ON THE HOCHSCHILD–KOSTANT–ROSENBERG MAP FOR GRADED MANIFOLDS

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Abstract. We show that the Hochschild–Kostant–Rosenberg map from the space of multivector fields on a graded manifold N (endowed with a Berezinian volume) to the cohomology of the algebra of multidifferential operators on N (as a subalgebra of the Hochschild complex of $C^\infty(N)$) is an isomorphism of Batalin–Vilkovisky algebras. These results generalize to differential graded manifolds.

1. Introduction

The multivector fields on a smooth manifold M can be seen as multidifferential operators on the algebra $\mathcal{C}^\infty(M)$ of smooth functions on M. This assignment is a particular case of the following general construction: given a graded associative and commutative algebra A, one defines the Hochschild–Kostant–Rosenberg map

$$\mathrm{HKR}\colon \mathcal{V}^\bullet(A) \to \mathsf{Hoch}^\bullet(A)$$

from the space of multivector fields $\mathcal{V}^\bullet(A) := S^\bullet(\mathsf{Der}(A)[-1])[1]$ to the Hochschild complex $\mathsf{Hoch}^\bullet(A)$, as the map which regards a multiderivation of A as a multilinear operator. Actually the image of HKR is contained in the subcomplex $\mathcal{D}^\bullet(A) \subset \mathsf{Hoch}^\bullet(A)$ of multidifferential operators.

If one considers $\mathcal{V}^\bullet(A)$ as a complex with trivial differential, then the HKR map is a morphism of complexes, and the classical Hochschild–Kostant–Rosenberg Theorem [10] states that when A is a smooth algebra, e.g., a polynomial algebra, the HKR map induces isomorphisms in cohomology $\mathcal{V}^\bullet(A) \simeq \mathsf{H}^\bullet(\mathcal{D}^\bullet(A)) \simeq \mathsf{HHoch}^\bullet(A)$. In this paper we are primarily concerned with the case in which A is the algebra of smooth functions on a graded manifold N. In this case it is known that HKR still induces an isomorphism $\mathcal{V}^\bullet(N) \simeq \mathsf{H}^\bullet(\mathcal{D}^\bullet(N))$, where we used the short-hand notations $\mathcal{V}^\bullet(N)$ for $\mathcal{V}^\bullet(\mathcal{C}^\infty(N))$ and $\mathcal{D}^\bullet(N)$ for $\mathcal{D}^\bullet(\mathcal{C}^\infty(N))$; for a proof, see [25] in case N is an ordinary manifold and [2] for the general case.

Many interesting algebraic structures can be defined on the objects introduced above. It is well known that $\mathcal{V}^\bullet(A)$ and $\mathsf{HHoch}^\bullet(A)$ are Gerstenhaber algebras [8], that $\mathsf{H}^\bullet(\mathcal{D}^\bullet(A))$ is a sub-Gerstenhaber algebra of $\mathsf{HHoch}^\bullet(A)$, and that HKR preserves these structures. Moreover, when A is a finite dimensional algebra endowed with a non-degenerate symmetric inner product compatible with the multiplication of A, then $\mathcal{V}^\bullet(A)$, $\mathsf{H}^\bullet(\mathcal{D}^\bullet(A))$ and $\mathsf{HHoch}^\bullet(A)$ become Batalin–Vilkovisky (BV) algebras [24]. The purpose of this paper is to extend this construction to the case in which A is the algebra of smooth functions on a graded manifold N. In this case

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the algebra is not finite dimensional but we can remedy when N has a Berezinian volume. We prove in fact the following

Theorem 5.3. *Let N be a graded manifold endowed with a fixed Berezinian volume v and whose body is a closed smooth manifold. Then $\mathcal{V}^\bullet(N)$ and $\mathsf{H}^\bullet(\mathcal{D}^\bullet(N))$ can be endowed with BV algebra structures compatible with their classical Gerstenhaber structures. Moreover* HKR *is a map of BV algebras.*

The BV algebra structure on multidifferential operators is inspired by [24], whereas the BV structure on $\mathcal{V}^\bullet(N)$ is the standard one on the space of multivector fields of a graded manifold N. Both structures depend on the choice of a Berezinian volume on N [12]. The HKR map lifts to an L_∞ map [11, 2] and, at least in the non graded case, to a G_∞ map [22] between complexes. One may conjecture that it also lifts to a BV_∞ map [23]. This would be the analogue, for a graded manifold, of Kontsevich's cyclic formality conjecture [20].

In the second part of the paper, we generalize our results to differential graded manifolds (N, Q). From an algebraic point of view, this corresponds to considering differential graded commutative associative algebras (A, d). In this case, the Hochschild complex is actually a bicomplex with differentials δ_0 and δ_1, and the Hochschild cohomology will be the cohomology of the total complex. The Hochschild bicomplex and its cohomology will be denoted by $\mathsf{Hoch}^\bullet_{\mathsf{DG}}(A)$ and $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A)$ to distinguish them from the Hochschild complex and cohomology of A seen as a graded algebra. The differential d gives rise to the differential $\{\mathrm{d}, \cdot\}$ on the space $\mathcal{V}^\bullet(A)$ of multivector fields; the HKR map $(\mathcal{V}^\bullet(A), \{\mathrm{d}, \cdot\}, 0) \to (\mathsf{Hoch}^\bullet_{\mathsf{DG}}(A), \delta_0, \delta_1)$ (see Lemma 6.1) is a map of bicomplexes. We show by an example that the induced map in cohomology is not an isomorphism in general. In particular we consider the differential graded manifold $N = T[1]M$, where M is a smooth manifold, with d given by the de Rham differential, so that $\mathcal{C}^\infty(T[1]M)$ is the de Rham algebra $\Omega^\bullet(M)$ of M, and we prove the following

Theorem 6.2. *If M is a simply connected closed oriented smooth manifold of positive dimension, then the HKR map $\mathsf{H}^\bullet(\mathcal{V}^\bullet(\Omega^\bullet(M)), \{\mathrm{d}, \cdot\}) \to \mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$ is not an isomorphism.*

The key ingredient of the proof is the isomorphism [4] between the (shifted) homology $H_\bullet(\mathcal{L}M)[\dim M]$ of the free loop space $\mathcal{L}M$ of M and the Hochschild cohomology of the differential graded algebra $\Omega^\bullet(M)$. We remark that when only ordinary smooth manifolds are considered, it is not known whether the space of multivector fields is quasi-isomorphic to the Hochschild cohomology. Up to our knowledge, only a partial result in this direction is known [16], namely, when M is a smooth manifold, $\mathcal{V}^\bullet(M)$ is quasi-isomorphic to the topological Hochschild complex $\mathsf{HHoch}_{\mathsf{top}}(\mathcal{C}^\infty(M))$ consisting of continuous multilinear homomorphisms (with respect to the Fréchet topology).

