant under the involution

$$\iota : \alpha \longleftrightarrow -\alpha, \quad \beta \longleftrightarrow -\beta, \quad u_i \longleftrightarrow s_i, \quad k_1 \longleftrightarrow k_2.$$

Proof of Theorem 0.1: The first equality in each of the expressions (6.2) and (6.3) below follows from the expressions for $\bar{\mathcal{A}}_1^+ f$ and $\bar{\mathcal{C}}_1 f$ in (4.10), whereas the second equalities follow from (5.5) and the third equality from the expressions for $\mathcal{A}_1^+ \mathcal{C}_1 f$, $\mathcal{A}_2^+ \mathcal{C}_1 f$, $\mathcal{C}_2 \mathcal{A}_1^+ f$ in (4.10):

$$\begin{aligned} \bar{\mathcal{C}}_1 \log \frac{\tau_{k_1+1, k_2}}{\tau_{k_1-1, k_2}} &= \frac{\partial}{\partial \bar{t}_1} \log \frac{\tau_{k_1+1, k_2}}{\tau_{k_1-1, k_2}} + 2\alpha J_{1m} \\ &= \frac{\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} \log \tau_{k_1, k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2}} + 2\alpha J_{1m} \\ &= \frac{\left(\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} + 2\alpha J_{1m} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \right) \log \tau_{k_1, k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2}} = -\frac{H_2^+}{F^+} \end{aligned} \quad (6.2)$$

$$\begin{aligned} \bar{\mathcal{A}}_1^+ \log \frac{\tau_{k_1+1, k_2}}{\tau_{k_1-1, k_2}} &= \frac{\partial}{\partial \bar{s}_1} \log \frac{\tau_{k_1+1, k_2}}{\tau_{k_1-1, k_2}} - \alpha J_{mm} \\ &= -\frac{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_2} \log \tau_{k_1, k_2}}{\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2}} - \alpha J_{mm} \\ &= -\frac{H_1^+ - X}{F^+} - \alpha J_{mm} \end{aligned} \quad (6.3)$$

where F^+ , X , H_i^+ can be functions of $\log \tau_{k_1 k_2}$, which can also be expressed in terms of the actual probability \mathbb{P}_n , taking into account Lemma 8.1,

$$\begin{aligned} F^+ &= \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \log \tau_{k_1, k_2} = \bar{\mathcal{A}}_1^+ \bar{\mathcal{C}}_1 \log \tau_{k_1, k_2} + k_1 J_{1m} = \bar{\mathcal{A}}_1^+ \bar{\mathcal{C}}_1 \log \mathbb{P}_n + k_1 J_{1m} \\ X &= \frac{1}{2} \bar{\mathcal{C}}_1 \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} = \frac{1}{2} \bar{\mathcal{C}}_1 \frac{\partial}{\partial \beta} \log \mathbb{P}_n \\ H_1^+ &= \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_2} \log \tau_{k_1, k_2} + \frac{1}{2} \frac{\partial}{\partial \beta} \bar{\mathcal{C}}_1 \log \tau_{k_1, k_2} \\ &= \left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m} \frac{\partial}{\partial \alpha} \right) \log \tau_{k_1, k_2} - \alpha(k_1 - k_2) J_{1m} J_{mm} \\ &= \left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m} \frac{\partial}{\partial \alpha} \right) \log \mathbb{P}_n - 2\alpha k_1 J_{1m} J_{mm} + \frac{k_1 k_2}{\alpha} J_{1m} \end{aligned}$$

$$\begin{aligned}
H_2^+ &= - \left(\frac{\partial^2}{\partial \bar{t}_2 \partial \bar{s}_1} + 2\alpha J_{1m} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{s}_1} \right) \log \tau_{k_1, k_2} \\
&= - (\bar{\mathcal{C}}_2 + 2\alpha J_{1m} \bar{\mathcal{C}}_1) \bar{\mathcal{A}}_1^+ \log \tau_{k_1, k_2} - \alpha(k_1 + k_2)(J_{1m})^2 \\
&= - (\bar{\mathcal{C}}_2 + 2\alpha J_{1m} \bar{\mathcal{C}}_1) \bar{\mathcal{A}}_1^+ \log \mathbb{P}_n
\end{aligned}$$

Thus, $\bar{\mathcal{A}}_1^+$ of the right hand side of (6.2) must equal $\bar{\mathcal{C}}_1$ of the right hand side of (6.3) and furthermore noticing that $\bar{\mathcal{C}}_1 \alpha J_{mm} = 0$, one finds:

$$\bar{\mathcal{A}}_1^+ \frac{H_2^+}{F^+} = \bar{\mathcal{C}}_1 \left(\frac{H_1^+}{F^+} - \frac{X}{F^+} \right)$$

or in Wronskian notation

$$\{X, F^+\}_{\bar{\mathcal{C}}_1} = \{H_1^+, F^+\}_{\bar{\mathcal{C}}_1} - \{H_2^+, F^+\}_{\bar{\mathcal{A}}_1} = G^+ \quad (6.4)$$

and

$$-\{X, F^-\}_{\bar{\mathcal{C}}_1} = \{H_1^-, F^-\}_{\bar{\mathcal{C}}_1} - \{H_2^-, F^-\}_{\bar{\mathcal{B}}_1} = G^- \quad (6.5)$$

upon applying the involution ι to the first equation. Equations (6.4) and (6.5) yield a linear system of equations in X and $\bar{\mathcal{C}}_1 X$, leading to

$$\begin{aligned}
X &= \frac{G^- F^+ + G^+ F^-}{-F^-(\bar{\mathcal{C}}_1 F^+) + F^+(\bar{\mathcal{C}}_1 F^-)} \\
\bar{\mathcal{C}}_1 X &= \frac{G^-(\bar{\mathcal{C}}_1 F^+) + G^+(\bar{\mathcal{C}}_1 F^-)}{-F^-(\bar{\mathcal{C}}_1 F^+) + F^+(\bar{\mathcal{C}}_1 F^-)}
\end{aligned}$$

Subtracting the second equation from $\bar{\mathcal{C}}_1$ of the first equation yields the following:

$$\begin{aligned}
&\left(F^+ \bar{\mathcal{C}}_1 G^- + F^- \bar{\mathcal{C}}_1 G^+ \right) \left(F^+ \bar{\mathcal{C}}_1 F^- - F^- \bar{\mathcal{C}}_1 F^+ \right) \\
&- \left(F^+ G^- + F^- G^+ \right) \left(F^+ \bar{\mathcal{C}}_1^2 F^- - F^- \bar{\mathcal{C}}_1^2 F^+ \right) = 0.
\end{aligned}$$

