Nonlinear Gravitons, Null Geodesics, and Holomorphic Disks

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Abstract

We develop a global twistor correspondence for pseudo-Riemannian conformal structures of signature $(++--)$ with self-dual Weyl curvature. Near the conformal class of the standard indefinite product metric on $S^2 \times S^2$, there is an infinite-dimensional moduli space of such conformal structures, and each of these has the surprising global property that its null geodesics are all periodic. Each such conformal structure arises from a family of holomorphic disks in $\mathbb{CP}_3$ with boundary on some totally real embedding of $\mathbb{RP}^3$ into $\mathbb{CP}_3$. An interesting sub-class of these conformal structures are represented by scalar-flat indefinite Kähler metrics, and our methods give particularly sharp results in this more restrictive setting.

1 Introduction

Twistor correspondences, as pioneered by Roger Penrose [42], provide a way of understanding certain differential geometries as fundamentally arising from moduli spaces of compact complex curves in a complex manifold. It has emerged only recently, however, that an analogous pattern of phenomena can also be expected to arise from moduli spaces of compact complex curves-with-boundary in a complex manifold, where the boundaries of the curves are constrained to lie in a maximal totally real submanifold. Our previous work in this direction [31] focused on spaces of holomorphic disks in $\mathbb{CP}_2$, with boundaries on a totally real embedding of $\mathbb{RP}^2$. In the present article, we will see that a similarly rich geometric story arises from the moduli space of holomorphic disks in $\mathbb{CP}_3$ with boundaries on a totally real $\mathbb{RP}^3$.

Penrose's original twistor correspondence, which he called the *nonlinear graviton*, hinged on the idea that self-dual conformal metrics on 4-manifolds tend to arise from suitable holomorphic families of $\mathbb{CP}_1$'s in complex 3-manifolds. Penrose's formulation of these ideas involved local analytic continuations of real-analytic geometries into the complex domain, thereby making the metric signature essentially irrelevant. Nonetheless, it was specifically the positive-definite realm of Riemannian geometry that witnessed the most intensive subsequent cultivation of these ideas, a

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development largely attributable to the elegant and definitive global Riemannian reformulation of the Penrose correspondence discovered by Atiyah, Hitchin, and Singer [2]. By contrast, however, the present article will focus entirely on 4-manifolds with *split-signature* metrics, meaning pseudo-Riemannian metrics of signature $(++--)$; these have elsewhere been called *neutral metrics* [27], and are characterized by the fact that they have components

$$\begin{bmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

in a suitably chosen basis for any given tangent space. What we will develop here is a global twistor correspondence for self-dual split-signature 4-manifolds in which *every null geodesic is a simple closed curve*. Such metrics will turn out to naturally arise as moduli spaces for holomorphic disks in $\mathbb{CP}_3$ with boundary on a fixed totally real submanifold.

As in the Riemannian case, a split-signature metric g on an oriented 4-manifold M is said to be *self-dual* if its Weyl (or conformal) curvature tensor, considered as a bundle-valued 2-form, is its own Hodge star; cf. §3 below. This is a conformally invariant condition, and should therefore primarily be thought of as a constraint on the conformal class $[g] = \{fg \mid f \neq 0\}$ of the metric. Notice that any locally conformally flat split-signature metric on an oriented 4-manifold is automatically self-dual.

For us, the protypical example is the indefinite product metric

$$g_0 = \pi_1^* h - \pi_2^* h$$

on $S^2 \times S^2$, where $\pi_1, \pi_2 : S^2 \times S^2 \to S^2$ are the two factor projections and h is the standard homogeneous metric on S^2. This metric is actually conformally flat, since, thinking of $S^2 \times S^2$ as the locus

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad y_1^2 + y_2^2 + y_3^2 = 1,$$

in $\mathbb{R}^{3,3} = \mathbb{R}^3 \times \mathbb{R}^3$, and introducing 'stereographic' coordinates by

$$\begin{aligned} \mathfrak{x}_1 &= \frac{x_1}{2(x_3 - y_3)} & \qquad \mathfrak{x}_2 &= \frac{x_2}{2(x_3 - y_3)} \\ \mathfrak{y}_1 &= \frac{y_1}{2(x_3 - y_3)} & \qquad \mathfrak{y}_2 &= \frac{y_2}{2(x_3 - y_3)} , \end{aligned} \tag{1}$$

the metric can be re-expressed in the form

$$g_0 = \frac{d\mathfrak{x}_1^2 + d\mathfrak{x}_2^2 - d\mathfrak{y}_1^2 - d\mathfrak{y}_2^2}{\mathfrak{x}_1^2 + \mathfrak{x}_2^2 + [\mathfrak{y}_1^2 + \mathfrak{y}_2^2 - \mathfrak{x}_1^2 - \mathfrak{x}_2^2 + \frac{1}{4}]^2} .$$

However, this example has a second fundamental property that will play a crucial rôle in this paper. Indeed, the null geodesics of $(S^2 \times S^2, g_0)$ are all *embedded circles*, since each is obtained by simultaneously traversing a great circle in each S^2 with equal speed. Following Guillemin [16], we will use the word *Zollfrei* to describe pseudo-Riemannian metrics with this property; for a detailed discussion, see §2 below. The Zollfrei condition is also conformally invariant, so that we may consider it as yet another property of the conformal class $[g]$.

Among all split-signature metrics on a given manifold, the Zollfrei condition is highly non-generic. It may therefore seem surprising that it becomes an *open* condition when restricted to the subspace of self-dual metrics:

Theorem A *Let (M, g) be a self-dual Zollfrei 4-manifold. Then, with respect to the C^2 topology, there is an open neighborhood of g in the space of pseudo-Riemannian metrics on M such that every self-dual metric contained in this neighborhood is also Zollfrei.*

For the purpose of studying the moduli of self-dual conformal structures, it thus seems reasonable to focus for the present on understanding those self-dual metrics which are also Zollfrei.

But this point of view immediately prompts us to ask, "Which 4-manifolds admit self-dual Zollfrei metrics?" We have just seen that $S^2 \times S^2$ is one such manifold. Another example is given by the projective quadric

$$\mathbb{M}^{2,2} = \left\{ [x_1 : x_2 : x_3 : y_1 : y_2 : y_3] \in \mathbb{RP}^5 \;\middle|\; |\vec{x}|^2 - |\vec{y}|^2 = 0 \right\} ,$$

which may be viewed as the quotient of $(S^2 \times S^2, g_0)$ by the isometric $\mathbb{Z}_2$-action generated by the double antipodal map

$$(\vec{x}, \vec{y}) \mapsto (-\vec{x}, -\vec{y}).$$

However, we will show in §5 that these are the only topological possibilities:

Theorem B *Let (M, g) be a connected oriented split-signature 4-manifold which is both Zollfrei and self-dual. Then M is homeomorphic to either $S^2 \times S^2$ or $\mathbb{M}^{2,2}$.*

This topological rigidity, however, is by no means symptomatic of any kind of underlying *geometric* rigidity. To the contrary, our central purpose here is to prove the following *flexibility* result:

Theorem C *There is a natural one-to-one correspondence between*

- *equivalence classes of smooth self-dual split-signature conformal structures on $S^2 \times S^2$; and*
- *equivalence classes of totally real embeddings $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}_3$,*

at least in a neighborhood of the standard conformal metric $[g_0]$ and the standard embedding of $\mathbb{RP}^3$.

Here, two conformal structures are considered to be equivalent iff one is the pull-back of the other via some orientation-preserving self-diffeomorphism of $S^2 \times S^2$; two embeddings $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}_3$ are considered to be equivalent iff they are interrelated by a reparameterization of $\mathbb{RP}^3$ and/or the action of $PSL(4, \mathbb{C})$ on $\mathbb{CP}_3$. In particular, the moduli space of self-dual Zollfrei conformal structures on $S^2 \times S^2$ is infinite-dimensional; and, roughly speaking, the general such conformal structure depends on 3 free functions of 3 variables. The correspondence between the two kinds of structures depends on the existence of an $(S^2 \times S^2)$-family of holomorphic disks with boundary on a given totally real $\mathbb{RP}^3 \subset \mathbb{CP}_3$.

By contrast, the same arguments also show that $(\mathbb{M}^{2,2}, [g_0])$ has no non-trivial self-dual deformations. Indeed, by analogy with the Blaschke conjecture [5, 31], we are tempted to speculate that, up to conformal isometry, $(\mathbb{M}^{2,2}, [g_0])$ might well be the only non-simply-connected self-dual Zollfrei 4-manifold.

Finally, it is interesting to observe that g_0 may be viewed as an indefinite scalar-flat Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$, and that, conversely, any indefinite scalar-flat Kähler metric on a complex surfaces is automatically self-dual. In this regard, our techniques lead to the following:

Theorem D *The only complex surface (M, J) admitting Zollfrei scalar-flat indefinite Kähler metrics is $\mathbb{CP}_1 \times \mathbb{CP}_1$. Every such metric arises from a family of analytic disks in $\mathbb{CP}_3$ with boundary on a totally real $\mathbb{RP}^3$. Near the standard metric g_0, moreover, indefinite scalar-flat Kähler metrics of fixed total volume are in one-to-one correspondence with those totally real embeddings $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}^3 - Q$ on which the pull-back of the 3-form*

$$\phi = \Im m \frac{z_1 dz_2 \wedge dz_3 \wedge dz_4 - \cdots - z_4 dz_1 \wedge dz_2 \wedge dz_3}{(z_1^2 + z_2^2 + z_3^2 + z_4^2)^2}$$

vanishes. Here Q denotes the quadric surface $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$.

In particular, the moduli space of such metrics is once again infinite-dimensional.

In the special setting of metrics with circular symmetry, indefinite scalar-flat Kähler metrics on $\mathbb{CP}_1 \times \mathbb{CP}_1$ were previously investigated by Tod [49] and, independently, by Kamada [21], both of whom discovered that infinite-dimensional families of such metrics can be written down in closed form by means of the Lorentzian analogue of the first author's hyperbolic ansatz [29]. We thus believe that the chief interest of the present article must be found, not in the mere infinite-dimensionality of the relevant moduli space, but, rather, in the manner in which our holomorphic disk picture allows one to explore this interesting, geometric, non-linear ultra-hyperbolic second-order equation in terms of a first-order elliptic boundary-value problem.

2 Zollfrei Metrics

If (M, g) is an indefinite pseudo-Riemannian manifold, a geodesic $\gamma \subset M$ is said to be a *null geodesic* if $g(v, v) = 0$ for any vector v tangent to γ. We will primarily consider these null geodesics as *unparameterized* curves, even though g endows them with a preferred class of so-called affine parameters. The reason behind this point of view is that the null geodesics of a pseudo-Riemannian manifold (M, g) are *conformally invariant* as unparameterized curves; that is, $f g$ has the same null geodesics as g, for any non-zero function f on M. Indeed, let $\mathscr{H} \subset T^*M$ be the hypersurface of non-zero null co-vectors, and notice that $\mathscr{H}$ is foliated by a unique system of curves tangent to $\ker(\omega|_{\mathscr{H}})$, where ω is the usual symplectic form on T^*M. The Hamiltonian formalism then tells us that that the projections into M of these integral curves are precisely the null geodesics of g. The conformal invariance of null geodesics is thus an immediate consequence of the conformal invariance of $\mathscr{H}$.

Manifolds for which every null geodesic is a simple closed curve will play a central rôle in this paper, and, following Guillemin [16], we will therefore introduce some convenient terminology to describe such spaces:

Definition 2.1 *An indefinite pseudo-Riemannian manifold* (M, g) *will be called* Zollfrei *if the image of each of its maximally extended null geodesics is an embedded circle* $S^1 \subset M$.

Notice that this condition is conformally invariant, so it makes perfectly good sense to say that $(M, [g])$ is Zollfrei, where

$$[g] = \{f g \mid f : M \to \mathbb{R}^\times\}$$

denotes the conformal class determined by the metric g.

Guillemin's definition [16] is actually a good deal more stringent than Definition 2.1. Let $\mathcal{Q} \subset \mathbb{P}T^*M$ denote the quotient $\mathscr{H}/\mathbb{R}^\times$, where $\mathbb{R}^\times = \mathbb{R} - \{0\}$ acts by scalar multiplication on T^*M. The lifts of null geodesics then define a foliation of $\mathcal{Q}$ by curves. Guillemin's definition then amounts to the following:

Definition 2.2 *Let* (M, g) *be a Zollfrei manifold. We will say that* M *is* strongly Zollfrei *if the foliation of* $\mathcal{Q}$ *by lifted null geodesics is a (locally trivial) circle fibration.*

Since this condition is obviously also conformally invariant, the strongly Zollfrei condition will also be considered as primarily pertaining to the conformal class $[g]$ rather than to the particular metric g representing it.

If $(M, [g])$ is a strongly Zollfrei n-manifold, we may defined its *space of null geodesics* N to be the leaf-space of the null-geodesic foliation of $\mathcal{Q}$. Because the foliation is assumed to be a locally trivial circle fibration, N is then automatically a smooth manifold of dimension $2n - 3$. The symplectic description of the foliation endows N with a *contact structure*, meaning a maximally non-integrable codimension-1 sub-bundle $C \subset TN$ of the tangent bundle. Concretely, the tangent space $T_\gamma N$ of N at a null geodesic γ is locally represented on M as equivalence classes of solutions w of Jacobi's equation

$$\nabla_v \nabla_v w = R_{vw}(v)$$

subject to the constraint

$$g(v, w) = \text{constant}$$

and the equivalence relation

$$w \sim w + (a + bt)v,$$

where R is the curvature tensor of g, t is a local affine parameter for γ, $v = d/dt$, and a and b are constants; in these terms, the contact sub-bundle $C \subset TN$ then corresponds to those Jacobi fields w which satisfy the constraint

$$g(v, w) = 0.$$

If $\mathcal{L} \to N$ is the line-subundle of T^*N consisting of the 1-forms which annihilate C, then $\mathcal{L}^\times := \mathcal{L} - 0_N$ is a symplectic submanifold of T^*N, called [1] the *symplectification* of the contact manifold (N, C). However, the pull-back of $\mathcal{L}^\times$ to $\mathcal{Q}$ can be canonically identified with $\mathscr{H}$, and the Marsden-Weinstein reduction of $\mathscr{H} \subset T^*M$ is therefore globally well defined. This shows that the null geodesic foliation must actually be periodic up on $\mathscr{H}$, and not just down on $\mathcal{Q}$. Thus, no matter which metric g we choose in the conformal class $[g]$, the the null geodesics of a strongly Zollfrei manifold are all automatically *periodic* with respect to their affine parameters. (This conclusion should be contrasted with the closed but non-periodic null geodesics [18] of the Taub-NUT metric and related examples.) For this reason, Definition 2.2 is logically equivalent to the definition used by Guillemin in [16].

3 Self-Duality

Suppose that M is an oriented 4-manifold, and that g is a split-signature pseudo-Riemannian metric. Then, as in the Riemannian case, the Hodge star operator $\star : \Lambda^2 \to \Lambda^2$ satisfies $\star^2 = +\mathbf{1}$, so there is an invariant splitting

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

of the 2-forms into the (± 1)-eigenspaces of $\star$. The inner product induced by g is of Lorentz signature on both $\Lambda^\pm$, reflecting the fact that $SO_+(2,2)$ is a double cover of $SO_+(1,2) \times SO_+(1,2)$. Sections of Λ^+ (respectively, Λ^-) are called *self-dual* (respectively, *anti-self-dual*) forms. Thinking of the curvature tensor $\mathcal{R}$ of g as a linear map $\mathcal{R} : \Lambda^2 \to \Lambda^2$, we thus obtain a decomposition

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

of the Riemann tensor into simpler pieces. Here W_+ and W_- are the trace-free pieces of the appropriate blocks, and are called the *self-dual* and *anti-self-dual Weyl curvatures*, respectively. The scalar curvature s is understood to act by scalar multiplication, whereas $\mathring{r}$ is a disguised form of the trace-free part of the Ricci curvature tensor.

Definition 3.1 *An oriented split-signature pseudo-Riemannian* 4*-manifold* (M, g) *is called* self-dual *if it satisifes* $W_- \equiv 0$.

This condition is conformally invariant, in the sense that if g is self-dual, so is the metric fg, where $f : M \to \mathbb{R}^\times$ is any non-zero function. Thus the self-duality condition should fundamentally be understood as pertaining to a conformal class

$$[g] = \{fg \mid f : M \to \mathbb{R}^\times\}$$

rather than to a particular metric g representing it.

If (M, g) is a pseudo-Riemannian manifold, we will say that a real linear subspace Π of a tangent space T_xM is *isotropic* if it consists entirely of null vectors. Notice that this is a conformally invariant condition. If (M, g) is an oriented split-signature 4-manifold, then the space of isotropic 2-planes in TM has two connected components, each of which is a circle bundle over M. Indeed, if $\Pi \subset T_xM$ is an isotropic 2-plane, then $\wedge^2\Pi$ corresponds, by index-lowering, to a null 1-dimensional subspace of either Λ^+ or Λ^-. In the first case, one says that Π is an *α-plane*, whereas in the second case one says that Π is a *β-plane*. We will henceforth use $\mathfrak{p} : F \to M$ to denote the circle-bundle of β-planes over an oriented split-signature 4-manifold (M, g).

Definition 3.2 *An immersed connected surface in $S \looparrowright M$ will be called a* proto-β-surface *if its tangent space T_xS is a β-plane for all $x \in S$. If, in addition, the proto-β-surface S is maximal, in the sense that it is not a proper subset of a larger proto-β-surface, we will say that S is a* β-surface.

Lemma 3.3 *Let $(M, [g])$ be an oriented 4-manifold with split-signature conformal metric, and let $S \looparrowright M$ be any proto-β-surface. Then the second fundamental form of S vanishes. Consequently, S is totally geodesic.*

Proof. The tangent bundle of any proto-β-surface S is locally spanned by vector fields v and w with $[v, w] = 0$ and

$$g(v, v) = g(w, w) = g(v, w) = 0.$$

Now notice $\nabla_v w = \nabla_w v$, and hence

$$g(v, \nabla_v w) = g(v, \nabla_w v) = \tfrac{1}{2} w g(v, v) = 0,$$

whereas we also have

$$g(w, \nabla_v w) = \tfrac{1}{2} v g(w, w) = 0.$$

However, $TS = TS^\perp$ with respect to g, so we conclude that

$$\nabla_v w \in TS.$$

Similarly,

$$g(w, \nabla_w w) = \tfrac{1}{2} w g(w, w) = 0,$$

and

$$g(v, \nabla_w w) = w g(v, w) - g(\nabla_v w, w) = 0,$$

so we must have

$$\nabla_w w \in TS,$$

too. Thus

$$\nabla : \Gamma(TS) \times \Gamma(TS) \to \Gamma(TS).$$

In other words, the second fundamental form

$$\begin{aligned} \mathrm{I\!I} : TS \times TS &\to TM/TS \\ (v, w) &\mapsto \nabla_v w \bmod TS \end{aligned}$$

vanishes. Equivalently, S is totally geodesic, in the sense that any geodesic tangent to S at some point necessarily remains within S. ■

This observation then allows one to prove the following:

Lemma 3.4 *Let $(M, [g])$ be an oriented 4-manifold with split-signature conformal structure. Then the following are equivalent:*

(*i*) *$[g]$ is self-dual;*

(*ii*) *every β-plane $\Pi \subset TM$ is tangent to some proto-β-surface;*

(*iii*) *if $\Pi \subset TM$ is any β-plane, and if $v, w \in \Pi$, then $\mathcal{R}_{vw}v \in \Pi$, too.*

Proof. Suppose that $\Pi \subset TM$ is a β-plane, and let v and w span Π. Then, using g to freely identify vectors and 1-forms via index lowering, $v \wedge w \in \Lambda^-$ and $\langle v \wedge w, v \wedge w\rangle = 0$. Hence

$$\begin{aligned} g(w, \mathcal{R}_{vw}v) &= \langle v \wedge w, \mathcal{R}(v \wedge w)\rangle \\ &= \left\langle v \wedge w, \left(W_- + \frac{s}{12}\right)(v \wedge w)\right\rangle \\ &= \langle v \wedge w, W_-(v \wedge w)\rangle . \end{aligned}$$

On the other hand,

$$g(v, \mathcal{R}_{vw}v) = 0$$

by the Bianchi identities. Since $\Pi = \Pi^\perp$ with respect to g, and because W_- is a trace-free quadratic form on Λ^-, it therefore follows that (*iii*) is equivalent to requiring that $W_- \equiv 0$. Hence (*i*) $\Longleftrightarrow$ (*iii*).

Now if S is any proto-β-surface in M, and if v and w are any vector fields on S, then $\nabla_v w \in TS$ by Lemma 3.3. The Riemann curvature tensor of $\mathcal{R}$ of g thus satisfies

$$\mathcal{R}_{vw}v = \nabla_v\nabla_w v - \nabla_w\nabla_v v - \nabla_{[v,w]}v \in TS$$

whenever S is a proto-β-surface and $v, w \in TS$. When every β-plane Π can be expressed as T_xS for some proto-β-surface, it thus follows that condition (*iii*) holds. Hence (*ii*) $\Longrightarrow$ (*iii*).

Conversely, suppose that (*iii*) holds. Let $\Pi \subset T_xM$ be a β-plane, and let γ be any null geodesic through x tangent to Π. Let $\mathit{\Pi} \to \gamma$ be the rank-2 sub-bundle of $TM|_\gamma$ obtained from Π by parallel transport of along γ, and notice that each fiber of $\mathit{\Pi}$ is the unique β-plane containing $v = \gamma'$. Hypothesis (*iii*) therefore guarantees that

$$w \in \mathit{\Pi} \Longrightarrow \mathcal{R}_{vw}v \in \mathit{\Pi},$$

and a solution of Jacobi's equation

$$\nabla_v\nabla_v w = \mathcal{R}_{vw}v \tag{2}$$

along γ is therefore a section of $\mathit{\Pi}$ iff $w|_x, (\nabla_v w)|_x \in \Pi$. Now let $U \subset \Pi$ be an open disk about $0 \in \Pi$ which is sufficiently small so as to be mapped diffeomorphically to a surface $S = \exp(U)$ with $T_xS = \Pi$ by the exponential map of g. Then S is a union of null geodesics γ through x, and along each such γ we have

$$T_{\tilde{x}}S = \left\{ w|_{\tilde{x}} \;\middle|\; w \text{ solves (2) along } \gamma, w|_x = 0, (\nabla_v w)|_x \in \Pi \right\}$$

for each $\tilde{x} \neq x$. The above argument thus shows that the tangent spaces of S are precisely the β-planes obtained from Π by parallel transport along radial null geodesics. In particular, S is a proto-β-surface tangent to the given β-plane Π.

Thus $(iii) \Longrightarrow (ii)$, and our proof is complete. ■

Now given an oriented split-signature 4-manifold (M, g), let us consider the bundle $\mathfrak{p} : F \to M$ of β-planes. We may define a 2-dimensional real distribution $E \subset TF$ by declaring that its value at Π is the horizontal lift of $\Pi \subset TM$ to $T_\Pi F$. Then every proto-β-surface in M has a canonical lift as an integral surface of E, and conversely every integral surface of E projects to a proto-β-surface in M. In this way, we see that E is integrable iff (M, g) is self-dual. In particular, when (M, g) is a self-dual, there is a foliation $\mathscr{F}$ of F tangent to E, and we can then obtain (maximal) β-surfaces in (M, g) by projecting the leaves of $\mathscr{F}$ into M via $\mathfrak{p} : F \to M$. Lemma 3.4 thus implies

Proposition 3.5 *The following assertions regarding an oriented split-signature* 4*-manifold* (M, g) *are logically equivalent:*

- g *is self-dual;*
- *each* β*-plane* $\Pi \subset TM$ *is tangent to a unique* β*-surface* $S \looparrowright M$*;*
- *the distribution of* 2*-planes* $E \to F$ *is Frobenius integrable.*

Since Lemma 3.3 tells us that each β-surface S is totally geodesic, the Levi-Civita connection ∇ of the ambient metric g induces a torsion-free connection $\triangledown$ on S whose geodesics of $\triangledown$ are precisely those null geodesics of (M, g) which are contained in S. But, as we saw in §2, null geodesics are conformally invariant as unparameterized curves, so the *projective class* $[\triangledown]$ of this induced connection [31, 44] therefore depends only on the ambient conformal class $[g]$.

Proposition 3.6 *Let* (M, g) *be a self-dual split-signature* 4*-manifold, let* $S \looparrowright M$ *be any* β*-surface, and let* $\triangledown$ *be the connection induced on* S *by restriction of the Levi-Civita connection* ∇ *to* S*. Then* $\triangledown$ *is* projectively flat*. Indeed, there is a local diffeomorphism* $\phi : \tilde{S} \to \mathbb{RP}^2$*, where* $\tilde{S}$ *is the universal cover of* S*, which maps each geodesic to a portion of some projective line.*

Proof. Locally, we have a 3-parameter family of β-surfaces in M, and, by taking the derivatives at S, these define a 3-dimensional space of sections of the normal bundle TM/TS of S. These are the covariantly constant local sections for the natural flat connection on the normal bundle TF/TS of lift of S to F induced by the integrable distribution E; and we obtain a natural 3-dimensional space of local sections of TM/TS by pushing forward parallel local sections of TF/TS via the derivative of $\mathfrak{p}$. Moreover, these sections can be taken to be *global* sections on the universal cover $\tilde{S}$ of S, since the pull-back of TF/TS to $\tilde{S}$ is not only flat, but actually has trivial holonomy.