If we further assume that (N, Q) is an SQ-manifold, i.e., that the vector field Q is divergence-free, then a BV structure is induced on the cohomology $\mathsf{H}^\bullet(\mathcal{V}^\bullet(A), \{\mathrm{d}, \cdot\})$ and on the Hochschild cohomology $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A)$, and the HKR map is a morphism of BV algebras (although, as remarked above, not an isomorphism in general). An example is the de Rham algebra $(\Omega^\bullet(M), \mathrm{d})$ of a closed manifold M. In this case, the

BV structure on $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$ corresponds to the one found in [3] on the homology of the free loop space [5, 15], whereas the BV structure on $\mathsf{H}^\bullet(\mathcal{V}^\bullet(\Omega^\bullet(M), \{\mathrm{d}, \cdot\}))$ is the trivial one.

The plan of the paper is as follows. We begin by constructing the BV structure on the space of multivector fields in Section 2. Next we recall some facts on Hochschild cohomology in Section 3. Then we discuss BV structures on the space of multidifferential operators in Section 4, and in Section 5 we define the HKR map, describe its main properties, and prove Theorem 5.3. Finally in Sections 6 and 7 we present a generalization of these results to the case of differential graded manifolds and prove Theorem 6.2.

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2. BV structure on multivector fields

Let A be a graded commutative and associative algebra and let $\mathsf{Der}(A) = \oplus_{j\in\mathbb{Z}}\mathsf{Der}^j(A)$ be the graded Lie algebra of derivations of A, namely $\mathsf{Der}^j(A)$ consists of linear maps $\phi\colon A \to A$ of degree j such that $\phi(ab) = \phi(a)b + (-1)^{j\,|a|}a\phi(b)$ and the bracket is $\{\phi, \psi\} = \phi\circ\psi - (-1)^{|\phi||\psi|}\psi\circ\phi$.

The space of multiderivations $\mathcal{V}^\bullet(A) := S^\bullet(\mathsf{Der}(A)[-1])[1]$ can be endowed with a Gerstenhaber structure, with the wedge product and the bracket which is the extension of the graded commutator $\{\cdot,\cdot\}$ on $\mathsf{Der}(A)$ to $\mathcal{V}^\bullet(A)$ by the Leibnitz rule. Since A is graded, the space $\mathcal{V}^\bullet(A)$ has a natural double grading given by

$$\mathcal{V}^{i,j}(A) = \{\phi \in S^i(\mathsf{Der}(A)[1])[-1] \mid \deg(\phi) = j\}.$$

We want to construct an operator Δ on $\mathcal{V}^\bullet(A)$ which makes this Gerstenhaber algebra into a BV algebra. We will use as an auxiliary tool the complex $\mathcal{I}^\bullet(A)$ of integral forms of A, closely following [7]; a different approach to the BV algebra structures on $\mathcal{V}^\bullet(A)$ can be found in [12]. Denote by $\Omega^1(A)$ the space of 1-forms of A, namely, the space $\mathrm{Hom}(\mathcal{V}^1(A), A)$, and assume that the Berezinian $\mathrm{Ber}(\Omega^1(A))$ is free and generated by one element v. To a *divergence operator* div, viz. an even linear map $\mathrm{div}\colon \mathsf{Der}(A) \to A$ satisfying

$$\mathrm{div}(fX) = f\mathrm{div}(X) + (-1)^{|f||X|}X(f),$$

we associate a linear operator $L\colon \mathcal{V}^1(A)\otimes_A \mathrm{Ber}(\Omega^1(A)) \to \mathrm{Ber}(\Omega^1(A))$ by the rule

$$L(X\otimes v) = \mathrm{div}(X)\,v.$$

Observe that for every $f \in A$ and every $X \in \mathsf{Der}(A)$, we have $L_X(fv) = X(f)\,v + (-1)^{|f||X|}f\,L_X(v)$ where we are using the notation $L_X(v) := L(X\otimes v)$.

We now introduce the space $\mathcal{I}^\bullet(A)$ of integral forms [7] as the A-module generated by the elements of $\mathrm{Ber}(\Omega^1(A))$ and by the operations ι_X with $X \in \mathcal{V}^1(A)$, acting on the left and subject to the rules $[\iota_X, \iota_Y] = 0$ and $\iota_{fX} = f\iota_X$. The action of L_X is extended to $\mathcal{I}^\bullet(A)$ by the rule $[L_X, \iota_Y] = \iota_{\{X,Y\}}$ One can define an exterior derivative d on $\mathcal{I}^\bullet(A)$ by imposing $\mathrm{d}v = 0$ and forcing Cartan's identity $\mathrm{d}\iota_X + \iota_X\mathrm{d} = L_X$. Indeed, a consequence of Cartan's formula is that $\mathrm{d}(\iota_{X_1}\cdots\iota_{X_k}v) = L_{X_1}(\iota_{X_2}\cdots\iota_{X_k}v) - \iota_{X_1}\mathrm{d}(\iota_{X_2}\cdots\iota_{X_k}v)$, and the action of d on elements of $\mathcal{I}^\bullet(A)$ can be computed inductively. The exterior derivative d defined by this procedure is a differential precisely when $[L_X, L_Y] = L_{\{X,Y\}}$. This is equivalent to the vanishing of the curvature of div; namely,

$$\mathrm{div}(\{X,Y\}) - X(\mathrm{div}(Y)) + (-1)^{|X|\,|Y|}Y(\mathrm{div}(X)) = 0.$$

Once the generator v of $\mathrm{Ber}(\Omega^1(A))$ is fixed, iterated "contractions" ι_X induce an isomorphism

$$\mathcal{V}^\bullet(A) \xrightarrow{\sim} \mathcal{I}^\bullet(A)$$

and the differential d induces on the space of multivector fields an operator Δ of degree -1 such that $\Delta^2 = 0$. An easy computation shows that $\Delta(X) = \mathrm{div}(X)$ for any $X \in \mathsf{Der}(A)$, and that Δ satisfies the seven term relation

$$\begin{aligned}\Delta(a\wedge b\wedge c) + \Delta(a)\wedge b\wedge c + (-1)^{|a|}a\wedge\Delta(b)\wedge c + (-1)^{|a|+|b|}a\wedge b\wedge\Delta(c) =\\ = \Delta(a\wedge b)\wedge c + (-1)^{|a|}a\wedge\Delta(b\wedge c) + (-1)^{(|a|+1)|b|}b\wedge\Delta(a\wedge c)\end{aligned} \tag{2.1}$$

and the compatibility with the bracket

$$\{a,b\} := (-1)^{|a|}\Big(\Delta(a\wedge b) - \Delta(a)\wedge b - (-1)^{|a|}a\wedge\Delta(b)\Big). \tag{2.2}$$

Therefore we have proved

Lemma 2.1. *If the Berezinian* $\mathrm{Ber}(\Omega^1(A))$ *is a free A-module of rank one and* div *is a curvature-free divergence operator, then the operator* Δ *defined as above endows* $\mathcal{V}^\bullet(A)$ *with a BV structure compatible with the usual Gerstenhaber structure.*