The second way of expressing these equations is to write a system of 4 equations, consisting of the system (6.4) and (6.5) and that same system acted upon by $\bar{\mathcal{C}}_1$:

$$\begin{aligned}
0 &= -\{X, F^+\}_{\bar{\mathcal{C}}_1} + G^+ \\
0 &= -\{X, F^-\}_{\bar{\mathcal{C}}_1} - G^- \\
0 &= -\bar{\mathcal{C}}_1 \{X, F^+\}_{\bar{\mathcal{C}}_1} + \bar{\mathcal{C}}_1 G^+ \\
0 &= -\bar{\mathcal{C}}_1 \{X, F^-\}_{\bar{\mathcal{C}}_1} - \bar{\mathcal{C}}_1 G^-,
\end{aligned}$$

or, in matrix notation,

$$\begin{pmatrix} G^+ & \bar{C}_1 F^+ & -F^+ & 0 \\ -G^- & \bar{C}_1 F^- & -F^- & 0 \\ \bar{C}_1 G^+ & \bar{C}_1^2 F^+ & 0 & -F^+ \\ -\bar{C}_1 G^- & \bar{C}_1^2 F^- & 0 & -F^- \end{pmatrix} \begin{pmatrix} 1 \\ X \\ \bar{C}_1 X \\ \bar{C}_1^2 X \end{pmatrix} = 0$$

and thus the matrix must be singular, leading to the second formulation and ending the proof of Theorem 0.1. \blacksquare

7 The PDE for the transition probability of the Pearcey process

Given $E_\ell := \bigcup_{i=1}^r [x_{2i-1}^{(\ell)}, x_{2i}^{(\ell)}] \subset \mathbb{R}$, define the “*space*” and “*time*” operators \mathcal{X}_k and \mathcal{T}_k related to Brownian motion, together with a mixed “*space-time*” operator $\tilde{\mathcal{X}}_{-1}$,

$$\mathcal{X}_k := \sum_{\ell} \sum_{i=1}^{2r_\ell} (x_i^{(\ell)})^{k+1} \frac{\partial}{\partial x_i^{(\ell)}}, \quad \mathcal{T}_j = \sum_{\ell} s_\ell^{j+1} \frac{\partial}{\partial s_\ell}, \quad \tilde{\mathcal{X}}_{-1} = \sum_{\ell} s_\ell \sum_{i=1}^2 \frac{\partial}{\partial x_i^{(\ell)}}.$$

We shall also need the intermediate operators

$$\mathcal{X}_k(x^{(\ell)}) := \sum_{i=1}^{2r_\ell} (x_i^{(\ell)})^{k+1} \frac{\partial}{\partial x_i^{(\ell)}}.$$

Define a new function \mathbb{Q}_z

$$\log \mathbb{P}_n(\alpha, c_1, \dots, c_{n-1}; b^{(1)}, \dots, b^{(m)}) \Big|_{n=\frac{2}{z^4}} = \mathbb{Q}_z(s_1, \dots, s_m; x^{(1)}, \dots, x^{(m)}) \quad (7.1)$$

by means of the change of variables, defined earlier

$$\begin{aligned}
\alpha &= \frac{2}{z} \frac{\sqrt{s_m - s_{m-1}}}{\sqrt{(1 - s_m z^2)(1 - s_{m-1} z^2)}} \\
c_j &= \begin{cases} \sqrt{\frac{(1+s_1 z^2)(s_3 - s_2)}{(1+s_2 z^2)(s_3 - s_1)}} & \text{for } j = 1 \\ \sqrt{\frac{(s_j - s_{j-1})(s_{j+2} - s_{j+1})}{(s_{j+1} - s_{j-1})(s_{j+2} - s_j)}} & \text{for } 2 \leq j \leq m - 2 \\ \sqrt{\frac{(s_{m-1} - s_{m-2})(1 - s_m z^2)}{(s_m - s_{m-2})(1 - s_{m-1} z^2)}} & \text{for } j = m - 1 \end{cases} \\
b_i^{(\ell)} &= \begin{cases} 2x_i^{(1)} \sqrt{\frac{1+s_2 z^2}{(1+s_1 z^2)(s_2 - s_1)}}, & \text{for } \ell = 1 \\ 2x_i^{(\ell)} \sqrt{\frac{s_{\ell+1} - s_{\ell-1}}{(s_{\ell} - s_{\ell-1})(s_{\ell+1} - s_{\ell})}}, & \text{for } 2 \leq \ell \leq m - 1 \\ 2x_i^{(m)} \sqrt{\frac{1 - s_{m-1} z^2}{(s_m - s_{m-1})(1 - s_m z^2)}}, & \text{for } \ell = m \end{cases}
\end{aligned} \tag{7.2}$$

with inverse map

$$\begin{aligned}
s_i &= s_i(\alpha, c; z) = \frac{1}{z^2} \frac{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} + 2}{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} - 2} \\
x_k^{(i)} &= b_k^{(i)} U_i(s(\alpha, c; z); z),
\end{aligned} \tag{7.3}$$

with

$$U_i(s(\alpha, c; z); z) = \frac{-1}{J_{mi}} \left(\frac{\alpha z \frac{J_{m1} J_{mi}}{J_{1i}}}{\alpha^2 z^4 \frac{J_{m1} J_{mi}}{J_{1i}} - 2} \right).$$

Then

$$\begin{aligned}
\frac{\partial x_k^{(i)}}{\partial \alpha} &= \frac{\partial (b_k^{(i)} U_i)}{\partial \alpha} = b_k^{(i)} \frac{\partial U_i}{\partial \alpha} = b_k^{(i)} U_i \frac{\partial \log U_i}{\partial \alpha} = x_k^{(i)} \frac{\partial \log U_i}{\partial \alpha} \\
\frac{\partial x_k^{(i)}}{\partial c_j} &= \frac{\partial (b_k^{(i)} U_i)}{\partial c_j} = b_k^{(i)} \frac{\partial U_i}{\partial c_j} = b_k^{(i)} U_i \frac{\partial \log U_i}{\partial c_j} = x_k^{(i)} \frac{\partial \log U_i}{\partial c_j}.
\end{aligned}$$

