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Twistor actions for non-self-dual fields; a new foundation for twistor-string theory

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Abstract: Twistor space constructions and actions are given for full Yang-Mills and conformal gravity using almost complex structures that are not, in general, integrable. These are used as the basis of a derivation of the twistor-string generating functionals for tree level perturbative scattering amplitudes of Yang-Mills and conformal gravity. The derivation follows by expanding and resumming the classical approximation to the path integral obtained from the twistor action. It provides a basis for exploring whether the equivalence can be made to extend beyond tree level and allows one to disentangle conformal supergravity modes and super Yang-Mills modes from the standard Yang-Mills modes.

Keywords: Twistor-string theory, QCD scattering amplitudes, Twistor theory.

Contents

1. Introduction

Twistor-string theory provides a dramatic reformulation of perturbative $N = 4$ super-Yang-Mills and conformal super-gravity scattering amplitudes in terms of integrals over moduli spaces of algebraic curves in super-twistor space (a supersymmetric version $\mathbb{CP}^{3|4}$ of complex projective three space, $\mathbb{CP}^3$), Witten (2004). Whilst it is widely believed that the twistor-string formulation is correct at tree level, no systematic proof is known. The purpose of this article is to provide a derivation of these formulae from first principles. It starts with the space-time action, and proceeds via a twistor space action associated to a corresponding twistor construction for fields that are not necessarily self-dual. The price we pay for this extra generality over and above the standard twistor correspondences for self-dual fields is that the twistor almost complex structures are no longer generally integrable. This limits the applicability of the constructions to problems in classical geometry and reflects the lack of integrability of the classical equations. Nevertheless, the constructions are sufficent to provide a derivation of twistor-string theory. In particular we give a formal proof that the twistor-string generating functionals for perturbative scattering amplitudes are correct at the classical limit, i.e., for tree diagrams; more work is required to extend the approach rigorously to loop diagrams, although it provides a platform from which one can investigate the problem. This approach also disentangle Yang-Mills from conformal gravity and the supersymmetric theories from their bosonic constituents, thus making the study of loops much more straightforward; from the twistor-string point of view, a higher genus contribution automatically includes conformal supergravity modes and supersymmetric partners in the loops. The proof is formal to the extent that it relies on an expansion of the classical limit of the path integral and infrared divergences are not addressed.

In twistor-string theory, scattering amplitudes for gluons in helicity eigenstates are given by integrals over the moduli space of algebraic curves in twistor space of degree d, where $d = q - 1 + l$, l is the number of loops and q is the number of external gluons of helicity $+1$. The genus of the curves is also bounded by the number of loops. Most of the investigations have been confined to tree diagrams and hence are concerned with moduli spaces of rational curves (genus 0). There have been, roughly speaking, two approaches to twistor-string theory. Cachazo, Svrcek and Witten (2004) consider integrals over the moduli space of maximally disconnected curves, i.e., d lines, whereas Roiban, Spradlin and Volovich (2004) consider integrals over moduli spaces of connected rational curves. In the former approach, the lines must be connected into a tree by holomorphic Chern-Simons propagators although these are absent in the latter approach for tree diagrams. Gukov, Motl and Nietzke (2004) argue that the two approaches are equivalent.

Perhaps the most elegant formula in the subject is that for the on-shell generating functional for tree-level scattering amplitudes $\mathcal{A}[a,g]$ in the Roiban, Spradlin & Volovich approach (here (a,g) are the on-shell twistor fields, both being $(0,1)$-forms on a region in $\mathbb{CP}^3$ with values in the endomorphisms of some given smooth bundle $E \to \mathbb{CP}^3$, but with g having homogeneity degree -4 and a homogeneity degree 0). In this case, the twistor on-shell fields (a,g) define a $\bar\partial$-operator $\bar\partial^s$ on a bundle E over $\mathbb{CP}^{3|4}$. The generating functional for processes with $d+1$ external fields of helicity $+1$ is then:

$$\mathcal{A}^d[a,g] = \int_{\mathscr{M}^d} \det(\bar\partial^s)\, \mathrm{d}\mu \tag{1.1}$$

where $\mathrm{d}\mu$ is a natural measure on the moduli space $\mathscr{M}^d$ of connected rational curves in $\mathbb{CP}^{3|4}$ of degree d.[1] For the CSW version, extra terms associated with the holomorphic Chern-Simons theory need to be incorporated also as described in §3 and this is the version that is proved here. We appeal to Gukov, Motl and Nietzke (2004) for the proof that this implies the connected formulation given above.

Whilst the extensions to the $N = 4$ supersymmetric versions of Yang-Mills and conformal gravity are likely to be straightforward, they are nevertheless complicated and ommitted here. We work in Euclidean signature throughout and ignore infrared divergences.

A summary of the rest of the article follows.

In §2, the Chalmer & Siegel Lagrangian for the anti-self-dual sector of Yang-Mills on space-time and its generalisation to full Yang-Mills is set out. Witten's twistor space reformulation of the anti-self-dual sector as a holomorphic Chern-Simons theory is then reviewed. We then give a twistorial formulation of the extra term, I, required in the action on twistor space to generalise to full Yang-Mills. This term is a two-point integral on

[1] see the transparencies from Witten's lectures posted at www.maths.ox.ac.uk/∼lmason/Tws.

twistor space. It is then shown that the full action correctly reproduces full Yang-Mills theory on space-time by virtue of a generalisation of the Ward construction for anti-self-dual gauge fields to gauge fields that are not anti-self-dual. The action is a functional of a $\bar{\partial}$ operator on a bundle $E \to \mathbb{PT}$ where $\mathbb{PT}$ is a region in $\mathbb{CP}^3$ and a homogeneous $(0,1)$-form g with values in $\mathrm{End}(E)$. The construction relies on the fact that the restriction of a $\bar{\partial}$-operator to a Riemann sphere, $\mathbb{CP}^1$, is automatically integrable. Not only does the twistor action reproduce the correct equations of motion, but it also takes the same value as the space-time action when evaluated on a solution to the field equations. In §2.3 the twistor space action is expressed more explicitly and the field equations derived and solved in terms of an arbitrary solution to the Yang-Mills equations on space-time; the general solution is gauge equivalent to such a solution.

§3 contains the main derivation of the the twistor-string on-shell generating functionals for tree-level scattering amplitudes. In §3.1 it is shown that the the extra term I has simple alternative expressions when written in terms of integrals over super twistor space $\mathbb{CP}^{3|4}$. Then in §3.2 the general definition of on-shell generating functionals is reviewed and expressed as the classical limit of a path integral. In §3.3 the twistor-string generating functionals are reviewed and that appropriate to the Cachazo, Svrcek and Witten's approach is presented. This is then expanded and resummed to show equivalence with the appropriate formulae from the twistor Lagrangians derived in the previous section. Since the classical approximation uses only the value of the action and this takes on the same value as the space-time action, this shows that the twistor-string formulae provide the correct generating function for Yang-Mills scattering theory at tree level.

In §4 the same process is worked through for conformal gravity. First we review the analogues of the Chalmer & Siegel Lagrangians appropriate to conformal gravity and their twistor space reformulations as given by Berkovits and Witten (2004). Then we move on to finding the extra (non-local) term required in the twistor action to extend to the full theory. Finally in analogy with the Yang-Mills case, we expand and resum the path integral to obtain the relevant twistor-string formulae for the generating function for perturbative scattering amplitudes.

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2. The Yang-Mills actions

For the purposes of this paper, we will Wick rotate to euclidean signature and use the appropriate euclidean signature conventions. It is not clear that euclidean signature is

[2]See www.maths.ox.ac.uk/∼lmason/Tws for the collected slides from the talks.

essential for all of what follows, but it helps avoid a number of technical difficulties. Thus $\mathbb{M}$ will denote $\mathbb{R}^4$ but with the standard flat euclidean metric η. We will take coordinates x^a, $a = 0, \ldots 3$ on $\mathbb{M}$ and will use the metric η_{ab} to raise and lower indices as usual. We will denote self dual spinors with a primed upper case roman index, e.g., $\pi_{A'}$, $A' = 0', 1'$. Anti-self-dual spinors will be denoted by ω^A, $A = 0, 1$. In Euclidean signature, the reality structure is quaternionic

$$\omega^A \to \hat{\omega}^A = (\bar{\omega}^1, -\bar{\omega}^0)$$

so that $\hat{\hat{\omega}}^A = -\omega^A$. We can represent a vector index as a pair of spinor indices, so the coordinates x^a on $\mathbb{M}$ can be represented as $x^{AA'}$ and we define $\partial_{AA'} = \partial/\partial x^{AA'}$.

2.1 The action on space-time

The basic variable for Yang-Mills theory is a 1-form A on Minkowski space $\mathbb{M}$ with values in the Lie algebra of some gauge group. Let $F = \mathrm{d}A + [A, A]$ be the associated curvature; it is a Lie-algebra valued 2-form. Then the standard action for the Yang-Mills equations is

$$S[A] = \int_{\mathbb{M}} \mathrm{tr}(F \wedge F^*)$$

where F^* is the Hodge dual of F (in indices, $F^*_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}$) and tr is an ad-invariant inner product on the Lie algebra. Since $\int \mathrm{tr}(F^2)$ is a topological invariant, for perturbative purposes, one can add any multiple of this into the action without changing the perturbative theory and this allows one to rewrite the action as

$$S[A] = \int_{\mathbb{M}} \mathrm{tr}(F^+ \wedge F^+)$$

where $F^+ = \frac{1}{2}(F + F^*)$ is the self-dual part of F satisfying $F^{+*} = +F^+$. (Here we have used the fact that $F = F^+ + F^-$ and $F^+ \wedge F^- = 0$ automatically.)