We can describe these local sections more concretely by exploiting our fixed metric g in the self-dual conformal class $[g]$. Indeed, since TS is maximally isotropic with respect to g, our metric induces a non-degenerate pairing

$$TS \times (TM/TS) \to \mathbb{R},$$

thus giving us an isomorphism between the cotangent bundle T^*S and the normal bundle TM/TS of S. Thus a section of the normal bundle precisely corresponds to a 1-form φ on S. We claim that a 1-form arises from a 1-parameter family of β-surfaces iff it satisfies the *generalized Killing equation*

$$\nabla\varphi = \frac{1}{2}d\varphi\ , \tag{3}$$

which, according to ones taste, can be rewritten either as

$$\nabla_j\varphi_k + \nabla_k\varphi_j = 0$$

or as

$$(\nabla\varphi)(v, v) = 0 \quad \forall v.$$

Let us first demonstrate the 'only if' direction of this assertion. Suppose we have a 1-parameter family of proto-β-surfaces obtained by moving some open subset $U \subset S$. Then we can foliate these surfaces by null geodesics in a smooth manner, say with tangent vector field v. The vector field u representing the variation then satisfies $[u, v] = 0$, and projects to the section of TM/TS along U which represents the first variation of the family. The 1-form φ on U representing the first variation is then given by

$$\varphi(w) = g(u, w) \quad \forall w \in TS.$$

But now

$$\begin{aligned}(\nabla\varphi)(v, v) &= g(\nabla_v u, v)\\ &= g(\nabla_u v, v)\\ &= \tfrac{1}{2}ug(v, v)\\ &= 0.\end{aligned}$$

Since v can be chosen to point in any direction at any point of U, it follows that $\nabla\varphi$ must be skew-symmetric, and φ is therefore a solution of (3).

Now (3) is an over-determined equation, and a solution φ is completely determined by its 1-jet at a point of S. To see this, observe that we certainly have

$$\text{Alt }(\nabla\nabla\varphi) = 0,$$

since S does not carry any non-zero 3-forms. Using (3), however, this six-term identity can be rewritten as the three-term identity

$$\nabla_j\nabla_k\varphi_\ell = \nabla_\ell\nabla_k\varphi_j - \nabla_k\nabla_\ell\varphi_j,$$

and we may then notice that the right-hand side is just a curvature term. Along a null geodesic $\gamma \subset S$ with tangent field v, φ therefore satisfies the ordinary differential equation

$$\nabla_v\nabla_v\varphi = \varphi(\mathcal{K}_{v\bullet}v), \tag{4}$$

where $\mathcal{K}$ is the curvature tensor of ∇, and where the right-hand-side denotes the 1-form

$$w \mapsto \varphi(\mathcal{K}_{vw}v).$$

Since a solution of (4) is completely determined by the value of φ and $\nabla_v\varphi$ at one point, it follows that solutions of (3) are completely determined by the value of φ and $\nabla\varphi$ at one point of a convex subset $U \subset S$. But (3) then tells us that a solution is consequently determined by the value of the 1-form φ and the 2-form $d\varphi$ at one point. This shows that the space of solutions is at most 3-dimensional. But since the codimension of the leaves in F is exactly 3, we conclude that the space of solutions of (3) must be exactly 3-dimensional up on the universal cover $\tilde{S}$ of S.

Thus, let $\mathbb{V} \cong \mathbb{R}^3$ be the space of solutions of (3) on $\tilde{S}$, and let $\mathbb{P}(\mathbb{V}) \cong \mathbb{RP}^2$ be the corresponding real projective space. For each $x \in \tilde{S}$, set

$$\mathbb{L}_x = \{\text{solutions } \varphi \text{ of (3) on } \tilde{S} \text{ for which } \varphi|_x = 0\},$$

and notice that this is a 1-dimensional subspace of $\mathbb{V}$, since the freedom in choosing such a solution amounts to specifying the value of the 2-form $d\varphi$ at x. We may thus define a map

$$\begin{array}{rcl} \phi : \tilde{S} & \longrightarrow & \mathbb{P}(\mathbb{V}) \\ x & \mapsto & \mathbb{L}_x. \end{array}$$

Let us then first notice that any geodesic γ is sent to a projective line by this map, because equation (3) implies that $\varphi(v) = \text{constant}$, where v is an autoparallel tangent field for γ; thus $\phi(\gamma) \subset \mathbb{P}(\mathbb{V}_0)$, where $\mathbb{V}_0 \subset \mathbb{V}$ is the plane defined by $(\varphi|_x)(v) = 0$ for some arbitrary $x \in \gamma$. Now let t be an affine parameter along γ, with $v = d/dt$, and let w be a parallel vector field along γ which is linearly independent from v. Then the restriction of an element of $\mathbb{V}_0$ to γ satisfies $\varphi(v) \equiv 0$ and $\varphi(w) = f(t)$, where f is a solution of the second order linear ordinary differential equation

$$\frac{d^2 f}{dt^2} + \kappa f = 0, \tag{5}$$

where $\kappa(t) = \mathcal{K}^2{}_{121}$ with respect to the frame $e_1 = v$, $e_2 = w$. If $\{f_1, f_2\}$ is a basis for the solution space of (5), then ϕ sends γ to $\mathbb{P}(\mathbb{V}_0) \cong \mathbb{RP}^1$ by $t \mapsto [f_2(t) : -f_1(t)]$. However, equation (5) implies that the Wronskian $W = f_1 f_2' - f_1' f_2$ is a non-zero constant along γ. Thus at least one of the expressions

$$\frac{d}{dt}\left(\frac{-f_1}{f_2}\right) = \frac{W}{f_2^2} \quad \text{and} \quad \frac{d}{dt}\left(\frac{f_2}{-f_1}\right) = -\frac{W}{f_1^2}$$

is defined and non-zero at each point of γ, and ϕ thus sends γ to the projective line $\mathbb{P}(\mathbb{V}_0) \subset \mathbb{P}(\mathbb{V})$ via a smooth immersion. Since the geodesic γ is arbitrary, it follows that $\phi : \tilde{S} \to \mathbb{P}(\mathbb{V})$ is an equidimensional smooth immersion sending each geodesic to a portion of a projective line. In particular, the connection $\triangledown$ induced on the β-surface S is projectively flat. ■

The above proof is loosely based on a spinor argument given in [28]. The careful reader may notice that, in its present form, the proof is not manifestly conformally invariant. However, it is not terribly difficult to embellish the argument so as to achieve this end. The main point is that the g-induced map $TM/TS \to T^*S$ actually carries a conformal weight, so that the 1-form fields φ under discussion may better be described as 1-forms with values in a line bundle.

It is also worth pointing out that the above result depends quite strongly on the assumption that M is self-dual. Indeed, it is not difficult to construct non-self-dual 4-manifolds with isolated β-surfaces on which the induced connection is *not* projectively flat. For example, let (Σ, h) be an oriented Riemannian 2-manifold of *non-constant* Gauss curvature. Since [44] a torsion-free connection ∇ on a surface is projectively flat iff its Ricci curvature ρ satisfies

$$2\nabla_j\rho_{k\ell} - 2\nabla_k\rho_{j\ell} + \nabla_j\rho_{\ell k} - \nabla_l\rho_{\ell j} = 0,$$

it follows that the Riemannian connection of such a generic metric h is not projectively flat. Now give $\Sigma \times \Sigma$ the indefinite product metric $\pi_1^* h - \pi_2^* h$, and observe that the diagonal $\Sigma \hookrightarrow \Sigma \times \Sigma$ becomes a β-surface if we endow $\Sigma \times \Sigma$ with the non-product orientation. However, the induced connection on this β-surface is just the Riemannian connection of h, so this β-surface is *not* projectively flat. Of course, this example in no way contradicts Proposition 3.6, since the 4-manifold in question is non-self-dual.

4 Projectively Flat Surfaces

Proposition 3.6 reveals that surfaces with flat projective connections play an important rôle in the theory of split-signature self-dual manifolds. The systematic study of surfaces with flat projective structures, also known as $\mathbb{RP}^2$-structures, was begun by Kuiper [24], who in particular observed that if $(S, [\nabla])$ is any projectively flat surface, the locally trivial nature of the geometry always gives rise to a developing map $\phi : \tilde{S} \to \mathbb{RP}^2$, defined on the universal cover $\tilde{S}$ of S, as well as a representation of $\phi : \pi_1(S) \to PGL(3, \mathbb{R})$, both of which are unique up to an overall $PGL(3, \mathbb{R})$ transformation. Crucial explorations of this idea by Sullivan and Thurston [47] eventually allowed Choi and Goldman [7] to develop a substantially complete theory of flat projective structures on *compact* surfaces.

In this article, we will be specifically interested in the case when the relevant projective structure is *Zoll*, meaning [31] that every geodesic is a simple closed curve. It seems quite plausible that a connected surface which admits a Zoll projective connection must necessarily be compact, but, to our knowledge, this still seems to be an open problem. Fortunately, however, the *projectively flat* case of the problem is a bit more manageable.

Lemma 4.1 *Let $(S, [\nabla])$ be a connected surface with flat projective structure, and suppose that every maximal geodesic of $[\nabla]$ is a simple closed curve in S. Then S is compact.*

Proof. It suffices to consider the case when S is *orientable*, since otherwise we may pass to an oriented double cover without sacrificing the assumption that every geodesic is a simple closed curve.

Since S is assumed to be Zoll, every (maximal) geodesic $\gamma \subset S$ is an embedded circle; and because we may now assume that S is orientable, any such γ has an open neighborhood $U \subset S$ diffeomorphic to an annulus $S^1 \times (-\epsilon, \epsilon)$. Now develop the universal cover $\tilde{U}$ of U onto the 2-sphere in such a manner that γ is sent to some portion of the equator $x_3 = 0$, sending a chosen base-point to $(1, 0, 0)$.We orient the equator in the usual west-to-east manner, and give γ the corresponding orientation. Let $I \subset \tilde{U}$ be an arc (that is, an embedded closed interval) such that $\gamma \subset U \subset S$ is obtained from I by identifying its two endpoints via the covering map $\tilde{U} \to U$, and such that the initial end-point is a pre-image of the chosen base-point for S. Then the restriction of the developing map to some open neighborhood of $I \subset \tilde{U}$ lifts to the universal cover V of $S^2 - \{(\pm 1, 0, 0)\}$, where V may may be explicitly identified with $\mathbb{R} \times (-\pi/2, \pi/2)$ through the use of spherical coordinates

$$(x_1, x_2, x_3) = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi), \qquad (\theta, \varphi) \in \mathbb{R} \times (-\pi/2, \pi/2).$$

This lift of the development then takes I diffeomorphically onto a closed interval, say, $\tilde{I} = [0, L] \times \{0\}$ in V. Thus, a perhaps smaller neighborhood U' of $\gamma \in S$ can be obtained from a neighborhood V' of $\tilde{I} \subset V$ by identifying some neighborhood V_1 of $(0, 0)$ with a neighborhood V_2 of $(L, 0)$; moreover, this identification is carried out via a lift of the action of some $A \in SL(3, \mathbb{R})$ of $S^2 = (\mathbb{R}^3 - 0)/\mathbb{R}^+$. Notice that this transformation A must send the equator to itself, in an orientation-preserving manner. Hence $(0, 0, 1)$ must be an eigenvector of A^t, with eigenvalue $\lambda > 0$.

Let us now examine the action of A^t on the space $\mathbb{RP}^{2*} = \mathbb{P}(\mathbb{R}^{3*})$ of great circles in S^2. We have just observed that $[0, 0, 1]$ must be a fixed point of this action. But our hypotheses also preclude the existence of a point $p \in \mathbb{RP}^{2*}$, $p \neq [0, 0, 1]$, such that $\lim_{n\to\infty}(A^t)^n(p) = [0, 0, 1]$. Indeed, if there were such p, the great circles corresponding to the iterates $(A^t)^n(p)$ would, for $n \gg 0$, link up end-to-end via A to form part of a geodesic $\gamma' \neq \gamma$ in the annulus $U \subset S$ which spiraled into γ; every point of γ would then be an accumulation point of γ', and as the Zoll hypothesis implies that γ' must be a closed subset of S, this would imply that $\gamma \subset \gamma'$, contradicting the fact that γ is a maximal geodesic. We can also run this argument backwards in parameter time by replacing A with A^{-1}, and thereby deduce that there cannot be any point $p \in \mathbb{RP}^{2*}$, $p \neq [0, 0, 1]$, such that $\lim_{n\to\infty}(A^t)^{-n}(p) = [0, 0, 1]$. The complex eigenvalues of A must therefore all have the same modulus. Moreover, there cannot be a vector $v \in \mathbb{R}^{3*}$ such that $A^t(v) = \lambda v + (0, 0, 1)$. Hence $A \in SL(3, \mathbb{R})$ can be put in one of the normal forms

$$(a) \quad \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad (b) \quad \begin{bmatrix} \pm 1 & 1 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by an appropriate change of basis of the x_1x_2-plane.

Now suppose that A has normal form (a). Then any geodesic with initial point and tangent direction close to that of γ will remain in our annular neighborhood U; indeed, every such geodesic is explicitly represented in our (θ, φ) coordinates as the union of the graphs

$$\varphi = \tan^{-1}(t\sin(\theta - \theta_0 + k\psi)), \quad \theta \in [0, \psi], \quad k \in \mathbb{Z},$$

for t and θ_0 given constants, with t is sufficiently small, and where the ostensible mod-2π ambiguity of ψ has been remedied by setting $\psi = L$. But the Zoll condition stipulates that every geodesic is a simple closed curve, and a simple closed curve in an annulus necessarily has winding number one. Thus the Zoll assumption guarantees that ψ is a multiple of 2π, and the developing map will therefore be well defined on a neighborhood of any such geodesic $\gamma \subset S$, even though general principles had led us to expect that it would merely be defined up on the universal cover $\tilde{S}$. Moreover, the subset of $\mathbb{P}(TS)$ consisting of directions tangent to geodesics γ with this normal form is *open*.

On the other hand, if A has normal form (b), then the $\pm$ sign must be $+$; if not, an annular neighborhood of the given geodesic γ would contain closed geodesics with self-crossings, obtained by gluing together great circles near the equator with their reflections through the x_3-axis. Thus, the transformation A must take the normal form

$$(a)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad (b)\quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we will henceforth say that a given geodesic γ is type (a) or type (b) depending on which one of these normal forms occurs.

Now recall that the developing-map construction gives us an immersion $\tilde{\phi}: \tilde{S} \to S^2$ and a group homomorphism $\phi: \pi_1(S) \to SL(3,\mathbb{R})$ such that the deck-transformation action of $\pi_1(S)$ on the universal cover $\tilde{S}$ is compatible with the action of $SL(3,\mathbb{R})$ on S^2. In particular, we may define a natural intermediate cover $\hat{S}$ of S by setting $\hat{S} = \tilde{S}/\ker\phi$, so that S is then obtained from $\hat{S}$ by dividing by the action of a matrix group $G = \phi[\pi_1(S)] \subset SL(3,\mathbb{R})$, and such that we still have a developing map $\hat{\phi}: \hat{S} \to S^2$ which correctly intertwines the effective actions of G on $\tilde{S}$ and S^2. Let $\varpi: \hat{S} \to S$ be the covering map. If $\gamma \subset S$ is a geodesic of type (a), then $\varpi^{-1}(\gamma) = \coprod_j \hat{\gamma}_j$, where each $\gamma_j \subset \hat{S}$ is a closed geodesic of type (a), and where $\varpi|_{\hat{\gamma}_j}: \hat{\gamma}_j \to \gamma$ is a diffeomorphism for each j; this follows immediately from the fact that every non-trivial deck transformation of $\hat{S}$ must act non-trivially on S^2, whereas non-trivial coverings of a closed geodesic of type (a) are invisible to the developing map.

On the other hand, if $\gamma' \subset S$ is a geodesic of type (b), then $\varpi^{-1}(\gamma') = \coprod_j \hat{\gamma}'_j$, where each $\hat{\gamma}'_j \approx \mathbb{R}$ is a non-closed geodesic in $\hat{S}$; moreover, the conjugates of the image of $[\gamma] \in \pi_1(S)$ in G give us deck transformations of $\hat{S}$ which roll up the various $\hat{\gamma}'_j$ into copies of γ, while simultaneously acting on S^2 via linear transformations of normal form (b). Such a matrix acts on S^2 in a manner fixing a great circle, and on this great circle there is a preferred antipodal pair of points, given by $(\pm 1, 0, 0)$ for the standard model, which are the accumulation points of the non-closed orbits, and which we will refer to as the *goals* of $\hat{\gamma}'_j$. Since S is paracompact, $G = \pi_1(S)/\ker\phi$ is countable, so it follows that only countably many points of S^2 occur as goals of geodesics of type (b).

Now let $\hat{x} \in \hat{S}$ be any point that is not sent to a goal, and let $x = \varpi(\hat{x})$ be its projection to S. Then only countably many geodesics through x can be of type (b), since any such geodesic would develop onto a great circle joining $\hat{\phi}(x)$ to a goal. Moreover, the set of directions in $\mathbb{P}(T_xS) \approx S^1$ which are tangent to geodesics of type (a) is open. Thus the set $B \subset \mathbb{P}(T_xS)$ of directions tangent to geodesics of type (b) is a countable closed subset of the circle. We claim that $B = \emptyset$. If not, choose a base-point for $\mathbb{P}(T_xS)$ which is not in B and use the counter-clockwise angle from this direction to define a homeomorphism $\mathbb{P}(T_xS) \approx [0,\pi]/\{0,\pi\}$ which sends the base-point to the equivalence class $\{0,\pi\}$. Then B then becomes a non-empty countable closed subset $Б \subset (0,\pi)$. Let $б = (\sup Б) \in Б$, and let $b \in \mathbb{P}(T_xS)$ be the corresponding direction. Let $\mathscr{I} \subset \mathbb{P}(T_xS)$ be the open subset corresponding to the open interval $(б,\pi)$. Every direction in $\mathscr{I}$ is tangent to a geodesic of type (a), and since every such geodesic γ has an annular neighborhood which looks like a finite covering of a band around the equator, all the geodesics

γ_t tangent to elements of $\mathscr{I}$ form a smooth family of maps of the circle, and in particular are all homotopic to one another. We can then uniquely lift this family of geodesics of type (a) as a smooth family $\hat{\gamma}_t$, $t \in (б, \pi)$, of closed geodesics though $\hat{x} \in \hat{S}$. Let $Y \subset \hat{S}$ be the union of these curves $\hat{\gamma}_t$, and notice that $\varpi|_Y$ is injective, since all the γ_t are homotopic. If $a \in G - \{1\}$, it thus follows that $a(Y) \cap Y = \emptyset$. Now let $\gamma' \subset S$ be the geodesic of type (b) through x with tangent b, and let $\hat{\gamma}'$ be its lift through $\hat{x}$. By composing $\hat{\phi}$ with an element of $SL(3, \mathbb{R})$ if necessary, we can henceforth assume that $\hat{\phi}[\hat{\gamma}']$ is a subset of the equator $z = 0$, that $\hat{\phi}(\hat{x}) = (0, 1, 0)$, and that there is an element $a \in G$ which sends $\hat{\gamma}'$ to itself, while acting on S^2 by $(x, y, z) \mapsto (x + y, y, z)/\|(x + y, y, z)\|$. Let $\sigma \subset \hat{\gamma}'$ consist of those points of $\hat{\gamma}'$ for which every neighborhood meets every $\hat{\gamma}_t$ for $t \in (б, б+\epsilon)$, where $\epsilon > 0$ is allowed to depend on the neighborhood. Now the developing map $\hat{\phi}$ is a local diffeomorphism, and carries Y onto $\{(0, \pm 1, 0)\} \cup \{y < z \cot б, z > 0\} \cup \{y > z \cot б, z < 0\}$ by a finite covering map. It follows that the non-empty subset $\sigma \subset \hat{\gamma}'$ is therefore both open and closed. Hence $\sigma = \hat{\gamma}'$. In particular, any point of $\hat{\gamma}'$ which is sent to the semi-circle $\{z = 0, y > 0\}$ is contained in an open disk which intersects Y in an open half-disk consisting of points south of $\hat{\gamma}'$. Hence each of the iterates $a^n(x)$, $n > 0$, has an open neighborhood U_n such that $a(U_n \cap Y) \cap (U_{n+1} \cap Y) \neq \emptyset$. But this means that $a(Y) \cap Y \neq \emptyset$, which is a contradiction. Hence $Б = \emptyset$, and every geodesics through x is of type (a).

The set of all geodesics though x therefore forms a smooth family of embedded circles. If $\tilde{X} \subset \mathbb{P}(TS)$ denotes the union of all the lifts of geodesics through x, then $\tilde{X}$ is a smooth compact surface—in fact, a Klein bottle. Moreover, $\mathbb{P}(T_x S)$ is a subset of $\tilde{X}$, and this circle has non-orientable normal bundle. Let X be the smooth compact surface—actually, a projective plane—obtained from $\tilde{X}$ by blowing this circle down to a point $x_0 \in X$. The projection $\tilde{X} \to S$ then induces a smooth proper map $f : X \to S$ such that $f_* : T_{x_0} X \to T_x S$ is an isomorphism. But, by assumption, any geodesic passes though x only once. Thus $f^{-1}(x) = \{x_0\}$, and the mod-2 degree of f is therefore $1 \in \mathbb{Z}_2$. If f were not onto, this would be a contradiction, since any regular value must have an odd number of points in its pre-image. Hence f is onto, and $S = f(X)$ is compact, as claimed. ■

We therefore obtain the following useful result:

Theorem 4.2 *Let $(S, [\nabla])$ be a connected surface with flat projective structure, and suppose that every maximal geodesic of $[\nabla]$ is a simple closed curve in S. Then, up to diffeomorphism, $(S, [\nabla])$ is either S^2 or $\mathbb{RP}^2$, equipped with the standard projective connection.*

Proof. By Lemma 4.1, S is a compact Zoll manifold. Hence [31, Lemma 2.8] tells us that S is diffeomorphic to either S^2 or $\mathbb{RP}^2$. In particular, the universal cover $\tilde{S}$ of S is compact, so the developing map $\tilde{\phi} : \tilde{S} \to S^2$ must be a covering map. Hence $\tilde{\phi}$ a diffeomorphism.

If $S \approx S^2$, $\tilde{\phi}$ is now a diffeomorphism $S \to S^2$ which sends the given flat projective structure to the standard one, and we are done.

If $S \approx \mathbb{RP}^2$, the non-trivial deck transformaton of $\tilde{S} \approx S^2$ defines a linear involution for which $+1$ is not an eigenvalue. But the only such involution is -1. Thus we actually obtain a developing map induces $S \to \mathbb{RP}^2$, and this gives us the promised diffeomorphism sending the given projective structure to the standard one. ■

In the next section, we will see that this has some interesting ramifications for the theory of Zollfrei self-dual 4-manifolds.

5 Topological Implications

In this section, we will show that, up to homeomorphism, the only oriented 4-manifolds which admit self-dual Zollfrei metrics are $S^2 \times S^2$ and the real projective quadric $\mathbb{M}^{2,2} = [S^2 \times S^2]/\mathbb{Z}_2$. We begin our proof with the following observation:

Lemma 5.1 *Let $(M, [g])$ be a Zollfrei self-dual 4-manifold. Then every β-surface $S \looparrowright M$ is an embedded S^2 or $\mathbb{RP}^2$ in M. Moreover, every two points of such a β-surface S are joined by a null geodesic γ.*

Proof. Let $(M, [g])$ be a Zollfrei self-dual 4-manifold, and let $S \looparrowright M$ be a β-surface. By Lemma 3.3, S is totally geodesic, so the immersion $S \looparrowright M$ is injective outside a discrete subset, where the various tangent spaces of S must be transverse to each other. Moreover, ∇ induces a connection $\triangledown$ on S. Proposition 3.6 asserts that the associated projective structure $[\triangledown]$ is *flat*. But the geodesics of $(S, [\triangledown])$ are all null geodesics of $[g]$, so the assumption that $(M, [g])$ is Zollfrei implies that $(S, [\triangledown])$ is a projectively flat surface in which every geodesic is a simple closed curve. Theorem 4.2 therefore tells us that S is diffeomorphic to either S^2 or $\mathbb{RP}^2$, in such a manner that $[\triangledown]$ becomes the standard projective structure. In particular, every pair of points of S can be joined by a geodesic of $[\triangledown]$. Since the restriction of $S \looparrowright M$ to any geodesic yields an immersion which is one-to-one outside a finite number of double-points with distinct tangents, the assumption that every null geodesic of $[g]$ is a simple closed curve therefore implies that $S \looparrowright M$ is actually an embedding. ■

When we say that $(M, [g])$ is self-dual, it is already implicit that M is oriented. However, $O(2,2)$ has *four* components; indeed, the inclusion $O(2) \times O(2) \hookrightarrow O(2,2)$ is a homotopy equivalence. We will say that an oriented split-signature pseudo-Riemannian 4-manifold is *space-time-orientable* if its structure group can be reduced to the identity component $SO_+(2,2)$ of $O(2,2)$. Obviously this is automatically the case if $H^1(M, \mathbb{Z}_2) = 0$, and so in particular holds whenever M is simply connected. If M is *not* space-time-orientable, there is always a double cover $\tilde{M} \to M$ which *is* space-time orientable with respect to the pull-back of the metric. Moreover, $\tilde{M}$ will then be Zollfrei if M is, since all the null geodesics of $\tilde{M}$ are at worst double covers of those in M.

