An ambitwistor Yang-Mills Lagrangian

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Abstract

We introduce a Chern-Simons Lagrangian for Yang-Mills theory as formulated on ambitwistor space via the Ward, Isenberg, Yasskin, Green, Witten construction. The Lagrangian requires the selection of a codimension-2 Cauchy-Riemann submanifold which is naturally picked out by the choice of space-time reality structure and we focus on the choice of Euclidean signature. The action is shown to give rise to a space-time action that is equivalent to the standard one, but has just cubic vertices. We identify the ambitwistor propagators and vertices and work out their corresponding expressions on space-time and momentum space. It is proposed that this formulation of Yang-Mills theory underlies the recursion relations of Britto, Cachazo, Feng and Witten and provides the generating principle for twistor diagrams for gauge theory.

1 Introduction

Ambitwistor space $\mathbb{A}$ is the space of complex null geodesics in complexified Minkowski space. It has complex dimension five and can be represented as the quadric $Z \cdot W = 0$ inside $\mathbb{PT} \times \mathbb{PT}^*$ where Z are homogeneous coordinates on projective twistor space $\mathbb{PT} = \mathbb{CP}^3$ and W are homogeneous coordinates on its dual $\mathbb{PT}^*$. It has been known for many years that it is possible to reformulate 4-dimensional Yang-Mills fields onto ambitwistor space via a generalization of the Ward transform. A Yang-Mills connection on space-time is encoded into a holomorphic vector E bundle over some subset of $\mathbb{A}$. The Yang-Mills equations can be expressed as the condition that the holomorphic vector bundle $E \to \mathbb{A}$ extends to a certain 'third formal neighbourhood' of the natural embedding of $\mathbb{A}$ into $\mathbb{PT} \times \mathbb{PT}^*$, Isenberg *et. al.* (1978). The construction can be stated more elegantly for $\mathcal{N} = 3$ super-Yang-Mills fields since the field equations hold automatically as a consequence of integrability along super light rays. Such super-Yang-Mills fields correspond to holomorphic vector bundles over $\mathcal{N} = 3$ super-ambitwistor space $\mathbb{A}_{[3]} = \{Z \cdot W + \xi \cdot \eta = 0; ([Z, \xi], [W, \eta]) \in \mathbb{PT}_{[3]} \times \mathbb{PT}^*_{[3]}\}$, where $\mathbb{PT}_{[3]} = \mathbb{CP}^{3|3}$ is $\mathcal{N} = 3$ super-twistor space, Witten (1978), Harnad et. al. (1985) and Manin (1988) (and Ferber (1978) for super-twistors).

These constructions have not, so far, been particularly useful as a tool for studying solutions to the full Yang-Mills equations. However, there has been recent interest arising from progress in twistor-string theory, Witten (2004), and its spin-offs in perturbative gauge theory, see Cachazo & Svrcek (2005) for a review. In addition to twistor-string theory in twistor space, a twistor-string theory in ambitwistor space was also briefly proposed in Witten

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(2004), and a number of authors, Aganagic & Vafa (2004), Neitzke & Vafa (2004) and Kumar & Policastro (2004) have argued that there should be a mirror symmetry relation between the string theories in twistor space and in ambitwistor space. Furthermore, recurrence relations for tree-level perturbative QCD scattering amplitudes were discovered by Britto *et. al.* (2005). There it was proposed that, since the recursion allowed one to generate arbitrary tree-level amplitudes from trivalent ones in an ambidextrous way, the relations might be understood as arising from a twistor-string theory in ambitwistor space.

Whilst it is expected that such a twistor-string theory should be equivalent to a holomorphic Chern-Simons theory on (super) ambitwistor space, it has been unclear as to how to formulate such a theory. The Chern-Simons form wedged against the natural super-Calabi-Yau form only yields an $(8|6)$-form, whereas the space is $(10|6)$ dimensional. Here we get around this problem by restricting to a naturally defined 8-dimensional Cauchy-Riemann (CR) submanifold $\mathbb{A}_{\mathbb{E}}$ of ambitwistor space $\mathbb{A}$ consisting of those complex null geodesics that intersect a given real slice, here taken to be the Euclidean slice $\mathbb{E}$. Such a Cauchy-Riemann (CR) manifold has a naturally defined analogue $\bar{\partial}_B$ of the $\bar{\partial}$-operator and associated $\bar{\partial}_B$-Dolbeault cohomology, and the cohomology is subject to the standard Penrose-transform with fields on space-time. Similarly one can define CR vector bundles over $\mathbb{A}_{\mathbb{E}}$ and, subject to topological triviality on the fibres, these will have a Ward correspondence with gauge fields on $\mathbb{E}$.

An analytic CR vector bundle on $\mathbb{A}_{\mathbb{E}}$ naturally has a unique extension to a holomorphic vector bundle on a full neighbourhood of this submanifold in ambitwistor space, so this subspace is sufficient to determine the full ambitwistor theory for analytic fields. However, when the fields are non-analytic, as will generically be the case off shell or when the field equations are satisfied in Lorentz or split signature, there will not be any extension and the ambitwistor theory must necessarily restrict to one on this 8-dimensional submanifold.

We give a holomorphic Chern-Simons Lagrangian for a $\bar{\partial}_B$-operator on a bundle E over the supersymmetric $\mathbb{A}_{[3]\mathbb{E}}$. This can be extended straightforwardly to a holomorphic Lagrangian on the CR analogue $\mathbb{PT}_{[3]} \times_{\mathbb{E}} \mathbb{PT}^*_{[3]}$ of $\mathbb{PT}_{[3]} \times \mathbb{PT}^*_{[3]}$ for $\mathbb{A}_{[3]\mathbb{E}}$. We also give the corresponding Lagrangians in the non-supersymmetric case and it is these Lagrangians that we spend most time analyzing in this paper.

The Lagrangians gives a mechanism for writing down a perturbation expansion for Yang-Mills theory involving the ambitwistor version of the fields. In this perturbative context, the on-shell linearised fields can be understood as arising from standard twistor and dual twistor cohomology classes. We give a preliminary examination of the ingredients of the Feynman rules: in particular we give formulae for the propagators and vertices on ambitwistor space and their transforms to space-time and momentum space. We note that, being based on Chern-Simons theory, its Feynman diagram expansion has only trivalent vertices and this suggests that this is indeed the expansion responsible for the BCFW recursion procedure that is generated by trivalent vertices. It is likely that the Feynman diagrams for this action will lead to a generating priniciple for the twistor diagram approach to scattering amplitudes as developed in Hodges (2005).