The anti-self-dual sector of the theory is the case when $F^+ = 0$, but in perturbing away from this to first order, one would want to introduce G, a Lie algebra valued self-dual 2-form, so that ϵG represents the infinitesimal value of F^+. Chalmers & Siegel have proposed the following action for the anti-self-dual sector of the Yang-Mills equations on Minkowski space $\mathbb{M}$:

$$S_{\mathrm{asd}}[A, G] = \int_{\mathbb{M}} \mathrm{tr}(G \wedge F)$$

The Euler-Lagrange equations imply that F is anti-self-dual, and G is covariantly closed.

To obtain the full Yang Mills equations we add the term $-\frac{\epsilon}{2}I[G]$ where

$$I[G] = \int_{\mathbb{M}} \mathrm{tr}(G \wedge G)\,.$$

The action for full Yang-Mills is then

$$S_{\mathrm{YM}} = S_{\mathrm{asd}} - \frac{\epsilon}{2}I[G]\,.$$

When $\epsilon \neq 0$, the Euler-Lagrange equations imply that F^+, the self-dual part of F is ϵG, which is in turn covariantly closed, so that the full Yang-Mills equations are satisfied.

Forming a perturbation series in ϵ around $\epsilon = 0$ therefore gives a way of perturbing full Yang-Mills theory around its anti-self-dual sector.

We also note that the value of this last action on a solution to the field equations is, up to an overall multiplicative factor, the same as for the standard Yang-Mills equations at least perturbatively so that the topological term does not contribute.

2.2 The actions on twistor space

We first review twistor space geometry and notational conventions that we will use. Twistor space $\mathbb{PT}$ will be taken to be some neighbourhood of a line in complex projective 3-space, $\mathbb{CP}^3$. We will work in a space-time of euclidean signature. In this case, we can choose our neighbourhood so that $\mathbb{PT}$ fibres over an open set $U \subset \mathbb{M}$, $p : \mathbb{PT} \to U$, with fibre the Riemann sphere, $\mathbb{CP}^1$. This fibre is best thought of as the projectivisation of the space $\mathbb{C}^2$ of the self dual spinors, $\pi_{A'}$, $A' = 0', 1'$ at $x \in U$. Thus $(x^{AA'}, \pi_{A'})$ are coordinates on non-projective twistor space $\mathbb{T}$, and the projective space is obtained by modding out the scale of $\pi_{A'}$.

Homogeneous coordinates on twistor space are provided by $Z^\alpha = (\omega^A, \pi_{A'})$ where $\omega^A = x^{AA'}\pi_{A'}$. We note that the complex conjugation on spinors induces a similar conjugation $Z^\alpha \to \hat{Z}^\alpha$ with $\hat{\hat{Z}}^\alpha = -Z^\alpha$. This conjugation restricts to give the antipodal map on each $\mathbb{CP}^1$ fibre of $\mathbb{PT} \to \mathbb{M}$. The coordinates $(\omega^A, \pi_{A'})$ are holomorphic coordinates for the standard complex structure on $\mathbb{PT}$. In terms of these coordinates, the projection p is given by

$$p(\omega^A, \pi_{A'}) = \{x^{AA'} = \frac{1}{\hat{\pi}^{C'}\pi_{C'}}\left(\omega^A\hat{\pi}^{A'} - \hat{\omega}^A\pi^{A'}\right)\}\,.$$

The complex structure can also be represented in terms of the distribution of $(0,1)$ vectors $\mathrm{D} = \{\partial/\partial\hat{\pi}_{A'}, \pi^{A'}\partial_{AA'}\}$, where $\partial_{AA'} = \partial/\partial x^{AA'}$. It can also be represented by the $\bar{\partial}$-operator (written here on the non-projective space)

$$\bar{\partial} = \frac{1}{\pi^{A'}\hat{\pi}_{A'}}\mathrm{d}x^{AA'}\hat{\pi}_{A'}\pi^{B'}\partial_{AB'} + \mathrm{d}\hat{\pi}_{A'}\frac{\partial}{\partial\hat{\pi}_{A'}}$$

The connection 1-form, A is a connection on a bundle E which, with an abuse of notation, can be pulled back to give a smooth bundle $E \to \mathbb{PT}$. The connection then allows one to define a d-bar operator $\bar{\partial}_a = \bar{\partial} + a$ on $E \to \mathbb{PT}$ where a is a $(0,1)$-form with values in $\mathrm{End}(E)$ and is the $(0,1)$-part of the pullback of A to $\mathbb{PT}$. We will see that G corresponds to a $(0,1)$-form g with values in $\mathrm{End}(E)\otimes\mathcal{O}(-4)$ where $\mathcal{O}(-1)$ is the tautological bundle over $\mathbb{CP}^3$. Witten (2004) shows that S_{asd} has a direct analogue on twistor space in the form of the spin-1 part

$$S_{\mathrm{asd}}[a,g] = \int_{\mathbb{PT}} \mathrm{tr}(g\wedge f)\wedge\Omega \qquad (2.1)$$

of a super Chern-Simons Lagrangian where $f := \bar{\partial}a + a\wedge a = \bar{\partial}_a^2$ is the $(0,2)$-part of the curvature of a connection with $(0,1)$-part a and $\Omega = \varepsilon_{\alpha\beta\gamma\delta}Z^\alpha\mathrm{d}Z^\beta\mathrm{d}Z^\gamma\mathrm{d}Z^\delta \in \Gamma(\mathbb{PT}, \Omega^{(3,0)}(4))$ is the (weighted) holomorphic volume form (here as usual $\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{[\alpha\beta\gamma\delta]}$, $\varepsilon_{0123} = 1$).

The correspondence is precise for classical fields modulo their appropriate gauge freedoms: the Euler-Lagrange equations from this action imply that $f = 0$ and $[\bar{\partial} + a, g] = 0$.

The Lagrangian is invariant under the usual group of gauge transformations (automorphisms) of E together with $g \to g + \bar\partial_a \chi$ for smooth sections χ of $\mathrm{End}(E)(-4)$. Thus, modulo gauge freedoms, the first equation implies that $\bar\partial + a$ defines a holomorphic structure on E up to gauge transformations, and g defines a cohomology class in $H^1(U, \mathrm{End}(E)(-4))$. Holomorphic vector bundles E correspond to anti-self-dual Yang-Mills gauge connections A by the Ward transform, and elements $g \in H^1(U, \mathrm{End}(E)(-4))$ corresponds to covariantly closed self-dual 2-forms with values in the Lie algebra of the gauge group by a standard generalisation of the Penrose transform as follows.

In the abelian case, the Penrose tranform $g \to G$ is implemented by

$$G = p_*(g \wedge \Omega) \tag{2.2}$$

that is, integrate over the fibres of p to obtain a 2-form on $\mathbb{M}$. This necessarily provides a self-dual 2-form, since in $(x^{AA'}, \pi_{A'})$ coordinates we can write

$$\Omega = \mathrm{D}\pi \wedge \pi_{B'}\pi_{C'}\varepsilon_{BC}\mathrm{d}x^{BB'} \wedge \mathrm{d}x^{CC'}\,, \qquad \text{where} \quad \mathrm{D}\pi = \pi^{A'}\mathrm{d}\pi_{A'}$$

and so if we set

$$G_{A'B'}(x) = \int_{L(x)} \pi_{A'}\pi_{B'} g \wedge \mathrm{D}\pi$$

then the above formula gives

$$G = G_{A'B'}\varepsilon_{AB}\mathrm{d}x^{AA'} \wedge \mathrm{d}x^{BB'}\,,$$

which is necessarily self-dual. It is easily seen that G must be closed since $g \wedge \Omega$ is.

In order to formulate full Yang-Mills on twistor space, we need to find the appropriate twistor version of the $I = \int_{\mathbb{M}} \mathrm{tr}(G \wedge G)$ term. It follows from the above that in order to express $I[G]$ in terms of g in the abelian case, we can consider the integral

$$I[g] = \int_{\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT}} \mathrm{tr}(g(Z_1) \wedge g(Z_2)) \wedge \Omega(Z_1) \wedge \Omega(Z_2)\,, \tag{2.3}$$

where $\mathbb{PT} \times_{\mathbb{M}} \mathbb{PT} = \{(Z_1, Z_2) \in \mathbb{PT} \times \mathbb{PT} | p(Z_1) = p(Z_2) \in \mathbb{M}\}$ is the fibrewise product of $\mathbb{PT}$ with itself over $\mathbb{M}$ with fibre $\mathbb{CP}^1 \times \mathbb{CP}^1$.[3]

To make sense of (2.3) in the nonabelian case, we must find some way of comparing the fibre of $\mathrm{End}(E)$ at Z_1 with that at Z_2. In the integral we have already restricted to $\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT}$ and so Z_1 and Z_2 both lie on the line $L(x)$ where $x = p(Z_1) = p(Z_2)$. Although the $\bar\partial_a$ operator is not a-priori integrable, it is nevertheless necessarily integrable on restriction to lines. We make the assumption that $(E, \bar\partial_a)$ is holomorphically trivial along such lines $L(x)$ for $x \in \mathbb{M}$; this will be the case for small a and hence perturbatively. We therefore define $\mathrm{tr}_a(g(Z_1) \wedge g(Z_2))$ to be the trace taken in such a frame that is globally holomorphic along the line from Z_1 to Z_2. We now generalize equation (2.3) to the non-abelian case as

$$I[g,a] = \int_{\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT}} \mathrm{tr}_a(g(Z_1) \wedge g(Z_2)) \wedge \Omega(Z_1) \wedge \Omega(Z_2)\,. \tag{2.4}$$

[3]This is the first point at which we needed to have specified a real slice $\mathbb{M}$ of complex Minkowski space $\mathbb{CM}$ (as indeed one must for the ordinary action principle).