Now suppose that (M, g) is a space-time-orientable split-signature self-dual 4-manifold. Then we may express TM as a direct sum $T_+ \oplus T_-$, where T_+ and T_- are mutually orthogonal with respect to g, and where the restriction of g to T_+ (respectively, to T_-) is positive (respectively, negative) definite; for example, this may be done by choosing some background Riemannian metric h on M, and then diagonalizing g with respect to h at each point. A space-time orientation for M then amounts to a choice of orientations for the bundles $T_\pm$. Notice that this then allows us to express TM as the sum of two complex line bundles; indeed, a reduction of the structure group of (M, g) to $SO(2) \times SO(2) = U(1) \times U(1)$ is equivalent [34] to the choice $(\mathfrak{J}, \tilde{\mathfrak{J}})$ of a pair of g-compatible almost-complex structures, where $\mathfrak{J}$ is compatible with the given orientation of M, and where $\tilde{\mathfrak{J}}$ is compatible with the opposite orientation. An isotropic 2-plane $\Pi \subset T_x M$ then becomes the graph of an isometry from $(T_{-x}, -g)$ to (T_{+x}, g), and such an isotropic subspace Π is then a β-plane iff this isometry is *orientation-reversing*. In particular, the orientation of T_- induces an orientation on every β-plane, and hence on any β-surface; what is more, any β-surface is a $\tilde{\mathfrak{J}}$-holomorphic curve in M. We thus obtain the following:

Lemma 5.2 *Let $(M, [g])$ be a space-time-orientable Zollfrei self-dual 4-manifold. Then every β-surface S in M is an embedded 2-sphere.*

Proof. A space-time orientation induces an orientation of each β-surface. Lemma 5.1 therefore tells us that each β-surface must be an embedded 2-sphere. ■

The following observation is therefore pertinent:

Lemma 5.3 *Suppose that $(M, [g])$ is a split-signature self-dual 4-manifold in which every β-surface is an immersed 2-sphere. Let $\wp : F \to M$ be the bundle of β-planes over M. Then the canonical foliation $\mathscr{F}$ of F by lifted β-surfaces is locally trivial, in the sense that every leaf has a neighborhood which is diffeomorphic to $S^2 \times \mathbb{R}^3$ in such a manner that each first-factor sphere $S^2 \times \{*\}$ is a leaf. Moreover, this diffeomorphism can be chosen in such a way that each great circle in each first-factor sphere projects to a null geodesic in M.*

Proof. Since every leaf of $\mathscr{F}$ is compact and simply connected, the holonomy of $\mathscr{F}$ around any leaf is trivial, and $\mathscr{F}$ is therefore a fibration [48]. In particular, we can choose a transversal U through a given leaf which meets every nearby leaf exactly once. Assume, without loss of generality, that $U \approx \mathbb{R}^3$, and let $V \approx U \times S^2$ be the corresponding neighborhood of the leaf. Since V is simply connected, the line bundle $\ker \wp$ becomes trivial when restricted to V, and we can therefore choose a non-zero vector field u on V which spans $\ker \wp$. The U-component of this vector field then defines a function $V \to (\mathbb{R}^3 - \{0\})$, and we thus get a map $V \to S^2$ by composing with the radial projection $(\mathbb{R}^3 - \{0\}) \to S^2$. Modulo the action of the $SL(3, \mathbb{R})$, however, the restriction of this map to any leaf S is really just the developing map $\phi : S \to \mathbb{RP}^2$ constructed in Proposition 3.6, lifted to the universal cover S^2 of $\mathbb{RP}^2$. Taking the Cartesian product with leaf projection $V \to U \approx \mathbb{R}^3$, we thus obtain a local trivialization $V \to S^2 \times \mathbb{R}^3$ of $\mathscr{F}$ in which every lifted null geodesic becomes a great circle in a first-factor S^2. ■

This gives us a more transparent understanding of the Zollfrei condition:

Theorem 5.4 *Let $(M, [g])$ be a space-time-orientable self-dual 4-manifold. Then the following conditions are equivalent:*

(*i*) *$(M, [g])$ is Zollfrei;*

(*ii*) *$(M, [g])$ is strongly Zollfrei;*

(*iii*) *every β-surface is an embedded 2-sphere in M.*

Proof. Definition 2.2 tells us that $(ii) \Longrightarrow (i)$, while Lemma 5.2 asserts that $(i) \Longrightarrow (iii)$. It therefore suffices to show that $(iii) \Longrightarrow (ii)$.

Thus, suppose that every β-surface of $(M, [g])$ is an embedded 2-sphere in M. Let $\mathcal{Q}$ be the bundle of null directions of $(M, [g])$, and notice that the bundle projection $\mathcal{Q} \to M$ factors through an S^1-fibration $\mathcal{Q} \to F$, since every non-zero null vector is an element of exactly one β-plane. But Lemma 5.3 tells us that the foliation of $\mathcal{Q}$ by lifted null geodesics simplifies when restricted to the null geodesics in a β-surface, where it just becomes the standard fibration $\mathbb{P}(TS^2) \to \mathbb{RP}^2$; moreover, this picture applies uniformly in a neighborhood of each leaf of the foliation $\mathscr{F}$ of F. Hence the foliation of $\mathcal{Q}$ by lifted null geodesics is a locally trivial circle fibration. Since each null geodesic lifts to a great circle in a leaf of $\mathscr{F}$, and each leaf embeds into M via $p : F \to M$, each null geodesic is also an embedded circle. It therefore follows that $(M, [g])$ is strongly Zollfrei, and we are done. ∎

We also obtain the following crucial fact:

Lemma 5.5 *Suppose that $(M, [g])$ is a space-time-orientable self-dual Zollfrei 4-manifold, and let $p : F \to M$ be the bundle of β-planes over M. Then there is a smooth 3-manifold P and a smooth proper submersion $q : F \to P$ whose fibers are exactly the leaves of the foliation $\mathscr{F}$.*

Proof. By Lemma 5.2 and Lemma 5.3, $\mathscr{F}$ must be a locally trivial fibration by 2-spheres, and the leaf space P is therefore a manifold. ∎

The situation is thus encapsulated by the diagram

$$\begin{array}{ccc} & F & \\ {}^{p}\swarrow & & \searrow^{q} \\ M & & P \end{array}$$

which we shall refer to as the (real) *double fibration* of $(M, [g])$.

Now since F is connected, so is $P = q(F)$, and we may therefore join any two distinct points of P by a smoothly embedded arc. Trivializing the restriction of q to this arc then results in a free homotopy of the corresponding leaves of $\mathscr{F}$. Finally, pushing this homotopy down via p produces a free homotopy of any two given β-surfaces in M. In particular, any two β-surfaces are homologous:

Lemma 5.6 *Let $(M, [g])$ be a space-time oriented Zollfrei self-dual 4-manifold. Then any two β-surface $S, S' \subset M$ are freely homotopic, and so, in particular, represent the same homology class in $H_2(M, \mathbb{Z})$.*

Now since M^4 is oriented, there is a well defined intersection form

$$H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z}$$

even if M is non-compact; for example, this reflects the fact that we always have a Poincaré-duality isomorphism $H_2(M) \cong H^2_c(M)$ as well as a natural homomorphism

$H^2_c(M) \to H^2(M)$. If S_1 and S_2 are compact embedded oriented surfaces in general position, one assigns a local intersection index of ± 1 to each intersection point $x \in S_1 \cap S_2$ so as to indicate whether the given orientations of $T_xS_1 \oplus T_xS_2$ and T_xM agree or disagree, and the homological intersection number $[S_1] \cdot [S_2]$ is then precisely the sum of these intersection indices. When S_1 and S_2 happen to be β-surfaces, we thus obtain the following:

Lemma 5.7 *If S_1 and S_2 are distinct compact embedded β-surfaces in a space-time-orientable self-dual 4-manifold $(M, [g])$, then their homological intersection number $[S_1] \cdot [S_2]$ equals $-\#(S_1 \cap S_2)$.*

Proof. Each β-surface is totally geodesic, so two distinct β-surfaces can never share the same tangent space. Since distinct β-planes in any tangent space are transverse, this shows that S_1 and S_2 are necessarily in general position. Now since any β-plane may be viewed as the graph of an orientation-reversing isometry $T_- \to T_+$, the intersection index assigned to each point of intersection is -1. Summing over intersection points thus yields $[S_1] \cdot [S_2] = -\#(S_1 \cap S_2)$. ■

When $(M, [g])$ is Zollfrei, we thus obtain the following:

Lemma 5.8 *Let $(M, [g])$ be a space-time-orientable Zollfrei self-dual manifold. Then any two β-surfaces in M have non-empty intersection. Moreover, any two distinct β-surfaces meet in exactly m points, where the homological self-intersection of any β-surface S is given by $[S] \cdot [S] = -m < 0$.*

Proof. Let S be a reference β-surface, and suppose that we wish to understand the intersection of two given β-surfaces S_1 and S_2. If $S_1 = S_2$, they certainly intersect, and there is nothing to prove. Otherwise, Lemma 5.6 tells us that $[S_1] = [S_2] = [S]$, and Lemma 5.7 then yields $\#(S_1 \cap S_2) = -[S_1] \cdot [S_2] = -[S] \cdot [S]$. In particular, the number m of points of interesection is independent of *which* pair of distinct β-surfaces we choose to consider. But since every β-plane is tangent to a β-surface, and since we have a circle's worth of different β-planes in each tangent space T_xM, we can certainly find pairs (S_1, S_2) with $S_1 \neq S_2$ and $S_1 \cap S_2 \neq \emptyset$. Thus $m > 0$, and we are done. ■

Lemma 5.9 *If $(M, [g])$ is a Zollfrei self-dual 4-manifold, then M is compact.*

Proof. By passing to a double cover if necessary, we may assume that (M, g) is space-time-oriented.

Fix a reference β-surface $S \subset M$. Then for any point $x \in M$, there is a β-surface through x that meets S; indeed, there certainly *are* β-surfaces through x, and Lemma 5.8 tells us that *any* of these must meet S. But this statement can be rewritten as the assertion that

$$M = \mathfrak{p}\left[\mathfrak{q}^{-1}\left(\mathfrak{q}\left[\mathfrak{p}^{-1}(S)\right]\right)\right].$$

Since $\mathfrak{p}$ and $\mathfrak{q}$ are both proper maps, and since S is compact, it therefore follows that M is compact, too. ■

If (M, g) is a space-time-oriented Zollfrei 4-manifold, each fiber $\mathfrak{p}^{-1}(x)$ of $F \to M$ is an oriented circle, and its image $\mathfrak{q}[\mathfrak{p}^{-1}(x)]$ in P may be thought of as a map $\gamma_x : S^1 \to P$, which we will call a *standard loop.*

Proposition 5.10 *Let M be space-time-orientable self-dual Zollfrei, and let P be its space of β-surfaces. Then P is diffeomorphic to $\mathbb{RP}^3$, and $\pi_1(P) \cong \mathbb{Z}_2$ is generated by any standard loop γ_x, $x \in M$. Moreover, any two distinct β-surfaces in M meet in exactly two points.*

Proof. Let $y \in P$ be any base point, and let $S \subset M$ be the corresponding β-surface. Every other β-surface in M meets S in m distinct points, where $[S] \cdot [S] = -m$. Moreover, through every point of S, there passes a circle's worth of β-surfaces, only one of which is S. Now recall that the structure group of $\mathfrak{p}$ is $O(1, 2) = PSL(2, \mathbb{R})$. Thus, by removing the one point representing TS from each fiber of $\mathfrak{p}^{-1}(S) \to S$, we obtain an affine $\mathbb{R}$-bundle L over $S \approx S^2$ which, via $\mathfrak{q}$, maps locally diffeomorphically onto $P - \{y\}$ in an m-to-1 fashion. Since any affine $\mathbb{R}$-bundle over S^2 is trivial, it follows that universal cover of $P - \{y\}$ is $L \approx S^2 \times \mathbb{R} = S^3 - \{2 \text{ points}\}$; and since the order of this covering is $m = -[S] \cdot [S]$, we also see that $|\pi_1(P - \{y\})| = m$. But $\pi_1(P) = \pi_1(P - \{y\})$, since removing a point from a 3-manifold doesn't change its fundamental group. The universal cover of $P - \{y\}$ is therefore gotten from the universal cover $\tilde{P}$ of P by removing $|\pi_1(P)| = m$ points. Since the universal cover of $P - \{y\}$ has m ends, whereas $S^2 \times \mathbb{R}$ has just two ends, it follows that $m = 2$. Thus $P = S^3/\mathbb{Z}_2$ for some free $\mathbb{Z}_2$-action, and a theorem of Livesay [32] tells us that $P \approx \mathbb{RP}^3$.

Finally, notice that the fiber of $L \to S$ defines a lift $\tilde{\gamma}_x$ of γ_x to $\tilde{P} \approx S^3$. Since this lift is not a loop, but rather is a curve joining the two pre-images of y, it follows that γ_x is non-trivial in $\pi_1(P)$. Thus $[\gamma_x]$ generates $\pi_1(P) \cong \pi_1(\mathbb{RP}^3) \cong \mathbb{Z}_2$, and we are done. ■

Imitating the proof of Lemma 5.3 now gives us the following:

Lemma 5.11 *If $(M, [g])$ is space-time-orientable and self-dual Zollfrei, then F is diffeomorphic to $\mathbb{RP}^3 \times S^2$ in such a manner that $\mathfrak{q}$ becomes the first-factor projection $\mathbb{RP}^3 \times S^2 \to \mathbb{RP}^3$.*

Proof. Let $\varpi : S(TP) \to P$ denote the sphere bundle defined by

$$S(TP) = (TP - 0_P)/\mathbb{R}^+,$$

where $0_P \subset TP$ denotes the zero section, and where the positive reals $\mathbb{R}^+$ act on TP by scalar multiplication. That is, $S(TP)$ may be thought of as the unit tangent bundle of P for any choice of Riemannian metric on P.

Let u be a non-zero vector field on F which spans $\ker \mathfrak{p}_*$ at each point; this is possible because the choice of a metric-compatible decomposition $TM = T_- \oplus T_+$ allows one to realize $\mathfrak{p} : F \to M$ as the principle $SO(2)$-bundle of orientation-reversing isometries $T_- \to T_+$. Since the fibers of $\mathfrak{p}$ and $\mathfrak{q}$ are nowhere tangent, we can therefore define a map

$$\begin{aligned} \Phi : F &\to S(TP) \\ z &\mapsto [\mathfrak{q}_* u] \end{aligned}$$

which makes the diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \Phi\ } & S(TP) \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \varpi} & \\ & P & \end{array}$$

commute. Over each point of P, the map Φ is just the lift of the developing map $\phi : S \to \mathbb{RP}^2$ constructed in Proposition 3.6 to the universal cover S^2 of $\mathbb{RP}^2$, and so is a diffeomorphism. Hence Φ is a bijection. Moreover, since q and ϖ are both submersions, it follows that Φ_* has maximal rank everywhere, and Φ is therefore a diffeomorphism. However, $P \approx \mathbb{RP}^3$ is parallelizable, so $F \approx S(TP) \approx \mathbb{RP}^3 \times S^2$, as claimed. ■

Theorem 5.12 *If $(M, [g])$ is a space-time-orientable self-dual Zollfrei 4-manifold, then M is homeomorphic to $S^2 \times S^2$.*

Proof. Since the standard loop $\gamma_x = q[p^{-1}(x)]$ generates $\pi_1(P)$, the pull-back map

$$q^* : H^1(P, \mathbb{Z}_2) \to H^1(F, \mathbb{Z}_2)$$

sends the generator of $H^1(P, \mathbb{Z}_2) \cong \mathbb{Z}_2$ to an element of $H^1(F, \mathbb{Z}_2)$ which is non-trivial on the fiber class of p. This shows that there is a double cover $\tilde{F} \to F$ which restricts to a double cover $S^1 \to S^1$ of each fiber of p.

Now choose any g-adapted, orientation compatible almost-complex structure $\mathfrak{J}$ on M. The S^1-bundle $p : F \to M$ can then be identified with the unit circle bundle of the canonical line bundle K of $(M, \mathfrak{J})$. The double cover $\tilde{F} \to F$ then becomes the unit circle bundle of a square-root $K^{1/2}$ of K. Hence $c_1(M, \mathfrak{J})$ is divisible by 2 in $H^2(M, \mathbb{Z})$. Because $w_2(M)$ is the mod-2 reduction [38] of $c_1(M, \mathfrak{J})$, and because the sequence

$$\cdots \to H^2(M, \mathbb{Z}) \xrightarrow{2} H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2) \to \cdots$$

is exact, it follows that $w_2(M) = 0$. Thus M is a spin manifold.

Since $F \approx \mathbb{RP}^3 \times S^2$, its universal cover must be $\tilde{F} \approx S^3 \times S^2$. Hence the long exact homotopy sequence [46]

$$\cdots \to \pi_2(S^1) \to \pi_2(\tilde{F}) \to \pi_2(M) \to \pi_1(S^1) \to \pi_1(\tilde{F}) \to \pi_1(M) \to 0$$

of the fibration $S^1 \to \tilde{F} \to M$ now tells us that $\pi_1(M) = 0$ and $\pi_2(M) = \mathbb{Z} \oplus \mathbb{Z}$. Thus M is a simply connected compact 4-manifold with $b_2 = 2$ and even intersection form. The Freedman classification of simply connected 4-manifolds [10] therefore tells us that M is homeomorphic to $S^2 \times S^2$. ■

In fact, it seems reasonable to conjecture that any space-time-orientable self-dual Zollfrei 4-manifold must actually be *diffeomorphic* to $S^2 \times S^2$. While we have not managed to prove this stronger statement in general, we will eventually see, in

Theorem 11.2 below, that it at least turns out to be true if $[g]$ is represented by an indefinite Kähler metric.

We now turn to the non-space-time-orientable case.

Proposition 5.13 *Let $(M,[g])$ be a Zollfrei self-dual 4-manifold which is* not *space-time-orientable. Then every β-surface in M is an embedded $\mathbb{RP}^2$, and every pair of distinct β-surfaces intersects in exactly one point.*

Proof. Notice that our definition of self-duality requires that M be orientable. Thus the set $\tilde{M}$ of orientation-compatible local space-time orientations of $(M,[g])$ is a double cover of M. Notice that $\tilde{M}$ is space-time-orientable and self-dual Zollfrei with respect to the pulled back metric. Let $a:\tilde{M}\to\tilde{M}$ be the non-trivial deck transformation.

If $S\subset\tilde{M}$ is any β-surface, then we claim that $a[S]=S$. Indeed, suppose not. Then $a[S]=S'$ would be a different β-surface, and hence $S\cap S'$ would consist of exactly two points by Proposition 5.10; and since $a[S\cap S']=a[S]\cap a[a[S]]=S'\cap S$, these two points would necessarily be interchanged by the fixed-point-free involution a. On the other hand, all the other points of S would be moved to the complement of S by a. Hence the image of S in $M=\tilde{M}/\langle a\rangle$ would be an imersed sphere with a single self-intersection. But this contradicts Lemma 5.1. Thus every β-surface in $\tilde{M}$ must be sent to itself by a.

It follows that every β-surface in $\tilde{M}$ is the double cover of a β-surface in M. Since all the β-surfaces in $\tilde{M}$ are 2-spheres by Lemma 5.2, and since every β-surface in M must be the image of a β-surface in $\tilde{M}$, it follows that every β-surface in M must be an $\mathbb{RP}^2$. Moreover, since $\tilde{M}\to M$ is a double cover, and since β-surfaces in $\tilde{M}$ intersect in pairs of points, pairs of distinct β-planes in M must always intersect in a unique point. ■

Theorem 5.14 *Let $(M,[g])$ be a self-dual split-signature 4-manifold. Then the following are equivalent:*

- (i) *$(M,[g])$ is Zollfrei;*
- (ii) *$(M,[g])$ is strongly Zollfrei;*
- (iii) *exactly one of the following holds:*
  - (a) *every β-surface is an embedded $S^2\subset M$; or*
  - (b) *every β-surface is an embedded $\mathbb{RP}^2\subset M$.*

Proof. Notice that $(iii)\Longrightarrow(i)$ by Proposition 3.6 and the uniqueness [24] of the flat projective structures on $\mathbb{RP}^2$ and S^2. Thus Lemma 5.2 and Proposition 5.13 tell us that $(iii)(a)$ can only occur if $(M,[g])$ is space-time orientable, whereas $(iii)(b)$ can only occur if $(M,[g])$ is *not* space-time orientable.

If $(M,[g])$ is space-time orientable, the desired equivalence is therefore given by Theorem 5.4.

If, on the other hand, $(M,[g])$ is not space-time orientable, then $(ii)\Longrightarrow(i)$ by Definition 2.2, and $(i)\Longrightarrow(iii)$ by Proposition 5.13. On the other hand,

$(iii)(b) \implies (ii)$, too. Indeed, the space-time-orientable double cover $\tilde{M}$ of M is Zollfrei, and hence strongly Zollfrei by Theorem 5.4. The non-trivial deck transformation a of $\tilde{M} \to M$ must therefore send each null geodesic to itself by the uniqueness [24] of the flat projective structure on $\mathbb{RP}^2$. ■

Proposition 5.13 also allows us to deduce the following:

Lemma 5.15 *Let $(M, [g])$ be a Zollfrei self-dual 4-manifold which is not space-time-orientable. Then M is non-spin.*

Proof. Let $\mathbf{b} \in H^2(M, \mathbb{Z}_2)$ denote the Poincaré dual of the $\mathbb{Z}_2$-homology class of any β-surface $S \subset M$. Since any two distinct β-surfaces are freely homotopic and intersect transversely in exactly one point, we have $\mathbf{b} \cdot \mathbf{b} = 1 \in \mathbb{Z}_2$, where

$$\cdot : H^2(M, \mathbb{Z}_2) \times H^2(M, \mathbb{Z}_2) \to \mathbb{Z}_2$$

is the intersection form of M with $\mathbb{Z}_2$ coefficients. But since M is orientable, Wu's formula [20] asserts that $w_2(M)$ satisfies

$$w_2 \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x}$$

for any $\mathbf{x} \in H^2(M, \mathbb{Z}_2)$, so we have

$$w_2 \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{b} = 1.$$

Thus $w_2(M) \neq 0$, and M is non-spin, as claimed. ■

Theorem 5.16 *Let $(M, [g])$ be a Zollfrei self-dual 4-manifold which is not space-time-orientable. Then M is homeomorphic to the real projective quadric $\mathbb{M}^{2,2}$.*

Proof. Freedman's topological classification of simply connected 4-manifolds has been extended to compact oriented 4-manifolds with finite cyclic fundamental group by Hambleton and Kreck [17, Theorem C]. They show that such manifolds are classified up to homeomorphism by their fundamental groups, their intersection forms on $H^2(\bullet, \mathbb{Z})/\text{torsion}$, their w_2-types, and their Kirby-Siebenmann invariants. The Kirby-Siebenmann invariant vanishes if a manifold admits a smooth structure. The w_2-type of a 4-manifold says whether the manifold and its universal cover are spin; namely, an oriented manifold M with universal cover $\tilde{M}$ is said to be of type (I) if $w_2(\tilde{M}) \neq 0$, type (II) if $w_2(M) = 0$, and type (III) if $w_2(\tilde{M}) = 0$, but $w_2(M) \neq 0$.

Now assume that $(M, [g])$ is a non-space-time-orientable self-dual Zollfrei manifold. Then M is smooth, and so has vanishing Kirby-Siebenmann invariant. Also, M is oriented, as is required by our definition of self-duality. Now recall that the double cover $\tilde{M}$ of M by its local space-time orientations is a space-time orientable Zollfrei self-dual 4-manifold, and so is homeomorphic to $S^2 \times S^2$ by Theorem 5.12. Since $S^2 \times S^2$ is simply connected, $\tilde{M}$ is actually the universal cover, and we therefore have $\pi_1(M) = \mathbb{Z}_2$. Moreover, the Euler characteristic of M must be $\chi(M) = \chi(S^2 \times S^2)/2 = 2$, so $H^2(M, \mathbb{Z})/\text{torsion} = 0$, and the intersection form of M must therefore be trivial. Finally, $w_2(M) \neq 0$ by Proposition 5.15, whereas

$w_2(\tilde{M}) = w_2(S^2 \times S^2) = 0$, so M is of type (III). Hambleton and Kreck therefore tell us that there is only one possible homeomorphism type for such an M. The quadric $\mathbb{M}^{2,2} \subset \mathbb{RP}^5$ therefore represents the only topological possibility. ■

Combining Theorems 5.12 and 5.16, we have thus proved **Theorem B**.