In §2 we first review the standard results for reformulating Yang-Mills theory on ambitwistor space. In §3 we introduce the CR submanifold $\mathbb{A}_{\mathbb{E}}$ and discuss the basic CR manifold theory, Ward transform and geometry. In §4 we introduce the ambitwistor gauge theory Lagrangian and explain how it fits in with the standard results. In §5 we give a systematic tranform of the action to space-time and show that the action is in fact equivalent to the following gauge theory action

$$S[A,G] = \int_{\mathbb{E}} \mathrm{tr}(F \wedge G - \frac{\epsilon}{2} G \wedge {}^*G) \tag{1}$$

where A is a Yang-Mills connection, F its curvature, G a Lie algebra-valued 2-form and $\epsilon = 1/g^2$ where g is the coupling constant. The equations of motion are $\epsilon G = {}^*F$ and $DG = 0$ and so it gives rise to the standard field equations. This gives a space-time explanation for how we can get away with a Feynman diagram expansion using only trivalent vertices. Note that it is an ambidextrous analogue of the Chalmers & Siegel (1996) Lagrangian that bears a close relationship with twistor string theory in twistor space, see Witten (2004), Mason (2005). In §6 we examine the perturbation theory arising from the ambitwistor Lagrangian. We obtain expressions for the free field inner products, twistor transform, propagators and vertices and their transforms to position space and momentum space. The position and momentum space formulae for propagators and vertices are those from the Lagrangian (1).

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2 The standard ambitwistor construction

An analytic Yang-Mills connection on a region in some real slice of complex Minkowski space can be analytically continued to a connection A on a vector bundle $E' \to U$ where U is a complex Stein neighbourhood[1] of the given region in complex Minkowski space $\mathbb{CM}$.

Let $\mathbb{A}_U$ be the subset of ambitwistor space consisting of complex null geodesics with non-trivial intersection with U. We construct a holomorphic vector bundle $E \to \mathbb{A}_U$ by defining the fibre E_l of E at $l \in \mathbb{A}_U$ to be the vector space of covariantly constant sections of E' along the corresponding null geodesic. We have:

Theorem 1 (Witten (1978), Isenberg, Yasskin, Green (1978)) *The bundle $E \to \mathbb{A}_U$ determines and is determined by A. Furthermore any such holomorphic vector bundle E with trivial first and second Chern classes determines a bundle $E' \to U$ with connection A.*

Briefly, the reconstruction works by defining $E' \to U$ to be the bundle whose fibre at $x \in U$ is the vector space of global sections of E over the corresponding quadric Q_x of null directions in $\mathbb{A}$. We can define parallel propagation along a light ray l from x to y by identifying E'_x with E_l where $l \in Q_x$ and then to E'_y since $l \in Q_y$ also. It turns out that this definition of parallel propagation arises from a connection on U.

In order to express the field equations we need to consider the natural embedding of ambitwistor space as a quadric hypersurface in $\mathbb{PT} \times \mathbb{PT}^*$ where $\mathbb{PT}$ is projective twistor space, the projectivisation of twistor space $\mathbb{T} \equiv \mathbb{C}^4$, and $\mathbb{PT}^*$ is dual twistor space, the projectivisation of $\mathbb{T}^*$. If (Z^α, W_β), $\alpha, \beta = 0, \ldots, 3$ are homogeneous coordinates on $\mathbb{PT}\times\mathbb{PT}^*$, then $\mathbb{A}$ is the subset $Z^\alpha W_\alpha = 0$.

The field equations are then expressed as follows:

Theorem 2 (Witten (1978), Isenberg, Yasskin, Green (1978)) *The Yang-Mills connection A satisfies the Yang-Mills equations iff the bundle E admits an extension to a third order formal neighbourhood $\mathbb{A}_{(3)}$ of $\mathbb{A}$ in $\mathbb{PT} \times \mathbb{PT}^*$.*

There are a number of ways of saying what this means explicitly. In the following, this will take this to mean that there exists a smooth bundle E over $\mathbb{PT}\times\mathbb{PT}^*$ with smooth $\bar\partial$-operator $\bar\partial_a$ satisfying $\bar\partial_a^2 = O((Z \cdot W)^4)$ at $\mathbb{A}$.

There is also a formulation for super Yang-Mills. Super-twistor space $\mathbb{PT}_{[3]}$ is

$$\mathbb{CP}^{3|3} = \mathbb{C}^{4|3}/(Z, \xi) \sim (\lambda Z, \lambda\xi) \qquad \lambda \in \mathbb{C}^*$$

and here ξ^i, $i = 1, 2, 3$ are the odd coordinates with Z as before. Similarly (W, η) are homogeneous coordinates on $\mathbb{PT}^*_{[3]}$ and

$$\mathbb{A}_{[3]} = \{((Z, \xi), (W, \eta)) \in \mathbb{PT}_{[3]} \times \mathbb{PT}^*_{[3]} | Z \cdot W + \xi \cdot \eta = 0\}.$$

We have

[1] we can and will require that the intersection of every complex null geodesic with U be connected and simply connected.

Theorem 3 (Witten (1978)) *The field equations for $\mathcal{N} = 3$ super Yang-Mills is equivalent to the condition that the connection is integrable along super light-rays and this is equivalent to the existence of a transform to a holomorphic vector bundle E over $\mathcal{N} = 3$ super-ambitwistor space $\mathbb{A}_{[3]}$. Conversely, such a bundle determines a super Yang-Mills connection satisfying the integrability along super light rays, and hence the constraints and hence the field equations.*

We note that the full details of this construction are quite complicated, see Harnad *et. al.* (1985), and for the most part we will restrict attention to the non-supersymmetric version.

3 The CR ambitwistor space $\mathbb{A}_{\mathbb{E}}$ for Euclidean space

Let $\mathbb{E} = \mathbb{R}^4$ be real affine Euclidean 4-space inside complex Minkowski space. Define the 8 dimensional submanifold $\mathbb{A}_{\mathbb{E}}$ of $\mathbb{A}$ to be the space of complex null geodesics that intersect $\mathbb{E}$. $\mathbb{A}_{\mathbb{E}}$ is naturally fibred over $\mathbb{E}$, $p : \mathbb{A}_{\mathbb{E}} \to \mathbb{E}$ since null geodesics can only intersect $\mathbb{E}$ in one point. The fibres are the space of complex null directions at the point which is a complex 2-quadric $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$ in the projectivised complexified tangent space. Thus, topologically, $\mathbb{A}_{\mathbb{E}} = \mathbb{R}^4 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. We give an explicit coordinatisation below. Had we chosen a real slice of complex Minkowski space of Lorentzian or split signature, the picture would not be so simple as some complex geodesics intersect the real slice in more than just one point and the corresponding points in $\mathbb{A}_{\mathbb{M}}$ will generically be singular. This is why we restrict attention to Euclidean signature in the following.

In Euclidean signature, complex conjugation on complexified space-time sends an α-plane Z to another, $\hat{Z}$, and the complex conjugation on non-projective twistor space is in fact quaternionic in the sense that $\hat{\hat{Z}} = -Z$ (i.e., complex conjugation defines a second complex structure anti-commuting with the standard one). The conjugation therefore has no fixed points on the projective space (it will be given explicitly below).