This defines the appropriate additional term, but is now a functional of a also, $I := I[g,a]$.

This is not the most helpful form of $I[g,a]$ and we rewrite it as follows. First note that $\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT} = \mathbb{M}\times\mathbb{CP}^1\times\mathbb{CP}^1$ and we coordinatize it by (x,π_1,π_2) by setting

$$(Z_1,Z_2) = \Big((x^{AA'}\pi_{1A'},\pi_{1A'}),(x^{BB'}\pi_{2B'},\pi_{2B'})\Big)\,.$$

In these coordinates

$$\Omega(Z_1)\wedge\Omega(Z_2) = (\pi_1\cdot\pi_2)^2\mathrm{D}\pi_1\wedge\mathrm{D}\pi_2\,\mathrm{d}^4x \tag{2.5}$$

where

$$\mathrm{D}\pi = \pi^{B'}\mathrm{d}\pi_{B'}\,,\qquad\text{and}\qquad \pi_1\cdot\pi_2 = \pi_1^{A'}\pi_{2A'}\,.$$

Thus

$$I[g,a] = \int_{\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT}} \mathrm{tr}_a(g(Z_1)\wedge g(Z_2))(\pi_1\cdot\pi_2)^2\mathrm{D}\pi_1\wedge\mathrm{D}\pi_2\wedge\mathrm{d}^4x\,.$$

In the non-abelian case, the integral formula for G in terms of g is:

$$G_{A'B'}(x) = \int_{\omega^A = ix^{AA'}\pi_{A'}} \pi_{A'}\pi_{B'}g\wedge\pi_{A'}\mathrm{d}\pi^{A'} \tag{2.6}$$

as before, but the integral must be performed in a holomorphic trivialisation of E over the Riemann sphere $p^{-1}(x)$. The 2-form

$$G = G_{A'B'}\varepsilon_{AB}\mathrm{d}x^{AA'}\wedge\mathrm{d}x^{BB'} = \int_{p^{-1}(x)} g\wedge\Omega$$

with the same proviso concerning the frame for E and so we can see that $I[g,a] = I[G]$.

We will therefore consider the twistor action

$$S_T = S_{\mathrm{asd}}[a,g] - \frac{\epsilon}{2}I[g,a].$$

The gauge symmetry of this action is the group of gauge transformations of the bundle $E\to\mathbb{PT}$ together with $g\to g+\bar\partial_a\chi$. It is easily seen that the action is invariant under the group of gauge transformations of E. To see invariance under $g\to g+\bar\partial_a\chi$ for $I[g,a]$, note that in the frame that is holomorphic up the fibres of p in which the trace is taken, $\bar\partial_a = \bar\partial$ on restriction to the fibres of p and so the integral over one of the $\mathbb{CP}^1$ factors of a fibre will give zero if we replace the corresponding g by $\bar\partial\chi$. The invariance of S_{asd} is elementary.

Proposition 2.1 *The action $S_T = S_{\mathrm{asd}}[a,g] - \frac{\epsilon}{2}I[a,g]$ is equivalent at the classical level to S_{YM}. This is true both in the sense that gauge equivalence classes of solutions to the Euler Lagrange equations on twistor space are in $1:1$ correspondence with gauge equivalence classes of solutions to the Yang-Mills equations on space-time, and in the sense that the twistor action takes the same values on (a,g) as the space-time action does on the corresponding (A,G).*

Proof: In this subsection we give a quick but inexplicit proof of this theorem. In the next we will develop more notation so as to be more explicit.

Given $(E \to \mathbb{PT}, a, g)$ satisfying the variational equations of the action S_T, we wish to construct $(E \to \mathbb{M}, A, G)$ satisfying the Yang-Mills equations. We first define the bundle $E \to \mathbb{M}$ to be the bundle whose fibre at x is the space of global $\bar{\partial}_a$-holomorphic sections of $E \to p^{-1}(x) \subset \mathbb{PT}$ (recalling that the bundle is assumed to be trivial on such lines).

We then note that we can define $G \in \Omega^{2+} \otimes \mathrm{End}(E)$ at each x to be the two form obtained by pushing down $g \wedge \Omega$ to $\mathbb{M}$ in the associated global holomorphic frame of $\mathrm{End}(E)$ over $p^{-1}(x)$. This necessarily provides a self-dual 2-form as before and we see that

$$I[g, a] = I[G]\,.$$

The classical equations of motion obtained by varying g are

$$\bar{\partial} a + a \wedge a = \epsilon \int_{Z' \in p^{-1}(p(Z))} g(Z')\Omega(Z')\,. \tag{2.7}$$

where the left hand side is evaluated at Z and as usual the integration is in a global holomorphic trivialisation of E over $p^{-1}(p(Z))$. The integral therefore yields the projection $G_{(0,2)}$ of $p^*G(p(Z))$ onto the $(0,2)$-forms at Z. Thus

$$\bar{\partial}_a^2 = G_{(0,2)}\,.$$

It follows that $f = \bar{\partial}_a^2$ has no component up the fibres of p. This allows us to define a connection A on $E \to \mathbb{M}$ as follows. Pull back a section s of $E \to \mathbb{M}$ to $E \to \mathbb{PT}$, then $\bar{\partial}_a s$ is holomorphic up the fibre of $p : \mathbb{PT} \to \mathbb{M}$. More concretely on $\mathbb{PT}$, $\pi^{A'}\partial_{AA'} \lrcorner\, \bar{\partial}_a s$ is therefore holomorphic in $\pi_{A'}$. It is also global with homogeneity degree 1 over the Riemann sphere with homogeneous coordinates $\pi_{A'}$, and depends linearly on s. We can deduce from a generalization of Liouville's theorem that $\pi^{A'}\partial_{AA'} \lrcorner\, \bar{\partial}_a s = \pi^{A'}(\partial_{AA'} + A_{AA'})s$ for some connection 1-form $A = A_{AA'}\mathrm{d}x^{AA'}$ on $\mathbb{M}$. This is in effect the standard Ward argument for constructing a connection A on $E \to \mathbb{M}$ from a. The $\bar{\partial}_a$ operator can therefore be represented in a frame pulled back from $E \to \mathbb{M}$ as

$$\bar{\partial}_a = \frac{1}{\pi^{C'}\hat{\pi}_{C'}}\mathrm{d}x^{AB'}\hat{\pi}_{B'}\pi^{A'}(\partial_{AA'} + A_{AA'}) + \mathrm{d}\hat{\pi}_{C'}\frac{\partial}{\partial \hat{\pi}_{C'}}\,.$$

Now we claim that $S_{\mathrm{asd}}[a, g] = S_{\mathrm{asd}}[A, G]$. This follows by using the gauge invariance on twistor space to use a gauge pulled back from $E \to \mathbb{M}$. In this gauge, a is the projection onto $(0,1)$-forms of $A_{AA'}\mathrm{d}x^{AA'}$ and f is the projection onto $(0,2)$-forms of F^+. Thus $S_{\mathrm{asd}}[a, g] = \int \mathrm{tr}(F^+ \wedge g) \wedge \Omega$. Integrating over the fibres of p then gives directly that $S_{\mathrm{asd}}[a, g] = S_{\mathrm{asd}}[A, G]$.

We have now reduced the desciption to that of the Chalmers & Siegel Lagrangian and so we have obtained the appropriate field equations as claimed. □

We have not in fact proved everything here: we have only provided a map from solutions to the field equations associated to S_T to those of S_{YM}. To see that it is $1:1$ and onto gauge equivalence classes we need to work more explicitly which we do in the next subsection.

2.3 The twistor action, field equations and solutions

We can therefore take the full twistor-space Lagrangian to be

$$S_T[a,g] = S_{\rm asd}[a,g] - \frac{\epsilon}{2} I[g,a] \,.$$

The equation of motion obtained by varying g is given in equation (2.7) but that obtained by varying a is more complicated, and we now make the $I[g,a]$ term more explicit in order to calculate the equations of motion. We also show in this subsection how every solution to the Yang-Mills equations on space-time gives rise to a solution to the Euler-Lagrange equations of the twistor action and that every solution to the Euler-Lagrange equations of the twistor action is gauge equivalent to a space-time solution arising in this way.

In the following our expressions will be functions of two or more twistors, Z_1, Z_2, ... or π spinors, π_1, π_2, ... We will adopt the convention that g_1 will denote a function of Z_1 and so on.

We first introduce global holomorphic frames $F(x,\pi)$ over the line L_x corresponding to $x \in \mathbb{M}$ by $F(x,\pi) : E_Z \to \mathbb{C}^r$, ($r$ is the rank of the bundle E) where $\bar\partial_a F|_{L_x} = 0$ and $F(x,\pi)$ is unique up to $F(x,\pi) \to F(x,\pi)\gamma(x)$ where γ is a function on $\mathbb{M}$ with values in the gauge group. We can then write $I[g,a]$ as

$$I[g,a] = \int_{\mathbb{M}\times\mathbb{CP}^1\times\mathbb{CP}^1} \mathrm{tr}(F_1^{-1} g_1 F_1 \wedge F_2^{-1} g_2 F_2) \wedge \Omega_1 \wedge \Omega_2 \,,$$

where F_1 and g_1 are evaluated at π_1 and Z_1 with $Z_1 = (x^{AA'}\pi_{1A'}, \pi_{1A'})$, etc..