The main example of this construction is when $A = \mathcal{C}^\infty(N)$, N being a graded manifold endowed with a Berezinian volume v. In this case the operators L_X and ι_X are just the classical Lie derivatives and contraction operators, and the complex $\mathcal{I}^\bullet(N)$ is the complex of integral forms of the graded manifold. Since the Berezinian is a line bundle and v is a nowhere zero section, there exists an operator div defined uniquely by the equation $L_Y(v) = \mathrm{div}(Y)\,v$, which is indeed a divergence operator whose curvature vanishes. Observe that in the case when N is an oriented smooth manifold, this amounts to choosing an ordinary volume form v. In the case when $N = T[1]M$, with M an oriented smooth manifold, there is a canonical Berezinian volume v characterized by

$$\int_N \alpha\, v = \int_M \alpha, \qquad \forall\alpha \in C^\infty(N) = \Omega^\bullet(M).$$

Remark 2.2. The geometry of $T[1]M$ is closely related to the geometry of the formal neighborhood of M inside its cotangent bundle T^*M. Namely, the Liouville volume form on T^*M induces a curvature-free divergence operator Δ on $\mathcal{V}^\bullet(T^*M)$, which makes it a BV algebra. The algebra $A = \Gamma(S^\bullet TM)$ of smooth functions on T^*M which are polynomial along the fibers is a BV subalgebra of $\mathcal{V}(T^*M)$; it can be considered as the algebra of multivector fields on T^*M which are "infinitesimal in the cotangent direction". As a consequence of the "Fourier transform" [2, 17], the Gerstenhaber algebras $\mathcal{V}^\bullet(T[1]M)$ and $\mathcal{V}^\bullet(A)$ are isomorphic. But it can be easily verified that they are also isomorphic as BV algebras.

Remark 2.3. For a smooth manifold M, integral forms are just ordinary differential forms and $\mathcal{I}^\bullet(M)$ is naturally identified with $\Omega^\bullet(M)$. On the other hand, for a graded manifold N which is non trivial in odd degrees, the complex $\mathcal{I}^\bullet(N)$ of integral forms is not isomorphic to the de Rham complex of N (see [7] for details).

3. BV structure on Hochschild cohomology

The aim of this Section is to recall some standard facts about Hochschild cohomology and fix notations for the rest of the paper. We address the reader to [13] and [24] for a comprehensive treatment.

3.1. **Hochschild cohomology.** Let $A = \oplus_{j\in\mathbb{Z}} A_j$ be a graded algebra over $\mathbb{R}$, with a graded commutative associative product μ and a unit $\mathbf{1}$. We also suppose that A is endowed with a non degenerate symmetric inner product compatible with the algebra multiplication, namely such that $\langle a,b\rangle = (-1)^{|a|\,|b|}\langle b,a\rangle$ and $\langle \mu(a\otimes b), c\rangle = \langle a, \mu(b\otimes c)\rangle$. Finally, a graded bimodule B over the algebra A is given.

Let us set $T(A) := \bigoplus_{k\geq 0} A^{\otimes k}$ and $T^B(A) := \mathbb{R}\oplus\bigoplus_{k,l\geq 0} A^{\otimes k}\otimes B\otimes A^{\otimes l}$. It is well known that $T(A)$ is a coalgebra and $T^B(A)$ a bi-comodule over $T(A)$ with the coproducts

$$T(A)\to T(A)\otimes T(A)$$
$$(a_1,\dots,a_n)\mapsto \sum_{i=0}^{n}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n)$$

and

$$T^B(A)\to (T(A)\otimes T^B(A))\oplus(T^B(A)\otimes T(A))$$
$$(a_1,\dots,a_k,b,a_{k+1},\dots,a_n)\mapsto \sum_{i=0}^{k}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,b,\dots,a_n)+ + \sum_{i=k}^{n}(a_1,\dots,b,\dots,a_i)\otimes(a_{i+1},\dots,a_n).$$

Hence we can define the space $\mathsf{Coder}(T(A),T^B(A))$, of coderivations from $T(A)$ to $T^B(A)$, with respects to the above coproducts.

The Hochschild cochain complex of A with values in B is defined as

$$\mathsf{Hoch}^\bullet(A,B) := \mathsf{Coder}(T(A[1]), T^{B[1]}(A[1]))[-1]$$

where by $A[1]$ we mean the graded algebra obtained by shifting the degrees of A by 1; namely, $A[1] = \oplus_{j\in\mathbb{Z}}(A[1])_j$ with $(A[1])_j := A_{j+1}$. As usual one can make the identification

$$\mathsf{Hoch}^\bullet(A,B) = \prod_n \mathsf{Hom}(A[1]^{\otimes n}, B[1])[-1] = \prod_n \mathsf{Hom}(A^{\otimes n}, B)[-n].$$

Let us denote by $\widetilde{\mu^B}$ and $\widetilde{\mu}$ the lifts of the bimodule structure $\mu^B\colon A\otimes B\otimes A\to B$ and of the multiplication $\mu\colon A\otimes A\to A$ to coderivations of $T(A[1])$ with values in $T^{B[1]}(A[1])$. Then, on the Hochschild cochain complex we can define a degree 1 differential $\delta^B\colon \mathsf{Hoch}^\bullet(A,B)\to\mathsf{Hoch}^\bullet(A,B)$, by setting $\delta^B(f) := \widetilde{\mu^B}\circ f - (-1)^{|f|} f\circ\widetilde{\mu}$. It is easy to check that $(\delta^B)^2 = 0$; the cohomology of the Hochschild complex with respect to the differential δ^B is called Hochschild cohomology of A with values in B and it is denoted by $\mathsf{HHoch}^\bullet(A,B)$. When $B = A$ with the canonical bimodule structure we write $\mathsf{HHoch}^\bullet(A)$ for $\mathsf{HHoch}^\bullet(A,A)$; moreover δ^A is simply denoted by δ.

Remark 3.1. Since A and B are graded objects, the Hochschild complex $\mathsf{Hoch}(A,B)$ is a bigraded object: in the identification $\mathsf{Hoch}^\bullet(A,B) = \prod_n \mathsf{Hom}(A[1]^{\otimes n}, B[1])[-1]$, the horizontal degree is provided by the number of A-factors, and the vertical degree by the degree of the maps:

$$\mathsf{Hoch}^{i,j}(A,B) = \{f \in \mathsf{Hom}(A[1]^{\otimes i}, B[1])[-1] |\ \deg(f) = j\}.$$

The differential δ^B is a horizontal differential, since it increases the number of factors by one, leaving the degree of the maps unchanged. So one can think of the Hochschild complex as a bicomplex, with horizontal differential $\delta_1^B(f) := \widetilde{\mu^B} \circ f - (-1)^{|f|} f \circ \widetilde{\mu}$ and trivial vertical differential $\delta_0^B := 0$, and to consider δ^B as the total differential $\delta^B = \delta_0^B + \delta_1^B$. We will come back to this point of view when we will be discussing the Hochschild cohomology of differential graded algebras in Section 7.

3.2. **Operations on the Hochschild cochain complex.** On the Hochschild cochain complex $\mathsf{Hoch}^\bullet(A)$ one can define various operations. First, there is a composition $f \circ g$ whose graded antisymmetrization $\{f, g\} := f \circ g - (-1)^{|f|\,|g|} g \circ f$ gives rise to a graded odd Lie bracket of degree $+1$, also known as the Gerstenhaber bracket. Notice that the associativity of the product μ of A is equivalent to $\{\widetilde{\mu}, \widetilde{\mu}\} = 0$, which immediately implies that the Hochschild differential $\delta(f) = \{\widetilde{\mu}, f\}$ indeed squares to zero. Similar relations holds for $\widetilde{\mu^B}$ and δ^B.