The space $\mathbb{A}_{\mathbb{E}}$ is the subset $Z \cdot \hat{W} = 0$ inside $\mathbb{A}$. We give a coordinate based derivation below. To see this using twistor geometry, we note that the condition $Z \cdot W = 0$ is the condition that the α-plane corresponding to Z and the β-plane corresponding to W intersect in a complex null geodesic in complexified space-time. The condition that $Z \cdot \hat{W} = 0$ implies that Z actually lies on the line formed by the intersection of the two planes $Z \cdot W = 0 = Z \cdot \hat{W}$. But this line corresponds to the intersection in space-time of the β-planes corresponding to W and $\hat{W}$ which is necessarily a point of $\mathbb{E}$ (or infinity) so the complex null geodesic corresponding to (Z, W) must be incident with this real point since Z and W both are.

From its embedding in $\mathbb{A}$, $\mathbb{A}_{\mathbb{E}}$ inherits a CR-structure, i.e., it has an inherited complex 3-dimensional integrable distribution $\mathcal{D}$ of $(0, 1)$-vectors that are the $(0, 1)$-vectors on $\mathbb{A}$ whose real and imaginary parts are tangent to $A_{\mathbb{E}}$. On a CR manifold, there is a standard construction of Dolbeault cohomology as follows. First define the space of $(1, 0)$-forms $\Omega^{(1,0)}$ to be the complex 1-forms that annihilate $\mathcal{D}$ and we define the $(0, p)$-forms,

$$\Omega_B^{(0,p)} = \Omega^p / \{\Omega^{(1,0)} \wedge \Omega^{p-1}\}$$

where Ω^p are the complex p-forms on $\mathbb{A}_{\mathbb{E}}$. The subscript B in these definitions stands for *boundary* as most studies of CR manifolds arise in the context of a real codimension-1 boundary of a complex domain. In our situation, however, $\mathbb{A}_{\mathbb{E}}$ has real codimension-2. Note the asymmetry in the definitions, $\Omega^{(1,0)}$ is 5 complex dimensional and is the restriction of $\Omega^{(1,0)}$ from $\mathbb{A}$, whereas $\Omega_B^{(0,1)}$ is 3 complex dimensional and defined as a quotient. Define the $\bar{\partial}_B$ operator to be the restriction of the exterior derivative d to $\Omega^{(0,p)}$; $\bar{\partial}_B^2 = 0$ from the integrability of $\mathcal{D}$. Thus we can define $\bar{\partial}_B$-cohomology

$$H_B^p(\mathbb{A}_{\mathbb{E}}) = \{\text{Ker } \bar{\partial}_B / \text{Im } \bar{\partial}_B\} \cap \Omega^{(0,p)} .$$

We can also consider the analogue of holomorphic vector bundles which will be a complex vector bundle E with a $\bar{\partial}$-operator $\bar{\partial}_a = \bar{\partial}_B + a$ where a is a $(0, 1)$-form with values in the endomorphisms of E and such that $(\bar{\partial}_a)^2 = 0$. The simplest bundles on $\mathbb{A}_{\mathbb{E}}$ are the line bundles $\mathcal{O}(p,q)$ which are the restrictions of tensor product of the pullback of $\mathcal{O}(p)$ from $\mathbb{PT}$ and $\mathcal{O}(q)$ from $\mathbb{PT}^*$ to $\mathbb{A}$. The Penrose-Ward transform can be applied in the usual way to give a correspondence between cohomology classes or bundles on $\mathbb{A}_{\mathbb{E}}$ and fields or bundles with connections on $\mathbb{E}$ entirely analogously with the standard ambitwistor correspondences.

We do not here give a complete derivation of the Ward transform in this context, but give an indication of the main argument. Given a topologically trivial $\bar{\partial}_B$-holomorphic vector bundle over $\mathbb{A}_{\mathbb{E}}$, it must be analytically trivial over each of the fibres of $\mathbb{A}_{\mathbb{E}} \to \mathbb{E}$ since the only topologically trivial holomorphic vector bundle over the quadric is the trivial one. We can define $E' \to \mathbb{E}$ to be the bundle whose fibre at $x \in \mathbb{E}$ is the corresponding space of global sections of $E \to p^{-1}(x)$. There is just one $(0,1)$ vector transverse to Q and, by integrability of the CR-structure it varies holomorphically over Q. It naturally has a lift to act on $\mathbb{E}$ and this lift must also be holomorphic and an explicit calculation shows that a generalization of Liouville's theorem applies to show that it must arise from a connection on $E' \to \mathbb{E}$ very much as in the standard case.

If a $\bar{\partial}_B$ cohomology class is analytic, then it extends naturally to a cohomology class on a neighbourhood of $\mathbb{A}_{\mathbb{E}}$ in $\mathbb{A}$ so cohomology classes on such subsets of $\mathbb{A}$ are determined by their restrictions to $A_{\mathbb{E}}$ and the same is true of holomorphic bundles. This can be seen by examining the ambitwistor correspondence: bundles or cohomomology classes on $\mathbb{A}_{\mathbb{E}}$ correspond to fields on the appropriate region in $\mathbb{E}$. The field is analytic iff the corresponding cohomology class or bundle has an analytic representative. However, if one does have such an extension from $\mathbb{A}_{\mathbb{E}}$ to a neighbourhood in $\mathbb{A}$, the Penrose-Ward transform will give a field on a complex thickening of $\mathbb{E}$ in $\mathbb{CM}$ and will therefore imply that the corresponding field on $\mathbb{E}$ was analytic. Since this is not necessarily the case (unless one is working in Euclidean signature and field equations are satisfied) everything is defined only on $A_{\mathbb{E}}$ in the first instance.

The above characterisations of the field equations suggest that we will also need to consider the embedding of $\mathbb{A}_{\mathbb{E}}$ into a suitable real codimension-2 subset $\mathbb{PT} \times^{\mathbb{E}} \mathbb{PT}^*$ of $\mathbb{PT} \times \mathbb{PT}^*$. Clearly the subset $Z \cdot \hat{W} = 0$ extends smoothly across $\mathbb{PT} \times \mathbb{PT}^*$ and we take this to be the definition of $\mathbb{PT} \times^{\mathbb{E}} \mathbb{PT}^*$. Note that the equation $Z \cdot \hat{W} = 0$ constitutes two real non-holomorphic conditions and so it defines a CR manifold of real codimension-2 type in $\mathbb{PT} \times \mathbb{PT}^*$. With this definition, $A_{\mathbb{E}}$ is the subset of $\mathbb{PT} \times^{\mathbb{E}} \mathbb{PT}^*$ on which $Z \cdot W = 0$.

3.1 The fibrations over $\mathbb{E}$ and explicit coordinatization

Here we develop further the geometry of the embedding of $\mathbb{A}_{\mathbb{E}}$ in $\mathbb{PT} \times^{\mathbb{E}} \mathbb{PT}^*$ in the context of the projections of $\mathbb{PT}$, $\mathbb{PT}^*$ and $\mathbb{A}_{\mathbb{E}}$ to euclidean space $\mathbb{E}$ and its conformal one point compactification S^4.