Thus, setting (7.3) in the right hand side of (7.1) and using the chain rule, one computes

$$\begin{aligned}
\frac{\partial \log \mathbb{P}_n}{\partial \alpha} \Big|_{n=\frac{2}{z^4}} &= \sum_{i=1}^m \frac{\partial s_i}{\partial \alpha} \frac{\partial Q_z}{\partial s_i} + \sum_{i=1}^m \sum_{k=1}^{2r_i} \frac{\partial x_k^{(i)}}{\partial \alpha} \frac{\partial Q_z}{\partial x_k^{(i)}} \\
&= \left(\sum_{i=1}^m \frac{\partial s_i}{\partial \alpha} \frac{\partial}{\partial s_i} + \sum_{i=1}^m \frac{\partial \log U_i}{\partial \alpha} \sum_{k=1}^{2r_i} x_k^{(i)} \frac{\partial}{\partial x_k^{(i)}} \right) \mathbb{Q}_z \\
&= \sum_{i=1}^m \left(\frac{\partial s_i}{\partial \alpha} \frac{\partial}{\partial s_i} + \frac{\partial \log U_i}{\partial \alpha} \mathcal{X}_0(x^{(i)}) \right) \mathbb{Q}_z
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log \mathbb{P}_n}{\partial c_j} \Big|_{n=\frac{2}{z^4}} &= \sum_{i=1}^m \frac{\partial s_i}{\partial c_j} \frac{\partial Q_z}{\partial s_i} + \sum_{i=1}^m \sum_{k=1}^{2r_i} \frac{\partial x_k^{(i)}}{\partial c_j} \frac{\partial Q_z}{\partial x_k^{(i)}} \\
&= \left(\sum_{i=1}^m \frac{\partial s_i}{\partial c_j} \frac{\partial}{\partial s_i} + \sum_{i=1}^m \frac{\partial \log U_i}{\partial c_j} \sum_{k=1}^{2r_i} x_k^{(i)} \frac{\partial}{\partial x_k^{(i)}} \right) \mathbb{Q}_z \\
&= \sum_{i=1}^m \left(\frac{\partial s_i}{\partial c_j} \frac{\partial}{\partial s_i} + \frac{\partial \log U_i}{\partial c_j} \mathcal{X}_0(x^{(i)}) \right) \mathbb{Q}_z
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{-1}(E_i) \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} &= \sum_{k=1}^{2r_i} \frac{\partial \log \mathbb{P}_n}{\partial b_k^{(i)}} \Big|_{n=\frac{2}{z^4}} \\
&= \sum_{k=1}^{2r_i} \frac{\partial x_k^{(i)}}{\partial b_k^{(i)}} \frac{\partial Q_z}{\partial x_k^{(i)}} \\
&= \left(U_i \sum_{k=1}^{2r_i} \frac{\partial}{\partial x_k^{(i)}} \right) \mathbb{Q}_z \\
&= U_i \mathcal{X}_{-1}(x^{(i)}) \mathbb{Q}_z
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_0(E_i) \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} &= \sum_{k=1}^{2r_i} b_k^{(i)} \frac{\partial \log \mathbb{P}_n}{\partial b_k^{(i)}} \Big|_{n=\frac{2}{z^4}} \\
&= \sum_{k=1}^{2r_i} b_k^{(i)} \frac{\partial x_k^{(i)}}{\partial b_k^{(i)}} \frac{\partial \mathbb{Q}_z}{\partial x_k^{(i)}} \\
&= \left(\sum_{k=1}^{2r_i} x_k^{(i)} \frac{\partial}{\partial x_k^{(i)}} \right) \mathbb{Q}_z \\
&= \mathcal{X}_0(x^{(i)}) \mathbb{Q}_z.
\end{aligned}$$

Using the information above, one computes ($\varepsilon = \pm 1$)

$$\begin{aligned}
\bar{\mathcal{A}}_1^\varepsilon \log \mathbb{P}_n(\alpha, c_1, \dots, c_{n-1}; b^{(1)}, \dots, b^{(m)}) \Big|_{n=\frac{2}{z^4}} \\
&= -\frac{1}{2} \left(\sum_{j=1}^m J_{mj} \mathcal{D}_{-1}(E_j) + \varepsilon \frac{\partial}{\partial \alpha} \right) \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} \\
&= -\frac{1}{2} \sum_{j=1}^m \left(J_{mj} U_j \mathcal{X}_{-1}(x^{(j)}) + \varepsilon \frac{\partial s_j}{\partial \alpha} \frac{\partial}{\partial s_j} + \varepsilon \frac{\partial \log U_j}{\partial \alpha} \mathcal{X}_0(x^{(j)}) \right) \mathbb{Q}_z \\
&=: A_1^\varepsilon \mathbb{Q}_z \tag{7.4}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{C}}_1 \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} &= \sum_{j=1}^m J_{1j} \mathcal{D}_{-1}(E_j) \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} \\
&= \left(\sum_{j=1}^m J_{1j} U_j \mathcal{X}_{-1}(x^{(j)}) \right) \mathbb{Q}_z \\
&=: C_1 \mathbb{Q}_z \tag{7.5}
\end{aligned}$$

$$\begin{aligned}
& \bar{\mathcal{A}}_2 \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} \\
&= \frac{1}{2} \left(\mathcal{D}_0(b^{(m)}) - \alpha \frac{\partial}{\partial \alpha} - c_{m-1} \frac{\partial}{\partial c_{m-1}} \right) \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} \\
&= -\frac{1}{2} \sum_{j=1}^m \left(\left(c_{m-1} \frac{\partial s_j}{\partial c_{m-1}} + \alpha \frac{\partial s_j}{\partial \alpha} \right) \frac{\partial}{\partial s_j} \right. \\
&\quad \left. + \left(\left(c_{m-1} \frac{\partial \log U_j}{\partial c_{m-1}} + \alpha \frac{\partial \log U_j}{\partial \alpha} \right) - \delta_{jm} \right) \mathcal{X}_0(x^{(j)}) \right) \mathbb{Q}_z \\
&=: A_2 \mathbb{Q}_z \tag{7.6}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{C}}_2 \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} &= \left(-\mathcal{D}_0(b^{(1)}) + c_1 \frac{\partial}{\partial c_1} \right) \log \mathbb{P}_n \Big|_{n=\frac{2}{z^4}} \\
&= \sum_{j=1}^m \left(c_1 \frac{\partial s_j}{\partial c_1} \frac{\partial}{\partial s_j} + \left(c_1 \frac{\partial \log U_j}{\partial c_1} - \delta_{j1} \right) \mathcal{X}_0(x^{(j)}) \right) \mathbb{Q}_z \\
&=: C_2 \mathbb{Q}_z \tag{7.7}
\end{aligned}$$