Next, using the identification of $\mathsf{Hoch}^\bullet(A)$ with $\prod_{n\geq 0} \mathsf{Hom}(A^{\otimes n}, A)[-n]$ we define a product between $\phi \in \mathsf{Hom}(A^{\otimes k}, A)[-k]$ and $\psi \in \mathsf{Hom}(A^{\otimes l}, A)[-l]$ as

$$(\phi \cup \psi)(a_1 \otimes \cdots \otimes a_{k+l}) := (-1)^\epsilon \mu(\phi(a_1 \otimes \cdots \otimes a_k) \otimes \psi(a_{k+1} \otimes \cdots \otimes a_{k+l})),$$

where $\epsilon = l(|a_1| + \cdots + |a_k| + k)$. This associative product is non-commutative but it gives rise to a graded commutative product in cohomology. The cup product and the Gerstenhaber bracket satisfy in cohomology the graded Leibnitz rule

$$\{a, b \cup c\} = \{a, b\} \cup c + (-1)^{(|a|+1)|b|} b \cup \{a, c\}.$$

Therefore $(\mathsf{HHoch}^\bullet(A), \cup, \{\cdot,\cdot\})$ is a Gerstenhaber algebra [8]. In addition, on the complex $\mathsf{Hoch}^\bullet(A, A^*)$ one has an operator β given by the dual to Connes' B-operator [6]. More explicitly, one defines $\beta\colon \mathsf{Hoch}^\bullet(A, A^*) \to \mathsf{Hoch}^{\bullet-1}(A, A^*)$ as

$$(\beta(f)(a_1, \ldots, a_n))(a_{n+1}) := \sum_{i=1}^{n+1} (-1)^\epsilon (f(a_i, \ldots, a_{n+1}, a_1, \ldots, a_{i-1}))(\mathbf{1})$$

where $\mathbf{1}$ is the unit of A and $\epsilon = |f| + |a_1| + \cdots + |a_{n+1}| + (|a_i| + \cdots + |a_n|)(|a_1| + \cdots + |a_{i-1}|)$.

The inner product on A gives rise to an injection $P\colon A \to A^*$ which is an A-bimodule map, and, by composing the Hochschild cochains with the injection P, one obtains an injective map $\wp\colon \mathsf{Hoch}^\bullet(A) \to \mathsf{Hoch}^\bullet(A, A^*)$. If moreover $\wp$ is a quasi-isomorphism, i.e., induces an isomorphism $H(\wp)$ in cohomology, then we can define an operator Δ_β of degree -1 on $\mathsf{HHoch}^\bullet(A)$ by setting $\Delta_\beta = H(\wp)^{-1} \circ \beta \circ H(\wp)$. As shown in [24] (see also [14]), the operator Δ_β squares to zero in cohomology and is compatible with the Gerstenhaber structure on $\mathsf{HHoch}^\bullet(A)$ in the sense that (cf. equation (2.1))

$$\begin{aligned}\Delta_\beta(a \cup b \cup c) + \Delta_\beta(a) \cup b \cup c + (-1)^{|a|} a \cup \Delta_\beta(b) \cup c + (-1)^{|a|+|b|} a \cup b \cup \Delta_\beta(c) = \\ = \Delta_\beta(a \cup b) \cup c + (-1)^{|a|} a \cup \Delta_\beta(b \cup c) + (-1)^{(|a|+1)|b|} b \cup \Delta_\beta(a \cup c)\end{aligned}$$

and (cf. equation (2.2))

$$\{a, b\} = (-1)^{|a|} \Big(\Delta_\beta(a \cup b) - \Delta_\beta(a) \cup b - (-1)^{|a|} a \cup \Delta_\beta(b) \Big).$$

In other words $(\mathsf{HHoch}^\bullet(A), \cup, \{\cdot,\cdot\}, \Delta_\beta)$ is a BV algebra. Summing up, we have

Proposition 3.2. *If the map $\wp\colon \mathsf{Hoch}^\bullet(A) \to \mathsf{Hoch}^\bullet(A, A^*)$ induced by the inner product of A is a quasi-isomorphism, then $\mathsf{HHoch}^\bullet(A)$ is endowed with a BV algebra structure, compatible with its Gerstenhaber structure.*

A trivial example is when A is finite dimensional, and hence $\wp$ is an isomorphism. A more interesting case is the algebra of functions on a graded manifold N endowed with a Berezinian volume v. In this case the pairing is defined by

$$\langle f_1, f_2\rangle = \int_N f_1 f_2\, v. \tag{3.1}$$

In general, when N is a graded manifold, $\mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N))$ is not necessarily quasi-isomorphic to $\mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N), \mathcal{C}^\infty(N)^*)$, and hence we do not know whether we can define a BV structure on $\mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N))$. However we will see in Section 4 that a version of Proposition 3.2 can be applied to a certain subcomplex of the Hochschild complex, namely to the subcomplex of multidifferential operators.

4. BV structure on multidifferential operators

The Hochschild complex of A has a sub-Gerstenhaber algebra $\mathcal{D}^\bullet(A)$ consisting of multidifferential operators, namely sums of cochains of the form $(a_1, \ldots, a_n) \mapsto \prod_{i=1}^n \phi_i(a_i)$ where ϕ_i are compositions of derivations. The bigrading on the Hochschild complex induces a bigrading on the subalgebra of multidifferential operators:

$$\mathcal{D}^{i,j}(A) := \mathcal{D}^\bullet(A) \cap \mathsf{Hoch}^{i,j}(A).$$

We now want to discuss under which conditions the cohomology of $\mathcal{D}^\bullet(A)$ admits a natural BV structure. As above we are assuming that there exists a non degenerate symmetric inner product on A compatible with the multiplication, and hence an injective map $\wp\colon \mathsf{Hoch}^\bullet(A) \to \mathsf{Hoch}^\bullet(A, A^*)$. The point is to determine when the Connes cyclic β-operator $\beta\colon \mathsf{Hoch}^\bullet(A, A^*) \to \mathsf{Hoch}^{\bullet-1}(A, A^*)$ induces an operator $\Delta_\beta\colon \mathcal{D}^\bullet(A) \to \mathcal{D}^{\bullet-1}(A)$ making the diagram

$$\begin{tikzcd}
\mathcal{D}^\bullet(A) \arrow[r, "\wp"] \arrow[d, dashed, "\Delta_\beta"'] & \mathsf{Hoch}^\bullet(A, A^*) \arrow[d, "\beta"] \\
\mathcal{D}^{\bullet-1}(A) \arrow[r, "\wp"] & \mathsf{Hoch}^{\bullet-1}(A, A^*)
\end{tikzcd}$$

commutative. To answer this question, we look at the problem from a more general perspective; namely, let $C^\bullet(A)$ be any sub-Gerstenhaber algebra of $\mathsf{Hoch}^\bullet(A)$ whose $\wp$-image in $\mathsf{Hoch}^\bullet(A, A^*)$ is closed under β. Since $\wp$ is injective, β induces a well-defined operator Δ_β on the complex $C^\bullet(A)$. Following [24] and [14], the operator Δ_β squares to zero in the cohomology of $C^\bullet(A)$, and endows $\mathsf{H}^\bullet(C^\bullet(A))$ with a BV algebra structure compatible with its Gerstenhaber structure.