To reformulate this further, we note that the greens function $K_{12} := K(x,\pi_1,\pi_2)$ for the d-bar operator $\bar\partial_a|_{L_x}$ on sections of $E \otimes \mathcal{O}(-1)|_{L_x}$ is, for $Z_1, Z_2 \in L_x$,

$$K_{12} = \frac{1}{2\pi i} \frac{F_1 F_2^{-1}}{\pi_1 \cdot \pi_2}$$

thus using equation (2.5) and ignoring certain multiples of $2\pi i$ (which can be absorbed into the definition of ϵ we can put

$$I[g,a] = \int_{\mathbb{M}\times\mathbb{CP}^1\times\mathbb{CP}^1} \mathrm{tr}(K_{21} g_1 K_{12} g_2)(\pi_1\cdot\pi_2)^4 \mathrm{D}\pi_1 \mathrm{D}\pi_2 \, \mathrm{d}^4 x \tag{2.8}$$

where $D\pi = \pi^{A'} \mathrm{d}\pi_{A'}$ and we use the fact that $\Omega_1 \wedge \Omega_2 = (\pi_1\cdot\pi_2)^2 \mathrm{d}^4 x \wedge D\pi_1 \wedge \mathrm{D}\pi_2$ as above.

The variation of K with respect to a is given by

$$\delta K_{12} = \int K_{13} \delta a_3 K_{32} \, \mathrm{D}\pi_3 \,.$$

We can use this to calculate the variation of $I[g,a]$ with respect to a and hence the Euler-Lagrange equation obtained by varying a in the action. This yields, after some manipulation,

$$\bar\partial_{a_3} g_3 = \epsilon \int_{\mathbb{CP}_1\times\mathbb{CP}_1} [K_{31} g_1 K_{13}, K_{32} g_2 K_{23}] (\pi_1\cdot\pi_2)^3 \pi_{1(A'}\pi_{2B')} \, \mathrm{D}\pi_1 \mathrm{D}\pi_2 \, \mathrm{d}^2 x^{A'B'}_{(0,2)} \,. \tag{2.9}$$

In this notation, the equation of motion from varying g (2.7) is

$$\bar{\partial} a_1 + a_1 \wedge a_1 = \epsilon \int_{\mathbb{CP}_1} K_{12} g_2 K_{21} (\pi_1 \cdot \pi_2)^2 \pi_{2A'} \pi_{2B'} \mathrm{D}\pi_2 \, \mathrm{d}^2 x^{A'B'}_{(0,2)} \,. \tag{2.10}$$

where in both the above two equations $\mathrm{d}^2 x^{A'B'}_{(0,2)}$ denotes the $(0,2)$-part of $\mathrm{d}^2 x^{A'B'} := \varepsilon_{AB} \mathrm{d}x^{AA'} \wedge \mathrm{d}x^{BB'}$ which is $\mathrm{d}^2 x^{A'B'}_{(0,2)} = \mathrm{d}^2 x^{C'D'} \hat{\pi}_{C'} \hat{\pi}_{D'} \pi^{A'} \pi^{B'} / (\pi \cdot \hat{\pi})^2$.

As a check on these equations, it is helpful to see how they can be solved in terms of the standard space-time data for a solution to the full Yang-Mills equations. Thus, let A be a connection 1-form on $\mathbb{M}$ for a solution to the full Yang-Mills equations and let $G_{A'B'}$ be the self-dual part of its curvature. The space-time field equations are

$$\partial^A_{(A'} A_{B')A} + A^A_{(A'} A_{B')A} = \epsilon G_{A'B'} \,, \qquad \nabla^{A'}_A G_{A'B'} = 0 \,,$$

where $\nabla_{AA'} = \partial_{AA'} + A_{AA'}$ is the gauge covariant derivative and is understood to act in the standard way on the adjoint representation. Using the standard Euclidean fibration $p : \mathbb{PT} \to \mathbb{M}$, we define a to be the $(0,1)$ part of the pull-back of A to $\mathbb{PT}$ and using Woodhouse (1985), we define

$$\begin{aligned} a &= \frac{1}{\pi \cdot \hat{\pi}} A_{AA'} \pi^{A'} \hat{\pi}_{B'} \mathrm{d}x^{AB'} \,, \\ g &= \frac{1}{(\pi \cdot \hat{\pi})^4} \left(3 G_{A'B'} \hat{\pi}^{A'} \hat{\pi}^{B'} \mathrm{D}\hat{\pi} + \nabla_{AA'} G_{B'C'} \hat{\pi}^{A'} \hat{\pi}^{B'} \hat{\pi}^{C'} \hat{\pi}_{D'} \mathrm{d}x^{AD'} \right) \,. \end{aligned} \tag{2.11}$$

It can now be checked that if the Yang-Mills equations hold, we have, in this gauge,

$$\begin{aligned} \bar{\partial} a + a \wedge a &= \epsilon G_{A'B'} \pi^{A'} \pi^{B'} \mathrm{d}^2 x_{(0,2)} \\ \bar{\partial}_a g &= \frac{\epsilon}{(\pi \cdot \hat{\pi})^2} \left(\hat{\pi}^{A'} \hat{\pi}^{B'} [G^{E'}_{A'}, G_{B'E'}] \mathrm{d}^2 x_{(0,2)} \right) \,, \end{aligned}$$

where $\mathrm{d}^2 x_{(0,2)}$ is the $(0,2)$-form with values in $\mathcal{O}(-2)$

$$\mathrm{d}^2 x_{(0,2)} := \frac{1}{(\pi \cdot \hat{\pi})^2} \varepsilon_{AB} \hat{\pi}_{A'} \hat{\pi}_{B'} \mathrm{d}x^{AA'} \wedge \mathrm{d}x^{BB'}$$

In this gauge, the matrix F can be taken to be the identity, and $K_{12} = 1/2\pi i(\pi_1 \cdot \pi_2)$ and equations (2.10) and (2.9) can be verified using equation (2.6).

We finally note that since both the action and field equations are invariant under the full group of gauge transformations on twistor space

$$(\bar{\partial}_a, g) \to (\bar{\partial}_{a'}, g') = (H^{-1} \bar{\partial}_a H, H^{-1}(g + \bar{\partial}_a \chi) H)$$

where H is an arbitrary smooth complex gauge transformation of the bundle $E \to \mathbb{PT}$ and χ an arbitrary smooth section of $E \otimes \mathcal{O}(-4)$ over $\mathbb{PT}$, given an arbitrary solution to equations (2.10,2.9), we can find a gauge transformation to a frame that is holomorphic on the fibres of $p : \mathbb{PT} \to \mathbb{M}$ (so that a vanishes on restriction to the fibres of p) and so that g is a harmonic representative on each of the fibres of p, see Woodhouse (1985). If, furthermore, (a, g) are solutions to (2.10,2.9) then we know from the previous subsection that they correspond to a solution of the Yang-Mills equations and that a and the vertical part of g have the form given above. It is then straightforward to see that the solution is precisely as given above up to a space-time gauge transformation.

We therefore see that the solutions to the Euler-Lagrange equations of the twistor action are in $1:1$ correspondence with gauge equivalence classes of solutions to the space-time Yang-Mills equations.

3. Twistor-string Yang-Mills generating functionals from the twistor action

The twistor-string formulae refer to a holomorphic Chern-Simons theory on super twistor space $\mathbb{PT}_s$ which is an appropriate subset of $\mathbb{CP}^{3|4}$. This space is obtained by introducing odd homogeneous twistor coordinates ψ_i, $i = 1, \ldots, 4$ in addition to the standard bosonic homogeneous coordinates Z^α so that $\mathbb{CP}^{3|4}$ is the space of non-zero $(Z^\alpha, \psi_i) \in \mathbb{C}^{4|4}$ modulo the equivalence relation $(Z^\alpha, \psi_i) \sim (\lambda Z^\alpha, \lambda\psi_i), \lambda \in \mathbb{C}^*$.

Chiral Super-Minkowski space $\mathbb{M}_s$ is then $\mathbb{R}^{4|8}$ with coordinates $(x^{AA'}, \theta^{iA'})$ and there is the incidence relation with supertwistor space given by

$$(\omega^A, \pi_{A'}, \psi_i) = (x^{AA'}\pi_{A'}, \pi_{A'}, \theta_i^{A'}\pi_{A'})\,. \tag{3.1}$$

Since we have stripped out all the superpartners except those of helicity ± 1, we set $\Psi = \psi_1\psi_2\psi_3\psi_4$ and this will be the supersymmetric quantity that we will use most.

In the following our expressions will be functions of two or more twistors, Z_1, Z_2, ... or π spinors, π_1, π_2, ... We will again adopt the convention that g_1 will denote a function of Z_1 and so on.