In euclidean signature, twistor space and its dual have projections to S^4. The most primitive definition of a (dual) twistor is as a totally null (anti) self-dual 2-plane or α (β)-planes in complex Minkowski space. These intersect S^4 at precisely one point and this leads to fibrations $p : \mathbb{PT} \to S^4$, $p : \mathbb{PT}^* \to S^4$ where we have abused notation to call all such fibrations p. The fibres of these fibrations are $\mathbb{CP}^1$s and can naturally be identified with projective (anti) self-dual spinors. The non-projective twistor space $\mathbb{T} - \{0\}$ can be identified with the total space of the bundle of self-dual spinors (minus the zero-section) and similarly $\mathbb{T}^* - \{0\}$ can be identified with the complement of the zero-section in the bundle of anti-self-dual spinors. In Atiyah, Hitchin & Singer (1978), twistor spaces were defined in a similar way as the the total spaces of bundles of metric and orientation compatible almost complex structures. There they were represented as self-dual 2-forms of unit length, whereas here we use the representation of a metric compatible almost complex structure by a spinor. See Woodhouse (1985) for an introduction to the approach used here and a review of basic twistor theory in this context.

We introduce coordinates $(x^a, y^b), a, b = 0, 1, 2, 3$ on $\mathbb{E} \times \mathbb{E}$ and coordinates $(Z^\alpha, W_\beta) = ((\omega^A, \pi_{A'}), (\lambda_B, \mu^{B'}))$ on $\mathbb{PT} \times \mathbb{PT}^*$, $A = 0, 1$ and $A' = 0', 1'$. We have $Z^\alpha W_\alpha = \omega^A \lambda_A +$

$\pi_{A'}\mu^{A'}$. The incidence relations with spacetime are

$$\omega^A = x^{AA'}\pi_{A'}\,, \qquad \mu^{A'} = -y^{AA'}\lambda_A\,,$$

where $x^{AA'} = \sigma_b^{AA'}x^b$ and $\sigma_b^{AA'}$ are the standard Van de Waerden symbols.

The Euclidean complex conjugation induces the spinor conjugation $\omega^A \to \hat{\omega}^A = (\bar{\omega}^1, -\bar{\omega}^0)$ and $\pi_{A'} \to \hat{\pi}_{A'} = (\bar{\pi}_{1'}, -\bar{\pi}_{0'})$. This extends to the conjugations

$$Z \to \hat{Z} = (\hat{\omega}^A, \hat{\pi}_{A'})\,, \qquad W \to \hat{W} = (\hat{\mu}^{A'}, \hat{\lambda}_A)$$

With this notation, the fibration $p : \mathbb{PT}\times\mathbb{PT}^* \to \mathbb{E}\times\mathbb{E}$ is given by

$$p(Z^\alpha, W_\beta) = \left(\frac{1}{\hat{\pi}^{B'}\pi_{B'}}(\omega^A\hat{\pi}^{A'} - \hat{\omega}^A\pi^{A'}), \frac{-1}{\hat{\lambda}^B\lambda_B}(\mu^{A'}\hat{\lambda}^A - \hat{\mu}^{A'}\lambda^A)\right)\,.$$

The functions $Z\cdot W = (x^{AA'} - y^{AA'})\pi_{A'}\lambda_A$, and $Z\cdot\hat{W} = (x^{AA'} - y^{AA'})\pi_{A'}\hat{\lambda}_A$.

The fibres of $p : \mathbb{PT}\times\mathbb{PT}^* \to \mathbb{E}\times\mathbb{E}$ are the cartesian product of the Riemann spheres parametrised by homogeneous coordinats $\pi_{A'}$ and λ_A, so we can equivalently use the non-holomorphic coordinates $\Big((x^{AA'}, \pi_{A'}), (y^{BB'}, \lambda_B)\Big)$ on $\mathbb{PT}\times\mathbb{PT}^*$. The distribution $\mathcal{D}$ defining the (0,1) vectors is given by

$$\mathcal{D} = \langle \pi^{A'}\frac{\partial}{\partial x^{AA'}}, \lambda^A\frac{\partial}{\partial y^{AA'}}, \frac{\partial}{\partial\hat{\pi}_{A'}}, \frac{\partial}{\partial\hat{\lambda}^A}\rangle$$

The distribution can be equivalently defined as being those vectors that are orthogonal to $D^3Z\wedge D^3W$ where

$$D^3Z = \mathrm{D}\pi\wedge\pi_{B'}\pi_{C'}\mathrm{d}^2x^{B'C'} \quad\text{ and }\quad D^3W = \mathrm{D}\lambda\wedge\lambda_B\lambda_C\mathrm{d}^2y^{BC}$$

where we define $\mathrm{D}\pi = \pi_{A'}\mathrm{d}\pi^{A'}$, $\mathrm{D}\lambda = \lambda_A\mathrm{d}\lambda^A$, and

$$\mathrm{d}^2x^{B'C'} = \varepsilon_{BC}\mathrm{d}x^{BB'}\wedge\mathrm{d}x^{CC'}\,, \quad \mathrm{d}^2y^{BC} = \varepsilon_{B'C'}\mathrm{d}y^{BB'}\wedge\mathrm{d}y^{CC'}\,.$$

On $\mathbb{A}_\mathbb{E}$ we have $x^a = y^a$ and hence homogeneous coordinates $(x^a, \pi_{A'}, \lambda_A)$ (we leave it to the reader to check that $Z\cdot W = Z\cdot\hat{W} = 0$ implies that $x^a = y^a$). The distribution defining the $(0,1)$ vectors non-projectively is $\{\pi^{A'}\lambda^A\partial/\partial x^{AA'}, \partial/\partial\hat{\pi}^{A'}, \partial/\partial\hat{\lambda}^A\}$. The $(1,0)$ forms are spanned by $\{\pi_{A'}\mathrm{d}x^{AA'}, \lambda_A\mathrm{d}x^{AA'}, \mathrm{d}\pi_{A'}, \mathrm{d}\lambda_A\}$.

In this notation we define

$$\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^* = \{(Z,W)\in\mathbb{PT}\times\mathbb{PT}^* | p(Z,W)\subset\mathbb{E}\times\mathbb{E}, Z\cdot\hat{W} = 0\}\,.$$

We emphasize that the notation $\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*$ is *not* intended to indicate the fibre-wise product. Whereas $p : \mathbb{A}_\mathbb{E}\to\mathbb{E}$, $p : \mathbb{PT}\times\mathbb{PT}^* \to S^4\times S^4$ induces a fibration $p : \mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*\to\mathbb{E}\times\mathbb{E}$. On $\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*$ we have that the fibres over $\mathbb{E}\times\mathbb{E}$ are $\mathbb{CP}^1\times\mathbb{CP}^1$ as above on $\mathbb{A}_\mathbb{E}\to\mathbb{E}$ where here $\mathbb{E}\subset\mathbb{E}\times\mathbb{E}$ as the diagonal, but away from the diagonal, the fibres jump to $\mathbb{CP}^1$ since the constraint $Z\cdot\hat{W} = 0$ gives $s^{AA'}\pi_{A'}\hat{\lambda}_A = 0$ where $2s^a = x^a - y^a$. When $s^{AA'}\neq 0$, we can solve for λ_A up to scale

$$\lambda^A \propto s^{AA'}\hat{\pi}_{A'}\,.$$

Thus, projectively, we can coordinatise the fibre with just $\pi_{A'}$. The $(1,0)$-forms are still spanned by $\{\pi_{A'}\mathrm{d}x^{AA'}, \lambda_A\mathrm{d}x^{AA'}, \mathrm{d}\pi_{A'}, \mathrm{d}\lambda_A\}$.