We now specialize to the case when $A = C^\infty(N)$, where N is a graded manifold endowed with a Berezinian volume v. In order to prove that the cohomology $\mathsf{H}^\bullet(\mathcal{D}^\bullet(N))$ of the algebra of multidifferential operator admits a natural BV structure, we only need to prove that $(\beta \circ \wp)(\mathcal{D}^\bullet(N)) \subseteq \wp(\mathcal{D}^\bullet(N))$ with $\wp$ induced by the pairing (3.1). We first need the following "integration-by-parts" Lemma.

Lemma 4.1. *Let D be a multidifferential operator. Then there exist a multidifferential operator $\tilde{D}$ such that*

$$\langle D(f_1, \ldots, f_n), \mathbf{1}\rangle = \langle \tilde{D}(f_1, \ldots, f_{n-1}), f_n\rangle$$

Then we observe that for every $D \in \mathcal{D}^n(N)$ and for every $i = 1, \ldots, n$, the operator

$$D_i(f_1, \ldots, f_n) := D(f_i, \ldots, f_n, f_1, \ldots, f_{i-1}), \qquad f_1, \ldots, f_n \in A,$$

is still in $\mathcal{D}^n(N)$. Finally

$$(\beta \circ \wp(D))(f_1, \ldots, f_{n-1})(f_n) = \sum_{i=1}^{n} (-1)^{\epsilon} \langle D(f_i, \ldots, f_n, f_1, \ldots, f_{i-1}), \mathbf{1}\rangle =$$

$$= \sum_{i=1}^{n} (-1)^{\epsilon} \langle D_i(f_1, \ldots, f_n), \mathbf{1}\rangle = \sum_{i=1}^{n} (-1)^{\epsilon} \langle \tilde{D}_i(f_1, \ldots, f_{n-1}), f_n\rangle =$$

$$= \wp\left(\sum_{i=1}^{n} (-1)^{\epsilon} \tilde{D}_i\right)(f_1, \ldots, f_{n-1})(f_n).$$

Proof of Lemma 4.1. The proof is by induction on the order of the multidifferential operator D. If D is homogeneous of order zero,

$$D(f_1, \ldots, f_n) = \lambda f_1 \cdots f_n$$

for some constant λ, so that

$$\langle D(f_1, \ldots, f_n), \mathbf{1}\rangle = \int_N \lambda f_1 \cdots f_n \, v = \langle \lambda f_1 \cdots f_{n-1}, f_n\rangle$$

and we are done. Now assume the claim proved for operators up to order k and prove it for order $k+1$ operators by the following argument. A homogeneous component of an order $k+1$ multidifferential operator can be written as

$$D(f_1, \ldots, f_n) = D_0(f_1, \ldots f_{i-1}, X(f_i), f_{i+1}, \ldots, f_n)$$

for a suitable multidifferential operator D_0 of order k, some index i and some vector field X. We compute

$$\langle D(f_1, \ldots, f_n), \mathbf{1}\rangle = \langle D_0(f_1, \ldots, X(f_i), \ldots, f_n), \mathbf{1}\rangle$$

Here we have to distinguish two cases. If $i \neq n$, by the induction hypothesis applied to D_0, we can write

$$\langle D_0(f_1, \ldots, X(f_i), \ldots, f_n), \mathbf{1}\rangle = \langle \tilde{D}_0(f_1, \ldots, X(f_i), \ldots, f_{n-1}), f_n\rangle$$

and we are done. If $i = n$ then the induction hypothesis gives

$$\langle D_0(f_1, \ldots, f_{n-1}, X(f_n)), \mathbf{1}\rangle = \langle \tilde{D}_0(f_1, \ldots, f_{n-1}), X(f_n)\rangle.$$

For any vector field Y, Cartan's formula gives $L_Y(v) = \mathrm{d}i_Y(v) + i_Y\mathrm{d}(v) = \mathrm{d}i_Y(v)$, since $\mathrm{d}(v) = 0$ [7]. Hence, by Stokes' Theorem we have that

$$0 = \int_N \mathrm{d}i_Y(f\, v) = \int_N Y(f)\, v + (-1)^{|f|\,|Y|} \int_N f\, L_Y(v).$$

Recall for Section 2 that there exists an operator div defined uniquely by the equation $L_Y(v) = \mathrm{div}(Y)\, v$. Therefore

$$\langle Y(f), \mathbf{1}\rangle = \int_N Y(f)\, v = -(-1)^{|f|\,|Y|} \int_N f\, \mathrm{div}(Y)\, v = -\langle \mathrm{div}(Y), f\rangle. \tag{4.1}$$

Going back to our problem with D_0, we apply the previous formula to the vector field $Y = \tilde{D}_0(f_1, \ldots, f_{n-1})X$ and obtain

$$\begin{aligned}\langle \tilde{D}_0(f_1, \ldots, f_{n-1}), X(f_n)\rangle &= \int_N \tilde{D}_0(f_1, \ldots, f_{n-1})X(f_n)\, v \\ &= \langle \mathrm{div}(\tilde{D}_0(f_1, \ldots, f_{n-1})X), f_n\rangle.\end{aligned}$$

The map $(f_1, \cdots, f_{n-1}) \mapsto \mathrm{div}(\tilde{D}_0(f_1, \ldots, f_{n-1})X)$ is a multidifferential operator, and the Lemma is proved by setting $\tilde{D}(f_1, \ldots, f_{n-1}) = \mathrm{div}(\tilde{D}_0(f_1, \ldots, f_{n-1})X)$. □

5. The Hochschild–Kostant–Rosenberg map

The Hochschild–Kostant–Rosenberg (HKR) map is defined as follows:

$$\begin{array}{ccc} \mathcal{V}^\bullet(A) & \longrightarrow & \mathsf{Hoch}^\bullet(A) \\ \phi_1 \wedge \cdots \wedge \phi_n & \mapsto & \dfrac{1}{n!} \displaystyle\sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\ \phi_{\sigma(1)} \cup \cdots \cup \phi_{\sigma(n)}. \end{array} \tag{5.1}$$

Note that the HKR map is actually a map of bigraded vector spaces: $\mathcal{V}^{i,j}(A) \to \mathsf{Hoch}^{i,j}(A)$. We have already observed that both $\mathcal{V}^\bullet(A)$ and $\mathsf{HHoch}^\bullet(A)$ are Gerstenhaber algebras, and it is well known that the HKR map in fact preserves these structures. More explicitly

Theorem 5.1. *If $\mathcal{V}^\bullet(A)$ is endowed with the zero differential, then* HKR *is a morphism of complexes. Moreover the induced map in cohomology is a morphism of Gerstenhaber algebras.*

Proof. This is a standard result: the fact that HKR respects the product structures in cohomology follows directly from the fact that the cup product is commutative in cohomology [8]. An easy check shows that for $X, Y \in \mathsf{Der}(A)$ we have

$$\{\mathrm{HKR}(X), \mathrm{HKR}(Y)\} - \mathrm{HKR}(\{X, Y\}) = 0$$

and hence, by the compatibility between the bracket and the product, HKR induces in cohomology a map of Gerstenhaber algebras. □

The classical Theorem of Hochschild, Kostant and Rosenberg [10] states that when A is a smooth algebra (e.g. for the coordinate ring of a smooth affine algebraic variety) then the HKR map is a quasi-isomorphism, i.e., induces an isomorphism $\mathcal{V}^\bullet(A) \xrightarrow{\sim} \mathsf{HHoch}^\bullet(A)$.