Lemma 7.1 *The following expansions hold near $z \sim 0$:*

$$\begin{aligned}
\alpha \frac{\partial s_i}{\partial \alpha} &= \frac{1}{z^2} - s_i^2 z^2 \\
\frac{\partial s_i}{\partial \alpha} &= \frac{1}{2z\sqrt{s_m - s_{m-1}}} \left(1 - \frac{z^2}{2}(s_{m-1} + s_m) + O(z^4) \right)
\end{aligned}$$

$$\begin{aligned}
\alpha \frac{\partial}{\partial \alpha} \log U_i(s(\alpha, c; z), z) &= -z^2 s_i \\
\frac{\partial}{\partial \alpha} \log U_i(s(\alpha, c; z), z) &= O(z^3)
\end{aligned}$$

$$\begin{aligned}
c_1 \frac{\partial}{\partial c_1} s_i &= \frac{1}{2(s_2 - s_1)} \left(\frac{1}{z^4} + \frac{s_1 + s_2 - 2s_i}{z^2} + O(1) \right) \\
c_{m-1} \frac{\partial}{\partial c_{m-1}} s_i &= \frac{1}{s_m - s_{m-1}} \left(\frac{1}{z^4} + \frac{s_m + s_{m-1} - 2s_i}{z^2} + O(1) \right)
\end{aligned}$$

$$c_1 \frac{\partial}{\partial c_1} \log U_i = -\frac{1}{2(s_2 - s_1)z^2} + O(1)$$

$$c_{m-1} \frac{\partial}{\partial c_{m-1}} \log U_i = -\frac{1}{2(s_m - s_{m-1})z^2} + O(1)$$

$$J_{mi}U_i = \frac{1}{4\sqrt{s_m - s_{m-1}}} \left(-\frac{1}{z^2} + \frac{1}{2}(s_m + s_{m-1} - 2s_i) + O(z^2) \right)$$

$$J_{1i}U_i = \frac{1}{4\sqrt{s_2 - s_1}} \left(-\frac{1}{z^2} - \frac{1}{2}(s_1 + s_2 - 2s_i) \right. \\ \left. + \frac{z^2}{8}((s_2 - s_1)^2 + 4s_1(s_1 + s_2)) + O(z^4) \right)$$

$$\alpha = \frac{2}{z}\sqrt{s_m - s_{m-1}} \left(1 + \frac{z^2}{2}(s_m + s_{m-1}) + O(z^4) \right).$$

Proof: These series follow from expanding the expressions given in Lemma 3.3 near $z \sim 0$. ■

Lemma 7.2 *The operators A_1^ε , A_2 , C_1 , C_2 , as defined in (7.4), (7.6), (7.5), (7.7) and as acting on the function $\mathbb{Q}_z(s_1, \dots, s_m; x^{(1)}, \dots, x^{(m)})$ admit the following expansions in $z \sim 0$, in terms of the operators \mathcal{X}_k , \mathcal{T}_k , $\tilde{\mathcal{X}}_{-1}$,*

$$A_1^\varepsilon = \frac{1}{8\sqrt{s_m - s_{m-1}}} \left\{ \begin{array}{l} \frac{1}{z^2}\mathcal{X}_{-1} - \frac{2\varepsilon}{z}\mathcal{T}_{-1} + \left(\tilde{\mathcal{X}}_{-1} - \frac{(s_{m-1} + s_m)}{2}\mathcal{X}_{-1} \right) \\ + \varepsilon z(s_{m-1} + s_m)\mathcal{T}_{-1} + O(z^2) \end{array} \right\}$$

$$C_1 = \frac{1}{4\sqrt{s_2 - s_1}} \left\{ \begin{array}{l} -\frac{1}{z^2}\mathcal{X}_{-1} + \left(\tilde{\mathcal{X}}_{-1} - \frac{(s_1 + s_2)}{2}\mathcal{X}_{-1} \right) \\ + z^2 \left(\frac{(s_2 - s_1)^2}{8}\mathcal{X}_{-1} + \frac{(s_1 + s_2)}{2}\tilde{\mathcal{X}}_{-1} \right) + O(z^4) \end{array} \right\}$$

$$A_2 = \frac{1}{4(s_m - s_{m-1})} \left\{ -\frac{\mathcal{T}_{-1}}{z^4} + \frac{1}{z^2}(\mathcal{X}_0 + 2\mathcal{T}_0 + (s_{m-1} - 3s_m)\mathcal{T}_{-1}) + O(1) \right\}$$

$$C_2 = \frac{1}{2(s_2 - s_1)} \left\{ \frac{\mathcal{T}_{-1}}{z^4} - \frac{1}{z^2}(\mathcal{X}_0 + 2\mathcal{T}_0 - (s_1 + s_2)\mathcal{T}_{-1}) + O(1) \right\}.$$

Proof: These formulae are an immediate consequence of formulae (7.4), (7.6), (7.5), (7.7), combined with the series in Lemma 7.1. ■

Lemma 7.3 *In the new coordinates $s_1, \dots, s_m, x^{(1)}, \dots, x^{(m)}$, the quantities appearing in the basic equation (0.3) for \mathbb{P}_n admit the following expansions in $z \sim 0$, setting $k = n/2 = 1/z^4$:*

$$\begin{aligned}
F^\varepsilon &= A_1^\varepsilon C_1 \mathbb{Q}_z + \frac{J_{1m}}{z^4} \\
&= \frac{1}{32\sqrt{(s_2 - s_1)(s_m - s_{m-1})}} \\
&\quad \left\{ \begin{aligned} &-\frac{16}{z^6} - \frac{1}{z^4}(\mathcal{X}_{-1}^2 \mathbb{Q}_z + 8(s_1 + s_2 - s_{m-1} - s_m)) \\ &+ \frac{2\varepsilon}{z^3} \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z + \frac{1}{z^2} \left[\frac{1}{2}(s_{m-1} + s_m - s_1 - s_2) \mathcal{X}_{-1}^2 \mathbb{Q}_z \right. \\ &\quad \left. + 2((s_2 - s_1)^2 + (s_m - s_{m-1})^2 + 2(s_1 + s_2)(s_{m-1} + s_m)) \right] \\ &-\frac{\varepsilon}{z}((s_{m-1} + s_m - s_1 - s_2) \mathcal{X}_{-1} + 2\tilde{\mathcal{X}}_{-1}) \mathcal{T}_{-1} \mathbb{Q}_z + O(1) \end{aligned} \right\} \quad (7.8)
\end{aligned}$$