4 The Lagrangian

Let $E\to\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*$ be a smooth complex vector bundle. The field variable is a $(0,1)$-form a with values in the endomorphisms of E so as to define an extension of $\bar{\partial}_B$ to an operator

$\bar{\partial}_a$ on sections of E (we drop the subscript B as all $\bar{\partial}$ operators in the following will be $\bar{\partial}_B$ operators). We give two versions of the action, the first being

$$S[a] = \int_{\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*} \mathrm{tr}\left(F^{(0,2)} \wedge F^{(0,2)}\right) \wedge \frac{D^3 Z \wedge D^3 W}{(Z\cdot W)^4} \tag{2}$$

where $D^3 Z = \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha \mathrm{d}Z^\beta \wedge \mathrm{d}Z^\gamma \wedge \mathrm{d}Z^\delta$, $D^3 W$ is defined similarly and $F^{(0,2)} = \bar{\partial}_a^2$. This action is related to a holomorphic Chern-Simons action with an integration by parts since $\bar{\partial}_B \mathrm{CS}(a) = \mathrm{tr}(F^{(0,2)})^2$ where $\mathrm{CS}(a) = \frac{1}{2}\,\mathrm{tr}(a \wedge \bar{\partial} a + \frac{2}{3} a^3)$, so formally

$$S[a] = \int_{\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}} \mathrm{CS}(a) \wedge \delta'''(Z\cdot W) D^3 Z D^3 W\,. \tag{3}$$

where $\delta'''(Z\cdot W) = \bar{\partial}\left(\frac{1}{(Z\cdot W)^4}\right)$ is the third holomorphic derivative of the delta function.

Here we are defining, for a complex variable z,

$$\delta(z) = \bar{\partial}\left(\frac{1}{z}\right) = 2\pi i \delta(\Re z)\delta(\Im z)\mathrm{d}\bar{z}\,, \quad \text{and} \quad \delta'(z) = \frac{\partial}{\partial z}\delta(z)$$

and so on. It can be checked that the homogeneity properties of δ-functions are such that $\delta(\lambda z) = \delta(z)/\lambda$ and so on for the derivatives. Thus, since $Z\cdot W$ takes values in the line bundle $\mathcal{O}(1,1)$, $\delta(Z\cdot W)$ makes sense as a $(0,1)$-form with values in $\mathcal{O}(-1,-1)$. Similarly $\delta'''(Z\cdot W)$ is a $(0,1)$-form with values in $\mathcal{O}(-4,-4)$.

This second formulation leads to the ambitwistor formulation of the Yang-Mills equations of theorem 2 directly in the form

$$F^{(0,2)} \wedge \delta'''(Z\cdot W) = 0\,,$$

i.e., $F^{(0,2)} = O(Z\cdot W^4)$, vanishing to 3rd order about $Z\cdot W = 0$.

Note that, since the action integral is supported to 3rd order on $\mathbb{A}_{\mathbb{E}}$, any variation $a \to a + \delta a$ with $\delta a = O(Z\cdot W^4)$ is a gauge symmetry over and above the standard gauge symmetry.

4.1 The supersymmetric version

It is also interesting to consider a supersymmetric version of this action. We start with the $\mathcal{N} = 3$ supertwistor space $\mathbb{T}_{[3]} = \mathbb{C}^{4|3}$ with coordinates (Z^α, ξ^i), $i = 1,2,3$ where ξ^i are Grassmann-odd. The dual supertwistor space $\mathbb{T}^*_{[3]}$ likewise has coordinates (W_β, η_j) and we projectivise with the usual equivalence $(Z^\alpha, \xi^i) \sim (\lambda Z^\alpha, \lambda\xi^i)$, $\lambda \in \mathbb{C}^*$. Super ambitwistor space is then defined as

$$\mathbb{A}_{[3]} = \left\{([Z^\alpha, \xi^i], [W_\beta, \eta_j]) \in \mathbb{PT}_{[3]} \times \mathbb{PT}^*_{[3]}\,;\, Z^\alpha W_\alpha + \xi^i \eta_i = 0\right\}.$$

Super ambitwistor space is (for $\mathcal{N} = 3$) a Calabi-Yau supermanifold as it possesses a global holomorphic measure

$$\Omega = \oint \frac{D^3 Z \wedge \mathrm{d}^3\xi \wedge D^3 W \wedge \mathrm{d}^3\eta}{Z\cdot W + \xi\cdot\eta},$$

where the contour is taken around any S^1 encircling $\mathbb{A} \in \mathbb{PT}\times\mathbb{PT}^*$. The integrand has weight $(0,0)$ under the scaling of homogeneous coordinates (since $\mathrm{d}\xi \mapsto \lambda^{-1}\mathrm{d}\xi$ by the rules of Berezinian integration) and has a pole on $Z\cdot W + \xi\cdot\eta = 0$. Hence it localizes on $\mathbb{A}_{[3]}$ after performing the contour integral, and defines a holomorphic measure.

As above, we can write our action on the ambient space $\mathbb{PT}_{[3]} \times^{\mathbb{E}} \mathbb{PT}^*_{[3]}$

$$S[a_s] = \int_{\mathbb{PT}_{[3]}\times^{\mathbb{E}}\mathbb{PT}^*_{[3]}} \mathrm{tr}(F^{(0,2)})^2 \wedge \frac{D^3 Z \wedge \mathrm{d}^3\xi \wedge D^3 W \wedge \mathrm{d}^3\eta}{Z\cdot W + \xi\cdot\eta},$$

where now $a_s \in \Omega^{(0,1)}_B \otimes \mathrm{End}(E)$ defines a $\bar\partial_B$-operator on the smooth complex bundle $E \to \mathbb{PT}_{[3]} \times^{\mathbb{E}} \mathbb{PT}^*_{[3]}$. Integrating by parts as above, we obtain a holomorphic Chern-Simons theory directly on $\mathbb{A}_{[3]\mathbb{E}}$ with action

$$S[a_s] = \int_{\mathbb{A}_{[3]\mathbb{E}}} \mathrm{CS}(a_s) \wedge \Omega$$

where we have used $\bar\partial (Z\cdot W + \xi\eta)^{-1} = \delta(Z\cdot W + \xi\cdot\eta)$ and integrated out the δ-function. Thus, in the supersymmetric theory, all the information is present on $\mathbb{A}_{[3]\mathbb{E}}$ and we do not have to consider the ambient $\mathbb{PT}_{[3]} \times^{\mathbb{E}} \mathbb{PT}^*_{[3]}$. Essentially, this is because the Grassmann-odd directions mimic the effect of the extension to the third formal neighbourhood, Eastwood & LeBrun (1986). We can recover the non-supersymmetric theory by considering only those terms in $\mathcal{A}$ which are independent of ξ and η, or equivalently, by setting the superpartners of the standard spin-1 gauge field to zero. In this case, the only terms which survive the Grassmann integration come from the term $(\xi\cdot\eta)^3/(Z\cdot W)^4$ in the expansion of $(Z\cdot W + \xi\cdot\eta)^{-1}$, and we recover our non-supersymmetric action.