3.1 Supersymmetric D-instanton reformulation of twistor action

To make closer contact with twistor string formulae, we can consider the $(\pi_1 \cdot \pi_2)^4$ term in equations (2.8)to arise from a superspace integral using the identity

$$\int \Psi_1\Psi_2 \, \mathrm{d}^8\theta = (\pi_1 \cdot \pi_2)^4\,,$$

where $\Psi_1 = \Pi_{i=1}^4 \theta_i^{A'}\pi_{1A'}$ and similarly for Ψ_2. Therefore we can write:

$$I[g,a] = \int_{\mathbb{M}_s} \int_{L_{(x,\theta)} \times L_{(x,\theta)}} \mathrm{tr}(K_{21}\Psi_1 g_1 K_{12}\Psi_2 g_2)\, \mathrm{D}\pi_1 \mathrm{D}\pi_2\, \mathrm{d}^4x\, \mathrm{d}^8\theta$$

We now wish to reformulate the Green's functions K_{12} in terms of vacuum expectation values of fermion currents. We use a device introduced in Mason, Singer & Woodhouse (2002) in the context of the Ward construction for integrable systems. Introduce fermion spinor fields α and β (i.e., fields of homogeneity -1) on each L_x taking values in E and E^* respectively with action

$$S[\alpha, \beta] = \int_{L_{(x,\theta)}} \beta\bar{\partial}_a\alpha \wedge \mathrm{D}\pi$$

where $L(x,\theta)$ is the line in super twistor space given by holding (x,θ) fixed in equation (3.1). These will be the D-instantons of twistor-string theory. Then

$$K_{12} = \langle \alpha_1 \beta_2 \rangle$$

where $\langle \mathcal{O} \rangle$ denotes the vacuum expectation of the operator $\mathcal{O}$ associated to the quantum field theory of the fermions α and β on $\mathbb{CP}^1$. In the above we are taking α_1 and β_2 to be associated to the same line, $L(x,\theta)$; if they are taken to be associated to $L(x,\theta)$ and $L(x',\theta')$ respectively, we would have instead K_{12} if $(x,\theta) = (x',\theta')$ or zero otherwise—there are no singular terms in x. With this, we can express $I[g,a]$ as follows:

$$\begin{aligned} I[g,a] &= \int_{\mathbb{M}_s} \left\langle \int_{L_{(x,\theta)} \times L_{(x,\theta)}} \operatorname{tr}(J_{a1}\Psi_1 g_1) \operatorname{tr}(J_{a2}\Psi_2 g_2) \right\rangle \mathrm{d}^4x \mathrm{d}^8\theta \\ &= \int_{\mathbb{M}_s} \mathrm{d}^4x \mathrm{d}^8\theta \left\langle \left(\int_{L_{(x,\theta)}} \operatorname{tr}(J_a \Psi g) \right)^2 \right\rangle \end{aligned} \tag{3.2}$$

where $J_a =: \alpha\beta : \mathrm{D}\pi$ is the current associated to α and β (the :: denoting wick-ordering and the subscript a denotes dependence on a) and the subscript 1 or 2 denotes evaluation at a point of the first or second factor of $L(x,\theta) \times L(x,\theta)$.

3.2 A digression concerning generating functions for scattering amplitudes

We now wish to use this twistor form of the Yang-Mills action to show that the conjectured formulae for the generating functional $\mathcal{A}_{\mathrm{TS}}[a,g]$ for tree level QCD amplitudes from twistor string theory is the same as the generating functional $\mathcal{A}_{\mathrm{YM}}$ obtained from the standard Yang-Mills action. We first need to establish some basic facts about these generating functions.

We are concerned here with on-shell generating functionals rather than the more common off-shell generating functional, usually denoted $Z[J]$, which is a functional of a source usually denoted J which is an arbitrary function on space-time. Instead, both the twistor-string and and the standard Yang-Mills generating functionals $\mathcal{A}[a,g]$ are functionals of linearised free gluon fields, which we will represent by their on-shell twistor data (a,g) where on shell means that they satisfy the linearised equations at $\epsilon = 0$: a and g are therefore both linear dolbeault cohomology classes of homogeneity 0 and -4 respectively.

The $\mathcal{A}[a,g]$ are generating functionals in the sense that n-point scattering amplitudes are obtained as functionals of positive frequency linear fields by taking the n'th functional derivative of $\mathcal{A}_{\mathrm{TS}}$ with respect to (a,g) in the directions of the given positive frequency linear fields and evaluating at $(a,g) = (0,0)$. The fact that we are considering a functional only of positive frequency fields means that the generating functional generates diagrams with no incoming fields, just outgoing fields. This is sufficient as crossing symmetry then allows one to construct all other processes.

For a generic quantum field theory of a field theory for a field ϕ with action, say $S[\phi] = \frac{1}{2}(\partial_a\phi\partial^a\phi - m^2\phi^2) - \lambda V(\phi)$, such a generating functional would have the path-integral expression

$$\mathcal{A}[\phi] = \int \mathrm{D}\tilde{\phi} \, \exp \frac{i}{\hbar} S[\tilde{\phi}]$$

where the functional integration is understood to be over fields $\tilde{\phi}$ such that, as $t \to +\infty$, the negative frequency part of $\tilde{\phi} - \phi$ tends to zero, and as $t \to -\infty$, the positive frequency part of $\tilde{\phi} - \phi$ tends to zero, Faddeev & Slavnov (1991). Thus $\mathcal{A}[\phi]$ is the 'wave functional' of the theory.

Given the complications associated with defining even the perturbation series for a functional integral, it is difficult to deduce rigourously that the quantum theory will be correctly reproduced after a manipulation of the exponential of the action in the path integral. However, more can be said about the classical limit.

The classical limit $\mathcal{A}^{\mathrm{cl}}[\phi]$ generates all the tree diagrams. It can be obtained by first constructing the classical solution $\tilde{\phi}$ that is appropriately asymptotic to ϕ as above by iterating the integral form of the field equations to produce a sequence of fields ϕ_n such that

$$\phi_{n+1}(x) = \phi(x) + \lambda \int \Delta_F(x, x') V(\phi_n(x')) \mathrm{d}^4 x'$$

where Δ_F is the Feynman propagator that inverts $\partial_a \partial^a + m^2$ and $\phi_0 = \phi$. We can then define $\tilde{\phi} = \lim_{n\to\infty} \phi_n$ as a power series in λ. Then we have

$$\mathcal{A}^{\mathrm{cl}} = \exp \frac{i}{\hbar} S[\tilde{\phi}].$$

In such perturbative studies, the free field ϕ is taken to be a plane wave. Such fields are entire on complex Minkowski space (but are singular at infinity in the conformal compactification). It follows from the analyticity properties of the Feynman propagator that the corresponding solution $\tilde{\phi}$ can be analytically continued to the Euclidean section from the Minkowski signature section. We can therefore assume that all integrals are over the Euclidean section $\mathbb{M}$.

Following the above, the generating functional for tree level Yang-Mills amplitudes $\mathcal{A}_{\mathrm{YM}}[a, g]$ is given by

$$\mathcal{A}_{\mathrm{YM}}[a, g] = \exp \frac{i}{\hbar} \left(S_{\mathrm{asd}}[\tilde{a}, \tilde{g}] + \frac{\epsilon^2 \hbar}{2i} I[\tilde{g}, \tilde{a}] \right) , \tag{3.3}$$

where it is undertood that $(\tilde{a}, \tilde{g}) = \lim_{n\to\infty}(a_n, g_n)$ where

$$(a_{n+1}, g_{n+1}) = (a, g) + \bar{\partial}^{-1} \left(-a_n \wedge a_n + \epsilon \ldots, -a_n \wedge g_n + \epsilon \ldots \right)$$

where the ... denotes the terms on the right hand sides of equations (2.10) and (2.9).

We deduce from this that any formal manipulation of the exponential of the action that preserves the value it takes on solutions to the equations will give rise to the correct tree diagrams in perturbation theory. This will nevertheless be somewhat formal as, even at tree level, these series exhibit infrared divergences. Infrared divergences are a standard problem in quantum field theory with a number of standard resolutions and we will not consider them further.

3.3 The twistor string generating functionals

The conjectures for the twistor-string form of the generating functional are only confident in the classical approximation corresponding to tree diagrams in perturbation theory and rational (genus 0) curves in twistor space; we restrict attention to this classical limit here and omit the 'cl' superscript on $\mathcal{A}$ in the following. There is some evidence that the conjecture might be valid for the full quantum field theory (which would correspond to loops in perturbation theory and curves of higher genus in twistor space in twistor-string theory) but we will not be able to say anything definitive here.

In its simplest form, the generating functional is given by:

$$\mathcal{A}_{\mathrm{TS}}[a,g] = \sum_d \int_{\mathscr{M}_s^d} \mathrm{Det}(\bar{\partial}_{a+\epsilon\Psi g}|_C)\, \mathrm{d}\mu \tag{3.4}$$

Here $C \in \mathscr{M}_s^d$ where $\mathscr{M}_s^d$ is a totally real submanifold (or contour) in the space of connected degree-d rational curves in super-twistor space, ϵ is a small parameter used to expand about the self-dual sector of the theory and $\mathrm{d}\mu$ is a naturally defined measure on $\mathscr{M}_s^d$. This determinant has a standard functional integral representation

$$\mathcal{A}_{\mathrm{TS}}[a,g] = \sum_d \int_{\mathscr{M}_s^d} \int \mathrm{D}\alpha \mathrm{D}\beta \exp\left(\int_C \beta \bar{\partial}_{a+\epsilon\Psi g} \alpha \right) ,$$

where the functional integral is over the space of αs and βs which are fermionic spinors on each C with values in E and E^* respectively. These conjectured forms are only confidently expected to be valid for tree diagrams when we consider connected rational curves and a special normal form of the Dolbeault representatives for (a,g).

Here we will only make contact with the MHV diagram formulation of twistor-string theory, Cachazo, Svrcek and Witten (2004), in which instead of connected rational curves, we consider maximally disconnected rational curves so that each C is the union of d lines (degree 1 curves) in super-twistor space. In this approach, the d lines need to be connected into a tree by Chern-Simons propagators. It has been argued that this disconnected formulation is equivalent to the connected formulation by Gukov, Motl and Nietzke (2004).