$$\begin{aligned}
\bar{C}_1 F^\varepsilon &= \frac{1}{128(s_2 - s_1)\sqrt{s_m - s_{m-1}}} \\
&\quad \left\{ \begin{aligned} &\frac{1}{z^6} \mathcal{X}_{-1}^3 \mathbb{Q}_z - \frac{2\varepsilon}{z^5} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z \\ &+ \frac{1}{z^4} \left((s_1 + s_2 - \frac{(s_{m-1} + s_m)}{2}) \mathcal{X}_{-1} - \tilde{\mathcal{X}}_{-1} \right) \mathcal{X}_{-1}^2 \mathbb{Q}_z \\ &+ \frac{\varepsilon}{z^3} ((s_{m-1} + s_m - 2(s_1 + s_2)) \mathcal{X}_{-1} + 4\tilde{\mathcal{X}}_{-1}) \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z + O\left(\frac{1}{z^2}\right) \end{aligned} \right\} \quad (7.9)
\end{aligned}$$

$$\begin{aligned}
\bar{C}_1^2 F^\varepsilon &= \frac{1}{512(s_2 - s_1)^{3/2} \sqrt{(s_m - s_{m-1})}} \\
&\quad \left\{ -\frac{\mathcal{X}_{-1}^4 \mathbb{Q}_z}{z^8} + \frac{2\varepsilon}{z^7} \mathcal{X}_{-1}^3 \mathcal{T}_{-1} \mathbb{Q}_z + O\left(\frac{1}{z^6}\right) \right\} \quad (7.10)
\end{aligned}$$

$$\begin{aligned}
\bar{A}_1 F^\varepsilon &= \frac{1}{256(s_m - s_{m-1})\sqrt{(s_2 - s_1)}} \\
&\quad \left\{ -\frac{\mathcal{X}_{-1}^3 \mathbb{Q}_z}{z^6} + \frac{4\varepsilon}{z^5} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z + O\left(\frac{1}{z^4}\right) \right\} \quad (7.11)
\end{aligned}$$

$$H_1^\varepsilon = \left(A_2 C_1 + \varepsilon J_{1m} \frac{\partial}{\partial \alpha} \right) \mathbb{Q}_z - \varepsilon \frac{J_{1m}}{z^4} \left(2\alpha J_{mm} - \frac{1}{z^4 \alpha} \right) \quad (7.12)$$

$$= \frac{1}{16(s_m - s_{m-1})\sqrt{s_2 - s_1}} \left\{ \begin{aligned} & -\frac{20\varepsilon}{z^9} - \frac{2\varepsilon}{z^7}(5(s_1 + s_2) - 10s_{m-1} + 6s_m) + \frac{1}{z^6} \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z \\ & + \frac{\varepsilon}{2z^5} (5(s_2 - s_1)^2 + 24s_{m-1}s_m + 4(s_1 + s_2)(5s_{m-1} - 3s_m)) \\ & + \frac{1}{z^4} \left(\begin{aligned} & 2\mathcal{X}_{-1} - \tilde{\mathcal{X}}_{-1} \mathcal{T}_{-1} - 2\mathcal{X}_{-1} \mathcal{T}_0 - \mathcal{X}_{-1} \mathcal{X}_0 \\ & + \left(\frac{s_1 + s_2}{2} - s_{m-1} + 3s_m \right) \mathcal{X}_{-1} \mathcal{T}_{-1} \end{aligned} \right) \mathbb{Q}_z + O\left(\frac{1}{z^3}\right) \end{aligned} \right\} \quad (7.13)$$

$$\bar{C}_1 H_1^\varepsilon = \frac{1}{64(s_2 - s_1)(s_m - s_{m-1})} \left\{ \begin{aligned} & -\frac{1}{z^8} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z + \frac{1}{z^6} \left(\begin{aligned} & 2\tilde{\mathcal{X}}_{-1} \mathcal{X}_{-1} \mathcal{T}_{-1} + 2\mathcal{X}_{-1}^2 \mathcal{T}_0 + \mathcal{X}_0 \mathcal{X}_{-1}^2 \\ & + (s_{m-1} - 3s_m - s_1 - s_2) \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \end{aligned} \right) \mathbb{Q}_z \\ & + O\left(\frac{1}{z^5}\right) \end{aligned} \right\} \quad (7.14)$$

$$-H_2^\varepsilon = (C_2 + 2\varepsilon \alpha J_{1m} C_1) A_1^\varepsilon \mathbb{Q}_z \\ = \frac{1}{16(s_2 - s_1)\sqrt{s_m - s_{m-1}}} \left\{ \begin{aligned} & \frac{\varepsilon}{z^7} \mathcal{X}_{-1}^2 \mathbb{Q}_z - \frac{1}{z^6} \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z \\ & + \frac{\varepsilon}{z^5} (-2\mathcal{T}_{-1}^2 + (s_1 + s_2 - \frac{s_{m-1} + s_m}{2}) \mathcal{X}_{-1}^2) \mathbb{Q}_z \\ & + \frac{1}{z^4} \left(\begin{aligned} & \mathcal{X}_{-1} - \mathcal{X}_0 \mathcal{X}_{-1} - 2\mathcal{X}_{-1} \mathcal{T}_0 + 3\tilde{\mathcal{X}}_{-1} \mathcal{T}_{-1} \\ & + \left(\frac{1}{2}(s_{m-1} + s_m) - s_1 - s_2 \right) \mathcal{X}_{-1} \mathcal{T}_{-1} \end{aligned} \right) \mathbb{Q}_z \\ & + O\left(\frac{1}{z^3}\right) \end{aligned} \right\}$$

$$-\bar{A}_1^\varepsilon H_2^\varepsilon = \frac{1}{128(s_2 - s_1)(s_m - s_{m-1})} \left\{ \begin{aligned} & \frac{\varepsilon}{z^9} \mathcal{X}_{-1}^3 \mathbb{Q}_z - \frac{3}{z^8} \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z \\ & + \frac{\varepsilon}{z^7} (\tilde{\mathcal{X}}_{-1} + (s_1 + s_2 - s_{m-1} - s_m) \mathcal{X}_{-1}) \mathcal{X}_{-1}^2 \mathbb{Q}_z \\ & + \frac{1}{z^6} (-\mathcal{X}_{-1}^2 \mathcal{X}_0 - 2\mathcal{X}_{-1}^2 \mathcal{T}_0 + 2\tilde{\mathcal{X}}_{-1} \mathcal{X}_{-1} \mathcal{T}_{-1} \\ & + 4\mathcal{T}_{-1}^3 + 3(s_m + s_{m-1} - s_1 - s_2) \mathcal{X}_{-1}^2 \mathcal{T}_{-1}) \mathbb{Q}_z \\ & + O\left(\frac{1}{z^5}\right) \end{aligned} \right\}$$