One sees from equation (5.1) that the HKR map actually takes its values in the subcomplex $\mathcal{D}^\bullet(A)$ of multidifferential operators. For a smooth algebra A, the inclusion $\mathcal{D}^\bullet(A) \hookrightarrow \mathsf{Hoch}^\bullet(A)$ is a quasi-isomorphism, so the classical Hochschild-Kostant-Rosenberg theorem can then be stated as follows.

Theorem 5.2. *If A is a smooth algebra, then* $\mathrm{HKR}\colon \mathcal{V}^\bullet(A) \to \mathsf{H}^\bullet(\mathcal{D}^\bullet(A))$ *is an isomorphism of Gerstenhaber algebras.*

Our main result is a version of Theorem 5.2 for graded manifolds, namely, we prove

Theorem 5.3. *Let N be a graded manifold endowed with a fixed Berezinian volume v and whose body is a smooth closed manifold. Then $\mathcal{V}^\bullet(N)$ and $\mathsf{H}^\bullet(\mathcal{D}^\bullet(N))$ can be endowed with BV algebra structures compatible with their classical Gerstenhaber structures. Moreover* $\mathrm{HKR}\colon \mathcal{V}^\bullet(N) \to \mathsf{H}^\bullet(\mathcal{D}^\bullet(N))$ *is an isomorphism of BV algebras.*

Proof. We have seen in Sections 2 and 4 that, in case $A = \mathcal{C}^\infty(N)$ is the algebra of smooth functions of a graded manifold N endowed with a Berezinian volume form, then both $\mathcal{V}^\bullet(N)$ and $\mathsf{H}^\bullet(\mathcal{D}^\bullet(N))$ are BV algebras in a way compatible with their classical Gerstenhaber structures.

We know from Theorem 5.1 that HKR induces in cohomology a morphism of Gerstenhaber algebras. Moreover we know from [2] that $\mathrm{HKR}\colon \mathcal{V}^\bullet(N) \to \mathcal{D}^\bullet(N)$ is a quasi-isomorphism. Therefore, by the compatibility between the BV Laplacian and the Gerstenhaber bracket, we only need to prove that for every vector field $X \in \mathcal{V}^1(N)$ on a graded manifold N, we have

$$\mathrm{HKR}(\Delta(X)) = \Delta_\beta(\mathrm{HKR}(X)).$$

To see this, consider the diagram

$$\begin{tikzcd}
\mathcal{V}^1(N) \arrow[r, "\mathrm{HKR}"] \arrow[d, "\Delta"'] & \mathcal{D}^1(N) \arrow[r, "\wp"] \arrow[d, "\Delta_\beta"'] & \mathsf{Hoch}^1(\mathcal{C}^\infty(N), \mathcal{C}^\infty(N)^*) \arrow[d, "\beta"'] \\
\mathcal{V}^0(N) \arrow[r, "\mathrm{HKR}"] & \mathcal{D}^0(N) \arrow[r, "\wp"] & \mathsf{Hoch}^0(\mathcal{C}^\infty(N), \mathcal{C}^\infty(N)^*)
\end{tikzcd}$$

Since the diagram on the right commutes and $\wp$ is injective, commutativity of the diagram on the left follows from the commutativity of the external diagram. This is indeed the case since on the one side, for $X \in \mathcal{V}^1(N)$ and $f \in \mathcal{C}^\infty(N)$, we have that

$$(\beta(\wp(\mathrm{HKR}(X))))\,(f) = -\langle X(f), \mathbf{1}\rangle, \tag{5.2}$$

on the other side

$$(\wp(\mathrm{HKR}(\Delta(X))))\,(f) = \langle \Delta(X), f\rangle. \tag{5.3}$$

By Section 2, $\Delta(X) = \mathrm{div}(X)$, and the right-hand sides of equations (5.2) and (5.3) coincide by means of equation 4.1. $\square$

6. The HKR theorem for differential graded manifolds

We now consider the more general case of differential graded manifolds, i.e., of graded manifolds N endowed with a degree 1 integrable vector field Q. Note that, since the degree of Q is 1, the integrability condition $\{Q, Q\} = 0$ is equivalent to $Q^2 = 0$. The algebraic counterpart of a differential graded manifold (N, Q) is a differential graded algebra (A, d), where d is a degree one differential on the graded algebra A. A classical example is given by the de Rham algebra $(\Omega^\bullet(M), \mathrm{d})$ of a differential manifold M with the de Rham differential. The corresponding graded manifold is $T[1]M$; the de Rham differential on differential forms corresponds to a degree 1 integrable vector field on $T[1]M$. Note that ordinary graded manifolds can be considered as differential graded manifolds with the trivial vector field $Q = 0$.

The construction of the Hochschild complex of a graded algebra A with values in B described in Section 3 generalizes to the case of a differential graded algebra (A, d). In this case one actually gets a nontrivial vertical differential by setting $\delta_0^B(f) := \widetilde{\mathrm{d}} \circ f - (-1)^{|f|} f \circ \widetilde{\mathrm{d}}$, where $\widetilde{\mathrm{d}}$ denotes the lift of the differential $\mathrm{d}\colon A \to A$, to coderivations of $T(A[1])$ with values in $T^{B[1]}(A[1])$. The horizontal differential δ_1^B is the same as in the case of graded algebras described in Section 3. One easily checks that the total differential $\delta^B = \delta_0^B + \delta_1^B$ squares to zero. We show this in the particular case $B = A$, the general case being similar. By definition,

$\delta_1 = \{\widetilde{\mu}, \cdot\}$ and $\delta_0 = \{\widetilde{\mathrm{d}}, \cdot\}$; the associativity of the product μ is equivalent to $\{\widetilde{\mu}, \widetilde{\mu}\} = 0$, the fact that d is a derivation for μ is equivalent to $\{\widetilde{\mathrm{d}}, \widetilde{\mu}\} = 0$, and $\mathrm{d}^2 = 0$ is equivalent to $\{\widetilde{\mathrm{d}}, \widetilde{\mathrm{d}}\} = 0$. These three properties immediately imply that the Hochschild differential $\delta(f) = \{\widetilde{\mu} + \widetilde{\mathrm{d}}, f\}$ indeed squares to zero.