To obtain a generating functional in the MHV diagram formulation, we write $C = \cup_{r=1}^d L(x_r, \theta_r)$ where (x_r, θ_r) are d points in super Minkowski space $\mathbb{M}_s$. The moduli space of such disconnected curves is therefore the d-fold product of super-Minkowski space $\mathbb{M}_s^d$. The generating function for this version of the theory needs to include the Chern-Simons action to give

$$\mathcal{A}_{\mathrm{TS}}[a,g] = \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{asd}}[\tilde{a},\tilde{g}]} \sum_d \int_{\mathbb{M}_s^d} \mathrm{d}\mu_d \int \mathrm{D}\alpha \mathrm{D}\beta\ \exp\left(\sum_{r=1}^d \int_{L(x_r,\theta_r)} \beta \bar{\partial}_{\tilde{a}+\epsilon\Psi\tilde{g}} \alpha \right) ,$$

where $\mathrm{d}\mu_d = \Pi_{r=1}^d \mathrm{d}^4 x_r \mathrm{d}^8 \theta_r$ and here, as in the previous subsection, $(\tilde{a}, \tilde{g})$ are understood to be the solutions to the classical field equations obtained by iterating the appropriate integral versions of the field equations with inhomogeneous terms given by (a,g).[4]

[4] In the main applications of twistor string theory (a,g) are taken to be the Penrose transform of plane waves which are entire on complex Minkowski space, but singular at infinity. Hence, (a,g) can be defined smoothly over the complement of the line in $\mathbb{PT}$ corresponding to the point at infinity in space-time. Wick rotation, using the analycity properties of the Feynman propagator and its counterpart on twistor space, can then be invoked to analytically continue the integrals over the Euclidean section. We will therefore assume that all integrals are over the Euclidean section $\mathbb{M}$.

Expanding this in ϵ, it is straightforward to see that the supersymmetric integrals over θ_r only give nontrivial contributions from terms in the expansion in which there are precisely two Ψs integrated over each set of θ_r. Thus $\mathcal{A}_{\rm TS} = \sum_d \epsilon^{2d} \mathcal{A}^d_{\rm TS}$ where

$$
\begin{aligned}
\mathcal{A}^d_{\rm TS} &= \mathrm{e}^{\frac{i}{\hbar}S_{\rm asd}[\tilde{a},\tilde{g}]} \int_{\mathbb{M}^d_s} \mathrm{d}\mu_d \int \mathrm{D}\alpha \mathrm{D}\beta\, \mathrm{e}^{\left(\sum_r \int_{L_{(x_r,\theta_r)}} \beta \bar{\partial}_{\tilde{a}} \alpha\right)} \frac{(2d)!}{2^d d!} \frac{\Pi_s \left(\int_{L_{(x_s,\theta_s)}} \beta \Psi \tilde{g} \alpha\right)^2}{(2d)!} \\
&= \mathrm{e}^{\frac{i}{\hbar}S_{\rm asd}[\tilde{a},\tilde{g}]} \int_{\mathbb{M}^d_s} \Pi^d_{r=1} \mathrm{d}^4 x_r \mathrm{d}^8 \theta_r \frac{\left\langle \left(\int_{L_{(x_r,\theta_r)}} \beta \Psi \tilde{g} \alpha\right)^2 \right\rangle}{2^d d!} \\
&= \mathrm{e}^{\frac{i}{\hbar}S_{\rm asd}[\tilde{a},\tilde{g}]} \frac{I[\tilde{g},\tilde{a}]^d}{2^d d!}\,,
\end{aligned} \tag{3.5}
$$

where the combinatorial factor in the first line comes from the number of choices of pairs of integrals over the ith copy of $\mathbb{M}_s$ over the $2d$ factors in the $2d$th term of the expansion of the exponential and in the second we have used the formula for $I[g,a]$ in equation (3.2). We can now resum over d to obtain

$$
\mathcal{A}_{\rm TS} = \exp \frac{i}{\hbar} \left(S_{\rm asd}[\tilde{a},\tilde{g}] + \frac{\epsilon^2 \hbar}{2i} I[\tilde{g},\tilde{a}] \right) . \tag{3.6}
$$

As can be seen, up to a redefinition of the expansion parameter, this gives rise to the classical Yang-Mills action generating functional as desired.

3.4 Extension to the full quantum field theory

The full quantum field theoretic generating functionals for Yang-Mills are expressed formally in terms of the path integral

$$
\mathcal{A}[a,g] = \int \mathrm{D}\tilde{a} \mathrm{D}\tilde{g} \exp \frac{i}{\hbar} \left(S_{\rm asd}[\tilde{a},\tilde{g}] + \epsilon I[\tilde{g},\tilde{a}] \right)
$$

It is clear that formally the expansion and resummation in (3.5)and (3.6) will be possible in the full quantum field theoretic path integral as in the generating functionals for tree diagrams discussed above to yield

$$
\mathcal{A}[a,g] = \int \mathrm{D}\tilde{a} \mathrm{D}\tilde{g} \mathrm{D}\alpha \mathrm{D}\beta\, \mathrm{e}^{\frac{i}{\hbar}S_{\rm asd}[\tilde{a},\tilde{g}]} \sum_d \int_{\mathbb{M}^d_s} \mathrm{d}\mu_d\, \mathrm{e}^{\sum_{r=1}^d \int_{L(x_r,\theta_r)} \beta \bar{\partial}_{\tilde{a}+\epsilon \Psi \tilde{g}} \alpha}
$$

However, in the full quantum field theory we would need to consider the gauge fixing. There is a useful Poincaré invariant twistor space gauge in which the $(0,1)$-forms (a,g) on twistor space are orthogonal to the fibres of the projection $\mathbb{PT} \to \mathbb{CP}^1$. This is particularly useful because it linearizes the Chern-Simons theory as $a \wedge a$ and $a \wedge g$ vanish identically. This is the gauge in which most of the twistor-string calculations have so far taken place. There is also a gauge that is adapted to the space-time description in which the cohomology classes are required to be harmonic up the fibres of the fibration $p : \mathbb{PT} \to \mathbb{M}$; this reduces (a,g) to the forms given in equations (2.11). In this latter gauge the quantum field theory will be equivalent to the standard space-time formulation because the Faddeev-Popov determinants will be independent of A and G. The task then is to use BRST to see that the quantum theory is the same in these two different gauges. In particular, we would like to see that all loops, and only Yang-Mills loops are obtained by some suitable interpretation or generalisation of equation (3.4) in which general algebraic curves of higher genus contribute to $\mathscr{M}_s^d$. We will discuss this problem in a subsequent paper.

4. Conformal gravity

Berkovits and Witten (2004) have analyzed the twistor-string formulation of conformal (super-)gravity. In this section we give the twistor construction, action and twistor-string reformulation for conformal gravity. This proceeds very much analogously to the corresponding ideas for Yang-Mills and so we will sketch the ideas relatively briefly in this section.

We take twistor space $\mathscr{PT}$ now to be a manifold diffeomorphic to $\mathbb{R}^4 \times S^2$ endowed with an almost complex structure $\mathcal{J}$, i.e., $\mathcal{J}$ is an endomorphism of the real tangent bundle satisfying $\mathcal{J}^2 = -1$. For the case of anti-self dual conformal gravity, Berkovits & Witten provide an analogue of the truncation of the Chern-Simons action on super twistor space which is a functional of the almost complex structure tensor $\mathcal{J}$, and a second tensor k.

As usual, one can use $\mathcal{J}$ to define subbundles $T^{(0,1)}$ and $T^{(1,0)}$ of the complexified tangent bundle $T_{\mathbb{C}}$ as the $-i$ and $+i$ eigenspaces of $\mathcal{J} : T_{\mathbb{C}} \to T_{\mathbb{C}}$ respectively and then define correspondingly subbundles $\Omega^{(p,q)}$ of the bundles Ω^{p+q} of complex differential forms. Similarly, ∂ and $\bar\partial$-operators can be defined as the projection of the exterior derivative d acting on sections of $\Omega^{p,q}$ onto $\Omega^{(p+1,q)}$ and $\Omega^{p,q+1}$ respectively. However, in general, we will have that $N := \bar\partial^2 \in \Omega^{(0,2)} \otimes T^{(1,0)}$ does not vanish.

In order for the ingredients to make contact with ordinary twistor theory, we must require that $\mathcal{J}$ is chosen so that canonical bundle $\Omega^{(3,0)}$ has Chern class -4 on the S^2 factors.

The tensor k is a section of $\Omega^{(1,1)} \otimes \Omega^{(3,0)}$. The twistor space Lagrangian for the anti-self dual field is

$$S[\mathcal{J}, k] = \int_{\mathscr{PT}} (\bar\partial^2, k)$$

where the pairing $(,)$ denotes both the contraction of the holomorphic $T^{(1,0)}$ index of $N = \bar\partial^2$ with the $\Omega^{(1,0)}$ index of k and the wedge product of the antiholomorphic form indices with each-other and the $\Omega^{(3,0)}$ indices. It is easily seen that the action is diffeomorphism invariant. Furthermore we have

Lemma 4.1 *The action $S[\mathcal{J}, k]$ is invariant under $k \to k + \bar\partial c$, where c is a compactly supported section of $\Omega^{(1,0)} \otimes \Omega^{(3,0)}$.*

Proof. This follows from an identity obeyed by N that arises as follows. In general the exterior derivative maps

$$\mathrm{d} : \Omega^{(p,q)} \to \Omega^{(p+2,q-1)} \oplus \Omega^{(p+1,q)} \oplus \Omega^{(p,q+1)} \oplus \Omega^{(p-1,q+2)} .$$