6 Stability of the Zollfrei Condition

We now turn to the important assertion that the Zollfrei condition is open among self-dual metrics. This phenomenon is actually a manifestation of aspects of the theory of foliations arising from Thurston stability for compact leaves of foliations [48]. The result we will need is originally due to Langevin and Rosenberg [26], although the formulation given here is actually that of Epstein and Rosenberg [9].

Theorem 6.1 (Langevin-Rosenberg) *Let $\varpi : X \to Y$ be a C^1 fiber bundle with compact fibers and compact base, where the fibers of ϖ have $b_1 = 0$ over $\mathbb{R}$. Let $\mathfrak{F}$ be the foliation of X by the fibers of ϖ. Then the foliation $\mathfrak{F}$ has a neighborhood $\mathscr{V}$ in the C^1 Epstein topology on the space of foliations of X such that every foliation $\mathfrak{F}' \in \mathscr{V}$ is of the form $\phi^*\mathfrak{F}$ for some C^1-diffeomorphism $\phi : X \to X$.*

Here two C^1 foliations of X are close in the C^1 *Epstein topology* [8] if there are finite atlases of trivializing charts for the two foliations which are close in the usual C^1 topology on the space of maps. The only thing that need concern us here is that two C^1 integrable distributions of k-planes which are C^1 close as sections of the Grassmann bundle $Gr_k(TX) \to X$ define foliations which are close in Epstein's sense.

Combining Theorem 6.1 with our results from §5, we thus obtain

Theorem A *Let (M, g) be a self-dual Zollfrei 4-manifold. Then any other self-dual metric g' on M that is sufficiently close to g in the C^2 topology is also Zollfrei.*

Proof. There is a C^0 neighborhood of g in the pseudo-Riemannian metrics in which every metric g' can be written as $g' = A^*g$ for a unique g-self-adjoint endomorphism $A : TM \to TM$ which is C^0 close to the identity. This endomorphism of TM allows one to identify the pseudo-orthonormal frame bundles of g and g'. Moreover, if g' is C^2 close to g, the corresponding principle connections are then C^1-close after this correspondence has been made. Using A to identify the bundle of β-planes for g' with the bundle $p : F \to M$ of β-planes for g, we then obtain two distributions E and E' on F which represent the horizontal lifts of the β-planes of g and g', respectively; and these two distributions will be C^1 close if we again assume that g and g' are C^2 close.

Now if g and g' are both self-dual, the distributions E and E' will both be integrable, and will be tangent to foliations $\mathscr{F}$ and $\mathscr{F}'$ that represent the canonical lifts of the β-surfaces of the two metrics. Moreover, $\mathscr{F}'$ will be C^1 close to $\mathscr{F}$ if we assume that g' is C^2 close to g. But if, in addition, g is Zollfrei, the leaves of the foliation $\mathscr{F}$ will exactly be the fibers of a fiber bundle $q : F \to P$. Now F is necessarily compact by Lemma 5.9, while Theorem 5.14 tells us that the fibers of q are spheres or projective planes. Since these are compact surfaces with $b_1 = 0$, we may therefore apply Theorem 6.1 to conclude that there is a C^1 diffeomorphism $\phi : F \to F$ which sends $\mathscr{F}$ to $\mathscr{F}'$. Thus, if g is self-dual and Zollfrei, and if g' is self-dual and C^2 close to g, then the β-surface of g' are either all spheres or all projective planes, and Theorem 5.14 therefore tells us that g' is Zollfrei, too, as claimed. ■

7 Constructing the Twistor Space

At this point, we have already achieved a certain level of intimacy with the bundle $p : F \to M$ of real β-planes over an oriented split-signature conformal 4-manifold $(M, [g])$. It is now time to introduce the bundle $\wp : \mathcal{Z} \to M$ of *complex* β-planes. Just as in the real case, a 2-dimensional complex subspace Π of a complexified tangent space $T_{\mathbb{C}}M|_x = \mathbb{C} \otimes T_xM$ of M is called *isotropic* if the complex-bilinear extension of g vanishes when restricted to Π. Such isotropic planes come in two flavors. The complex α-planes are precisely those complex 2-planes Π such that $\wedge^2\Pi$ corresponds by index lowering to a complex null line in $\Lambda^+_{\mathbb{C}}$; the complex β-planes instead correspond to null 1-dimensional subspaces of $\Lambda^-_{\mathbb{C}}$. Thus, the bundle of complex β-planes on M is exactly given by

$$\mathcal{Z} = \{[\varphi] \in \mathbb{P}(\Lambda^+_{\mathbb{C}}) \mid \langle \varphi, \varphi \rangle = 0\},$$

where $\langle \varphi, \psi \rangle = \frac{1}{2}\varphi_{ab}\psi_{cd}g^{ac}g^{bd}$ is the complex-bilinear extension to $\Lambda^2_{\mathbb{C}}$ of the inner-product on 2-forms induced by g. Since $\mathbb{P}(\Lambda^+_{\mathbb{C}})$ is a $\mathbb{CP}_2$-bundle over M, each fiber of $\mathcal{Z}$ is a non-degenerate conic in $\mathbb{CP}_2$, and so is intrinsically a $\mathbb{CP}_1$. Indeed, $\mathcal{Z}$ is precisely the $\mathbb{CP}_1$-bundle obtained from $F \to M$ by remembering that F has structure group $PSL(2, \mathbb{R})$, and that one can therefore construct an associated $\mathbb{CP}_1$-bundle over M by including $PSL(2, \mathbb{R})$ in $PSL(2, \mathbb{C})$ and considering the standard action of $PSL(2, \mathbb{C})$ on $\mathbb{CP}_1$. In particular, each fiber of $\wp : \mathcal{Z} \to M$ is a holomorphic curve. Let $\mathcal{V}^{0,1} \subset T_{\mathbb{C}}\mathcal{Z}$ be the $(0, 1)$-tangent bundle of the fibers.

Fix a metric g in the conformal class, and notice that g determines a connection on $\mathcal{Z}$, in the sense that g determines a notion of parallel transport of elements of $\mathcal{Z}$ along smooth curves in M. Let $\mathcal{H} \subset T\mathcal{Z}$ be the horizontal subspace of this connection, so that the derivative of the projection gives us a canonical isomorphism $\wp_* : \mathcal{H} \to \wp^*TM$. Let $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \otimes \mathbb{C}$. Then $\mathcal{Z}$ carries a unique distribution $\mathcal{E} \subset \mathcal{H}_{\mathbb{C}} \subset T_{\mathbb{C}}\mathcal{Z}$ of horizontal complex 2-planes such that

$$\wp_*(\mathcal{E}|_{\Pi}) = \Pi \subset T_{\mathbb{C}}M.$$

Set

$$\mathcal{D} = \mathcal{E} + \mathcal{V}^{0,1}.$$

Since $\mathcal{E}$ is horizontal and $\mathcal{V}^{0,1}$ is vertical, this sum is in fact a direct sum, and $\mathcal{D}$ is therefore a distribution of complex 3-planes on $\mathcal{Z}$.

Let us make this discussion more concrete by temporarily restricting our attention to an open subset $\mathscr{U} \subset M$ on which we can find an oriented pseudo-orthonormal frame $e_1, \ldots, e_4$ with

$$g(e_j, e_k) = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k \in \{1, 2\}, \\ -1 & \text{if } j = k \in \{3, 4\}. \end{cases}$$

We remark that if g is of differentiability class C^k, then such frames $e_1, \ldots, e_4$ of class C^k can locally be constructed by means of the Gramm-Schmidt procedure. This in turn determines a pseudo-orthonormal basis for $\Lambda^-|_{\mathscr{U}}$ by setting

$$\begin{aligned}\varphi_1 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^2 - e^3 \wedge e^4) \\ \varphi_2 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4) \\ \varphi_3 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)\end{aligned}$$

so that

$$\langle \varphi_j, \varphi_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k = 1 \\ -1 & \text{if } j = k \in \{2, 3\}. \end{cases}$$

We can then identify $\mathbb{CP}_1 \times \mathscr{U}$ with $\wp^{-1}(\mathscr{U}) \subset \mathcal{Z}$ by

$$([\zeta_1 : \zeta_2], x) \longmapsto \left[(\zeta_1^2 + \zeta_2^2)\ \varphi_1 + (\zeta_1^2 - \zeta_2^2)\ \varphi_2 - 2\zeta_1\zeta_2\ \varphi_3\right]\Big|_x,$$

and it is worth noting that in the process we have identified $\mathbb{RP}^1 \times \mathscr{U}$ with $p^{-1}(\mathscr{U}) \subset F \subset \mathcal{Z}$. In particular, an open dense subset of $\wp^{-1}(\mathscr{U})$ may be parameterized by $\mathbb{C} \times \mathscr{U}$, via the map

$$(\zeta, x) \longmapsto [(1 + \zeta^2)\ \varphi_1 + (1 - \zeta^2)\ \varphi_2 - 2\zeta\ \varphi_3]\Big|_x,$$

and in the process we sweep out an open dense subset of $p^{-1}(\mathscr{U})$ with $\mathbb{R} \times \mathscr{U}$. Notice that for each (ζ, x) with $\zeta \neq \pm i$, the corresponding β-plane is exactly

$$\Pi = \text{span}\left\{(\zeta^2 + 1)e_1 - 2\zeta e_3 + (\zeta^2 - 1)e_4\ ,\ (\zeta^2 + 1)e_2 + (\zeta^2 - 1)e_3 + 2\zeta e_4\right\}\Big|_x.$$

Now observe that we have

$$\nabla \varphi_j = \theta_j^k \otimes \varphi_k,$$

for an $\mathfrak{so}(1, 2)$-valued 1-form $[\theta_j^k]$:

$$\theta_2^1 = \theta_1^2, \quad \theta_3^1 = \theta_1^3, \quad \theta_3^2 = -\theta_2^3, \quad \theta_1^1 = \theta_2^2 = \theta_3^3 = 0.$$

When we then expand these 1-forms as $\theta_j^k = \theta_{j\ell}^k e^\ell$ the resulting functions $\theta_{j\ell}^k$ are just linear combinations of the components of the usual connection symbols of the frame, and so are of class C^{k-1} if our frame is of class C^k. The distribution $\mathcal{D}$ now becomes

$$\mathcal{D} = \text{span}\left\{\mathfrak{w}_1, \mathfrak{w}_2, \frac{\partial}{\partial \bar{\zeta}}\right\}$$

on $(\mathbb{C} - \{\pm i\}) \times \mathscr{U}$, where the vector fields

$$\begin{aligned}\mathfrak{w}_1 &= (\zeta^2 + 1)e_1 - 2\zeta e_3 + (\zeta^2 - 1)e_4 + Q_1(x, \zeta)\frac{\partial}{\partial \zeta} \\ \mathfrak{w}_2 &= (\zeta^2 + 1)e_2 + (\zeta^2 - 1)e_3 + 2\zeta e_4 + Q_2(x, \zeta)\frac{\partial}{\partial \zeta}\end{aligned}$$

are defined in terms of the functions

$$\begin{aligned}
Q_1(x,\zeta) &= \frac{1-\zeta^2}{2}\Big[(\zeta^2+1)\theta^3_{11} - 2\zeta\theta^3_{13} + (\zeta^2-1)\theta^3_{14}\Big] \\
&\quad +\zeta\Big[(\zeta^2+1)\theta^2_{11} - 2\zeta\theta^2_{13} + (\zeta^2-1)\theta^2_{14}\Big] \\
&\quad -\frac{1+\zeta^2}{2}\Big[(\zeta^2+1)\theta^2_{31} - 2\zeta\theta^2_{33} + (\zeta^2-1)\theta^2_{34}\Big] \\
Q_2(x,\zeta) &= \frac{1-\zeta^2}{2}\Big[(\zeta^2+1)\theta^3_{12} + (\zeta^2-1)\theta^3_{13} + 2\zeta\theta^3_{14}\Big] \\
&\quad +\zeta\Big[(\zeta^2+1)\theta^2_{12} + (\zeta^2-1)\theta^2_{13} + 2\zeta\theta^2_{14}\Big] \\
&\quad -\frac{1+\zeta^2}{2}\Big[(\zeta^2+1)\theta^2_{32} + (\zeta^2-1)\theta^2_{33} + 2\zeta\theta^2_{34}\Big]
\end{aligned}$$

The minutiæ of these expressions are of little importance, but three facts are worthy of emphasis. First of all, the components of $\mathfrak{w}_1$ and $\mathfrak{w}_2$ in the basis $e_1, \ldots, e_4, \partial/\partial\zeta$ are polynomial in ζ for any fixed $x \in \mathscr{U}$, and so, in particular,

$$\left[\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}_1\right] = \left[\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}_2\right] = 0.$$

Secondly, we have chosen the vector fields $\mathfrak{w}_1$ and $\mathfrak{w}_2$ to be real and *horizontal* along the locus F where ζ is real[1] . Finally, notice that $\mathscr{D}$ is spanned by C^{k-1} vector fields if g is of class C^k.

Proposition 7.1 *Let (M, g) be an oriented split-signature C^2 pseudo-Riemannian 4-manifold. Let $\wp : \mathcal{Z} \to M$ be the bundle of complex β-planes in $T_{\mathbb{C}}M$, and let $\mathscr{D} \subset T_{\mathbb{C}}\mathcal{Z}$ be the C^1 distribution of complex 3-planes defined above. Then $\mathscr{D}$ is involutive, in the sense that*

$$[C^1(\mathscr{D}), C^1(\mathscr{D})] \subset C^0(\mathscr{D}),$$

iff (M, g) is self-dual.

Proof. Let us begin by noticing that

$$\mathscr{D} \cap T_{\mathbb{C}}F = \mathscr{E}|_F = E \otimes \mathbb{C},$$

where the real distribution of 2-planes E on F is defined on page 9. Also recall that Proposition 3.5 tells us that E is Frobenius integrable iff g is self-dual.

Now, supose that $\mathscr{D}$ is involutive. Then both $T_{\mathbb{C}}F$ and $\mathscr{D}$ are closed under Lie brackets. Hence $\mathscr{D} \cap T_{\mathbb{C}}F = E \otimes \mathbb{C}$ is closed under Lie brackets, too. Thus E is Frobenius integrable, and Proposition 3.5 therefore tells us that g is self-dual.

Conversely, suppose that g is self-dual. Then Proposition 3.5 tells us that $E \to F$ is involutive. Let $\mathscr{U} \subset M$ be any open set on which there exists a pseudo-orthononormal frame $e_1, \ldots, e_4$, and consider the vector fields $\mathfrak{w}_1$ and $\mathfrak{w}_2$

[1]The $\mathfrak{w}_j$ could only be forced to be horizontal *everywhere* at the price of adding muliples of $\partial/\partial\overline{\zeta}$ to them. We have avoided doing so here because the relevant coefficients would generally *not* be holomorphic in ζ, and the Lie brackets of the $\mathfrak{w}_j$ with $\partial/\partial\overline{\zeta}$ would therefore no longer vanish.

constructed on an open dense subset of $\wp^{-1}(\mathscr{U})$ above. Along F, the vector fields $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are linearly independent sections of the involutive rank-2 bundle $E \subset TF$, so

$$[\mathfrak{w}_1, \mathfrak{w}_2] \wedge \mathfrak{w}_1 \wedge \mathfrak{w}_2 = 0 \qquad \text{when } \zeta = \overline{\zeta}.$$

However, relative to the frame $e_1, \dots e_4, \partial/\partial\zeta$, the components of $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are polynomial in ζ, so it follows that the components of the tensor field $[\mathfrak{w}_1, \mathfrak{w}_2] \wedge \mathfrak{w}_1 \wedge \mathfrak{w}_2$ are polynomial in ζ, too. But we have already seen that $[\mathfrak{w}_1, \mathfrak{w}_2] \wedge \mathfrak{w}_1 \wedge \mathfrak{w}_2$ vanishes when ζ is real. Hence $[\mathfrak{w}_1, \mathfrak{w}_2] \wedge \mathfrak{w}_1 \wedge \mathfrak{w}_2$ vanishes identically, and we therefore have

$$\left[\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}_1\right], \left[\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}_2\right], \left[\mathfrak{w}_1, \mathfrak{w}_2\right] \in \text{span}\left\{\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}_1, \mathfrak{w}_2\right\}.$$

Thus $\mathscr{D}$ is involutive on the region of $\wp^{-1}(\mathscr{U})$ parameterized by $(\mathbb{C} - \{\pm i\}) \times \mathscr{U}$, and the O'Neill tensor

$$\begin{array}{rcl} A_{\mathscr{D}} : \mathscr{D} \times \mathscr{D} & \longleftarrow & T_{\mathbb{C}}\mathcal{Z}/\mathscr{D} \\ (u, v) & \mapsto & [u, v] \bmod \mathscr{D} \end{array}$$

therefore vanishes on an open dense subset of $\wp^{-1}(\mathscr{U})$. But $A_{\mathscr{D}}$ is continuous, so it therefore vanishes on all of $\wp^{-1}(\mathscr{U})$. Since such subsets $\mathscr{U}$ cover all of M, it therefore follows that $\mathscr{D}$ is involutive on all of $\mathcal{Z}$. ■

Similar reasoning also shows the following:

Proposition 7.2 *Let $(M, [g])$ be an oriented split-signature self-dual 4-manifold. Then the involutive distribution $\mathscr{D}$ on $\mathcal{Z}$ is conformally invariant — that is, it depends only on the conformal class $[g]$, rather than on the metric $g \in [g]$.*

Proof. Since multiplying g by -1 does not change the metric connection, and therefore does not change $\mathscr{D} = \mathscr{E} \oplus \mathscr{V}^{0,1}$, it suffices to henceforth consider only conformally related pairs of metrics g and $\hat{g} = fg$ for which the factor f is positive.

Now the distribution E on F only depends on $[g]$, since it is tangent to the foliation $\mathscr{F}$ of F by lifted β-surfaces. Now consider two metrics g and $\hat{g} = fg$ in $[g]$, where $f > 0$. If $e_1, \dots, e_4$ is a pseudo-orthonormal frame for g on an open subset $\mathscr{U} \subset M$, then $f^{-1/2}e_1, \dots, f^{-1/2}e_4$ is a pseudo-orthonormal frame for $\hat{g}$. Let $\mathfrak{w}_j$ and $\hat{\mathfrak{w}}_j$ be the vector fields on $(\mathbb{C} - \{\pm i\}) \times \mathscr{U}$ constructed from these two frames and metrics. Then $\mathfrak{w}_j$ and $f^{1/2}\hat{\mathfrak{w}}_j$ coincide along F, since they are sections of E with the same projections. But the components of $\mathfrak{w}_j$ and $f^{1/2}\hat{\mathfrak{w}}_j$ (expressed, say, as linear combinations of the e_j and $\partial/\partial\zeta$) are polynomial in ζ. Since they coincide when ζ is real, we must therefore have $\mathfrak{w}_j \equiv f^{1/2}\hat{\mathfrak{w}}_j$. Hence the distribution $\mathscr{D}$ determined by g coincides with the distribution $\hat{\mathscr{D}}$ determined by $\hat{g}$ on an open dense subset of $\wp^{-1}(U)$, and we therefore have $\mathscr{D} \equiv \hat{\mathscr{D}}$ on $\wp^{-1}(U)$ by continuity. Since M can be covered with such open sets $\mathscr{U}$, it therefore follows that $\mathscr{D} = \hat{\mathscr{D}}$ on all of $\mathcal{Z}$, as claimed. ■

Actually, the conformal invariance of $\mathscr{D}$ holds even in the absence of the self-duality hypothesis, but we will never need this fact. It is also worth remarking that Proposition 7.1 could instead, for example, have been proved by imitating the arguments of Atiyah-Hitchin-Singer [2]. The route we have chosen is not arbitrary, however, but rather is specifically intended to prepare the reader for the proof of Proposition 10.1 below.

What is the 'real' geometrical meaning of a point of the bundle $\wp : \mathcal{Z} \to M$? Obviously, the points of $F \subset \mathcal{Z}$ are real totally null 2-planes, and there is not much more to be said. By contrast, a point of $\mathcal{Z} - F$ is a subspace $\Pi \subset T_xM \otimes \mathbb{C}$ with the property that $\Pi \cap \overline{\Pi} = 0$. Thus $\Pi \oplus \overline{\Pi} = T_xM \otimes \mathbb{C}$, and we can therefore define a unique almost-complex structure $\jmath : T_xM \to T_xM$ at x by declaring that Π is its $(+i)$-eigenspace. The requirement that Π be isotropic is then equivalent to the condition that $\jmath$ be an orthogonal transformation— i.e. that $\jmath^* g = g$. Finally, the requirement that Π be a β-plane, rather than an α-plane, is exactly that $\jmath$ determine the *given* orientation of M, rather than the opposite one. This last requirement concretely amounts to asking that there be an oriented pseudo-orthonormal basis $e_1, \ldots, e_4$ with $\jmath e_1 = e_2$ and $\jmath e_3 = e_4$. Notice that this formulation implicitly is associated with a decomposition $T_xM = T_+ \oplus T_-$, where $T_+ = \text{span}\,\{e_1, e_2\}$ and $T_- = \text{span}\{e_3, e_4\}$, and that $\jmath$ gives us a specific orientation of the maximally positive and negative subspaces T_+ and T_-.

Now suppose that $(M, [g])$ is space-time orientable. It then follows that $\mathcal{Z} - F$ has two connected components, depending on whether the associated orientation on T_- is the given one, or its reverse. Let $U \subset (\mathcal{Z} - F)$ be the open subset corresponding to $\jmath$ for which the induced orientation on T_- agrees with the previously chosen one. Then $\wp|_U : U \to M$ is an open disk bundle over M, and corresponds to the region $\Im m\ \zeta > 0$ in our explicit local description of $\mathcal{Z}$. Let $\mathcal{Z}_+ = U \cup F$ be the closure of U in $\mathcal{Z}$. Thus $\mathcal{Z}_+$ is a compact 6-manifold-with-boundary, and $\wp|_{\mathcal{Z}_+} : \mathcal{Z}_+ \to M$ is a bundle of closed oriented 2-disks.

Now F carries a foliation $\mathscr{F}$ by lifted β-surfaces. If we assume that our space-time-oriented self-dual 4-manifold $(M, [g])$ is also *Zollfrei*, then $\mathscr{F}$ becomes the system of fibers of the fibration $q : F \to P$, and Lemma 5.11 tells us, moreover, that $q : F \to P$ is a trivial 2-sphere bundle over $P \approx \mathbb{RP}^3$. We can thus give the disjoint union

$$Z = U \coprod P$$

the structure of a compact topological 6-manifold by endowing it with the quotient topology induced by the map

$$\Psi : \mathcal{Z}_+ \to Z,$$

where the restriction of Ψ to $\text{Int}\ \mathcal{Z}_+ = U$ is the identity map $U \to U$, and where the restriction of Ψ to $\partial \mathcal{Z}_+ = F$ is the fibration $q : F \to P$. Indeed, we may do this by using the 'polar coordinate' map

$$\begin{array}{ccc} P \times S^2 \times [0, \infty) & \longrightarrow & P \times \mathbb{R}^3 \\ (p, \vec{x}, t) & \longmapsto & (p, t\vec{x}) \end{array}$$

as our model for Ψ near $\partial \mathcal{Z}_+ = F$. Now if g is of class C^k, then $q : F \to P$ is of class C^{k-1}, and the diffeomorphism $\Phi : F \to P \times S^2$ of Lemma 5.11 is of class C^{k-2}, so this picture actually endows Z with the structure of a C^{k-2} manifold, in such a way that Ψ becomes a C^{k-2} map.

This said, we are now ready for one of the key constructions of this article:

Theorem 7.3 *Let $(M,[g])$ be a space-time-oriented self-dual Zollfrei manifold, where $[g]$ can be represented by a C^4 split-signature metric g. Let Z be the differentiable 6-manifold obtained from $\mathcal{Z}_+$ by collapsing $\partial\mathcal{Z}_+ = F$ down to P along the foliation $\mathscr{F}$. Then Z can be made into a compact complex 3-manifold in a unique way such that the quotient map $\Psi : \mathcal{Z}_+ \to Z$ satisfies*

$$\Psi_* \mathscr{D} \subset T^{0,1}Z.$$

Moreover, Ψ is C^∞ with respect to the associated complex-analytic atlas of Z if g is itself assumed to be C^∞.