5 Relation to standard Yang-Mills action

It is clear from the above that the solutions to the field equations arising from the action principles given above should give rise to solutions of the Yang-Mills equations. In this section we give an explicit proof and show that the actions can in fact be identified, at least on solutions to the equations of motion.

Given a Yang-Mills connection A on a bundle $E' \to \mathbb{E}$, we consider an arbitrary smooth extension $\mathcal{A}$ to $\mathbb{E}\times\mathbb{E}$ where $\Delta : \mathbb{E} \to \mathbb{E}\times\mathbb{E}$ is embedded as the diagonal. We will restrict the action to $\bar\partial_B$ operators on bundles over $\mathbb{PT} \times^{\mathbb{E}} \mathbb{PT}^*$ that are induced from such connections pulled back from $\mathbb{E}\times\mathbb{E}$. The reason we can assume that our $\bar\partial_B$-operators are obtained in this way is because the part of the field equation $\bar\partial_a^2 = O(Z\cdot W^2)$ on $\mathbb{A}_{\mathbb{E}}$, together with the gauge freedom to alter a at higher order away from $\mathbb{A}_{\mathbb{E}}$ will imply that the $\bar\partial_B$-operator is gauge equivalent to one obtained from a connection pulled back from $\mathbb{E}\times\mathbb{E}$ in this way, and so this will necessarily be the case when some of the ambitwistor field equations are satisfied (although the Yang-Mills equations themselves are not implied at this point).

The action then becomes:

$$S[A] = \int_{\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*} \mathrm{tr}\, \mathcal{F}^2 \wedge \frac{D^3Z\wedge D^3W}{(Z\cdot W)^4}$$

where $\mathcal{F}$ is the curvature of the connection $\mathcal{A}$. On $Z\cdot\hat W = 0$, we have that $\lambda^A = (x^{AA'} - y^{AA'})\hat\pi_{A'}$ up to scale and, since the integrand is scale invariant, we can eliminate λ_A and then integrate over the S^2 fibres away from the diagonal over $\mathbb{E}\times\mathbb{E}$. Introduce

$$s^a = \frac{1}{2}(x^a - y^a)\,, \qquad r^a = \frac{1}{2}(x^a + y^a)\,.$$

We have

$$\int_{s^{AA'}\lambda_A\hat\pi_{A'}=0} \mathrm{D}\lambda\wedge\mathrm{D}\pi \frac{\pi_{A'}\pi_{B'}\lambda_A\lambda_B}{(2s^{AA'}\lambda_A\pi_{A'})^4} = 16\pi i \frac{s_{A(A'}s_{B')B}}{3(s\cdot s)^3}$$

where the integral is over the S^2 fibres away from the diagonal over $\mathbb{E}\times\mathbb{E}$ (this follows in particular from §3 of Woodhouse 1985). Thus, performing the fibre integrals we obtain

$$\int_{Z\cdot\hat W=0} \frac{D^3Z\wedge D^3W}{(Z\cdot W)^4} = 16\pi i \frac{s_{AA'}s_{BB'}}{(s\cdot s)^3}\mathrm{d}^2x^{A'B'}\,\mathrm{d}^2y^{AB}$$

so that we obtain the action for the connection $\mathcal{A}$:

$$S[\mathcal{A}] = \int_{\mathbb{E}\times\mathbb{E}} \mathrm{tr}\mathcal{F}^2 \wedge \frac{s_{AA'}s_{BB'}\mathrm{d}^2x^{A'B'}\mathrm{d}^2y^{AB}}{(s\cdot s)^3}$$

on $\mathbb{E}\times\mathbb{E}$ (recall that we have assumed that the connection itself has no dependence on the fibre coordinates). We now choose a frame for the bundle, write $\mathrm{tr}\mathcal{F}^2 = \mathrm{dCS}(\mathcal{A})$ and integrate by parts. We have the relation

$$\mathrm{d}\left(\frac{s_{AA'}s_{BB'}\mathrm{d}^2x^{A'B'}\mathrm{d}^2y^{AB}}{(s\cdot s)^{-3}}\right) = \frac{8\pi^2}{3}\frac{\partial\delta^4(s)}{\partial s^{AA'}}\,\mathrm{d}s_{BB'}\wedge\mathrm{d}^2x^{A'B'}\wedge\mathrm{d}^2y^{AB}$$
$$= -2\pi^2\frac{\partial\delta^4(s)}{\partial s^{AA'}}\left(\mathrm{d}^3x^{A'}_B\,\mathrm{d}^2y^{AB} + \mathrm{d}^2x^{A'B'}\mathrm{d}^3y^A_{B'}\right)$$

where $\mathrm{d}^3x^a = *\,\mathrm{d}x^a = \frac{1}{6}\varepsilon^a{}_{bcd}\mathrm{d}x^b\wedge\mathrm{d}x^c\wedge\mathrm{d}x^d$; the formula follows from the derivative of the standard relation $\nabla_a\nabla^a(1/s\cdot s) = 4\pi^2\delta^4(s)$. Integrate by parts on the $\partial/\partial s^{AA'}$ derivatives to obtain the action (ignoring irrelevant overall constant factors)

$$S[\mathcal{A}] = \int_{\mathbb{E}\times\mathbb{E}}\frac{\partial\mathrm{CS}(\mathcal{A})}{\partial s^{AA'}}\wedge\delta^4(s)\,\mathrm{d}s_{BB'}\wedge\mathrm{d}^2x^{A'B'}\wedge\mathrm{d}^2y^{AB}$$
$$= \int_{\mathbb{E}\times\mathbb{E}}\mathrm{tr}\left(\frac{\partial\mathcal{A}}{\partial s^{AA'}}\wedge\mathcal{F} - \frac{1}{2}\mathrm{d}\left(\mathcal{A}\wedge\frac{\partial\mathcal{A}}{\partial s^{AA'}}\right)\right)\delta^4(s)\,\mathrm{d}s_{BB'}\wedge\mathrm{d}^2x^{A'B'}\wedge\mathrm{d}^2y^{AB}$$

We now integrate by parts on the second term to obtain an expression in the $\partial\delta^4(s)/\partial s^{CC'}$ and integrate by parts back on the $\partial/\partial s^{AA'}$ derivatives again to obtain

$$S[\mathcal{A}] = \int_{\mathbb{E}\times\mathbb{E}}\mathrm{tr}\left(\frac{\partial\mathcal{A}}{\partial s^{AA'}}\wedge\mathcal{F} - \frac{1}{2}\frac{\partial}{\partial s^c}\left(\mathcal{A}\wedge\frac{\partial\mathcal{A}}{\partial s^{AA'}}\right)\mathrm{d}s^c\right)\delta^4(s)\,\mathrm{d}s_{BB'}\wedge\mathrm{d}^2x^{A'B'}\wedge\mathrm{d}^2y^{AB}$$