The map $\mathrm{d} : \Omega^{(p,q)} \to \Omega^{(p+2,q-1)}$ is given by contraction with the vector index of $-N$ and wedge product over the form indices which can we write as $\alpha \to -N\lrcorner \wedge \alpha$. The map $\mathrm{d} : \Omega^{(p,q)} \to \Omega^{(p-1,q+2)}$ is similarly determined by $\bar{N}$. It is a consequence of $\mathrm{d}^2 = 0$ that for $\alpha \in \Omega^{(1,0)}$, $\bar\partial(N\lrcorner\,\alpha) - N\lrcorner \wedge \bar\partial\alpha = 0$. We can therefore see that if $k = \bar\partial(c \otimes \nu)$, where ν is a section of $\Omega^{(3,0)}$ and c a section of $\Omega^{(1,0)}$, then we have

$$(N, \bar\partial(c \otimes \nu)) = (N\lrcorner \wedge \bar\partial c) \wedge \nu + (N\lrcorner\, c) \wedge \bar\partial\nu = \mathrm{d}(N\lrcorner\, c \wedge \nu)$$

and this implies the appropriate gauge invariance.□

The field equations from this action are that $\bar\partial^2 = 0$, i.e., that $\mathscr{J}$ should be integrable and $\bar\partial k = 0$. Given the gauge invariance, k defines an element of $H^1(\mathscr{PT}, \Omega^{(1,0)} \otimes \Omega^{(3,0)})$. The standard nonlinear-graviton construction, Penrose (1976), applied to $\mathscr{PT}$ constructs a complex 4-manifold $\mathscr{M}$ with holomorphic conformal structure $[g]$ that has anti-self-dual Weyl curvature. $\mathscr{M}$ is the space of rational curves in $\mathscr{PT}$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (this requires either the existence of one rational curve in $\mathscr{PT}$ with normal bundel $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or that $\mathscr{J}$ be close to the standard complex structure on a neighbourhood of a line in $\mathbb{CP}^3$). The Penrose transform for k leads to a self-dual spinor $K_{A'B'C'D'}$ obeying the equation

$$(\nabla_A^{A'} \nabla_B^{B'} + \Phi_{AB}^{A'B'}) K_{A'B'C'D'} = 0\,, \tag{4.1}$$

which are the linearised self-dual conformal gravity equations with $K_{A'B'C'D'}$ playing the role of an infinitesimal self-dual Weyl spinor on the anti-self-dual background. We will see how this can be done explicitly in a somewhat more general context later. We note, following Atiyah, Hitchin and Singer (1978), that $\mathscr{M}$ admits a real slice M on which the conformal structure has Euclidean signature iff $\mathscr{PT}$ admits a conjugation $\hat{} : \mathscr{PT} \to \mathscr{PT}$ (i.e., it reverses the sign of $\mathscr{J}$) with no fixed points. The real space-time M is then the space of rational curves that are left invariant by the conjugation.

Following Berkovits & Witten, we note that the above action corresponds to the space-time action

$$S_{\mathrm{asd}}[g, K] = \int_M \psi^{A'B'C'D'} K_{A'B'C'D'} \mathrm{d}^4 x\,.$$

for anti-self-dual conformal gravity on space-time where g is a conformal structure and $\psi_{A'B'C'D'}$ is its self-dual Weyl spinor. The field equations implies the vanishing of $\psi_{A'B'C'D'}$ and equation (4.1) for $K_{A'B'C'D'}$.

4.1 The extension to non anti-self-dual fields

This ASD space-time action can be extended to full conformal gravity if we include the term

$$I[g, K] = \frac{\epsilon}{2} \int_M K^{A'B'C'D'} K_{A'B'C'D'} \mathrm{d}^4 x\,.$$

We will reformulate this on twistor space in terms of k to obtain $I[k, \mathscr{J}]$. We will take our integral to be that of a product of k_1 and k_2 over an 8-dimensional contour in $\mathscr{PT} \times \mathscr{PT}$. We must first develop the Penrose non-linear graviton construction in the case that the complex structure $\mathscr{J}$ is not integrable in order to define the ingredients that we will need.

We first introduce a conjugation $\hat{}\ : \mathscr{PT} \to \mathscr{PT}$, $\hat{}^{\,2} = 1$ that reverses $\mathcal{J}$, i.e., $\hat{}^*\mathcal{J} = -\mathcal{J}$. There are two types of such conjugations normally employed in twistor theory, depending on whether the conjugation has fixed points in twistor space or not. The latter case leads to Euclidean signature on space-time and we will assume that to be the case hereon.

We now consider the moduli space $\mathscr{M}$ of pseudo-holomorphic rational curves in $\mathscr{PT}$, i.e., the space of embedded S^2s in $\mathscr{PT}$ in the same topological class as the S^2 factors in $\mathscr{PT} = \mathbb{R}^4 \times S^2$, such that $\mathcal{J}$ leaves the tangent space invariant inducing a complex structure thereon. Theorems in McDuff and Salamon (2004) imply that $\mathscr{M}$ exists and is 8-dimensional if $\mathcal{J}$ is close to the standard complex structure on a neighbourhood of a line in $\mathbb{CP}^3$ (and we will assume this to be the case hereon). This follows from the ellipticity of the equations defining such a $\mathcal{J}$-holomorphic curve and the index theorem applied to its linearization. The conjugation $\hat{}$ induces a conjugation $\hat{}\ : \mathscr{M} \to \mathscr{M}$, $\hat{}^{\,2} = 1$ and we define M to be the (4-dimensional) fixed point set of $\hat{}$ on $\mathscr{M}$.

We take M to be our candidate space-time and we will have a projection $p : \mathscr{PT} \to M$ as a consequence of the fact that, with our assumptions, there will be a unique rational curve in $\mathscr{PT}$ through Z and $\hat{Z}$. The fibres of p are, by construction, Riemann spheres, $\mathbb{CP}^1$.

We define $\mathscr{T}$ to be the total space of the line bundle $\left(\Omega^{(3,0)}\right)^{1/4}$ (this 4th root exists as a consequence of our assumptions on the topology of $\mathscr{PT}$ and $\mathcal{J}$, in particular that $\Omega^{(3,0)}$ has Chern class -4). Since $\Omega^{(3,0)}$ is an almost complex line bundle, its total space and its powers are almost complex, so that $\mathscr{T}$ has an almost complex structure. We denote the complex line bundles $\left(\Omega^{(3,0)}\right)^{-n/4}$ by $\mathcal{O}(n)$. On restriction to each $\mathbb{CP}^1$ fibre, $\mathscr{T}$ will be a line bundle of degree -1 and is hence the tautological bundle on each $\mathbb{CP}^1$ fibre of p. Let $\tilde{p} : \mathscr{T} \to M$ denote the projection induced by p. The fibres of $\tilde{p}$ minus the zero section are canonically identifiable with the complement of the zero-section in a rank two vector bundle with structure group SU(2) over M and, with an abuse of notation, we will think of $\mathscr{T}$ as being this complex rank two vector bundle. Introduce linear coordinates $\pi_{A'}$, $A' = 0', 1'$ on the fibres of $\tilde{p}$. Define the Euler homogeneity operator $\Upsilon = \pi_{A'}\partial/\partial\pi_{A'}$.

We choose a frame for $\Omega^{(1,0)}(\mathscr{T})$ as follows. Choose 1-forms $D\pi_{A'}$ in $\Omega^{(1,0)}(\mathscr{T})$ of homogeneity degree 1 in $\pi_{A'}$, $\mathcal{L}_\Upsilon D\pi_{A'} = D\pi_{A'}$, and so that on restriction to the fibres of $\tilde{p}$, $D\pi_{A'} = \mathrm{d}\pi_{A'}$. In order to achieve this, in general $D\pi_{A'}$ will have to have non-holomorphic dependence on $\pi_{A'}$. The 1-form $\mathrm{D}\pi := \pi^{A'}\mathrm{D}\pi_{A'}$ descends to $\mathscr{PT}$ to give a 1-form with values in $\mathcal{O}(2)$. At each point we can find a pair of complex 1-forms, θ^A, $A = 0, 1$ homogeneous degree 1 in $\pi_{A'}$, such that θ^A are orthogonal to the fibres of p and to $T^{(0,1)}$. The condition that $\Omega^{(3,0)} = \mathcal{O}(-4)$ by definition means that we have a canonical section Ω of $\Omega^{(3,0)} \otimes \mathcal{O}(4)$. Thus, since θ^A are sections of $\mathcal{O}(1) \otimes \Omega^{(1,0)}$ we can also require $\Omega = \theta^0 \wedge \theta^1 \wedge \pi_{A'}\mathrm{d}\pi^{A'}$. Such θ^A can be chosen to be global and non-vanishing on $\mathscr{PT}$. This gives our basis $\theta^\alpha = (\theta^A, D\pi_{A'})$ of $\Omega^{(1,0)}$.

We can now study the Penrose transform of k by setting

$$k = (k^{A'} \wedge \mathrm{D}\pi_{A'} + k_A \wedge \theta^A) \otimes \Omega$$

where $k^{A'}$ and k_A are $(0, 1)$-forms of homogeneity degree -5. Note that $\Upsilon \lrcorner\, k = 0$ so

$k^{A'}\pi_{A'} = 0$ so that $k^{A'} = \pi^{A'}\varkappa$ for some $(0,1)$-form $\varkappa$ with values in $\mathcal{O}(-6)$. We can now finally define the indexed 2-form $K^{A'}_{B'}$ on M by

$$K^{A'}_{B'}(x) = \int_{L_x} \pi_{B'} k^{A'} \wedge \Omega = \int_{L_x} \pi_{B'}\pi^{A'}\varkappa \wedge \Omega \, .$$

Clearly $K^{A'}_{A'} = 0$. We then define

$$I[k, \mathcal{J}] = \int_M K^{A'}_{B'} \wedge K^{B'}_{A'} \, .$$

This can be expressed directly in terms of k and $\varkappa$ as follows. Let $\mathscr{PT} \times_M \mathscr{PT}$ be the 8-dimensional space which fibres over M with fibre $\mathbb{CP}^1 \times \mathbb{CP}^1$, the cartesian product of two copies of the fibre of $p : \mathscr{PT} \to M$. This has two projections p_1 and p_2 onto $\mathscr{PT}$, one on each factor. Let $k_1 = p_1^* k$ and $k_2 = p_2^* k$ and similarly $\varkappa_1 = p_1^*\varkappa$ etc.. Our integral, then, is

$$I[k, \mathcal{J}] = \int_{\mathscr{PT}\times_M \mathscr{PT}} (\pi_1 \cdot \pi_2)^2 \varkappa_1 \wedge \varkappa_2 \wedge \Omega_1 \wedge \Omega_2 \, .$$

where $(\pi_{1A'}, \pi_{2B'})$ are homogeneous coordinates on the $\mathbb{CP}^1 \times \mathbb{CP}^1$ fibres of $\mathscr{PT} \times_M \mathscr{PT} \to M$. This integral is invariant under $k \to k + \bar\partial l$ since this induces a variation of the integrand in $I[k, \mathcal{J}]$ that is exact on the fibres of p_1 and p_2.