Proof. By construction, Ψ is a diffeomorphism between $\mathcal{Z}_+ - F$ and $Z - P$. Since $\mathscr{D} \oplus \overline{\mathscr{D}} = T_\mathbb{C}\mathcal{Z}_+$ on $\mathcal{Z}_+ - F$, it follows that there a unique complex structure J on $Z - P$ with $T^{0,1} = \Psi_*\mathscr{D}$. Moreover, the assumption that g is C^4 guarantees that $\mathscr{D}$ is C^3. Since $\mathscr{D}$ is involutive by Proposition 7.1, the Malgrange version [33] of the Newlander-Nirenberg theorem [40] implies that this almost-complex structure is integrable, in the sense that Z admits local complex coordinates in which J becomes the standard complex structure on $\mathbb{C}^3$. Thus the crux of the theorem resides in understanding the behavior of $\Psi_*\mathscr{D}$ in the vicinity of P.

Now let us recall that the proof of Lemma 5.11 hinges on the introduction of a non-zero vector field u on F which spans $\ker \mathscr{p}_*$ at each point. By rescaling u by an appropriate function, we may now assume henceforth that $\mathscr{q}_*u$ is always a unit vector with respect to, say, the standard metric on $P \approx \mathbb{RP}^3$. With this convention, $S(TP)$ may be identified with the concrete S^2-bundle of unit vectors on $\mathbb{RP}^3$, and the C^2 diffeomorphism $\Phi : F \to S(TP)$ is just given by $\mathscr{q}_*u$. Now this vector field u is tangent to the boundary circles of the disk fibers of $\wp : \mathcal{Z}_+ \to M$, and the fiber-wise complex structure $\jmath$ of these disks then sends u to some vector field $v = \jmath u$ along $\partial\mathcal{Z}_+ = F$ which points inward at every boundary point of $\mathcal{Z}_+$. Extend this v to a C^2 vector field on a collar neighborhood of $\mathcal{Z}_+$ so that we have $v \in \ker \wp_*$ at every point of the collar, and then use the flow of v to identify a slightly smaller collar with $F \times [0,\epsilon)$. Using Φ and Ψ, we may thus construct a C^2 diffeomorphism between a tubular neighborhood of P and the ϵ-tube around the zero section of TP, in such a manner that the restriction of Ψ to our collar $F \times [0,\epsilon) \approx S(TP) \times [0,\epsilon)$ becomes the map

$$\begin{array}{rcl} S(TP) \times [0,\epsilon) & \to & TP \\ (\vec{v}, t) & \mapsto & t\vec{v} \end{array}$$

and so that our vector field v becomes the radial field $\vec{v}/\|\vec{v}\|$. In particular, this picture gives us a specific isomorphism

$$TZ|_P \cong TP \oplus TP,$$

where the first factor is tangent to P, and where the second factor is transverse to it. Moreover, this isomorphism has been constructed precisely so that $\Psi_*(\jmath u) = J\Psi_*(u)$ at each point of $F = \partial\mathcal{Z}_+$, provided that we take $J : TP \oplus TP \to TP \oplus TP$ to be the almost complex structure given by

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

where $I : TP \to TP$ denotes the identity map. Since the rank of $\Psi_* \mathcal{D}$ is just 1 along F, this choice of J therefore gives us $\Psi_* \mathcal{D} = T^{0,1}(Z, J)$ along P, as desired; moreover, this is the only choice of J with this property, since every unit element of $TP \subset TZ$ is of the form $\Psi_{*z} u$ for some $z \in \partial \mathcal{Z}_+$. Thus, in conjunction with our previous discussion of $Z - P$, we see that there is a unique almost-complex structure J on all of Z such that $\Psi_* \mathcal{D} \subset T^{0,1}(Z, J)$. However, it is not yet clear that this J is even continuous, much less integrable!

We will remedy this by next showing that J is actually *Lipschitz* continuous, relative to the C^2 structure with which we have provisionally endowed Z. Of course, this is is only an issue near P, since J has been constructed so as to be better than C^1 on $Z - P$. It therefore suffices to show that J is Lipschitz along each radial line segments $t \mapsto t\vec{v}$, t$\in [0, \epsilon)$ in our tubular neighborhood of P modeled on the ϵ tube in TP, provided we can also show in the process that the Lipschitz constants are uniformly bounded.

To this end, let us therefore recall that we have written down an explicit local basis $(\mathfrak{w}_1, \mathfrak{w}_2, \partial/\partial\overline{\zeta})$ for $\mathcal{D}$ such that $[\mathfrak{w}_j, \partial/\partial\overline{\zeta}] = 0$. Moreover, the $\mathfrak{w}_j$ are real along $F = \partial \mathcal{Z}_+$, where they span the distribution of 2-planes E tangent to the foliation $\mathscr{F}$ of F. Now, through a given point of $q^{-1}(y) \subset F$, there is a unique curve in the leaf $q^{-1}(y)$ with parameter t such that $d/dt = \mathfrak{w}_1$. For any C^2 function f on Z, we then have

$$\frac{d}{dt}\left[\Psi_*(\frac{\partial}{\partial\overline{\zeta}})f\right] = \frac{d}{dt}\frac{\partial}{\partial\overline{\zeta}}\Psi^* f = \mathfrak{w}_1 \frac{\partial}{\partial\overline{\zeta}}\Psi^* f = \frac{\partial}{\partial\overline{\zeta}}\mathfrak{w}_1 \Psi^* f = \frac{\partial}{\partial\overline{\zeta}}\left[\Psi_*(\mathfrak{w}_1) f\right].$$

Thus, setting $\zeta = \xi + i\eta$,

$$\frac{d}{dt}\left[\Psi_*(\frac{\partial}{\partial\overline{\zeta}})\right] = \frac{\partial}{\partial\overline{\zeta}}\left[\Psi_*(\mathfrak{w}_1)\right] = \frac{i}{2}\frac{\partial}{\partial\eta}\left[\Psi_*(\mathfrak{w}_1)\right]$$

at any $y \in P$, since $\Psi_*(\mathfrak{w}_1) \equiv 0$ along F, where $\eta = 0$. Here the right-hand side should be interpreted as the invariant derivative *at a zero* of a section of a vector bundle on the disk $D_x := \Psi[\wp^{-1}(x) \cap \mathcal{Z}_+] \approx \mathrm{D}^2$. On the other hand,

$$\Psi_*\left(\frac{\partial}{\partial\overline{\zeta}}\right) \in T^{0,1}_y(Z, J)$$

for all t, by our previous discussion, so it follows that

$$\frac{\partial}{\partial\eta}\left[\Psi_*(\mathfrak{w}_1)\right]\Big|_{\eta=0} \in T^{0,1}_y(Z, J).$$

The same argument, with $\mathfrak{w}_1$ replaced by $\mathfrak{w}_2$, tells us that

$$\frac{\partial}{\partial\eta}\left[\Psi_*(\mathfrak{w}_2)\right]\Big|_{\eta=0} \in T^{0,1}_y(Z, J),$$

too. Along any D_x, we therefore have, near an arbitrary point $y \in P \cap D_x$, three continuous sections of $T^{1,0}$ given by

$$\mathfrak{v}_j = \begin{cases} \left[\Psi_*(\mathfrak{w}_j)\right]/\eta & \eta \neq 0 \\ \frac{\partial}{\partial\eta}\left[\Psi_*(\mathfrak{w}_j)\right] & \eta = 0 \end{cases}$$

for $j = 1, 2$, and $\mathfrak{v}_3 = \Psi_*(\partial/\partial\overline{\zeta})$. These sections are linearly independent at every point, and so span $T^{1,0}_y$, because $\det(\Psi_*)$ only vanishes to second order at Z. Moreover, since Ψ appears to be C^2 in our coordinates, these sections are all continuously differentiable along D_x, with coordinate derivatives expressible in terms of partial derivatives of Ψ of order ≤ 2. In particular, J is Lipschitz along D_x, with Lipschitz constant controlled by the partial derivatives of Ψ of order ≤ 2. Since each radial line of our tube is contained in a disk D_x, and because a finite number of balls with compact closure within coordinate domains suffice to cover the compact manifold P, it therefore follows that the tensor field J on Z is Lipschitz near P, and hence on all of Z.

Since J is $C^{0,1}$ on Z, and better than C^1 on $Z - P$, the naïve coordinate partial derivatives of the components of J on $Z-P$ extend to Z as locally bounded measurable functions. Integration by parts, however, shows that these L^∞_{loc} functions are exactly the *distributional* partial derivatives of the components of J. The Nijenhuis tensor

$$N^\ell_{jk} = J_k{}^m \partial_m J_j{}^\ell - J_j{}^m \partial_m J_k{}^\ell + J_m{}^\ell \partial_j J_k{}^m - J_m{}^\ell \partial_k J_j{}^m$$

of our almost-complex structure J is therefore well-defined in the distributional sense, and has L^∞_{loc} components. Hence N vanishes in the distributional sense, since by construction $N = 0$ on a subset $Z - P$ of full measure. However, Hill and Taylor [19] have shown that the Newlander-Nirenberg theorem holds for Lipschitz almost-complex structures for which $N = 0$ in the distributional sense. Thus every point of Z has a neighborhood on which we can find a triple (z^1, z^2, z^3) of differentiable complex-valued functions with $dz^k \in \Lambda^{1,0}(Z, J)$ and $dz^1 \wedge dz^2 \wedge dz^3 \neq 0$. Taking these to be the complex coordinate systems gives Z the structure of a compact complex 3-fold. In particular, this gives Z a specific preferred C^∞ structure compatible with the C^1 structure we built by hand, so Ψ remains a differentiable map even with respect to this brand new atlas for Z.

Now, if g is actually C^∞, we claim that Ψ is actually a C^∞ map with respect to the tautological smooth structure on $\mathcal{Z}_+$ and the complex atlas of Z. Away from $F \to P$, this is an immediate consequence of the classical Newlander-Nirenberg theorem [40], so we need merely verify this assertion near P. To do this, let (x^1, x^2, x^3) be any smooth system of local coordinates on a region $\mathcal{V} \subset P$, and pull them these functions back to F as three smooth functions $\mathsf{q}^* x^j$ on $\mathsf{q}^{-1}(\mathcal{V}) \subset F = \partial\mathcal{Z}_+$ which are constant along the leaves of $\mathscr{F}$. These can then be extended [50] into $\mathcal{Z}_+$ as smooth complex-valued functions $\mathfrak{z}^j$ near $\partial\mathcal{Z}_+$ such that $\partial\mathfrak{z}^j/\partial\overline{\zeta}$ vanishes to infinite order along $\eta = 0$, and the $\mathfrak{w}_k z^j$ will then also vanish to infinite order along $\eta = 0$, too. Now the real and imaginary parts of the $\mathfrak{z}^j$ give us a differentiable coordinate system on Z, and in these coordinates we have

$$T^{0,1}Z = \text{span}\left\{\frac{\partial}{\partial\overline{\mathfrak{z}}^j} + a^k_j(\mathfrak{z})\frac{\partial}{\partial\mathfrak{z}^k}\right\}$$

where the smooth functions $a^k_j(\mathfrak{z}^1, \mathfrak{z}^2, \mathfrak{z}^3)$ vanish to infinite order along the locus P given by $\Im m\ \mathfrak{z}^j = 0$. If (z^1, z^2, z^3) is a system of holomorphic local coordinates on $\mathcal{U} \subset Z$, where $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, then $z^j(\mathfrak{z}^1, \mathfrak{z}^2, \mathfrak{z}^3)$ is therefore C^∞ by elliptic regularity. Since, by construction, each $\Psi^*\mathfrak{z}^k$ is a smooth function on $\mathcal{Z}_+$, it thus follows that the $\Psi^* z^j$ are smooth functions, too. Hence Ψ is smooth with respect to the complex coordinate atlas of Z, and we are done. ■

Definition 7.4 *The* twistor space *of a space-time-oriented C^4 Zollfrei self-dual* 4-*manifold* $(M,[g])$ *is the compact complex* 3-*manifold* (Z,J) *constructed from* $(M,[g])$ *via Theorem 7.3.*

Definition 7.5 *The twistor space of a non-space-time-orientable C^4 Zollfrei self-dual 4-manifold $(M,[g])$ is defined to be the twistor space (Z,J) of the space-time-oriented double cover $(\tilde{M},[g])$ of M.*

8 Unmasking the Twistor Space

Our construction of the twistor space of a self-dual Zollfrei 4-manifold may seem rather technical. However, the hidden motivation behind the entire construction is the observation that when $(M,[g])$ is one of our prototypical models, the associated twistor space (Z,J) is simply the familiar complex projective 3-space $\mathbb{CP}_3$. Let us now make this explicit:

Lemma 8.1 *If $(M,[g])$ is either $(S^2\times S^2,[g_0])$ or $(\mathbb{M}^{2,2},[g_0])$, then the twistor space (Z,J) of $(M,[g])$, in the sense of Definitions 7.4 and 7.5, is biholomorphic to $\mathbb{CP}_3$ in such a manner than $P\subset Z$ becomes the standard $\mathbb{RP}^3\subset\mathbb{CP}^3$.*

Proof. The relationship between Definitions 7.4 and 7.5 makes it sufficient to consider the case of $\mathbb{M}^{2,2}$. Now this may seem to be a strange choice, because Definition 7.5 ostensibly instructs us to pass up to the double cover $S^2\times S^2\to\mathbb{M}^{2,2}$ and then blow down $\partial\mathcal{Z}_+(S^2\times S^2)$ along the foliation $\mathscr{F}$. However, the quotient of $\mathcal{Z}_+(S^2\times S^2)$ by the covering map action of $\mathbb{Z}_2$ on $\partial\mathcal{Z}_+(S^2\times S^2)$ is just $\mathcal{Z}(\mathbb{M}^{2,2})$. Thus, Definition 7.5 can be restated as saying that Z is to be obtained from $\mathcal{Z}(\mathbb{M}^{2,2})$ by blowing down the hypersurface $F\subset\mathcal{Z}$.

In fact, there is a nice way of explicitly realizing of this blowing-down map. Let $\mathbb{V}\cong\mathbb{R}^4$ be a real 4-dimensional vector space, and let $\mathbb{V}_{\mathbb{C}}\cong\mathbb{C}^4$ be its complexification. Then $\mathbb{M}^{2,2}$ can be be identified with the real Klein quadric

$$Q_{\mathbb{R}}=\{[\psi]\in\mathbb{P}(\wedge^2\mathbb{V})\mid\psi\wedge\psi=0\}$$

in $\mathbb{P}(\wedge^2\mathbb{V})\cong\mathbb{RP}^5$ by choosing a diagonalizing basis for the signature $(+++---)$ quadratic form $(\psi,\chi)=\phi\wedge\chi$ on $\wedge^2\mathbb{V}$. For a suitable choice of orientation, the β-surfaces of $(\mathbb{M}^{2,2},[g_0])$ are exactly those projective planes $\mathbb{RP}^2\subset Q_{\mathbb{R}}\subset\mathbb{RP}^5$ which are of the form

$$\{[\psi]\in Q_{\mathbb{R}}\mid v\lrcorner\psi=0\}$$

for some $[v]\in\mathbb{P}(\mathbb{V}^*)\cong\mathbb{RP}^3$. Thus $F(\mathbb{M}^{2,2})$ may be concretely realized as the flag manifold

$$F_{2,3,4}=\{([\psi],[v])\in Q_{\mathbb{R}}\times\mathbb{P}(\mathbb{V}^*)\mid v\lrcorner\psi=0\}\subset Q_{\mathbb{R}}\times\mathbb{P}(\mathbb{V}^*)$$

in such a way that p and q become the tautological projections $F_{2,3,4}\to Q_{\mathbb{R}}=Gr_{2,4}$ and $\mathbb{P}(\mathbb{V}^*)=Gr_{3,4}$. However, $Q_{\mathbb{R}}$ is just a real slice of the complex 4-quadric

$$Q_{\mathbb{C}}=\{[\psi]\in\mathbb{P}(\wedge^2\mathbb{V}_{\mathbb{C}})\mid\psi\wedge\psi=0\},$$

so we have a canonical isomorphism $T_{\mathbb{C}}Q_{\mathbb{R}} = TQ_{\mathbb{C}}|_{Q_{\mathbb{R}}}$. Any *complex* β-plane $\Pi \subset T_{\mathbb{C}}Q_{\mathbb{R}}$ is then tangent to a unique *complex* β-surface $\mathbb{CP}_2 \subset Q_{\mathbb{C}} \subset \mathbb{CP}_5$ given by

$$\{[\psi] \in Q_{\mathbb{C}} \mid v \lrcorner \psi = 0\}$$

for some $[v] \in \mathbb{P}(\mathbb{V}^*_{\mathbb{C}}) \cong \mathbb{CP}_3$. Thus $\mathcal{Z}(\mathbb{M}^{2,2})$ may naturally be identified with the locus

$$\{([\psi],[v]) \in Q_{\mathbb{R}} \times \mathbb{P}(\mathbb{V}^*_{\mathbb{C}}) \mid v \lrcorner \psi = 0\} \subset F_{2,3,4}(\mathbb{C})$$

in such a way that the $\Psi : \mathcal{Z} \to Z$ is just becomes the tautological projection to $\mathbb{P}(\mathbb{V}^*_{\mathbb{C}}) \cong \mathbb{CP}_3$.

It remains to show that the constructed complex structure on Z coincides with that of $\mathbb{CP}_3$. To do this, we first recall that the distribution $\mathscr{D}$ is conformally invariant by Proposition 7.2. Passing to the stereographic coordinates of equation (1), it thus suffices do our computations for the flat metric $d\mathfrak{x}_1^2 + d\mathfrak{x}_2^2 - d\mathfrak{y}_1^2 - d\mathfrak{y}_2^2$ using the pseudo-orthonormal frame

$$e_1 = \frac{\partial}{\partial \mathfrak{x}_1},\ e_2 = \frac{\partial}{\partial \mathfrak{x}_2},\ e_3 = \frac{\partial}{\partial \mathfrak{y}_1},\ e_4 = \frac{\partial}{\partial \mathfrak{y}_2}.$$

Since the connection forms θ_j^k vanish for this frame, the distribution $\mathscr{D}$ is thus spanned by

$$\begin{aligned}
\mathfrak{w}_1 &= (\zeta^2+1)\frac{\partial}{\partial \mathfrak{x}_1} - 2\zeta\frac{\partial}{\partial \mathfrak{y}_1} + (\zeta^2-1)\frac{\partial}{\partial \mathfrak{y}_2} \\
\mathfrak{w}_2 &= (\zeta^2+1)\frac{\partial}{\partial \mathfrak{x}_2} + (\zeta^2-1)\frac{\partial}{\partial \mathfrak{y}_1} + 2\zeta\frac{\partial}{\partial \mathfrak{y}_2}
\end{aligned}$$

and $\partial/\partial\overline{\zeta}$. But the projection $\Psi : \mathcal{Z} \to \mathbb{CP}_3$ coming from the Klein quadric picture is just given by

$$\begin{aligned}
z_1 &= (\mathfrak{x}_1 + \mathfrak{y}_2) + (\mathfrak{y}_1 - \mathfrak{x}_2)\zeta \\
z_2 &= (\mathfrak{y}_1 + \mathfrak{x}_2) + (\mathfrak{x}_1 - \mathfrak{y}_2)\zeta \\
z_3 &= \zeta
\end{aligned}$$

in suitable affine coordinates (z_1, z_2, z_3) for $\mathbb{CP}_3$. Since $\mathfrak{w}_1$, $\mathfrak{w}_2$, and $\partial/\partial\overline{\zeta}$ all annihilate z_1, z_2 and z_3, it follows that the complex structure J we have constructed on $Z = \mathbb{CP}_3$ coincides with the usual one on an open dense set, and hence everywhere. Thus, for both $(\mathbb{M}^{2,2}, [g_0])$ and $(S^2 \times S^2, [g_0])$, the twistor space is just $\mathbb{CP}_3$, with its standard complex structure. ■

Now recall that the complex structure of $\mathbb{CP}_3$ is rigid, in the sense of Kodaira and Spencer [22]. In other words, because $H^1(\mathbb{CP}_3, \mathcal{O}(T^{1,0}\mathbb{CP}_3)) = 0$, any complex-analytic family of deformations of the complex structure is trivial for small values of the perturbation parameter. It might therefore seem reasonable to expect that the twistor space of any Zollfrei self-dual 4-manifold, in the sense of Definitions 7.4 and 7.5, will *always* turn out simply to be $\mathbb{CP}_3$, with its usual complex structure. Our goal in this section will be to show that this is indeed the case provided that suitable extra hypotheses are imposed. To this end, we will use a beautiful circle of characterizations of the standard complex structure on $\mathbb{CP}_3$ due to Nakamura [39]. One such result is the following:

Theorem 8.2 (Nakamura) *Let (Z, J) be a compact complex 3-manifold homeomorphic to $\mathbb{CP}_3$. If $H^q(Z, \mathcal{O}) = 0$ for all $q > 0$, and if $h^0(Z, \mathcal{O}(K^{-m})) \geq 2$ for some $m > 0$, then (Z, J) is biholomorphic to $\mathbb{CP}_3$.*

Nakamura then used this to show that any Moishezon 3-fold homeomorphic to $\mathbb{CP}_3$ must be biholomorphic to $\mathbb{CP}_3$ unless it is of general type. Recall that a compact complex n-fold Z is said to be *Moishezon* if there exist n meromorphic functions $f_1, \ldots, f_n : Z \dashrightarrow \mathbb{C}$ which give local complex coordinates near some point $z \in Z$; this is holds, in particular, if [51] there is some holomorphic line bundle $L \to Z$ with $h^0(Z, \mathcal{O}(L^m)) > cm^n$ for some $c > 0$ and all $m \gg 0$. Kollár [23] eventually improved Nakamura's result by excluding the possibility that Z might be of general type. Thus:

Theorem 8.3 (Nakamura/Kollár) *A Moishezon manifold is homeomorphic to $\mathbb{CP}_3$ iff it is biholomorphic to $\mathbb{CP}_3$.*

The following standard piece of folklore is a minor variation on one of Nakamura's results [39]. We include a proof here only because one does not seem to appear elsewhere in the literature.

Corollary 8.4 *Let J_t be a family of smooth, integrable almost-complex structures on a smooth compact 6-manifold Z, which, in the C^∞ topology, depends continuously on an auxiliary real variable $t \in [0, 1]$. If (Z, J_0) is biholomorphic to the standard $\mathbb{CP}_3$, so is (Z, J_1).*

Proof. Kuranishi [25] has shown that whenever two smooth complex structures are close enough in a sufficiently high Sobolev norm, they can be joined by a complex-analytic family in the sense of Kodaira-Spencer. Hence there is a finite subset $\{t_0 = 0, t_1, \ldots, t_\ell = 1\}$ of $[0, 1]$ such that, for each $j = 1, \ldots, \ell$, $(Z, J_{t_{j-1}})$ and (Z, J_{t_j}) both occur as fibers of a single holomorphic family of complex manifolds over over the unit disk $\subset \mathbb{C}$.

Now Kodaira-Spencer theory [22] tells us that if $(Z, J_{t_{j-1}})$ is biholomorphic to $\mathbb{CP}_3$, every nearby fiber is, too. Hence there is a non-empty open set in the disk for which every correspondiing fiber satisfies $h^0(\mathcal{O}(K^{-m})) > m^3$ for all $m > 0$. But, by the semi-continuity principle [3], the set of parameter values for which $h^0(\mathcal{O}(K^{-m})) > m^3$ for a particular m must be closed in the analytic Zariski topology — i.e. either discrete, or the whole disk. Hence every fiber must have $h^0(\mathcal{O}(K^{-m})) > m^3$ for all $m > 0$, and this conclusion applies, in particular, to (Z, J_{t_j}). Hence (Z, J_{t_j}) is Moishezon. Theorem 8.3 therefore shows that

$$(Z, J_{t_{j-1}}) \cong \mathbb{CP}_3 \implies (Z, J_{t_j}) \cong \mathbb{CP}_3.$$

Since (Z, J_0) is biholomorphic to $\mathbb{CP}_3$ by hypothesis, it therefore follows by induction on j that (Z, J_1) is also biholomorphic to $\mathbb{CP}_3$, as claimed. ■

Note that an analogous rigidity assertion also holds for any $\mathbb{CP}_n$, even if n is large, as a consequence of an entirely different circle of ideas due to Siu [45].