The total complex will be denoted by $\mathsf{Hoch}_{\mathsf{DG}}(A, B)$; its cohomology is called Hochschild cohomology of A with values in B and it is denoted by $\mathsf{HHoch}^{\bullet}_{\mathsf{DG}}(A, B)$, where the subscript DG means that we are working in the category of differential graded algebras. Clearly, one recovers the Hochschild cohomology of a graded algebra A by considering it as a differential graded algebra with trivial differential. When $B = A$ with the canonical bimodule structure, we write $\mathsf{HHoch}^{\bullet}_{\mathsf{DG}}(A)$ for $\mathsf{HHoch}^{\bullet}_{\mathsf{DG}}(A, A)$. As in the graded case, the differential graded Hochschild complex $\mathsf{Hoch}_{\mathsf{DG}}(A)$ has a graded Lie algebra structure, and both δ_0 and δ_1 are operators of adjoint type for this Lie algebra structure.

Since the vector field Q squares to zero, it induces a differential on the algebra of multivector fields of the differential graded manifold (N, Q). Algebraically, this amounts to saying that the operator $\{\mathrm{d}, \cdot\}$ acts as a differential on $\mathcal{V}^{\bullet}(A)$. We can therefore look at $\mathcal{V}^{\bullet}(A)$ as a bicomplex: the horizontal differential is zero, and the vertical differential is $\{\mathrm{d}, \cdot\}$. We have a HKR map $\mathcal{V}^{\bullet}(A) \to \mathsf{Hoch}_{\mathsf{DG}}(A)$, which is defined as in the case of differential algebras.

Lemma 6.1. *The HKR map* $(\mathcal{V}^{\bullet}(A), \{\mathrm{d}, \cdot\}, 0) \to (\mathsf{Hoch}_{\mathsf{DG}}(A), \delta_0, \delta_1)$ *is a map of bicomplexes.*

Proof. What we have said on the HKR map for graded algebras implies that $\mathrm{HKR}\colon (\mathcal{V}^{\bullet}(A), 0) \to (\mathsf{Hoch}_{\mathsf{DG}}(A), \delta_1)$ is a map of complexes. So we are left with checking the compatibility of $\{\mathrm{d}, \cdot\}$ with the differential δ_0. This follows from the following more general fact: given a vector field X and a multivector field Y, then $\mathrm{HKR}(\{X, Y\}) = \{\mathrm{HKR}(X), \mathrm{HKR}(Y)\}$, as one can easily verify. Note that for an arbitrary multivector field X, the above identity only holds up to homotopy. Since $\delta_0(\mathrm{HKR}(Y)) = \{\mathrm{HKR}(\mathrm{d}), \mathrm{HKR}(Y)\}$, this concludes the proof. □

Being compatible with the differentials, the HKR map induces a map between the cohomologies of the total complexes $\mathsf{H}^{\bullet}(\mathcal{V}^{\bullet}(A), \{\mathrm{d}, \cdot\}) \to \mathsf{HHoch}^{\bullet}_{\mathsf{DG}}(A)$, which is a map of graded Lie algebras. In contrast with the case of smooth algebras which are the subject of the classical HKR theorem, this map is not an isomorphism in general, as the next theorem shows.

Theorem 6.2. *If M is a simply connected closed oriented smooth manifold of positive dimension, then the HKR map* $\mathsf{H}^{\bullet}(\mathcal{V}^{\bullet}(\Omega^{\bullet}(M)), \{\mathrm{d}, \cdot\}) \to \mathsf{HHoch}^{\bullet}_{\mathsf{DG}}(\Omega^{\bullet}(M))$ *is not an isomorphism.*

We need the following Lemma, relating the $\{\mathrm{d}, \cdot\}$-cohomology of multivector fields on $T[1]M$ to the de Rham cohomology of M:

Lemma 6.3. *For any differential manifold M, there is an isomorphism*

$$\mathsf{H}^{\bullet}(\mathcal{V}^{\bullet}(\Omega^{\bullet}(M), \{\mathrm{d}, \cdot\})) \simeq \mathsf{H}^{\bullet}_{\mathrm{deRham}}(M).$$

Proof. Recall that $\mathcal{V}^{\bullet}(\Omega^{\bullet}(M))$ is the algebra of multivector fields on the graded manifold $T[1]M$. We fix local coordinates $\{x^i, \theta^j\}$ on $T[1]M$, where x^i are (even) coordinates on M and θ^j (odd) coordinates on the fibers. Consider the globally well-defined derivation ι_E which on the local generators of multivector fields acts as

$$\iota_E\left(x^i\right) = 0\,; \quad \iota_E\left(\theta^i\right) = 0\,; \quad \iota_E\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial \theta^i}\,; \quad \iota_E\left(\frac{\partial}{\partial \theta^i}\right) = 0\,.$$

The derivation $\{\mathrm{d}, \cdot\}$ acts as

$$\left\{\mathrm{d}, x^i\right\} = \theta^i\,; \quad \left\{\mathrm{d}, \theta^i\right\} = 0\,; \quad \left\{\mathrm{d}, \frac{\partial}{\partial x^i}\right\} = 0\,; \quad \left\{\mathrm{d}, \frac{\partial}{\partial \theta^i}\right\} = \frac{\partial}{\partial x^i}\,.$$

It follows that $L_E = \{\mathrm{d}, \cdot\} \circ \iota_E + \iota_E \circ \{\mathrm{d}, \cdot\}$ is a derivation on $\mathcal{V}(T[1]M)$ which, when restricted to the fields of degree m, is the multiplication by m; namely

$$L_E\left(x^i\right) = 0\,; \quad L_E\left(\theta^i\right) = 0\,; \quad L_E\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\,; \quad L_E\left(\frac{\partial}{\partial \theta^i}\right) = \frac{\partial}{\partial \theta^i}\,.$$

Now, suppose that Ψ is a $\{\mathrm{d}, \cdot\}$-closed multivector field of degree $m \geq 1$. Then it is also $\{\mathrm{d}, \cdot\}$-exact:

$$\begin{aligned}\Psi &= \frac{1}{m} L_E(\Psi) = \frac{1}{m}\{\mathrm{d}, \iota_E \Psi\} + \frac{1}{m}\iota_E(\{\mathrm{d}, \Psi\}) \\ &= \{\mathrm{d}, \frac{1}{m}\iota_E \Psi\}\end{aligned}$$

This shows that higher cohomology groups vanish, and we are left to prove that $\mathsf{H}^0(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) = \mathsf{H}^\bullet_{\mathrm{deRham}}(M)$. To see this, just notice that the 0-vector fields on $T[1]M$ are the differential forms on M and the action of $\{\mathrm{d}, \cdot\}$ on $\mathcal{V}^0(T[1]M)$ is precisely the action of the de Rham differential on $\Omega^\bullet(M)$. $\square$

Proof of Theorem 6.2. Let $\mathcal{L}M$ be the free loop space on M. On the one hand we have Chen's isomorphism [4, 9]

$$\mathsf{H}_\bullet(\mathcal{L}M)[\dim M] \simeq \mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M)).$$

On the other hand, we have the isomorphism

$$\mathsf{H}^\bullet(\mathcal{V}^\bullet(\Omega^\bullet(M), \{\mathrm{d}, \cdot\})) \simeq \mathsf{H}^\bullet_{\mathrm{deRham}}(M)$$

from Lemma 6.3. Finally, $\mathsf{H}_\bullet(\mathcal{L}M)[\dim M] \not\simeq \mathsf{H}^\bullet_{\mathrm{deRham}}(M)$ for any simply connected closed oriented smooth manifold of positive dimension [21]. $\square$

Remark 6.4. Observe that another way of proving Lemma 6.3 goes through the Gerstenhaber isomorphism described in Remark 2.2. In fact, it is not difficult to see that the image of the multivector field d under this isomorphism is the restriction to $A = \Gamma(S^\bullet TM)$ of the canonical Poisson bivector field on the symplectic manifold T^*M. Thus, $\mathsf{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\})$ is isomorphic to the Poisson cohomology of T^*M (restricted to functions polynomial along the fibers) which in turn (by nondegeneracy of the Poisson structure) is isomorphic to the de Rham cohomology of the total space and hence of the base.