The indices in the second term turn out to be skew over a and c and so no second derivative terms in s^a appear. We can now integrate out the s^a using the delta functions to obtain the action as a functional of $\mathcal{A}$ and its first derivative on $s^a = 0$. To simplify the calculation, choose an arbitrary gauge at $s^a = 0$ and parallel propagate the frame out along $s^a(\partial/\partial x^a - \partial/\partial y^a)$. Then set $\mathcal{A} = A^x_a\mathrm{d}x^a + A^y_a\mathrm{d}y^a$ and expand in s^a

$$A^x_a = \frac{1}{2}A_a + A^x_{ab}s^b + \dots \qquad A^y_a = \frac{1}{2}A_a + A^y_{ab}s^b + \dots \tag{4}$$

where $A_a(r^b)$ is a given gauge field on $\mathbb{E}$, and A^x_{ab} and A^y_{ab} are functions of r^a only and the gauge condition (which has already been used in the above) also implies that $A^x_{(ab)} - A^y_{(ab)} = 0$. The action reduces to a functional of A_a, A^x_{ab} and A^y_{ab}. Decomposing into irreducibles, we discover that some of the irreducible pieces of A^x_{ab} and A^y_{ab} decouple from the other fields and appear quadratically in the action so that the field equations make them vanish. Eliminating these we are left with A_a and a Lie algebra valued 2-form G with action

$$S[A,G] = \int \mathrm{tr}G\wedge F - \frac{1}{2}\int \mathrm{tr}\ G\wedge{}^*G\,, \tag{5}$$

where F is the curvature of the connection A. The field equations from this action are simply $G = {}^*F$ and $DG = 0$ and so we see that, eliminating G from the action, we finally obtain the standard Yang-Mills action.

6 Perturbation theory

The action allows us to consider perturbation theory. In perturbation theory we take the $(0,1)$-form a to be 'on-shell' in the sense that it satisfies the field equations of linear theory. In linear theory we can take a to be a Lie algebra valued element of $H^1(\mathbb{A}_{\mathbb{E}},\mathcal{O})$. The extension of such a cohomology class to 3rd order in $\mathbb{PT}\times^{\mathbb{E}}\mathbb{PT}^*$ implies extension to all orders (this can be seen by the sheaf theoretic arguments of Baston and Mason 1987). By the Kunneth formula (or simply by the Penrose transform and the fact that linear fields decompose into self-dual and anti-self-dual parts) we can write $a = a^-(Z) + a^+(W)$ and the notation indicates that $a^-(Z)$ is pulled back from a class in $H^1(\mathbb{PT}', \mathcal{O})$, i.e., $a^-(Z)$ is a $(0,1)$-form pulled back from twistor space and $a^+(W)$ is one pulled back from dual-twistor space. ($\mathbb{PT}' = p^{-1}(\mathbb{E})$ is $\mathbb{PT}$ with the line corresponding to the point at ∞ taken away—the cohomology of $\mathbb{PT}$ is at most finite dimensional.)

To construct Feynman diagrams from this action, we note that the vertices coming from the Chern-Simons action are all trivalent. (The possible 4-vertex coming from the $\mathrm{tr}(F^2)$ form of the action vanishes identically.)

6.1 The inner product, twistor transform and propagators

The symplectic inner product on such linear fields can be derived from the Lagrangian in the standard way as the boundary term in the variation of the action. This gives

$$\langle a_1, a_2 \rangle = \int_{\mathrm{Contour}} \mathrm{tr}\,(a_1 \wedge a_2)\, \delta'''(Z \cdot W) D^3 Z \wedge D^3 W \tag{6}$$

This is the integral of a closed 9-form over the contour obtained by the intersection of Euclidean space with Lorentzian space. Using the delta-function to restrict the integral to $\mathbb{A}_{\mathbb{E}}$, the contour can be taken to be the part of $\mathbb{A}_{\mathbb{E}}$ fibering over this intersection of the Euclidean with the Lorentzian slice. This symplectic inner product yields a positive definite inner product on positive frequency fields on Minkowski space: such fields extend over the $t \geq 0$ half of Euclidean space (where $t = 0$ is the intersection with Minkowski space) and taking a_1 to be such a positive frequency field, and a_2 to be its Minkowskian complex conjugate, so that it extends over $t \leq 0$, the above integral gives the standard norm.

In twistor theory there is an alternative representation of helicity ± 1 fields in terms of twistor functions of homogeneity degree -4, $g^+(Z) \in H^1(\mathbb{PT}', \mathcal{O}(-4))$ and $g^-(W) \in H^1(\mathbb{PT}^{*\prime}, \mathcal{O}(-4))$. These are related to the homogeneity degree 0 representation by

$$g^+(Z) = \int_{Z \cdot \hat{W} = 0} a^+(W) \wedge \delta'''(Z \cdot W) D^3 W\,.$$

This is an integral of a 5-form over the four-manifold $W \cdot \hat{Z} = 0$ and can be checked explicitly using the explicit Dolbeault representatives of Woodhouse (1985) in which $a^+(W) = A(y)_{AA'} \lambda^A \hat{\lambda}_B \mathrm{d}y^{BA'}$ and $g^+(Z) = \hat{\partial}(\phi_{A'B'}(x) \hat{\pi}^{A'} \hat{\pi}^{B'} / (\hat{\pi} \cdot \pi)^3)$, $\phi_{A'B'} = \nabla^A_{(A'} A_{B')A}$ and $\hat{\partial} = \mathrm{d}\hat{Z}^\alpha \partial / \partial Z^\alpha$ (here we use the notation of §3.1).

It is worth noting that this twistor transform relation also works in this context off-shell, i.e., $\bar{\partial} a$ does not need to vanish although, in that case, $\bar{\partial} g^+$ will not vanish either, but the push down of $g^+ \wedge D^3 Z$ from $\mathbb{PT}$ to $\mathbb{E}$ will be the self-dual 2-form corresponding to the potential A^+ on $\mathbb{E}$ whose pullback to $\mathbb{PT}^*$ and projection onto $\Omega^{0,1}$ determines $a^+(W)$.

Using the twistor transform, we can verify that the symplectic inner product gives the standard expression by taking a_1 to be pulled back from $\mathbb{PT}^*$ and a_2 from $\mathbb{PT}$ to give

$$\begin{aligned}\langle a_1(W), a_2(Z) \rangle &= \int_{t=0} \mathrm{tr}\,(a_1(W) \wedge a_2(Z))\, \delta'''(Z \cdot W) D^3 Z \wedge D^3 W \\ &= \int \mathrm{tr}\,(g_1(Z) \wedge a_2(Z)) \wedge D^3 Z\,.\end{aligned}$$

This is a standard expression for the inner product in twistor theory in which the $t = 0$ contour becomes PN, the space of null twistors in Lorentz signature that correspond to real light rays in Minkowski space.