$$G^\varepsilon = \{H_1^\varepsilon, F^\varepsilon\}_{C_1} - \{H_2^\varepsilon, F^\varepsilon\}_{A_1} \quad (7.15)$$

$$= \frac{1}{2 \cdot (32)^2 ((s_2 - s_1)(s_m - s_{m-1}))^{3/2}} \quad (7.16)$$

$$\left\{ \begin{array}{l} \frac{12\varepsilon}{z^{15}} \mathcal{X}_{-1}^3 \mathbb{Q}_z \\ + \frac{\varepsilon}{z^{13}} (-28\tilde{\mathcal{X}}_{-1} + (18(s_1 + s_2 - s_{m-1}) + 14s_m)\mathcal{X}_{-1})\mathcal{X}_{-1}^2 \mathbb{Q}_z \\ + \frac{1}{z^{12}} \left(\begin{array}{l} \frac{1}{2}\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z, \mathcal{X}_{-1}^2 \mathbb{Q}_z\}\mathcal{X}_{-1} \\ -16(\mathcal{T}_0 - 1 + \frac{1}{2}\mathcal{X}_0)\mathcal{X}_{-1}^2 \mathbb{Q}_z \\ +32(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}_z \end{array} \right) \\ +O\left(\frac{1}{z^{11}}\right) \end{array} \right\}$$

Proof: This is done by straightforward computation, using Lemma 7.2 and the series for the constants appearing in F^ε , H_1^ε , H_2^ε , G^ε , namely

$$\begin{aligned} J_{m1}(c) &= -\frac{1}{2z^2\sqrt{(s_2 - s_1)(s_m - s_{m-1})}} \left(1 + \frac{z^2}{2}(s_1 + s_2 - s_{m-1} - s_m) \right. \\ &\quad \left. - \frac{z^4}{8}((s_2 - s_1)^2 + (s_m - s_{m-1})^2 + 2(s_1 + s_2)(s_{m-1} + s_m)) + \mathbf{O}(z^4) \right) \\ (\alpha J_{m1}) &= -\frac{1}{\sqrt{s_2 - s_1}} \left(\frac{1}{z^3} + \frac{1}{2z}(s_1 + s_2) - \frac{z}{8}(s_2 - s_1)^2 + O(z^3) \right) \end{aligned}$$

$$\begin{aligned} &\left(\frac{k^2 J_{m1}}{\alpha} - 2\alpha k J_{1m} J_{mm} \right) \\ &= \frac{1}{4z^9(s_m - s_{m-1})\sqrt{s_2 - s_1}} \\ &\quad \left(-5 - \frac{z^2}{2}(5(s_1 + s_2) - 10s_{m-1} + 6s_m) \right. \\ &\quad \left. + \frac{z^4}{8}(5(s_2 - s_1)^2 + 24s_{m-1}s_m + 4(s_1 + s_2)(5s_{m-1} - 3s_m)) + O(z^6) \right) \end{aligned}$$

Proof of Theorem 0.2: In terms of T_i 's, defined by

$$\begin{aligned} T_1 &:= F^+ \bar{\mathcal{C}}_1 G^- + F^- \bar{\mathcal{C}}_1 G^+ = \bar{\mathcal{C}}_1 T_3 - T_5 \\ T_2 &:= \{F^-, F^+\}_{\bar{\mathcal{C}}_1} \\ T_3 &:= F^+ G_1^- + F^- G_1^+ \\ T_4 &:= \bar{\mathcal{C}}_1 T_2 = F^+ \bar{\mathcal{C}}_1^2 F^- - F^- \bar{\mathcal{C}}_1^2 F^+ \\ T_5 &:= G^- \bar{\mathcal{C}}_1 F^+ + G^+ \bar{\mathcal{C}}_1 F^-, \end{aligned}$$

the fundamental equation (0.3) can be written

$$0 = -T_1 \cdot T_2 + T_3 \cdot T_4 = \{T_2, T_3\}_{\bar{C}_1} + T_2 T_5$$

meaning that T_2 , T_3 and T_5 only are needed, which one checks to have the following series in z , using Lemma 7.3,

$$\begin{aligned} T_2 &= -\frac{\mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z}{64(s_2 - s_1)^{3/2}(s_m - s_{m-1})z^{11}} + O\left(\frac{1}{z^9}\right) \\ T_3 &= \frac{1}{128(s_2 - s_1)^2(s_m - s_{m-1})^2 z^{18}} \\ &\quad \left\{ \begin{aligned} &\frac{1}{16}\{\mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z, \mathcal{X}_{-1}^2 \mathbb{Q}_z\}_{\mathcal{X}_{-1}} \\ &+ (\mathcal{T}_0 - 1 + \frac{1}{2}\mathcal{X}_0)\mathcal{X}_{-1}^2 \mathbb{Q}_z \\ &- 2(\tilde{\mathcal{X}}_{-1} \mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1} \mathbb{Q}_z \\ &- \frac{3}{32}(\mathcal{X}_{-1}^2 \mathbb{Q}_z)(\mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z) \end{aligned} \right\} + O\left(\frac{1}{z^{17}}\right) \\ T_5 &= \frac{3(\mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z)(\mathcal{X}_{-1}^3 \mathbb{Q}_z)}{128^2(s_2 - s_1)^{5/2}(s_m - s_{m-1})^2 z^{20}} + O\left(\frac{1}{z^{19}}\right). \end{aligned} \quad (7.17)$$

Then remembering from Lemma 7.2,

$$C_1 = \frac{-1}{4\sqrt{s_2 - s_1}} \left(\frac{1}{z^2} \mathcal{X}_{-1} + O(1) \right),$$

one easily computes (letting $T_i^{(0)}$ be the leading coefficient of T_i in (7.17))

$$\begin{aligned} &\{T_2, T_3\}_{C_1} \\ &= \frac{1}{z^{31}} \{T_2^0, T_3^0\}_{C_1} + O\left(\frac{1}{z^{30}}\right) \\ &= \frac{1}{32^3(s_2 - s_1)^4(s_m - s_{m-1})^3 z^{31}} \\ &\quad \left\{ \begin{aligned} &\mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z, \frac{1}{16}\{\mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q}_z, \mathcal{X}_{-1}^2 \mathbb{Q}_z\}_{\mathcal{X}_{-1}} + \left(\mathcal{T}_0 - 1 + \frac{\mathcal{X}_0}{2}\right) \mathcal{X}_{-1}^2 \mathbb{Q}_z \\ &- 2(\tilde{\mathcal{X}}_{-1} \mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1} \mathbb{Q}_z - \frac{3}{32}(\mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}_z)(\mathcal{X}_{-1}^2 \mathbb{Q}_z) \end{aligned} \right\}_{\mathcal{X}_{-1}} + O\left(\frac{1}{z^{30}}\right) \end{aligned}$$

and then using below the trivial Wronskian relation $-a^2b' = \{a, ab\}$, one computes

$$\begin{aligned}
T_2T_5 &= \frac{1}{z^{31}}T_2^0T_5^0 + O\left(\frac{1}{z^{30}}\right) \\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3z^{31}} \left(-\frac{3}{32}(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z)^2(\mathcal{X}_{-1}^3\mathbb{Q}_z) \right) + O\left(\frac{1}{z^{30}}\right) \\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3z^{31}} \left\{ \mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z, \frac{3}{32}(\mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z)(\mathcal{X}_{-1}^2\mathbb{Q}_z) \right\}_{\mathcal{X}_{-1}} \\
&\quad + O\left(\frac{1}{z^{30}}\right).
\end{aligned}$$