Now the proof of Theorem 7.1 shows that two self-dual Zollfrei metrics which are close in the C^∞ topology will give rise to two complex structures on Z which are close in the C^∞ topology. If g_t is a continuous curve in the space of of C^∞ self-dual Zollfrei metrics, with the C^∞ topology, Corollary 8.4 then immediately implies that if one of the relevant twistor spaces is biholomorphic to $\mathbb{CP}_3$, so are all the others. When this happens, the smooth submanifold $P = \Psi(F)$ thus becomes a smoothly embedded totally real submanifold of $\mathbb{CP}_3$, and every fiber of $\mathcal{Z}_+ \to M$ is then sent by Ψ to an embedded holomorphic disk in $\mathbb{CP}_3$ with boundary on P. Thus:

Theorem 8.5 *Let $\mathcal{C}$ be the space of C^∞ self-dual Zollfrei conformal classes metrics on $S^2 \times S^2$, endowed with the smooth topology. Let $\mathcal{C}_0 \subset \mathcal{C}$ be the path component containing our protypical example $[g_0]$. Then, for each conformal class $[g] \in \mathcal{C}_0$, the corresponding twistor space (Z, J) is biholomorphically equivalent to $\mathbb{CP}_3$, equipped with its standard complex structure. In particular, every conformal class in $\mathcal{C}_0$ gives rise to a smooth totally real submanifold $P \approx \mathbb{RP}^3$ of $\mathbb{CP}^3$ and a 4-parameter family of embedded holomorphic disks $(D^2, \partial D^2) \hookrightarrow (\mathbb{CP}_3, P)$.*

Unfortunately, however, we cannot *a priori* expect an indefinite self-dual metric to be highly differentiable, as the relevant partial differential equation is ultra-hyperbolic rather than elliptic. It thus behooves us to see what we can say about solutions with comparatively little regularity. However, even trying to understand C^4 self-dual metrics will lead us to consider families of twistor spaces with so little regularity that the results of Kodaira-Spencer and Kuranishi cannot be invoked with confidence. Fortunately, however, Nakamura's results are more than enough to deal with the matter at hand:

Theorem 8.6 *Let g_0 be the standard indefinite product metric on $S^2 \times S^2$. Then g has a neighborhood $\mathscr{U}$ in space of C^4 pseudo-Riemannian metrics such that any self-dual metric $g \in \mathscr{U}$ is Zollfrei and has twistor space (Z, J) biholomorphic to $\mathbb{CP}_3$.*

Proof. By Theorem A, there is a C^2 neighborhood of g_0 in which every self-dual g is Zollfrei, and if g is also assumed to be C^4 close to g_0, then the proof of Theorem 7.1 shows that there is a diffeomorphism between the twistor spaces of g and g_0 such that the almost-complex structure J associated with g is close to the almost-complex structure J_0 associated with g_0 in the $C^{0,1}$ topology on tensor fields on Z. Choose a biholomorphism, once and for all, between (Z, J_0) and $\mathbb{CP}_3$. Then, by shrinking our neighborhood $\mathscr{U}$ if necessary, we may identify the (p, q)-forms for J with those of J_0 via the tautological projections, and it therefore makes sense to think of the operators D and D_0 given by $\overline{\partial} + \overline{\partial}^*$ associated to these two complex structures as being defined on the same spaces, even after twisting with any power of the canonical line bundle. Thus, for example, if we consider D and D_0 applied to $(0, 1)$-forms, then, for every $\varepsilon > 0$ there exists a $\mathscr{U}$ such that for every $g \in \mathscr{U}$ we have $\|(D - D_0)f\|^2 \leq \varepsilon(\|\nabla f\|^2 + \|f\|^2)$ for each and every smooth $(0, 1)$-form f, where $\| \ \|$ denotes the L^2 norm on $Z = \mathbb{CP}_3$ with respect to, say, the Fubini-Study metric. Now assume that such an elliptic operator D_0 has trivial kernel. By Gårding's inequality for D_0 we therefore have

$$\|(D - D_0)f\|^2_{L^2} \leq \varepsilon\|f\|^2_{L^2_1} \leq C\varepsilon\|D_0 f\|^2_{L^2}$$

so that

$$\|Df\|_{L^2} \geq (1 - \sqrt{C\varepsilon})\|D_0 f\|,$$

and we therefore D has trivial kernel, too, provided that we take $\varepsilon < 1/C$. Thus, by shrinking our neighborhood $\mathscr{U}$ if necessary, we may arrange that every associated twistor space has $H^1(Z, \mathcal{O}) = 0$, just like $\mathbb{CP}_3$. Similarly, we may arrange that $H^q(Z, \mathcal{O}) = 0$ and $H^q(Z, \mathcal{O}(K^{-1})) = 0$ for $q = 1, 2, 3$ by further shrinking $\mathscr{U}$. Since Z also has the same Chern classes as $\mathbb{CP}_3$, the index theorem then gives us $h^0(Z, \mathcal{O}(K^{-1})) = \binom{7}{4}$, so Nakamura's result certainly guarantees that there is a biholomorphism between Z and $\mathbb{CP}_3$. ■

The holomorphic rigidity of the twistor space implies the following geometric rigidity result:

Theorem 8.7 *Let g_0 be the standard conformally flat split-signature metric on $\mathbb{M}^{2,2} = (S^2 \times S^2)/\mathbb{Z}_2$. Then, in the C^4 topology on the space of pseudo-Riemannian metrics, g_0 has a neighborhood $\mathscr{U}$ such that any other self-dual metric $g \in \mathscr{U}$ is of the form $f\phi^* g_0$ for some diffeomorphism $\phi : \mathbb{M}^{2,2} \to \mathbb{M}^{2,2}$ and some function $f \neq 0$.*

Proof. If $\mathscr{U}$ is small enough, every self-dual $g \in \mathscr{U}$ is Zollfrei and has a twistor space (Z, J) which is biholomorphic to $\mathbb{CP}_3$ by the previous result. This twistor space can be obtained by blowing $\mathcal{Z}$ down along F. Complex conjugation in $\mathcal{Z}$ therefore induces an anti-holomorphic involution $\varrho : Z \to Z$ with fixed point set $P \approx \mathbb{RP}^3$. By a change of homogeneous coordinates, any such ϱ can be put into the standard form

$$[z_0 : z_1 : z_2 : z_3] \mapsto [\overline{z}_0 : \overline{z}_1 : \overline{z}_2 : \overline{z}_3],$$

as may be seen by considering the induced action on the sections of the hyperplane line bundle, thought of as meromorphic functions with simple poles along an invariant hyperplane. Thus P becomes the standard $\mathbb{RP}^3 \subset \mathbb{CP}_3$ in these coordinates. Let Q denote the quadric given in these coordinates by $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$, and observe that $[Q]$ now generates $H^2(\mathbb{CP}_3 - P, \mathbb{Z})$. However, any fiber disk of $\mathcal{Z}_+ \to \tilde{M}$ generates $H_2(\mathcal{Z}_+, \partial\mathcal{Z}_+; \mathbb{Z})$, where $\tilde{M} = S^2 \times S^2$ is the space-time-oriented double cover of $M = \mathbb{M}^{2,2}$. Since Ψ induces a homotopy equivalence between $\mathbb{CP}_3 - P$ and $\mathcal{Z}_+$, Poincaré duality now tells us that each of these holomorphic disks must meet Q in exactly one point. Thus $\Psi^{-1}(Q)$ is a section of $(\text{Int } \mathcal{Z}_+) \to \tilde{M}$. Moreover, the non-trivial deck transformation $\tilde{M} \to \tilde{M}$ acts on Q via the complex conjugation map ϱ, so we have constructed a diffeomorphism $\phi : (Q/\varrho) \to M$, and since Q is a complex submanifold of Int $\mathcal{Z}_+$, our construction of $\mathscr{D} = T^{0,1}(\text{Int } \mathcal{Z}_+)$ also shows that ϕ is of class $C^{k,\alpha}$ if g is of class $C^{k,\alpha}$. But the two holomorphic disks that make up $C_x = \Psi[\wp^{-1}(x)] \subset \mathbb{CP}_3$ have the same boundary along $P = \mathbb{RP}^3$, and their union is therefore a rational curve in $\mathbb{CP}_3$, for any $x \in M$. Each such curve meets Q in a conjugate pair of points; and since $Q \subset \mathbb{CP}_3$ has degree 2, this means that C_x has degree 1. Hence each C_x is a projective line $\mathbb{CP}_1 \subset \mathbb{CP}_3$. However, $P = \mathbb{RP}^3$ is the space of β-surfaces of $(M, [g])$, and, for any $x \in M$, $q[p^{-1}(x)] = C_x \cap \mathbb{RP}^3$. Thus any β surface in $M = Q/\varrho$ is obtained by choosing some point $y \in \mathbb{RP}^3$, looking at all the $\mathbb{RP}^2$-family of all ϱ-invariant projective lines in $\mathbb{CP}_3$ that pass through y, and tracing out the intersections of these lines with Q. But this same picture also, in particular, describes the β-surfaces of g_0. We have thus found a diffeomorphism ϕ between M and $Q/\varrho = (S^2 \times S^2)/\mathbb{Z}_2 = \mathbb{M}^{2,2}$ which sends β surfaces to β-surfaces. Since this last statement means that ϕ takes null vectors to null vectors, we have

$\phi^*[g_0] = [g]$, and hence $g = f\phi^* g_0$, as promised. ■

It will turn out that the situation on $S^2 \times S^2$ is far different. Nonetheless, we do get some interesting immediate geometric pay-off from the present discussion:

Theorem 8.8 *Let g_0 be be the standard indefinite product metric on $S^2 \times S^2 = \mathbb{CP}_1 \times \mathbb{CP}_1$. Then g_0 has a neighborhood $\mathscr{U}$ in the space of C^4 pseudo-Riemannian metrics such that any self-dual metric $g \in \mathscr{U}$ is of the form $g = \psi^* h$, where h is an indefinite* Hermitian *metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$, and where ψ is a self-diffeomorphism of $S^2 \times S^2$.*

Proof. The quadric $Q \subset \mathbb{CP}_3$ given by $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ does not meet the standard $\mathbb{RP}^3$. For every self-dual metric g close to g_0 in the C^4 topology, P will be C^1 close to the standard $\mathbb{RP}^3$, and so will also not meet Q if our neighborhood $\mathscr{U}$ is small enough. The inverse image of Q under $\Psi : \mathcal{Z}_+ \to M$ is therefore a complex submanifold of Int $\mathcal{Z}_+$. Moreover, the fibers of $\mathcal{Z}_+$ have intersection number 1 with Q, and as both Q and these disks are complex submanifolds, it follows that each fiber meets Q transversely in one point. Thus Q is the image of a smooth section $\mathfrak{J}$ of Int $\mathcal{Z}_+$. But this section is a bihomorphism between $(M, \mathfrak{J})$ and $Q \cong \mathbb{CP}_1 \times \mathbb{CP}_1$; in particular, $\mathfrak{J}$ is integrable. On the other hand, $\mathfrak{J}$ is, by construction, a g-compatible almost-complex structure. Thus what we have constructed is a diffeomorphism $\psi : \mathbb{CP}_1 \times \mathbb{CP}_1 \to M$ such that $\psi^* g$ is an indefinite Hermitian metric. ■

Finally, we observe that the smooth topology of the twistor space is always standard, even without restrictions on our Zollfrei self-dual 4-manifold. This will turn out to be quite useful in §11 below.

Theorem 8.9 *Let $(M, [g])$ be a self-dual Zollfrei* 4*-manifold, and let Z be the twistor space of $(M, [g])$, as defined in Definitions 7.4 and 7.5. Then Z is diffeomorphic to $\mathbb{CP}_3$ in such a manner that the Chern classes $c_j(Z, J)$ are sent to the usual Chern classes of $\mathbb{CP}_3$.*

Proof. By passing to a double cover if necessary, we may assume that M is space-time orientable. Thus M is homeomorphic to $S^2 \times S^2$, by Theorem 5.12. Let $Y \subset Z$ be the closure of a small tubular neighborhood of $P \approx \mathbb{RP}^3$, and let $X = Z - (\text{Int } Y)$. Thus $Y \approx \mathbb{RP}^3 \times D^3$, $X \cap Y \approx \mathbb{RP}^3 \times S^2$, and $X \approx \mathcal{Z}_+$.

Next, choose an almost-complex structure $\mathfrak{J}$ on M which is compatible with g and the space-time orientation. Then $\mathcal{Z}_+$ is diffeomorphic to the unit disk bundle in the anti-canonical line bundle $\Lambda^{0,2}(M, \mathfrak{J})$. In particular, $\mathcal{Z}_+$ deform retracts to a copy of M. Moreover, $T\mathcal{Z}_+|_M = TM \oplus \nu$, where the normal bundle ν of M is exactly the anti-canonical line bundle. Since we therefore have $c_1(\nu) = c_1(M, \mathfrak{J})$, so $w_2(T\mathcal{Z}_+)|_M = 2w_2(TM) = 0$. It follows that X is spin.

Now X is simply connected, and since the inclusion $X \cap Y \hookrightarrow Y$ induces an isomorphism of fundamental groups, the Seifert-van Kampen theorem tells us that Z is simply connected, too. Since the inclusion $\partial X \hookrightarrow X$ is homotopic to an S^1-bundle projection $\mathbb{RP}^3 \times S^2 \to M$, the Mayer-Vietoris sequence of $X \cup Y$ now becomes

$$\begin{array}{ccccc} & & \cdots & \to & H^1(\mathbb{RP}^3 \times S^2) \to \\ H^2(Z) & \to & H^2(\mathbb{RP}^3) \oplus H^2(S^2 \times S^2) & \to & H^2(\mathbb{RP}^3 \times S^2) \to \\ H^3(Z) & \to & H^3(\mathbb{RP}^3) \oplus H^3(S^2 \times S^2) & \to & H^3(\mathbb{RP}^3 \times S^2) \cdots \end{array}$$

and so tells us that $H^2(Z,\mathbb{Z}) = \mathbb{Z}$ and $H^3(Z,\mathbb{Z}) = 0$. In the same way, we also see that the inclusions $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ induce an injection

$$H^2(Z,\mathbb{Z}_2) \hookrightarrow H^2(X,\mathbb{Z}_2) \oplus H^2(Y,\mathbb{Z}_2),$$

so the fact that X and Y are both spin implies that that Z is spin, too.

Now a theorem of C.T.C. Wall [52] asserts the diffeotype of a simply connected compact spin 6-manifold with torsion-free H^2 and H^3 is completely determined by the ranks of these groups, the Pontrjagin class $p_1(TX)$, and the trilinear form

$$\smile : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}.$$

To finish the proof, it thus just remains to check that Z and $\mathbb{CP}_3$ have the same Pontrjagin class and trilinear form.

To this end, notice that, since M is homeomorphic to $S^2 \times S^2$, our almost-complex structure $\mathfrak{J}$ must have

$$\begin{aligned} c_1 &\equiv w_2 = 0 \bmod 2, \\ c_1^2 &= 2\chi + 3\tau = 8, \end{aligned}$$

and we must therefore have $c_1(M,\mathfrak{J}) = (2,2) \in \mathbb{Z} \oplus \mathbb{Z} = H^2(M,\mathbb{Z})$ after correctly orienting each factor S^2 of $S^2 \times S^2$. Since $c_1(\nu) = c_1(M,\mathfrak{J})$, the Poincaré dual of $M \subset Z$ has evaluation 2 on a factor S^2, and since the above Mayer-Vietoris sequence shows that this evaluation map $H^2(Z,\mathbb{Z}) \to H^2(S^2,\mathbb{Z})$ is an isomorphism, it follows that $[M] = 2\alpha$ for a generator $\alpha \in H^2(Z,\mathbb{Z}) \cong \mathbb{Z}$. But since $c_1(\nu) = c_1(M,\mathfrak{J}) = (2,2)$, it follows that $(2\alpha)^3 = [M]^3 = (2,2) \cdot (2,2) = 8$, so that $\alpha^3 = 1$. This shows that Z has the same trilinear form as $\mathbb{CP}_3$.

Now notice that $p_1(TZ|_M) = p_1(TM) + p_1(\nu)$. However, since M has an orientation-reversing homeomorphism, it has vanishing signature, and we therefore have $p_1(TM) = 0$ by the Hirzebruch signature theorem [38]. Thus $p_1(TZ) \cdot (2\alpha) = \langle p_1(TZ), [M]\rangle = [c_1(\nu)]^2 = 8$, and hence $p_1(TZ) = 4\alpha^2$. Since this is the same answer one obtains for $\mathbb{CP}_3$, Wall's theorem now allows us to conclude that $Z \approx \mathbb{CP}_3$. Moreover, this diffeomorphism can be chosen so that the pull-back of the hyper-plane class in $H^2(\mathbb{CP}_3,\mathbb{Z})$ is $\alpha \in H^2(Z,\mathbb{Z})$. Since we have also shown that $c_1(Z,J) = 4\alpha$, this diffeomorphism also takes the Chern classes of (Z,J) to those of the usual complex structure on $\mathbb{CP}_3$, as promised. ■

9 Families of Holomorphic Disks

In this section, we will show that every small perturbation of the standard embedding $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}_3$ gives rise to a self-dual Zollfrei conformal structure on $S^2 \times S^2$.

First let us recall that there is a standard $(S^2 \times S^2)$-family of holomorphic disks in $\mathbb{CP}_3$ with boundaries on the standard $\mathbb{RP}^3 \subset \mathbb{CP}_3$. Indeed, the boundary circles of these disks are exactly the real projective lines $\mathbb{RP}^1 \subset \mathbb{RP}^3$. Each such real projective line is contained in a unique complex projective line $\mathbb{CP}_1 \subset \mathbb{CP}_3$, and divides it into two hemispheres. A choice of orientation for such an $\mathbb{RP}^1$ then uniquely determines a hemisphere for which it is the oriented boundary. These hemispheres are the promised holomorphic disks.

A complex projective line $\mathbb{CP}_1 \subset \mathbb{CP}_3$ is the complexification of a real projective line $\mathbb{RP}^1 \subset \mathbb{RP}^3$ iff it ia ϱ-invariant, where $\varrho : \mathbb{CP}_3 \to \mathbb{CP}_3$ denotes the complex-conjugation map

$$\varrho([z_1 : z_2 : z_3 : z_4]) = [\bar{z}_1 : \bar{z}_2 : \bar{z}_3 : \bar{z}_4].$$

Now, for reasons of degree, every ϱ-invariant $\mathbb{CP}_1 \subset \mathbb{CP}_3$ must meet the standard quadric

$$\mathcal{Q} = \left\{ [z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}_3 \;\middle|\; z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \right\}$$

in a conjugate pair of points; and exactly one of these points will lie in each of the hemispheres into which the $\mathbb{CP}_1$ is divided by the fixed-point set $\mathbb{RP}^3$ of ϱ. Conversely, each point $z \in \mathcal{Q}$ is joined to its conjugate point $\varrho(z)$ by a unique ϱ-invariant $\mathbb{CP}_1$, and so is contained in exactly one of such hemisphere. Thus, the parameter space of our family may conveniently be identified with $\mathcal{Q} \approx S^2 \times S^2$. Moreover, the standard conformal structure on $S^2 \times S^2$ is completely encoded by this picture, in the sense that each β-surface is precisely the family of disks whose boundaries pass through some given point $y \in \mathbb{RP}^3$.

Although this entire story takes place in projective space, each of the individual disks in question actually lies in an affine subset. To see this, we once again let $[z_1 : z_2 : z_3 : z_4]$ be the standard homogeneous coordinates on $\mathbb{CP}_3$, so that standard $\mathbb{RP}^3 \subset \mathbb{CP}_3$ is represented by $z_1, \ldots, z_4$ real, and consider the affine chart $(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)$ on $\mathbb{CP}_3$ defined by

$$\mathfrak{z}_1 = \frac{z_1 - iz_2}{z_1 + iz_2} , \qquad \mathfrak{z}_2 = \frac{z_3}{z_1 + iz_2} , \qquad \mathfrak{z}_3 = \frac{z_4}{z_1 + iz_2} .$$

This chart realizes the complement of the line $z_1 = z_2 = 0$ in $\mathbb{RP}^3$ as the totally real submanifold B of $\mathbb{C}^3$ given by

$$\mathfrak{z}_1\bar{\mathfrak{z}}_1 = 1 , \qquad \mathfrak{z}_1\bar{\mathfrak{z}}_2 = \mathfrak{z}_2 , \qquad \mathfrak{z}_1\bar{\mathfrak{z}}_3 = \mathfrak{z}_3 . \tag{6}$$

For each $a, b \in \mathbb{C}$, the disk

$$|\mathfrak{z}_1| \leq 1 , \qquad \mathfrak{z}_2 = a + \bar{a}\mathfrak{z}_1 , \qquad \mathfrak{z}_3 = b + \bar{b}\mathfrak{z}_1$$

has boundary on B, and belongs to the family under discussion. Notice that, as promised, these unparameterized disks depend on 4 real parameters. Of course, each of these may in turn be realized as a parameterized holomorphic disk in a 3-parameter family of ways by also setting

$$\mathfrak{z}_1 = \frac{c\zeta + d}{\bar{c} + \bar{d}\zeta} , \quad |\zeta| \leq 1 , \quad |c|^2 - |d|^2 = 1.$$

In this manner, we actually obtain a 7-parameter family of *parameterized* disks. In any case, it will suffice for our purposes to primarily focus on the particular parameterized disk

$$\mathfrak{z}_1 = \zeta , \qquad |\zeta| \leq 1 , \qquad \mathfrak{z}_2 = \mathfrak{z}_3 = 0 ,$$

since all the other disks in the family can be obtained from this one via the action of $PSL(4, \mathbb{R})$ on $\mathbb{CP}_3$.

We will now appeal to some general results concerning holomorphic disks in $\mathbb{C}^n$ with boundary on a totally real submanifold. Suppose that $X^n \subset \mathbb{C}^n$ is a maximal totally real differentiable submanifold, in the sense that $T\mathbb{C}^n|_X = TX \oplus J(TX)$. The first result we will need is a regularity result [6]:

Lemma 9.1 (Chirka) *Suppose that $X^n \subset \mathbb{C}^n$ is a totally real submanifold of class $C^{\ell+1}$, $\ell \geq 2$, and that $\digamma : (D, \partial D) \to (\mathbb{C}^n, X)$ is a C^1-map which is holomorphic in the interior of the disk. Then $\digamma$ is actually a C^ℓ map.*

Now suppose that X is a is a maximal totally real submanifold of $\mathbb{C}^n$, and that $\digamma : (D, \partial D) \to (\mathbb{C}^n, X)$ is a holomorphic disk with boundary on X. Then $\digamma$ is said to have partial indices $\kappa_1, \ldots, \kappa_n$ if there is a map $A : D \to GL(n, \mathbb{C})$ which is holomorphic on the interior of D and continuous up to the boundary such that $TX|_{\digamma(\zeta)} \subset \mathbb{C}^n$ is the real span of the columns of the matrix

$$A(\zeta) \begin{bmatrix} \zeta^{\kappa_1/2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \zeta^{\kappa_n/2} \end{bmatrix}$$

for all $\zeta \in \partial D$. These partial indices turn out to be well defined up to permutation. Their sum

$$\kappa = \kappa_1 + \cdots + \kappa_n$$

is called the *Maslov index* of the holomorphic disk $\digamma$. An application of the Banach-space implicit function theorem to the Hilbert transform on the circle leads to the following result [14, 41]:

Proposition 9.2 (Globevnik/Oh) *Suppose that $\digamma : D \to \mathbb{C}^n$ is a holomorphic map of the unit disk whose boundary is contained in a totally real submanifold X of class $C^{2\ell+1}$. Suppose, moreover, that all the partial indices $\kappa_1, \ldots, \kappa_n$ of $\digamma$ satisfy $\kappa_j \geq -1$. Then, for any totally real submanifold X' of $\mathbb{C}^n$ which is sufficiently close to X in the $C^{2\ell+1}$-topology, there is a $(\kappa + n)$-real-parameter family of holomorphic embeddings $(D, \partial D) \hookrightarrow (\mathbb{C}^n, X')$, where $\kappa = \kappa_1 + \cdots + \kappa_n$ is the Maslov index of $\digamma$. This family is of class C^ℓ, depends in a C^ℓ manner on the choice of X', and sweeps out all holomorphic maps of the disk which satisfy the relevant boundary conditions and which are C^ℓ close to $\digamma$.*

Let us now apply these ideas to the case at hand. If we take X to be the submanifold $B = \mathbb{RP}^3 - \mathbb{RP}^1$ of $\mathbb{C}^3$ defined by (6), and consider the holomorphic disk $\digamma : D \to \mathbb{C}^3$ given by $\zeta \mapsto (\zeta, 0, 0)$ for $|\zeta| \leq 1$, then TB is spanned over $\mathbb{R}$ by the columns of the matrix

$$\begin{bmatrix} i\zeta & 0 & 0 \\ 0 & \zeta^{1/2} & 0 \\ 0 & 0 & \zeta^{1/2} \end{bmatrix}$$

for all $\zeta \in \partial D$. The partial indices of this disk are thus $\kappa_1 = 2$, $\kappa_2 = 1$, and $\kappa_3 = 1$, and its Maslov index is consequently $\kappa = 4$. Proposition 9.2 thus asserts that the 7-parameter family of perturbations of $\digamma$ we previously found by hand is actually stable under deformations of B. That is, for any B' represented by a a section of the normal bundle of $B \subset \mathbb{C}^3$ of small C^3 norm on a neighborhood of $f(S^1) \subset B$, we can find a C^1 family of parameterized holomorphic disks near $\digamma$ with boundary values in B' and nonetheless C^1 close to the boundary values of a neighborhood of $\digamma$ in our original 7-parameter family. Provided the norm of this section is small, each of the new disks will remain embedded, and will meet the hypersurface

$$\mathfrak{z}_1 + \mathfrak{z}_2^2 + \mathfrak{z}_3^2 = 0$$

that represents the quadric $\mathcal{Q}$ in our affine chart.