Proposition 4.1 *Solutions to the classical field equations up to diffeomorphism arising from the action $S_T[\mathcal{J}, k] = S_{asd}[\mathcal{J}, k] - \frac{\epsilon}{2} I[k, \mathcal{J}]$ on twistor space are in one to one correspondence with solutions to the conformal gravity equations up to diffeomorphism.*

Proof. We proceed as before for Yang-Mills and focus on the field equation that arises from varying k. Varying first with respect to k_A we find

$$\bar\partial^2 \lrcorner\, \theta^A = 0.$$

Varying $k^{A'}$ (or equivalently $\varkappa$) we obtain:

$$(\bar\partial^2 \lrcorner\, \mathrm{D}\pi)|_Z = \int_{p^{-1}(p(Z))} \pi^{B'}\pi_{A'}\pi^1_{B'} k_1^{A'} \wedge \Omega_1 = \pi_{A'}\pi^{B'} K^{A'(0,2)}_{B'} \qquad (4.2)$$

where the subscript $(0, 2)$ on a 2-form denotes projection onto the $(0, 2)$ part.

In particular, the (0,2)-form part of $\bar\partial^2$ annihilates vertical vectors. The normal bundle to the fibre L_x of p over x is therefore a holomorphic vector bundle on L_x. On each L_x, θ^A can therefore be chosen uniquely up to a fibrewise global $\mathrm{GL}(2, \mathbb{C})$ action on the A index to be holomorphic. The topological assumption that the canonical bundle of twistor space is $\mathcal{O}(-4)$ means that the normal bundle should have first Chern class 2, so that it has generic splitting type $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and must be isomorphic to this with our assumption that $\mathcal{J}$ is close to the standard one; θ^A can be defined to be this isomorphism. Since θ^A is global and holomorphic in $\pi_{A'}$ with homogeneity degree 1, there exists 1-forms $\theta^{AA'}$ on M such that $\theta^A = \theta^{AA'}\pi_{A'}$. This yields a conformal structure

$$\mathrm{d}s^2 = \varepsilon_{AB}\varepsilon_{A'B'}\theta^{AA'}\theta^{BB'} \, .$$

Similarly, $\mathrm{D}\pi_{A'}$ can be chosen to be holomorphic up the fibres and chosen globally up to a freedom $\mathrm{D}\pi_{A'} \to \mathrm{D}\pi_{A'} + \gamma_{AA'}\theta^A$ for some $\gamma_{AA'}$ that depends only on x. Equation (4.2) implies the vanishing of the conformally invariant part of the torsion of the connection determined by the horizontal subspaces defined by $\mathrm{D}\pi_{A'}$.

We can now see that M is a manifold with Riemmannian conformal structure and that $\mathscr{PT} \to M$ is its projective spin bundle with the standard twistorial almost complex structure as in Atiyah Hitchin and Singer (1978). For this almost complex structure, it is standard that

$$\bar{\partial}^2 = \pi_{A'}\psi^{A'}_{B'C'D'}\varepsilon_{CD}\theta^{CC'} \wedge \theta^{DD'}\frac{\partial}{\partial \pi_{B'}}$$

where $\psi_{A'B'C'D'}$ is the self-dual Weyl-spinor. Thus the field equation obtained by varying k implies that

$$\psi_{A'B'C'D'} = \epsilon K_{A'B'C'D'}$$

We can now see that the action reduces to the space-time action $S_{\mathrm{asd}}[g, K] + I[g, K]$ and so this action is in fact equivalent to that for conformal gravity.□

We note that in the above, the diffeomorphism freedom appropriate to full twistor space is broken to the diffeomeorphism freedom on space-time, together with the automorphisms of the bundle of self-dual spinors. In order to similarly reduce the gauge freedom for k, we can require that, on each fibre of p, it is a harmonic representative of the restriction of the cohomology class. As before, an explicit expression for k can be given in this gauge that satisfies the field equations arising from the twistor action when the conformal structure has vanishing Bach tensor.

4.2 Reformulation on supertwistor space

As in the Yang-Mills case, we can rewrite the $I[g, K]$ term in terms of an integral over M and the two copies of the fibre L_x of $\mathscr{PT} \to M$.

$$I[k, \mathcal{J}] = \int_{\mathscr{PT}\times_M \mathscr{PT}} (\pi_1 \cdot \pi_2)^4 \varkappa_1 \wedge \varkappa_2 \wedge \mathrm{D}\pi_1 \wedge \mathrm{D}\pi_2 \wedge \mathrm{d}^4x \tag{4.3}$$

As before, we can introduce $N = 4$ super-twistor space $\mathscr{PT}_s$ and its correspondence with super space-time, M_s as follows. Let ψ_i, $i = 1, \ldots, 4$ be anticommuting variables on the supersymmetric twistor space $\mathscr{PT}_s$ with values in $\mathcal{O}(1)$, and let $\theta_i^{A'}$ be the corresponding anti-commuting coordinates on the super space-time M_s with incidence relation $\psi_i = \pi_{A'}\theta_i^{A'}$. As before set $\Psi = \psi_1\psi_2\psi_3\psi_4$ and $\Omega^s = \Omega \wedge \mathrm{d}\psi_1 \wedge \ldots \wedge \mathrm{d}\psi_4$. Using again the relation $\int \mathrm{d}^8\theta\ \Psi_1\Psi_2 = (\pi_1 \cdot \pi_2)^4$ we can write

$$\begin{aligned}
I[k, \mathcal{J}] &= \int_{\mathscr{PT}_s\times_M \mathscr{PT}_s} (\pi_1 \cdot \pi_2)^{-2}(\Psi_1\varkappa_1) \wedge (\Psi_2\varkappa_2) \wedge \Omega_1^s \wedge \Omega_2^s \\
&= \int_{M_s\times L(x,\theta)\times L(x,\theta)} (\Psi_1\varkappa_1 \wedge \mathrm{D}\pi_1) \wedge (\Psi_2\varkappa_2 \wedge \mathrm{D}\pi_2) \wedge \mathrm{d}^4x \wedge \mathrm{d}^8\theta \\
&= \int_{M_s\times L(x,\theta)\times L(x,\theta)} (\Psi_1 k) \wedge (\Psi_2 k) \wedge \mathrm{d}^4x \wedge \mathrm{d}^8\theta \\
&= \int_{M_s} \mathrm{d}^4x\mathrm{d}^8\theta \left(\int_{L(x,\theta)} \Psi k\right)^2
\end{aligned} \tag{4.4}$$

where the second last identity follows simply from the fact that $k|_{L(x,\theta)} = \varkappa \wedge \mathrm{D}\pi$ and we now think of $k \in \Omega^{(1,0)}(-4)$ rather than as a 1-form with values in $\Omega^{(3,0)}$.

4.3 Twistor-string theory for conformal gravity

For simplicity we work with formal path integral formulae. Following the logic of §3 backwards now, we start with the twistor version of the path integral for conformal gravity and work towards a formulation along the lines of equation (3.4).

We have that the generating functional for conformal gravity scattering in terms of the twistor Lagrangians is

$$\mathcal{A}[\mathcal{J}, k] = \int \mathrm{D}\tilde{\mathcal{J}}\, \mathrm{D}\tilde{k}\; e^{S_{\mathrm{asd}}[\tilde{\mathcal{J}},\tilde{k}] - \frac{\epsilon^2}{2} I[\tilde{k},\tilde{\mathcal{J}}]}$$

where again the path integral is over fields $\tilde{\mathcal{J}}$ and $\tilde{k}$ that are suitably aymptotic to $\mathcal{J}$ and k.

We can manipulate this as before to obtain:

$$\begin{aligned}
\mathcal{A}[\mathcal{J}, k] &= \int \mathrm{D}\tilde{\mathcal{J}}\, \mathrm{D}\tilde{k}\; e^{S_{\mathrm{asd}}[\tilde{\mathcal{J}},\tilde{k}]} \sum_{d=0}^{\infty} \frac{\epsilon^{2d} I[\tilde{k}, \tilde{\mathcal{J}}]^d}{2^d d!} \\
&= \int \mathrm{D}\tilde{\mathcal{J}}\, \mathrm{D}\tilde{k}\; e^{S_{\mathrm{asd}}[\tilde{\mathcal{J}},\tilde{k}]} \sum_{d=0}^{\infty} \frac{\epsilon^{2d}}{2^d d!} \int_{M_s^d} \Pi_{r=1}^d \mathrm{d}^4 x_r \mathrm{d}^8 \theta_r \left(\int_{L(x_r,\theta_r)} \Psi_r \tilde{k}_r \right)^2 \\
&= \sum_{d=0}^{\infty} \int_{M_s^d} \Pi_{r=1}^d \mathrm{d}^4 x_r \mathrm{d}^8 \theta_r \int \mathrm{D}\tilde{\mathcal{J}}\, \mathrm{D}\tilde{k}\; e^{S_{\mathrm{asd}}[\tilde{\mathcal{J}},\tilde{k}] + \epsilon \sum_{r=1}^d \int_{L(x_r,\theta_r)} \Psi \tilde{k}}
\end{aligned} \tag{4.5}$$

This yields the coupling of the D1 instantons, $L(x_r, \theta_r)$, to the 1-form Ψk precisely as proposed in Berkovits and Witten (2004).

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