Adding these two contributions, one finds

$$\begin{aligned}
0 &= -T_1 \cdot T_2 + T_3 \cdot T_4 \\
&= \{T_2, T_3\}_{C_1} + T_2T_5 \\
&= \frac{1}{32^3(s_2-s_1)^4(s_m-s_{m-1})^3z^{31}} \\
&\quad \left\{ \mathcal{X}_{-1}^2\mathcal{T}_{-1}\mathbb{Q}_z, \frac{1}{16}\{\mathcal{X}_{-1}\mathcal{T}_{-1}\mathbb{Q}_z, \mathcal{X}_{-1}^2\mathbb{Q}_z\}_{\mathcal{X}_{-1}} \right. \\
&\quad \left. + \left(\mathcal{T}_0 - 1 + \frac{\mathcal{X}_0}{2} \right) \mathcal{X}_{-1}^2\mathbb{Q}_z - 2(\tilde{\mathcal{X}}_{-1}\mathcal{X}_{-1} - \mathcal{T}_{-1}^2)\mathcal{T}_{-1}\mathbb{Q}_z \right\}_{\mathcal{X}_{-1}} + O\left(\frac{1}{z^{30}}\right)
\end{aligned} \tag{7.18}$$

In [21], Tracy and Widom show that the extended kernel for the non-intersecting Brownian motions tend to the extended Pearcey kernel uniformly in each bounded interval. Since \mathbb{Q}_z is the log of the Fredholm determinant for that kernel, it follows that

$$\lim_{z \rightarrow 0} \mathbb{Q}_z = \mathbb{Q}_0.$$

Then taking the limit when $z \rightarrow 0$ in (7.18) leads to the PDE for \mathbb{Q}_0 , ending the proof of Theorem 0.2. \blacksquare

8 Appendix: evaluation of the integral over the full range

Lemma 8.1 *The following integral can be evaluated explicitly:*

$$\begin{aligned}
\tau_{k_1 k_2}(\mathbb{R}) &= \frac{1}{k_1! k_2!} \int_{(\mathbb{R}^m)^{k_1+k_2}} \Delta_{k_1+k_2}(y^{(1)}) \prod_{i=1}^m dy_i^{(\ell)} \\
&\quad \Delta_{k_1}(y^{(m)'}) \prod_{i=1}^{k_1} e^{-\frac{1}{2} \sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} + \alpha y_i^{(m)}} \\
&\quad \Delta_{k_2}(y^{(m)''}) \prod_{i=k_1+1}^{k_1+k_2} e^{-\frac{1}{2} \sum_{\ell=1}^m y_i^{(\ell)2} + \sum_{\ell=1}^{m-1} c_\ell y_i^{(\ell)} y_i^{(\ell+1)} - \alpha y_i^{(m)}} \\
&= \det \left(\begin{array}{c} (\mu_{i,j}(\alpha))_{\substack{0 \leq i \leq k_1-1 \\ 0 \leq j \leq n-1}} \\ (\mu_{i,j}(-\alpha))_{\substack{0 \leq i \leq k_2-1 \\ 0 \leq j \leq n-1}} \end{array} \right) \\
&= c_{k_1 k_2} \alpha^{k_1 k_2} \left(\prod_1^{m-1} c_i \right)^{-\frac{k_1+k_2}{2}} e^{-\frac{\alpha^2}{2}(k_1+k_2)J_{mm}} (J_{1m})^{\frac{1}{2}(k_1+k_2)^2}
\end{aligned} \tag{8.1}$$

where

$$\begin{aligned}
\mu_{ij}(\alpha) &= \int_{\mathbb{R}^m} x_1^j x_m^i e^{-\frac{1}{2}(\sum_1^m x_j^2 - 2 \sum_1^{m-1} c_j x_j x_{j+1}) + \alpha x_m} dx_1 \dots dx_m \\
c_{k_1 k_2} &= (-2)^{k_1 k_2} (2\pi)^{\frac{m}{2}(k_1+k_2)} \prod_0^{k_1-1} j! \prod_0^{k_2-1} j! .
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\bar{\mathcal{A}}_2 \bar{\mathcal{C}}_1 + J_{1m} \frac{\partial}{\partial \alpha} \right) \log \tau_{k_1 k_2}(\mathbb{R}) &= J_{1m} \frac{\partial}{\partial \alpha} \left(k_1 k_2 \log \alpha - \frac{\alpha^2}{2} (k_1 + k_2) J_{mm} \right) \\
&= J_{1m} \left(\frac{k_1 k_2}{\alpha} - \alpha (k_1 + k_2) J_{mm} \right), \\
- (\bar{\mathcal{C}}_2 + 2\alpha J_{1m} \bar{\mathcal{C}}_1) \mathcal{A}_1 \log \tau_{k_1 k_2}(\mathbb{R}) &= \frac{1}{2} c_1 \frac{\partial}{\partial c_1} \frac{\partial}{\partial \alpha} \left(k_1 k_2 \log \alpha - \frac{\alpha^2}{2} (k_1 + k_2) J_{mm} \right) \\
&= \alpha (k_1 + k_2) J_{1m}^2.
\end{aligned}$$