7. BV structures in the differential graded case

By forgetting the differential, i.e., by looking at a differential graded manifolds simply as a graded manifolds, we obtain a BV structure on the space of their multivector fields, as in Section 2. In general, this BV structure does not induce a BV structure on the $\{Q, \cdot\}$-cohomology of multivector fields. Indeed, the BV generator Δ is a derivation of the BV bracket, so it does not map $\{Q, \cdot\}$-closed vector fields to $\{Q,\cdot\}$-closed vector fields. Rather, if X is a $\{Q,\cdot\}$-closed vector field, then

$$\{Q,\Delta(X)\}=\{\Delta(Q),X\}.$$

Yet, this implies that, if the vector field Q is divergence-free, i.e., if $\Delta(Q)=0$ then Δ induces a BV structure on the $\{Q,\cdot\}$-cohomology, since

$$\Delta\{Q,X\}=-\{Q,\Delta(X)\}$$

and so $\{Q,\cdot\}$-exact multivector fields are mapped to $\{Q,\cdot\}$-exact multivector fields. Note that, since the divergence operator Δ we are considering in this paper is defined as the variation of the Berezinian volume form of N along a vector field, the condition $\Delta(Q)=0$ means that the volume form is Q-invariant. A differential graded manifold (N,Q) with a Q-invariant Berezinian volume form is usually called an SQ-manifold [18, 19].

Remark 7.1. In case N is an odd symplectic manifold and the vector field Q is Hamiltonian, one speaks of PQ-manifolds [1]. Note that, if $Q=H_S$, i.e., if S is the function on N whose Hamiltonian vector field is Q, then $\operatorname{div}(Q)=\Delta(S)$ and $\{Q,Q\}=H_{\{S,S\}}$ where on the right we have the odd Poisson bracket associated to the odd symplectic structure on N. Therefore, under the mild assumption that S has at least one critical point, the two equations $\{Q,Q\}=0$ and $\operatorname{div}(Q)=0$ imply the quantum master equation for S, namely $\Delta(S)+\frac{\sqrt{-1}}{2\hbar}\{S,S\}=0$.

As far as concerns the BV structures on Hochschild cohomology, the same construction we described in Section 3 also works in the differential graded case: if (A,d) is the differential graded algebra of functions on the SQ-manifold (N,Q), then we have a BV structure on $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A)$ under the hypothesis that $\wp\colon \mathsf{Hoch}^\bullet_{\mathsf{DG}}(A)\to \mathsf{Hoch}^\bullet_{\mathsf{DG}}(A,A^*)$ is a quasi-isomorphism. Moreover, by the same argument used in Section 5, the HKR map $\mathsf{H}^\bullet(\mathcal{V}^\bullet(A),\{\mathrm{d},\cdot\})\to \mathsf{Hoch}^\bullet_{\mathsf{DG}}(A)$ is a BV map in this case.

An example is given by the de Rham algebra $(\Omega^\bullet(M),\mathrm{d})$ of a smooth closed manifold M. In the coordinates $\{x^i,\theta^j\}$ on $T[1]M$, the de Rham differential on $\Omega^\bullet(M)$ is written

$$\mathrm{d}=\sum_{i=1}^{\dim M}\theta^i\frac{\partial}{\partial x^i},$$

so that its divergence is

$$\operatorname{div}(\mathrm{d})=\sum_{i=1}^{\dim M}\frac{\partial\theta^i}{\partial x^i}=0.$$

The pairing on $\Omega^\bullet(M)$ induced by the canonical Berezinian volume form on $T[1]M$ is the usual Poincaré duality pairing:

$$\langle\omega_1,\omega_2\rangle=\int_M\omega_1\wedge\omega_2.$$

The induced map $\wp\colon \mathsf{Hoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))\to \mathsf{Hoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M),\Omega^\bullet(M)^*)$ is a quasi-isomorphism [15], and so there exists a BV algebra structure on $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$. This BV algebra structure coincides, via Chen's isomorphism

$$\mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))\simeq \mathsf{H}_\bullet(\mathcal{L}M)[\dim M],$$

with the Chas–Sullivan BV structure on the homology of the free loop space of M [3, 4, 5, 9, 15, 24]. Also the $\{\mathrm{d},\cdot\}$-cohomology of $\mathcal{V}^\bullet(\Omega^\bullet(M))$ has a nice geometrical interpretation: we have shown in the proof of Lemma 6.3 that

$$\mathsf{H}^p(\mathcal{V}^\bullet(\Omega^\bullet(M)), \{\mathrm{d}, \cdot\}) = \begin{cases} 0 & \text{if } p \neq 0 \\ \mathsf{H}^\bullet_{\mathrm{deRham}}(M) & \text{if } p = 0. \end{cases}$$

Note that, since the $\{\mathrm{d}, \cdot\}$-cohomology of $\mathcal{V}^\bullet(\Omega^\bullet(M))$ is concentrated in degree zero, the BV structure on $\mathsf{H}^\bullet(\mathcal{V}^\bullet(\Omega^\bullet(M)), \{\mathrm{d}, \cdot\})$ is the trivial one. Finally, the BV map $\mathsf{H}^\bullet(\mathcal{V}^\bullet(\Omega^\bullet(M)), \{\mathrm{d}, \cdot\}) \to \mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$ is the natural map

$$\mathsf{H}^\bullet_{\mathrm{deRham}}(M) \simeq \mathsf{H}_\bullet(M)[\dim M] \to \mathsf{H}_\bullet(\mathcal{L}M)[\dim M]$$

induced by the natural embedding $M \hookrightarrow \mathcal{L}M$ which identifies the points of M with the constant loops in $\mathcal{L}M$.

Remark 7.2. The constructions of Section 4 also work in the differential graded case: a BV structure is defined on the total cohomology of any sub-Gerstenhaber algebra $C^\bullet_{\mathsf{DG}}(A)$ of $\mathsf{Hoch}^\bullet_{\mathsf{DG}}(A)$, whose $\wp$-image in $\mathsf{Hoch}^\bullet_{\mathsf{DG}}(A, A^*)$ is closed under β. This way we obtain a BV structure on the total cohomology of multidifferential operators on an SQ-manifold. Moreover, the HKR map $\mathsf{H}^\bullet(\mathcal{V}^\bullet(A), \{\mathrm{d}, \cdot\}) \to \mathsf{H}^\bullet_{\mathsf{DG}}(\mathcal{D}^\bullet(A))$ is a BV map.

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