Let us now give this assertion a more concrete geometrical interpretation. Suppose that $P \subset \mathbb{CP}_3$ be the image of a general C^∞ embedding of $\mathbb{RP}^3$ into $\mathbb{CP}_3$ which satisfies the sole constraint that, with respect to the C^3 topology on the space of maps, it lies in a sufficiently small neighborhood $\mathcal{X}$ of the standard embedding. By shrinking $\mathcal{X}$ if necessary, we may assume that every such P is totally real and does not meet the quadric $\mathcal{Q} \subset \mathbb{CP}_3$. Since the complement of Q is a tubular neighborhood of $\mathbb{RP}^3 \subset \mathbb{CP}_3$, any such P may be represented by a smooth section of the normal bundle of $\mathbb{RP}^3$; and since the complex structure J provides an isomorphism between the tangent and normal bundles of $\mathbb{RP}^3$, the freedom in choosing P amounts to that of choosing a vector field on $\mathbb{RP}^3$ of small C^3 norm. Proposition 9.2, in conjunction with Lemma 9.1, now tells us the following:

Proposition 9.3 *Suppose that $P \subset \mathbb{CP}_3$ is the image of a smooth embedding $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}_3$ which is sufficiently close to the standard one in the C^3 topology. Then P contains a uniquely determined smooth family of embedded oriented circles $\ell_x \subset P$, $x \in S^2 \times S^2$, each of which bounds an embedded holomorphic disk $D^2 \subset \mathbb{CP}_3$ whose relative homology class generates $H_2(\mathbb{CP}_3, P; \mathbb{Z}) \cong \mathbb{Z}$. The corresponding family of holomorphic disks is smooth, and the interiors of these disks smoothly foliate $\mathbb{CP}_3 - P$.*

In fact, the existence of a C^1 family of such holomorphic disks simply follows from elementary Fourier analysis and the inverse function theorem, and so may be rederived by essentially repeating the self-contained arguments given in [31]. Once this is known, one can then use Lemma 9.1 to conclude that each of the constructed disks is actually smooth, and the smoothness of the constructed family then follows from Proposition 9.2 by showing that it locally coincides with the families of disks obtained by perturbing any given smooth disk through disks of increasing regularity. A less elementary, but distinctly compelling, road to the same conclusion would be to appeal to the non-linear elliptic methods that are now standard in the theory of J-holomorphic curves [36].

10 Constructing Self-Dual Metrics

So far, we have associated a 4-dimensional space of embedded holomorphic disks with each small perturbation of $\mathbb{RP}^3 \subset \mathbb{CP}_3$. To finish our construction, we need to show that this 4-dimensional parameter space carries a natural self-dual split-signature conformal structure. This will be obtained via the following general mechanism:

Proposition 10.1 *Let M be a smooth connected 4-manifold, and let $\varpi : \mathcal{X} \to M$ be a smooth $\mathbb{CP}_1$-bundle. Let $\varrho : \mathcal{X} \to \mathcal{X}$ be an involution which commutes with the projection ϖ, and has as fixed-point set $\mathcal{X}_\varrho$ an S^1-bundle over M which disconnects $\mathcal{X}$ into two closed 2-disk bundles $\mathcal{X}_\pm$ with common boundary $\mathcal{X}_\varrho$. Suppose that $\Pi \subset T_{\mathbb{C}}\mathcal{X}$ is a distribution of complex 3-planes on $\mathcal{X}$ such that*

- *$\varrho_* \Pi = \overline{\Pi}$;*
- *the restriction of Π to $\mathcal{X}_+$ is smooth and involutive;*

- $\text{Д} \cap \overline{\text{Д}} = 0$ *on* $\mathcal{X} - \mathcal{X}_\varrho$*;*
- $\text{Д} \cap \ker \varpi_*$ *is the* $(0,1)$ *tangent space of the* $\mathbb{CP}_1$ *fibers of* ϖ*; and*
- *the restriction of* Д *to a fiber of* $\mathcal{X}$ *has* $c_1 = -4$ *with respect to the complex orientation.*

Then $E = \text{Д} \cap T\mathcal{X}_\varrho$ *is an integrable distribution of real 2-planes on* $\mathcal{X}_\varrho$*, and* M *admits a unique smooth split-signature self-dual conformal structure* $[g]$ *for which the* β*-surfaces are the projections via* ϖ *of the integral manifolds of* E*.*

Proof. Let us begin by noticing that, since $\text{Д} = \varrho^*\overline{\text{Д}}$ is continuous on the closed sets $\mathcal{X}_+$ and $\mathcal{X}_-$, it is continuous on all of $\mathcal{X}$.

Now let $V^{0,1}$ be the $(0,1)$ tangent space of the fibers. By hypothesis, $V^{0,1} \subset \text{Д}$, so that $\mho = \text{Д}/V^{0,1}$ is a well defined rank-2 complex vector bundle. Also notice that, since $\text{Д} \cap \ker \varpi_* = V^{0,1}$, the fibers of $\mho$ are carried injectively into $T_{\mathbb{C}}M$ by ϖ_*. We may therefore define a continuous map

$$\begin{array}{rcl} \psi : \mathcal{X} & \to & Gr_2(T_{\mathbb{C}}M) \\ z & \mapsto & \varpi_*(\mho|_z) = \varpi_*(\text{Д}|_z) \end{array}$$

which makes the diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & Gr_2(T_{\mathbb{C}}M) \\ & \searrow \quad \swarrow & \\ & M & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & Gr_2(T_{\mathbb{C}}M) \\ \varrho \downarrow & & \downarrow c \\ \mathcal{X} & \xrightarrow{\psi} & Gr_2(T_{\mathbb{C}}M) \end{array}$$

commute, where c denotes the map induced by complex conjugation $T_{\mathbb{C}}M \to T_{\mathbb{C}}M$.

Now let ζ be a smooth, fiber-wise holomorphic coordinate on $\mathcal{X}$, and notice that the corresponding vertical vector field $\partial/\partial\overline{\zeta}$ is both smooth and a section of Д. Next, near any point of the interior of $\mathcal{X}_+$, let $\mathfrak{w}_1$ and $\mathfrak{w}_2$ be any two local sections of Д which are linearly independent from $\partial/\partial\overline{\zeta}$ and from each other. Then the involutivity hypothesis $[C^\infty(\text{Д}), C^\infty(\text{Д})] \subset C^\infty(\text{Д})$ tells us that

$$\frac{\partial}{\partial\overline{\zeta}}(\varpi_*(\mathfrak{w}_j)) = \varpi_*(\mathcal{L}_{\frac{\partial}{\partial\overline{\zeta}}}\mathfrak{w}_j) = \varpi_*\left(\left[\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}_j\right]\right) \equiv 0 \bmod \Big\langle \varpi_*(\mathfrak{w}_1), \varpi_*(\mathfrak{w}_2)\Big\rangle,$$

and it follows that ψ is a fiber-wise holomorphic on the interior $\mathcal{X}_+$. Since $\psi = c \circ \psi \circ \varrho$, it thus follows that ψ is also fiber-wise holomorphic on the interior $\mathcal{X}_-$. However, ψ is also continuous across $\mathcal{X}_\varrho = \mathcal{X}_+ \cap \mathcal{X}_-$, so this implies that ψ is actually fiber-wise holomorphic on all of $\mathcal{X}$.

By construction, the restriction of $\mho$ to $\varpi^{-1}(x)$ is the pull-back, via ψ, of the universal bundle $\mathbb{U}$ over $Gr_2(T_{\mathbb{C}}M|_x) \cong Gr_2(\mathbb{C}^4)$. Now consider the Plücker embedding

$$\begin{array}{rcl} \mathfrak{P} : Gr_2(T_{\mathbb{C}}M) & \hookrightarrow & \mathbb{P}(\wedge^2 T_{\mathbb{C}}M) \\ \text{span}(w_1, w_2) & \mapsto & [w_1 \wedge w_2] \end{array}$$

and the induced map $\hat{\psi} = \mathfrak{P} \circ \psi : \mathcal{X} \to \mathbb{P}(\wedge^2 T_{\mathbb{C}} M)$. Since $\mathfrak{P}^* \O(-1) = \wedge^2 \mathbb{U}$, we must have $\hat{\psi}^* \O(-1) = \wedge^2 \mho$. But $V^{0,1}$ is the $(0,1)$ tangent space of $\varpi^{-1}(x)$, and hence $c_1(V^{0,1}) = -2$ on any fiber of ϖ. On the other hand, $c_1(\text{Д}) = -4$ on $\varpi^{-1}(x)$, by hypothesis. Adjunction therefore tells us that $c_1(\mho) = -2$ on any fiber. Thus the restriction of $\hat{\psi}$ to any fiber is a holomorphic map of degree 2 from $\mathbb{CP}_1$ to the 4-quadric $Q_4 \subset \mathbb{CP}_5$. There are only two possibilities for this map: either it is the inclusion of a non-degenerate plane conic Q_1 into Q_4, or else it is a ramified double cover of a projective line $\mathbb{CP}_1 \subset Q_4$ branched at two points.

The latter possibility, however, is excluded by our hypotheses. Indeed, any line $\mathbb{CP}_1 \subset Q_4 \subset \mathbb{CP}_5$ corresponds to the curve in $Gr_2(\mathbb{C}^4)$ given by the pencil of all 2-planes contained in a 3-dimensional subspace of $\mathbb{C}^4$ and containing some fixed line. If the image of $\varpi^{-1}(x)$ under $\hat{\psi}$ were a line, we would thus have

$$\varpi_*(\text{Д}_z) + \varpi_*(\text{Д}_{z'}) = \varpi_*(\mho|_z) + \varpi_*(\mho|_{z'}) \subsetneq T_{\mathbb{C}} M|_x$$

for all $z, z' \in \varpi^{-1}(x)$. However, since $\text{Д} + \overline{\text{Д}} = T_{\mathbb{C}} \mathcal{X}$ away from $\mathcal{X}_\varrho$, and because $\varrho^* \text{Д} = \overline{\text{Д}}$, we actually have

$$\varpi_*(\text{Д}|_z) + \varpi_*(\text{Д}|_{\varrho(z)}) = \varpi_*(\text{Д}|_z + \overline{\text{Д}}|_z) = T_{\mathbb{C}} M|_x$$

for all $z \in \varpi^{-1}(x)$ with $\varrho(z) \neq z$. This contradiction shows that $\hat{\psi}[\varpi^{-1}(x)]$ cannot be a line.

Thus $\hat{\psi}$ holomorphically includes each fiber of ϖ into $\mathbb{P}(\wedge^2 T_{\mathbb{C}} M)$ as a non-degenerate conic curve. For each x, this conic is cut out by a unique 3-plane $\Lambda_-^{\mathbb{C}}|_x \subset \wedge^2 T_{\mathbb{C}} M|_x$. The restriction of the wedge product

$$\begin{array}{rcl} \wedge^2 T_{\mathbb{C}} M \times \wedge^2 T_{\mathbb{C}} M & \to & \wedge^4 T_{\mathbb{C}} M \\ (\varphi, \omega) & \mapsto & \varphi \wedge \omega \end{array}$$

to $\Lambda_-^{\mathbb{C}}|_x$ is, moreover, always a non-degenerate bilinear form, since, by construction, $\mathbb{P}(\Lambda_-^{\mathbb{C}})$ always meets the quadric $\omega \wedge \omega = 0$ in the non-degenerate conic $\hat{\psi}[\varpi^{-1}(x)]$.

Now $\hat{\psi}$ is at least smooth on the interior of $\mathcal{X}_+$. By taking the images under $\hat{\psi}$ of three generic smooth local sections of ϖ which avoid $\mathcal{X}_\varrho$, we can thus locally span $\Lambda_-^{\mathbb{C}}$ by three smooth local sections of $\wedge^2 T_{\mathbb{C}} M$. Thus $\Lambda_-^{\mathbb{C}} \subset \wedge^2 T_{\mathbb{C}} M$ is a smooth sub-bundle. Moreover, essentially the same argument shows that $\hat{\psi}$ is smooth on all of $\mathcal{X}$.

Since $\psi \circ \varrho = c \circ \psi$, we must therefore have $\Lambda_-^{\mathbb{C}} = \mathbb{C} \otimes \Lambda_-$ for a unique, smooth real vector sub-bundle $\Lambda_- \subset \wedge^2 TM$ on which the wedge product is non-degenerate. However, the wedge product must be *indefinite* on every fiber of Λ_-, since $\mathcal{X}_\varrho$ meets every fiber of $\mathcal{X}$. Since

$$O(3,3)/[O(2,1) \times O(1,2)] = SL(4, \mathbb{R})/SO(2,2)$$

it follows that there is a unique smooth split-signature conformal metric $[g]$ on M for which Λ_- is the bundle of anti-self-dual bi-vectors for an appropriate orientation of M. For each metric g in this conformal class, Λ_- then corresponds via index-lowering to the bundle $\Lambda^- \subset \wedge^2$ of real anti-self-dual 2-forms.

Now consider the subset of the complex tangent bundle of $\mathcal{X}_\varrho$ defined by

$$E_{\mathbb{C}} = \text{Д} \cap T_{\mathbb{C}} \mathcal{X}_\varrho.$$

Since each fiber of $T_{\mathbb{C}}\mathcal{X}_\varrho$ has codimension 1 in $T_{\mathbb{C}}\mathcal{X}$, and since $T_{\mathbb{C}}\mathcal{X}_\varrho$ does not contain the 1-dimensional subspace $V^{0,1} \subset Д$, the subspace $Д$ is always in general position relative to $T_{\mathbb{C}}X_\varrho$. Hence $E_{\mathbb{C}}$ is a smooth distribution of complex 2-planes on $\mathcal{X}_\varrho$. However, ϱ acts trivially on $\mathcal{X}_\varrho$, and hence ϱ_* acts on $T_{\mathbb{C}}\mathcal{X}_\varrho$ via the identity. The assumption that $\varrho_* Д = \overline{Д}$ therefore implies that $\overline{E_{\mathbb{C}}} = E_{\mathbb{C}}$. Hence $E_{\mathbb{C}}$ is the complexification of a smooth distribution of real 2-planes

$$E = Д \cap T\mathcal{X}_\varrho$$

on $\mathcal{X}_\varrho$. Since $T\mathcal{X}_\varrho$ and $Д$ are both closed under Lie brackets, it follows that E is Frobenius integrable. Thus $\mathcal{X}_\varrho$ is foliated by 2-manifolds tangent to E. But the inclusion $E_{\mathbb{C}} \hookrightarrow Д$ induces an canonical isomorphism $E \otimes \mathbb{C} \to \mho|_{\mathcal{X}_\varrho}$, whereas ψ identifies $\mathcal{X}_\varrho$ with the bundle of real β-planes for $(M, [g])$. Thus each integral manifold of E is sent via ϖ to a β-surface of $(M, [g])$, and $[g]$ is therefore self-dual by Proposition 3.5. Moreover, $[g]$ is uniquely determined by this last prescription, since, at each point of M, the union of the tangent spaces of these β-surfaces is precisely the null cone of $[g]$, and the conformal class of any indefinite metric is completely determined by its null cone. ■

Theorem 10.2 *Let $P \subset \mathbb{CP}_3$ be a smooth, totally real submanifold which, in the C^3 topology, is close to the standard 'linear' $\mathbb{RP}^3 \subset \mathbb{CP}_3$; and, for clarity, fix a quadric $Q \subset \mathbb{CP}_3$ which is disjoint from P. For each $x \in Q \approx S^2 \times S^2$, let $D_x \subset \mathbb{CP}_3$ be the unique holomorphic disk of the family constructed in Proposition 9.3 which passes through x. For each $y \in P$, set*

$$S_y = \{x \in S^2 \times S^2 \mid y \in D_x\}.$$

Then there is a unique, smooth Zollfrei self-dual split-signature conformal structure $[g]$ on $Q \approx S^2 \times S^2$ whose β-surfaces are exactly the S_y, $y \in P$.

Proof. Let $M = Q \approx S^2 \times S^2$, and let $\mathcal{X}_+ \to M$ be the 2-disk bundle whose fiber over $x \in M$ is the holomorphic disk $D^2 \subset \mathbb{CP}_3$ of the family passing through x. Thus there is a tautological smooth map $\digamma : \mathcal{X}_+ \to \mathbb{CP}_3$ which sends the interior of $\mathcal{X}_+$ diffeomorphically onto $\mathbb{CP}_3 - P$, and which sends $\partial\mathcal{X}_+ \to P$. Recalling that $\digamma_* : T_{\mathbb{C}}\mathcal{X}_+ \to T_{\mathbb{C}}\mathbb{CP}_3$ denotes the derivative of this map, let ${\digamma_*}^{1,0} : T_{\mathbb{C}}\mathcal{X}_+ \to T^{1,0}\mathbb{CP}_3$ denote be the $(1,0)$-component of this derivative, and let

$$Д = \ker {\digamma_*}^{1,0} \subset T_{\mathbb{C}}\mathcal{X}_+$$

denote the kernel of this component. Since $\digamma$ is C^1 close to the corresponding map for the flat model, we may assume that ${\digamma_*}^{0,1}$ is everywhere of maximal rank, as in the flat case. Thus $Д$ is a smooth complex bundle of rank 3 on all of $\mathcal{X}_+$. Now if $V^{0,1}$ is the $(0,1)$-tangent space of the fibers of the D^2-bundle $\mathcal{X}_+ \to M$, then $V^{0,1} \subset Д$ because $\digamma$ is fiber-wise holomorphic. But because the 5-manifold $\partial\mathcal{X}_+$ is sent to the 3-manifold P by $\digamma$, each fiber

$$E = \ker \digamma_*|_{\partial\mathcal{X}_+}$$

has dimension ≥ 2, and since $(E \otimes \mathbb{C}) \oplus V^{0,1} \subset Д$, we conclude that E is in fact a smooth distribution of real 2-planes on $\partial\mathcal{X}_+$.

Now let $\mathcal{X}_-$ be a second copy of $\mathcal{X}_+$, and define $Д \to \mathcal{X}_-$ to be the push-forward of the distribution of complex 3-planes $\overline{Д} \to \mathcal{X}_+$ via the tautological diffeomorphism $\mathcal{X}_+ \to \mathcal{X}_-$. Similarly, let $V^{0,1} \to \mathcal{X}_-$ be the distribution of complex lines obtained from $\overline{V^{0,1}} \to \mathcal{X}_+$. Let

$$\mathcal{X} = \mathcal{X}_+ \cup_{\partial \mathcal{X}_+} \mathcal{X}_-$$

be the double of $\mathcal{X}_+$. Then we have a canonical projection $\varpi : \mathcal{X} \to M$ which makes $\mathcal{X}$ into a $\mathbb{CP}_1$-bundle with vertical $(0,1)$ tangent space $V^{0,1}$. Moreover, our two definitions of $Д$ agree along the hypersurface $\partial\mathcal{X}_+ = \partial\mathcal{X}_-$, because both coincide with $V^{0,1} \oplus (E \otimes C)$ along this locus. Moreover, $V^{0,1} = Д \cap \ker \varpi_*$ on all of $\mathcal{X}$.

Let $\varrho : \mathcal{X} \to \mathcal{X}$ be the map which interchanges $\mathcal{X}_\pm$ via the tautological diffeomorphism. This is an involution of $\mathcal{X}$ which commutes with ϖ, and its fixed-point set $\mathcal{X}_\varrho = \partial\mathcal{X}_+$ divides $\mathcal{X}$ into two disk bundles over M. By construction, we have $\varrho_* Д = \overline{Д}$. Moreover, $Д$ is smooth, involutive, and satisfies $Д \cap \overline{Д} = 0$ on Int $\mathcal{X}_+$, since the diffeomorphism $\digamma$ from Int $\mathcal{X}_+$ to $\mathbb{CP}_3 - P$ sends $Д$ to $T^{0,1}\mathbb{CP}_3$.

Finally, observe that any holomorphic disk $(D^2, S^1) \to (\mathbb{CP}_3, P)$ obtained by restricting $\digamma : \mathcal{X}_+ \to \mathbb{CP}_3$ to a fiber of $\mathcal{X}_+ \to M$ must have Maslov index $\kappa = 4$, since each such disk is obtained by deforming a disk with $\kappa = 4$ from our flat model, and the Maslov index invariant under deformations [36]. This index is by definition the winding number of $\wedge^3 TP$ in $\wedge^3 T^{1,0} T\mathbb{CP}_3$ along $S^1 = \partial D^2$, relative to any trivialization of $T^{1,0}\mathbb{CP}_3$ over D^2, remembering that the space of real lines in $\mathbb{C}$ is exactly $\mathbb{RP}^1 \approx S^1$. But recall that $\digamma_*^{1,0}$ is surjective, so that we can identify $\digamma^* T^{1,0}\mathbb{CP}_3$ with the quotient $T_{\mathbb{C}}\mathcal{X}_+/Д$. Since $\wedge^6 T_{\mathbb{C}}\mathcal{X}_+$ is the complexification of the trivial real bundle $\wedge^6 T\mathcal{X}_+$, it follows by adjunction that this Maslov index must be *minus* the Maslov index of $T\mathcal{X}_\varrho/E \subset Д$. However, the latter winding number is also exactly the degree of $\wedge^3 Д$ on a $\mathbb{CP}_1$ fiber, since, by construction, $Д$ is defined on the double of the disk precisely by gluing $Д|_{D^2}$ to $\overline{Д}|_{\overline{D^2}}$ so as to send $T\mathcal{X}_\varrho/E$ to itself. Thus the evaluation of $c_1(Д)$ on a fiber of ϖ is exactly $-\kappa = -4$.

The above arguments show that all the hypotheses of Proposition 10.1 are satisfied. Thus $M = S^2 \times S^2$ admits a unique self-dual split-signature metric $[g]$ for which the β-surfaces are the projections to M of the integral manifolds of $E \to \mathcal{X}_\varrho$. By construction, however, E is precisely the vertical tangent bundle of the smooth submersion

$$\digamma|_{\partial\mathcal{X}_+} : \partial\mathcal{X}_+ \to P,$$

so these β-surfaces are exactly of the form

$$S_y = \varpi[\digamma^{-1}(y)]$$

for $y \in P \subset \mathbb{CP}_3$, which is to say that

$$S_y = \{x \in M \mid y \in D_y\},$$

as promised.

Now $\mathcal{X}_+$ is diffeomorphic to the Chern class $(2,2)$ disk bundle over $S^2 \times S^2$. Thus $\mathcal{X}_\varrho = \partial\mathcal{X}_+$ is the Chern class $(2,2)$ circle bundle over $S^2 \times S^2$, and so is diffeomorphic to $\mathbb{RP}_3 \times S^2$. Moreover, since every disk of the family represents the generator of $H_2(\mathbb{CP}_3, P)$, the long exact homotopy sequence

$$\cdots \to H_2(\mathbb{CP}_3, P; \mathbb{Z}) \to H_1(P; \mathbb{Z}) \to H_1(\mathbb{CP}_3; \mathbb{Z}) \to \cdots$$

of the pair $(\mathbb{CP}_3, P)$ tells us that the boundary of each disk generates $\pi_1(P) = H_1(P) \cong \mathbb{Z}_2$. It follows that $\mathcal{F}|_{\mathcal{X}_\varrho}$ induces a surjection $\pi_1(\mathcal{X}_\varrho) \to \pi_1(P)$. But $\mathcal{F}|_{\mathcal{X}_\varrho}$ is a proper submersion, and therefore a smooth fibration, so we have the long exact homotopy sequence

$$\cdots \to \pi_2(P) \to \pi_1(S_y) \to \pi_1(\mathcal{X}_\varrho) \to \pi_1(P) \to \pi_0(S_y) \to \cdots$$

and it follows that the compact surface S_y is connected and simply connected. Thus every β-surface of $(M, [g])$ is a 2-sphere. Hence $[g]$ is Zollfrei by Theorem 5.4, and we are done. ■

Theorem C now follows from Theorems 8.5 and 10.2.

11 The Kähler Case

The protypical example which motivated our entire investiagtion of a Zollfrei self-dual manifold was the the indefinite product metric $g_0 = \pi_1^* h - \pi_2^* h$ on $S^2 \times S^2$. Notice, however, that this metric may be considered as an indefinite Kähler metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$, with Kähler form $\omega = \pi_1^*\mu - \pi_2^*\mu$, where μ denotes the area form of (S^2, h). Notice that, since h has constant Gauss curvature 1, the scalar curvature of g_0 is $s = s_{g_0} = \pi_1^* s_h - \pi_2^* s_h = 2 - 2 = 0$. A pseudo-Riemannian metric with this last property is said to be *scalar-flat*.

Now, more generally, suppose that we have a scalar-flat indefinite Kähler metric g on a compact complex surface $(M^4, \mathfrak{J})$. From the outset, we choose to give M the usual *complex* orientation, but we will also need to systematically consider the reverse-oriented version $\overline{M}$ of our manifold. To see why, observe that the Kähler form ω of $(M, g, \mathfrak{J})$ is a closed non-degenerate 2-form on M, and so may be considered as a *symplectic form.* However, such a form determines an orientation, and in the present case this orientation is the *opposite* of the complex-manifold orientation; thus, $(\overline{M}, \omega)$ becomes a symplectic 4-manifold, oriented according to standard symplectic conventions.