Proof: From the explicit evaluation of the zero moment⁵

$$\begin{aligned}\mu_{00}(\alpha, \gamma) &:= \int_{\mathbb{R}^m} e^{-\frac{1}{2}(\sum_1^m x_j^2 - 2\sum_1^{m-1} c_j x_j x_{j+1}) + \gamma x_1 + \alpha x_m} dx_1 \dots dx_m \\ &= \frac{(2\pi)^{m/2}}{\sqrt{\det J^{-1}(c_1, \dots, c_{m-1})}} e^{-\frac{1}{2}(J_{mm}\alpha^2 + 2\alpha\gamma J_{1m} + J_{11}\gamma^2)},\end{aligned}$$

one deduces the other moments by derivation,

$$\begin{aligned}\mu_{ij}(\pm\alpha) &= \left(\pm \frac{\partial}{\partial \alpha}\right)^i \left(\frac{\partial}{\partial \gamma}\right)^j \mu_{00}(\pm\alpha, \gamma) \Big|_{\gamma=0} \\ &= \mu_{00}(\pm\alpha, \gamma) (\pm 1)^i A^i B_{\pm}^j(1) \Big|_{\gamma=0} \\ &= \mu_{00}(\pm\alpha, \gamma) (\pm 1)^i A^i p_j(\pm\alpha),\end{aligned}$$

where

$$\begin{aligned}A &:= \mu_{00}(\pm\alpha, \gamma)^{-1} \frac{\partial}{\partial \alpha} \mu_{00}(\pm\alpha, \gamma) \Big|_{\gamma=0} \\ &= \frac{\partial}{\partial \alpha} - J_{mm}\alpha \\ B_{\pm} &:= \mu_{00}(\pm\alpha, \gamma)^{-1} \frac{\partial}{\partial \gamma} \mu_{00}(\pm\alpha, \gamma) \\ &= \frac{\partial}{\partial \gamma} - J_{11}\gamma \mp \alpha J_{1m} \\ p_j(\alpha) &:= B_{+}^j(1) \Big|_{\gamma=0}.\end{aligned}$$

The following holds:

$$p_{2i}(\alpha) = \text{even polynomial}, \quad p_{2i+1}(\alpha) = \text{odd polynomial of } \alpha,$$

which is used in equality ^{**} below, and

$$A^n p_k(\alpha) = p_k^{(n)} + \beta_1(\alpha) p_k^{(n-1)} + \beta_2(\alpha) p_k^{(n-2)} + \dots + \beta_n p_k,$$

⁵Using

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}\langle Qx, x \rangle + \langle \ell, x \rangle} dx_1 \dots dx_m = \frac{(2\pi)^{m/2}}{\sqrt{\det Q}} e^{\frac{1}{2}\langle Q^{-1}\ell, \ell \rangle},$$

for $Q := -J^{-1}$ and $\ell := (\gamma, 0, \dots, 0, \alpha)$.

where $p_k^{(n)} := \left(\frac{d}{d\alpha}\right)^n p_k$ and where $\beta_i(\alpha)$ are polynomials in α , independent of k ; this feature is used in equality $*$ below. The equality $***$ hinges on the identity

$$\det \begin{pmatrix} (-\alpha^{j-i})_{\substack{1 \leq i \leq k_1 \\ 1 \leq j \leq n}} \\ (\alpha^{j-i})_{\substack{1 \leq i \leq k_2 \\ 1 \leq j \leq n}} \end{pmatrix} = c''_{k_1 k_2} \alpha^{k_1 k_2}$$

for a constant $c''_{k_1 k_2}$ depending on k_1 and k_2 only. Then, setting

$$\mu := \mu_{00}(\pm\alpha, 0) = \frac{(2\pi)^{m/2}}{\sqrt{\det(-J^{-1}(c_1, \dots, c_{m-1}))}} e^{-\frac{1}{2}J_{mm}\alpha^2},$$

we compute:

$$\begin{aligned} & \tau_{k_1 k_2}(\mathbb{R}) \Big|_{t=s=u=\beta=0} \\ &= \mu^n \det \begin{pmatrix} (A^i p_j(\alpha))_{\substack{0 \leq i \leq k_1 - 1 \\ 0 \leq j \leq n - 1}} \\ ((-A)^i p_j(-\alpha))_{\substack{0 \leq i \leq k_2 - 1 \\ 0 \leq j \leq n - 1}} \end{pmatrix} \\ &= \mu^n (-1)^{\frac{k_2(k_2-1)}{2}} \det \begin{pmatrix} (A^i p_j(\alpha))_{\substack{0 \leq i \leq k_1 - 1 \\ 0 \leq j \leq n - 1}} \\ (A^i p_j(-\alpha))_{\substack{0 \leq i \leq k_2 - 1 \\ 0 \leq j \leq n - 1}} \end{pmatrix} \\ &\stackrel{*}{=} \mu^n (-1)^{\frac{k_2(k_2-1)}{2}} \det \begin{pmatrix} (p_j^{(i)}(\alpha))_{\substack{0 \leq i \leq k_1 - 1 \\ 0 \leq j \leq n - 1}} \\ (p_j^{(i)}(-\alpha))_{\substack{0 \leq i \leq k_2 - 1 \\ 0 \leq j \leq n - 1}} \end{pmatrix} \\ &\stackrel{**}{=} c_{k_1 k_2} \mu^n \det \begin{pmatrix} (((-J_{1m}\alpha)^{j-1})^{(i)})_{\substack{0 \leq i \leq k_1 - 1 \\ 1 \leq j \leq n}} \\ (((J_{1m}\alpha)^{j-1})^{(i)})_{\substack{0 \leq i \leq k_2 - 1 \\ 1 \leq j \leq n}} \end{pmatrix} \\ &= c_{k_1 k_2} \mu^n (J_{1m})^{\frac{n(n-1)}{2}} \det \begin{pmatrix} (-\alpha^{j-i})_{\substack{1 \leq i \leq k_1 \\ 1 \leq j \leq n}} \\ (\alpha^{j-i})_{\substack{1 \leq i \leq k_2 \\ 1 \leq j \leq n}} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&\stackrel{***}{=} c'_{k_1 k_2} \mu^n (J_{1m})^{\frac{n(n-1)}{2}} \alpha^{k_1 k_2} \\
&= c'_{k_1 k_2} \frac{(2\pi)^{nm/2}}{\Delta(c_1, \dots, c_{m-1})^{n/2}} e^{-\frac{n}{2} J_{mm} \alpha^2} (J_{1m})^{\frac{n(n-1)}{2}} \alpha^{k_1 k_2} \\
&= c'_{k_1 k_2} (2\pi)^{nm/2} \alpha^{k_1 k_2} \left(\prod_1^{m-1} c_i \right)^{-\frac{n}{2}} e^{-\frac{\alpha^2}{2} n J_{mm} (J_{1m})^{\frac{1}{2} n^2}}.
\end{aligned}$$

In order to evaluate the integer $c'_{k_1 k_2}$, it suffices to notice that this constant is independent of m , so that we may choose $m = 1$, which was done in ([5]). This ends the proof of Lemma 8.1. \blacksquare

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