The Chern-Simons propagator Δ is formally $\bar{\partial}^{-1}$ acting on $\Omega^{0,1}$ with appropriate boundary conditions. If we write $\bar{\partial} a = j$ where the current j is a closed $(0,2)$-form, then we have that currents and potentials are dual by

$$(j, a) = \int a \wedge j \wedge \delta'''(Z \cdot W) D^3 Z \wedge D^3 W\,.$$

The propagator then can be expressed as the integral kernel Δ satisfying the formal relation

$$j_1 \Delta j = (j_1, a) \ \ \forall j_1 \qquad \Leftrightarrow \qquad \bar{\partial} a = j. \tag{7}$$

We have the following formula for Δ

$$j_1 \Delta j_2 = \int j_1 \wedge j_2 \frac{D^3 Z \wedge D^3 W}{(Z \cdot W)^4} \,,$$

and (7) can now be verified directly by integrating by parts.

6.2 Space-time and Momentum space Feynman rules

It is most likely that the direct application of Feynman rules to this action yields a version of the twistor diagram formulation of scattering amplitudes; see Hodges (2005) for a recent discussion. Indeed it is to be hoped that this action provides a generating principle for twistor diagrams. Here it is simplest, however, to convert the diagram formalism into a set of position space and momentum space Feynman rules. We first consider the 3-vertex. This arises from the a^3 term in the Chern-Simons form and so the corresponding 3-vertex will arise if we put three on-shell linearized fields a_1, a_2 and a_3 into the formula

$$V(a_1, a_2, a_3) = \int \operatorname{tr}(a_1 \wedge a_2 \wedge a_3) \wedge \delta'''(Z \cdot W) D^3 Z \wedge D^3 W \,.$$

In order to obtain a non-trivial result, we cannot have that all of the as are functions of Z (or all functions of W) as the forms will wedge to give zero. Thus we can take without loss of generality either $a_1 = a_1(Z)$, $a_2 = a_2(Z)$ and $a_3 = a_3(W)$ to give the $--+$ vertex or $a_1 = a_1(W)$, $a_2 = a_2(W)$ and $a_3 = a_3(Z)$ to give the $++-$ vertex. We focus on the $--+$ case as clearly the $++-$ case works similarly. We first partially integrate using the twistor transform

$$g_3(Z) = \int_{\mathbb{PT} \times {}^{\mathbb{E}}\mathbb{PT}|_{Z=\text{const.}}} a_3(W) \wedge \delta'''(Z \cdot W) D^3 W$$

as above. This then gives the formula for the vertex as

$$V(a_1, a_2, a_3) = \int_{\mathbb{PT}} \operatorname{tr}(a_1 \wedge a_2 \wedge g_3) \wedge D^3 Z \,.$$

To evaluate this, we note first the standard integral formula for a self-dual Maxwell field G_3 in terms of a homogeneity degree -4 cohomology class g_3 is

$$G_3 = G_{3A'B'}(x) \mathrm{d}^2 x^{A'B'} = \int_{p^{-1}(x)} g_3 \wedge D^3 Z \,.$$

Secondly, the Woodhouse (1985) explicit representatives for a homogeneity degree 0 class $a_1(Z)$ is simply the $(0,1)$ part of $p^* A_1$ where A_1 is the corresponding potential 1-form on $\mathbb{E}$. But $a_1 \wedge D^3 Z = p^* A_1 \wedge D^3 Z$ because wedging with $D^3 Z$ projects out the holomorphic part of a 1-form. Thus we can write

$$V(a_1, a_2, a_3) = \int_{\mathbb{PT}} \operatorname{tr}(A_1 \wedge A_2 \wedge g_3) \wedge D^3 Z = \int_{\mathbb{E}} \operatorname{tr}(A_1 \wedge A_2 \wedge G_3) \,.$$

We note that this space time representative is precisely what we would have obtained from the space-time action (5).

This formula can now be evaluated on momentum eigenstates

$$A_1 = \mathrm{e}^{ip_1 \cdot x} p_{1A} \epsilon_{1A'} \mathrm{d}x^{AA'} \,, \quad A_2 = \mathrm{e}^{ip_2 \cdot x} p_{2A} \epsilon_{2A'} \mathrm{d}x^{AA'} \,, \quad G_3 = \mathrm{e}^{ip_3 \cdot x} \tilde{p}_{3A'} \tilde{p}3B' \mathrm{d}^2 x^{A'B'} \,,$$

where $p_1^{AA'} = p_1^A \tilde{p}_1^{A'}$, $\tilde{p}_1^{A'} \epsilon_{1A'} = 1$ etc.. We find, after some manipulations the standard result

$$V(a_1, a_2, a_3) = \delta^4(p_1 + p_2 + p_3) \frac{\langle p_1 \cdot p_2 \rangle^4}{\langle p_1 \cdot p_2 \rangle \langle p_2 \cdot p_3 \rangle \langle p_3 \cdot p_1 \rangle}$$

where $\langle p_1 \cdot p_2 \rangle = p_{1A} p_2^A$ etc., the (degenerate) MHV formula for the $- - +$ vertex. In order to obtain this result, we need to use the fact for 3 complex null vectors, the equation $p_1 + p_2 + p_3 = 0$ implies that either all the self-dual, or all the anti-self-dual spinor constituents of the momenta are proportional. That leads to relations that allow one to eliminate the polarization spinors ϵ_1 and ϵ_2 leaving the desired formula.

The propagator can also be represented on space-time following the calculation in §5. This gives the representation for the propagator as the two-point function

$$\frac{s_{AA'} s_{BB'} \mathrm{d}^2 x^{A'B'} \wedge \mathrm{d}^2 y^{AB}}{(s \cdot s)^3} .$$

This is a non-standard expression for the photon propagator because it is appearing in the field representation (i.e., on the 2-forms rather than the potential) and so it is in fact the second derivative of the standard photon propagator. This is related to the fact that the currents are also being represented as 2-forms, i.e., as potentials for their usual 3-form representation. On momentum space this propagator becomes $p_{A(A'} p_{B')B} / p \cdot p$.

7 Conclusions

We see that the propagators and vertices are identical to those of the space-time Lagrangian (1) and the computation of scattering amplitudes from these two theories should therefore be the same at tree level. However, more work is required to test the equivalence or otherwise at the level of loops. The generation of all amplitudes from trivalent vertices is suggestive of the BCFW relations.

The Chern-Simons theory is clearly suggestive of a B-model twistor-string theory in ambitwistor space in which a D7-brane is wrapped on $\mathbb{A}_{\mathbb{E}}$ (with $\bar{\xi} = 0 = \bar{\eta}$); this would, however, be a non-standard construction in string theory as in the B-model branes are usually wrapped on holomorphic cycles. We intend to return to the construction of an ambitwistor-string theory in a subsequent paper.

Clearly there is a good prospect of providing a firm basis to twistor diagram theory. The propagator and vertices are clearly those of twistor diagrams, but there are a number of differences as well. For example, in the twistor diagram approach both $\delta'''(Z \cdot W)$ and $(Z \cdot W)^{-4}$ are represented by the latter, but distinguished by the choice of contour. A more fundamental difference is that vertices have more twistor-functions ending on a twistor than are usually allowed. This is perhaps the most basic expression of infrared divergences. There is clearly much useful work to be done to make contact with the work of Hodges (2005) on the twistor diagrams for gauge theory.

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