Notice that while ω is a self-dual 2-form on $(\overline{M}, g)$, it is instead *anti-self-dual* on (M, g). With this potential source of confusion kept clearly in focus, standard Riemannian folklore [13] immediate tells us the following:

Lemma 11.1 *Let $(M^4, \mathfrak{J}, g)$ be a complex surface with an* indefinite *Kähler metric. Then (M, g) is self-dual iff g is scalar-flat.*

Proof. The curvature of any Kähler manifold, indefinite or not, is necessarily of type $(1, 1)$, so that the corresponding curvature operator $\mathcal{R}$ kills $\Lambda^{2,0} \oplus \Lambda^{0,2}$, and amounts to a linear map $\mathcal{R} : \Lambda^{1,1} \to \Lambda^{1,1}$. Now observe that if $(M^4, \mathfrak{J}, g)$ is an *indefinite* Kähler manifold, equipped with the complex orientation, we have

$$\begin{aligned} \Lambda^{1,1} &= \mathbb{C}\omega \oplus \Lambda^+ \\ \Lambda^-_{\mathbb{C}} &= \mathbb{C}\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} \end{aligned}$$

so that the

$$\left[W_- + \frac{s}{12}\right] : \Lambda^- \to \Lambda^-$$

block of the curvature operator kills $\Lambda^{2,0} \oplus \Lambda^{0,2}$, and sends $\mathbb{C}\omega$ to itself. Since $s/4$ is the trace of this block, and since W_- is its trace-free part, we therefore have

$$W_- = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

in an appropriate basis. Hence $W_- = 0$ iff $s = 0$, as claimed. ■

Now the structure group of an indefinite Kähler surface is $U(1,1)$, which is a connected Lie group. Every indefinite Kähler surface therefore carries a canonical space-time orientation. As a consequence, Theorem 5.12 tells us that any Zollfrei scalar-flat indefinite Kähler surface is homeomorphic to $S^2 \times S^2$. However, a much stronger assertion is actually true:

Theorem 11.2 *Suppose that* $(M^4, \mathfrak{J}, g)$ *be a complex surface with scalar-flat* Zollfrei *indefinite Kähler metric. Then* $(M, \mathfrak{J})$ *is biholomorphic to* $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Proof. Let S be any β-surface in (M, g). At each point of S the image of $\Lambda^2 TS$ in Λ^2 is then the span of a real, non-zero simple element of Λ^-. But, up to a positive constant, the general such simple 2-form can be written uniquely as $\omega + \phi + \overline{\phi}$, where $\phi \in \Lambda^{2,0}$ is any element of unit norm. It follows that the restriction of ω to S is non-zero at every point. Hence S is a symplectic submanifold of $(\overline{M}, \omega)$. Moreover, since S is orientable and Lemma 5.1 asserts that S is either a 2-sphere or a projective plane, we have $S \approx S^2$. Proposition 5.10 thus tells us that $[S] \cdot [S] = -2$ in M, and hence that $[S] \cdot [S] = +2$ in the reverse-oriented manifold $\overline{M}$. Hence $(\overline{M}, \omega)$ contains a symplectic 2-sphere S of positive self-intersection, and a fundamental result of McDuff [35] therefore tells us that $(\overline{M}, \omega)$ must be diffeomorphic to either $S^2 \times S^2$ or to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 0$. Since M is spin by Proposition 5.12, it therefore follows that M is therefore diffeomorphic to $S^2 \times S^2$.

In particular, this shows that M is a minimal complex surfae which admits a Riemannian metric of positive scalar curvature, and Seiberg-Witten theory [11, 30] therefore tells us $(M, \mathfrak{J})$ must have Kodaira dimension $-\infty$. By the Kodaira-Enriques classification [4, 15], our simply connected complex surface $(M, \mathfrak{J})$ is therefore *rational*, in the sense of being obtained from $\mathbb{CP}_2$ by blowing up and down, and since M is also spin, so it follows that $(M, \mathfrak{J})$ is an even Hirzebruch surface $\mathbb{P}[\text{Ø} \oplus \text{Ø}(2m)] \to \mathbb{CP}_1$. However, Kamada [21] has shown that the existence of scalar-flat indefinite Kähler metrics is obstructed for $m \neq 0$ because a generalized form of the Futaki invariant [12] is non-zero in all these cases. Hence $m = 0$, and $(M, \mathfrak{J})$ must be biholomorphic to $\mathbb{CP}_1 \times \mathbb{CP}_1$. ■

Remark. The above result would certainly become false if the Zollfrei hypothesis were dropped. For example, one can easily construct scalar-flat indefinite Kähler metrics on the product $\Sigma \times \Sigma$ of any Riemann surface Σ with itself, just by setting $g = \pi_1^* h - \pi_2^* h$, where h is a metric on Σ of constant sectional curvature. ♢

Theorem 11.3 *Let* $(M, g, \mathfrak{J})$ *be a scalar-flat Zollfrei indefinite Kähler metric. Then its twistor space* (Z, J)*, in the sense of definition 7.4, is biholomorphic to* $\mathbb{CP}_3$.

Moreover, this biholomorphism determines a preferred non-singular quadric $Q \subset \mathbb{CP}_3$ obtained by thinking of $\mathfrak{J}$ as section of $\mathcal{Z}_+ \to M$.

Proof. The complex structure $\mathfrak{J}$ defines a section of Int $\mathcal{Z}_+$, and because $\mathfrak{J}$ is parallel, the image of this section is tangent to $\mathscr{E} \subset \mathscr{D}$. The composition of Ψ with this section is therefore a holomorphically embedding of $(M, \mathfrak{J})$ into (Z, J). Moreover, as we saw in the proof of Theorem 8.9, the normal bundle ν of $(M, \mathfrak{J})$ has Chern class $c_1(\nu) = c_1(M, \mathfrak{J})$. Since Theorem 11.2 tells us that $(M, \mathfrak{J}) \cong \mathbb{CP}_1 \times \mathbb{CP}_1$, this therefore gives us a hypersurface $Q \subset Z$ cut out by a section of the corresponding divisor line bundle $L \to Z$, such that $Q \cong \mathbb{CP}_1 \times \mathbb{CP}_1$ and such that $L|_Q \cong \mathcal{O}(2,2)$. For each integer m, we therefore have an exact sequence

$$0 \to \mathcal{O}(L^{m-1}) \to \mathcal{O}(L^m) \to \mathcal{O}_Q(2m, 2m) \to 0$$

of sheaves on Z. Since

$$H^q(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(2m, 2m)) = 0 \quad \forall q, m > 0,$$

it follows that, as $m \to \infty$, $h^1(Z, \mathcal{O}(L^m))$ is decreasing, while $h^2(Z, \mathcal{O}(L^m))$ and $h^3(Z, \mathcal{O}(L^m))$ remain constant. Hence

$$\begin{aligned} \chi(Z, \mathcal{O}(L^m)) &= h^0(\mathcal{O}(L^m)) - h^1(\mathcal{O}(L^m)) + h^2(\mathcal{O}(L^m)) - h^3(\mathcal{O}(L^m)) \\ &= h^0(Z, \mathcal{O}(L^m)) + \text{const} \qquad \forall m \gg 0. \end{aligned}$$

However, Theorem 8.9 tells us that Z is diffeomorphic to $\mathbb{CP}_3$ in a manner sending the Chern classes of Z to the Chern classes of $\mathbb{CP}_3$. Since $c_1(L) = \frac{1}{2}c_1(Z)$, the Hirzebruch-Riemman-Roch theorem therefore tells us that

$$\chi(Z, \mathcal{O}(L^m)) = \chi(\mathbb{CP}_3, \mathcal{O}(2m)) = \frac{(2m+1)(2m+2)(2m+3)}{6}.$$

Hence $h^0(Z, \mathcal{O}(L^m))$ grows cubically in m. The complex 3-fold (Z, J) is therefore Moishezon. Since Z is also diffeomorphic to $\mathbb{CP}_3$, Theorem 8.3 therefore tells us that (Z, J) is biholomorphic to $\mathbb{CP}_3$. Moreover, $Q \subset Z$ is carried by this biholomorphism to a non-singular hypersurface of degree 2. ■

Now, which totally real submanifolds of $\mathbb{CP}_3$ correspond to scalar-flat self-dual metrics? The following result provides the key to the answer.

Proposition 11.4 *Let $(M, g, \mathfrak{J})$ be a Zollfrei indefinite scalar-flat Kähler manifold, let $Q \subset Z \cong \mathbb{CP}_3$ be the quadric constructed in Theorem 11.3, and let $P = \Psi(F)$ be the space of β-surfaces in M. Then there is a meromorphic 3-form Ω on Z which is holomorphic and non-zero on $Z - Q$ and which has the property that its pull-back to P is a real 3-form.*

Proof. Consider a pseudo-orthonormal frame $e_1, \ldots, e_4$ on some region of $M = \mathbb{CP}_1 \times \mathbb{CP}_1$ in which

$$\omega = \sqrt{2}\varphi_1 = e^1 \wedge e^2 - e^3 \wedge e^4.$$

Since $\nabla\omega = 0$, we have

$$\theta_1^2 = \theta_2^1 = \theta_1^3 = \theta_3^1 = 0,$$

so the connection on Λ^- is determined by a single 1-form

$$\theta = \theta^2_3 = -\theta^3_2.$$

The distribution $\mathcal{D}$ is thus spanned by

$$\begin{aligned}
\mathfrak{w}_1 &= (\zeta^2+1)e_1 - 2\zeta e_3 + (\zeta^2-1)e_4 \\
&\quad -\frac{1+\zeta^2}{2}\Big[(\zeta^2+1)\theta_1 - 2\zeta\theta_3 + (\zeta^2-1)\theta_4\Big]\frac{\partial}{\partial\zeta} \\
\mathfrak{w}_2 &= (\zeta^2+1)e_2 + (\zeta^2-1)e_3 + 2\zeta e_4 \\
&\quad -\frac{1+\zeta^2}{2}\Big[(\zeta^2+1)\theta_2 + (\zeta^2-1)\theta_3 + 2\zeta\theta_4\Big]\frac{\partial}{\partial\zeta}
\end{aligned}$$

and $\partial/\partial\overline{\zeta}$, where $\theta_j = \theta(e_j)$. However, $\varphi_2 + i\varphi_3$ is a unit section of the canonical line bundle of $(M, g, \mathfrak{J})$, and

$$\begin{aligned}
d\varphi_1 &= 0 \\
d\varphi_2 &= -\theta\wedge\varphi_3 \\
d\varphi_3 &= \theta\wedge\varphi_2
\end{aligned} \tag{7}$$

Hence $d\theta$ is just the Ricci form ρ of $(M, g, \mathfrak{J})$. But the Ricci form of any Kähler manifold is of type $(1,1)$, and in our case $\rho\wedge\omega = 0$, since $(M, g, \mathfrak{J})$ is assumed to be scalar-flat. We thus conclude that

$$d\theta\wedge\varphi_j = 0, \quad j = 1, 2, 3. \tag{8}$$

We remark in passing that this is simply a special case of a more general fact: namely, Λ^- has self-dual curvature on any scalar-flat self-dual 4-manifold.

Let us now set

$$\begin{aligned}
\Omega &= -\frac{[(1+\zeta^2)\varphi_1 + (1-\zeta^2)\varphi_2 - 2\zeta\varphi_3]\wedge[2d\zeta + (1+\zeta^2)\theta]}{(1+\zeta^2)^2} \\
&= (\varphi_1 + \cos t\ \varphi_2 + \sin t\ \varphi_3)\wedge(dt - \theta)
\end{aligned}$$

where

$$t = -2\tan^{-1}\zeta = i\log(1-i\zeta) - i\log(1+i\zeta).$$

The restriction of this form to $F = \partial\mathcal{Z}_+$ is a real, geometrically meaningful, and globally defined 3-form. Indeed, $\varphi_1 + \cos t\ \varphi_2 + \sin t\ \varphi_3$ is the tautological 2-form on F, thought of as the space of those real null anti-self-dual 2-forms for which the inner product with the Kähler form ω is $\sqrt{2}$; and $dt - \theta$ the principle-connection 1-form of the unit canonical bundle of $(M, g, \mathfrak{J})$. Since Ω is the unique analytic continuaton of $\Omega|_F$ up the fiber disks of $\mathcal{Z}_+ \to M$, this shows that Ω is globally defined on the $\mathcal{Z}_+ - Q$, where Q is the image of the section $\mathfrak{J}$, which is represented by $\zeta = i$.

Next, notice that Ω annihilates $\mathfrak{w}_1$, $\mathfrak{w}_2$, and $\partial/\partial\overline{\zeta}$. Thus Ω is a $(3,0)$-form on $Z - (P\cup Q)$. Moreover, equations (7) and (8) tell us that

$$\begin{aligned}
d\Omega &= d(\varphi_1 + \cos t\ \varphi_2 + \sin t\ \varphi_3)\wedge(dt-\theta) \\
&\quad +(\varphi_1 + \cos t\ \varphi_2 + \sin t\ \varphi_3)\wedge d(dt-\theta)
\end{aligned}$$

$$\begin{aligned}
&= (-\sin t\ dt\wedge\varphi_2+\cos t\ d\varphi_2+\cos t\ dt\wedge\varphi_3+\sin t\ d\varphi_3)\wedge(dt-\theta)\\
&\qquad -(\varphi_1+\cos t\ \varphi_2-\sin t\ \varphi_3)\wedge d\theta\\
&= (-\sin t\ dt\wedge\varphi_2-\cos t\ \theta\wedge\varphi_3+\cos t\ dt\wedge\varphi_3+\sin t\ \theta\wedge\varphi_2)\wedge(dt-\theta)\\
&= \sin t\ dt\wedge\varphi_2\wedge\theta-\cos t\ \theta\wedge\varphi_3\wedge dt-\cos t\ dt\wedge\varphi_3\wedge\theta+\sin t\ \theta\wedge\varphi_2\wedge dt\\
&= 0
\end{aligned}$$

so the $(3,0)$-form Ω is actually $\overline{\partial}$-closed on $Z-P$, where it is therefore a meromorphic 3-form with only a pole of order 2 along Q. Moreover, the restriction of Ω to $\partial\mathcal{Z}_+ = F$ is a real closed 3-form which kills the tangent space of the foliation $\mathscr{F}$, since it annihilates $\mathfrak{w}_1$ and $\mathfrak{w}_2$; thus $\Omega|_F$ is actually the pull-back of a real 3-form on P. This shows that Ω descends to a continuous 3-form on $Z-Q$ which is holomorphic on the complement of P. It is therefore holomorphic even across P, by an iterated application of the Weierstrass removable singularities theorem. Identifying Z with $\mathbb{CP}_3$ as in Theorem 11.3, Ω thus becomes a meromorphic 3-form on $\mathbb{CP}_3$ with a double pole at a quadric Q, and its pull-back to the totally real submanifold $P\subset\mathbb{CP}_3-Q$ is real, as promised. ■

Analogy with Pontecorvo's characterization [43] of the twistor spaces of positive-definite scalar-flat K ähler metrics would lead one to expect that the converse is also true. Fortunately, this is indeed the case:

Proposition 11.5 *Let $(M,[g])$ be a space-time-oriented Zollfrei self-dual 4-manifold whose twistor space (Z,J) is biholomorphic to $\mathbb{CP}_3$. Suppose that there is a quadric $Q\subset Z\cong\mathbb{CP}_3$ such that $P\cap Q=\emptyset$, and that there is a meromorphic 3-form Ω on Z which is holomorphic and non-zero on $Z-Q$ and which has the property that its pull-back to P is a real 3-form. Then Q determines an integrable complex structure $\mathfrak{J}$ on M such that $(M,\mathfrak{J})\cong\mathbb{CP}_1\times\mathbb{CP}_1$, and the conformal class $[g]$ contains a scalar-flat metric g which is indefinite Kähler with respect to the complex structure $\mathfrak{J}$. Moreover, this metric is uniquely determined up to an overall multiplicative constant.*

Proof. Since Q represents double the generator of $H^2(\mathbb{CP}_3,\mathbb{Z})$, it generates $H^2(\mathbb{CP}_3-P,\mathbb{Z})=H^2(\mathcal{Z}_+,\mathbb{Z})$, and so has intersection number 1 with any fiber disk in $\mathcal{Z}_+$. Thus Q represents a section of Int $\mathcal{Z}_+$, and may be interpreted as an almost-complex structure $\mathfrak{J}$. Moreover, the induced projection $Q\to M$ is a diffeomorphism, and the pull-back of $\mathfrak{J}$ to Q is exactly the given complex structure on $Q\cong\mathbb{CP}_1\times\mathbb{CP}_1$, so $\mathfrak{J}$ is, in particular integrable.

Near an arbitrary point of M, choose a local pseudo-orthonormal frame so that $e_2=\mathfrak{J}e_1$ and $e_4=\mathfrak{J}e_3$. Then Q is represented in the corresponding local coordinates on $\mathcal{Z}_+$ by $\zeta=i$. Now pull Ω back to $\mathcal{Z}_+$, and observe that we must then have

$$\Omega=-\frac{f}{(1+\zeta^2)^2}[(1+\zeta^2)\varphi_1+(1-\zeta^2)\varphi_2-2\zeta\varphi_3]\wedge[2d\zeta-(1-\zeta^2)\theta_1^3-2\zeta\theta_1^2+(1+\zeta^2)\theta_3^2]$$

for some function f on $\mathcal{Z}_+$, since Ω annihilates $\mathfrak{w}_1$, $\mathfrak{w}_2$, and $\partial/\partial\overline{\zeta}$. Moreover, this function f is holomorphic in $\zeta\neq i$, bounded on the entire half-plane, and real when ζ is real. Hence f is independent of ζ, by Liouville's Theorem and the reflection principle. In particular, the ζ-derivative of f vanishes at $\zeta=i$, so the residue ω of

Ω at $\zeta = i$ is a multiple of φ_1. However, this residue is also a closed nowhere-zero 2-form on M, as, up to an overall constant, it may instead be obtained by restricting Ω to $F = \partial \mathcal{Z}_+ = \Psi^{-1}(P)$ and integrating along the fibers of $\mathfrak{p} : F \to M$. But this means that ω is the Kähler form with respect to $\mathfrak{J}$ of an indefinite Kähler metric g in the self-dual conformal class $[g]$. Since such a metric must also be scalar-flat by Lemma 11.1, the claim therefore follows. ■

Theorem D now follows immediately from Propositions 11.4 and 11.5. since a projective transformation is all that is needed to arrange for the quadric Q to be given by $z_1^2+z_2^2+z_3^2+z_4^2 = 0$, and for the associated 3-form to be some real constant times

$$\Omega = \frac{\left(z_j \frac{\partial}{\partial z_j}\right) \lrcorner \, (dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4)}{[z_1^2 + z_2^2 + z_3^2 + z_4^2]^2}.$$

Of course, requiring that the pull-back of Ω to P be real has been re-interpreted in the statement of Theorem D as the condition that the pull-back of $\phi = \Im m \ \Omega$ should vanish.

It remains only to ask whether there are many submanifolds P near $\mathbb{RP}^3 \subset \mathbb{CP}_3$ on which $\phi = \Im m \ \Omega$ vanishes. In fact, the condition in question is a weakening of the *special Lagrangian* condition studied by McLean [37], and similar arguments will now show that such submanifolds exist in considerable profusion:

Proposition 11.6 *For any integer $k \geq 1$ and any $\alpha \in (0,1)$, the space of compact $C^{k,\alpha}$ totally real submanifolds $P \subset \mathbb{CP}_3 - Q$ near $\mathbb{RP}^3$ on which $\phi = \Im m \ \Omega$ vanishes is a Banach manifold whose tangent space at P consists of real $C^{k,\alpha}$ vector fields v on P for which div $v = 0$ with respect to the standard volume form on $\mathbb{RP}^3$.*

Proof. The normal bundle of $\mathbb{RP}^3 \subset \mathbb{CP}_3$ may be identified with $T\mathbb{RP}^3$ via J, so some tubular neighborhood of $\mathbb{RP}^3$ must be diffeomorphic to $T\mathbb{RP}^3$. In fact, we can even take this tubular to be all of $\mathbb{CP}_3 - Q$ by invoking the real-analytic diffeomorphism

$$\beth : T\mathbb{RP}^3 \longrightarrow \mathbb{CP}^3 - Q$$

given by

$$\pm(\vec{x}, \vec{y}) \mapsto \left[\vec{x} + i\frac{\vec{y}}{\sqrt{1+|\vec{y}|^2}}\right]$$

for $\vec{x}, \vec{y} \in \mathbb{R}^4$ with $|\vec{x}|^2 = 1$ and $\vec{x} \cdot \vec{y} = 0$. Thus, for any integer $k \geq 1$ and any $\alpha \in (0,1)$, each real-valued $C^{k,\alpha}$ vector field v on $\mathbb{RP}^3$ defines a new embedding

$$\begin{array}{rcl} \hbar_v : \mathbb{RP}^3 & \longrightarrow & \mathbb{CP}_3 - Q \\ y & \longmapsto & \beth(Jv_y) \end{array}$$

and every other compact submanifold of $\mathbb{CP}_3$ which is $C^{k,\alpha}$ close to $\mathbb{RP}^3$ can be so parameterized in a unique manner. Now let $\mathfrak{B}^{k,\alpha}$ be the Banach space of $C^{k,\alpha}$ vector fields on $\mathbb{RP}^3$, and let $\mathfrak{C}^{k,\alpha}$ be the Banach space of $C^{k,\alpha}$ real-valued 3-forms on $\mathbb{RP}^3$ with integral 0 on $\mathbb{RP}^3$. Let μ be the standard volume form on $\mathbb{RP}^3$. We may then define a smooth map of Banach manifolds

$$\begin{array}{rcl} \mho : \mathfrak{B}^{k,\alpha} \times \mathfrak{C}^{k,\alpha} & \longrightarrow & \mathfrak{C}^{k-1,\alpha} \times \mathfrak{B}^{k-1,\alpha} \\ (v, f\mu) & \longmapsto & (\hbar_v^* \phi, \text{curl } v + \text{grad } f) \end{array}$$

whose derivative at 0 is

$$(v, f) \mapsto (\text{div } v, \text{curl } v + \text{grad } f).$$

Now this is essentially just the elliptic operator $d+d^* : \Lambda^{\text{even}} \to \Lambda^{\text{odd}}$, and so has trivial kernel and cokernel because $H^2(\mathbb{RP}^3, \mathbb{R}) = 0$. The interior Schauder estimates for elliptic equations therefore imply that $\mathfrak{F}_{*0}$ is a Banach-space isomorphism. Hence the inverse function theorem for Banach spaces implies that $\mathfrak{F}$ becomes a diffeomorphism when restricted to some neighborhood $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \subset \mathfrak{B}^{k,\alpha} \times \mathfrak{C}^{k,\alpha}$ of the origin. Thus $(\mathfrak{F}|_{\mathcal{U}})^{-1}(\{0\} \times \mathfrak{B}^{k-1,\alpha})$ is a Banach manifold. By inspection, however, this set is of the form $\mathcal{M} \times \mathcal{U}_2$, where $\mathcal{M}$ is its projection to $\mathfrak{B}^{k,\alpha}$. Hence $\mathcal{M}$ is a Banach manifold, and represents the desired moduli space of solutions. Moreover,

$$T_0\mathcal{M} = \{v \in \mathfrak{B}^{k,\alpha} \subset \Gamma(T\mathbb{RP}^3) \mid \text{div } v = 0\},$$

precisely as claimed, so we are done. ■

Thus, while self-dual split-signature conformal structures on $S^2 \times S^2$ essentially depend on a vector field on $\mathbb{RP}^3$, scalar-flat Kähler metrics correspond to the case in which the vector field is *divergence free*. So far, though, this is just a abstract existence statement. Nonetheless, one can do much better in the real-analytic case. Indeed, let v be a divergence-free real-analytic vector field on $\mathbb{RP}^3$; in other words, let $v = \text{curl } w$ for w some real-analytic vector field on $\mathbb{RP}^3$. Then Jv corresponds to the section $Jv + iv$ of $(T^{1,0}\mathbb{CP}_3)|_{\mathbb{RP}^3}$. Because v is locally represented by power series, $Jv+iv$ can then be uniquely extended to some neighborhood $U \subset \mathbb{CP}_3 - Q$ of $\mathbb{RP}^3$ as a holomorphic vector field $\mathfrak{v}$, and we then have a real-analytic 1-parameter family $\{\psi_t \mid t \in (-\varepsilon, \varepsilon)\}$ of biholomorphisms from neighborhoods $U_t \subset U$ of $\mathbb{RP}^3$ to U obtained by following the integral curves of $\Re e\ \mathfrak{v}$. Notice, however, that $\mathcal{L}_{\mathfrak{v}}\Omega = 0$, since this expression is the analytic continuation of div iv from $\mathbb{RP}^3$ to U. The constructed biholomorphisms therefore satisfy $\psi_t^*\Omega = \Omega$. Hence $P = \psi_t(\mathbb{RP}^3)$ is a submanifold on which $\phi = \Im m\ \Omega$ vanishes.

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