Solitons and admissible families of rational curves in twistor spaces

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Abstract

It is well known that twistor constructions can be used to analyse and to obtain solutions to a wide class of integrable systems. In this article we express the standard twistor constructions in terms of the concept of an admissible family of rational curves in certain twistor spaces. Examples of of such families can be obtained as subfamilies of a simple family of rational curves using standard operations of algebraic geometry. By examination of several examples, we give evidence that this construction is the basis of the construction of many of the most important solitonic and algebraic solutions to various integrable differential equations of mathematical physics. This is presented as evidence for a principal that, in some sense, all soliton-like solutions should be constructable in this way.

1 Introduction

Penrose's nonlinear graviton construction [24] realizes any 4-dimensional conformal manifold with anti-self-dual Weyl tensor as a family of rational curves in a complex 3-manifold, $\mathcal{T}$, known as the twistor space. There are many generalizations, most notably the extension by Ward to the anti-self-dual Yang-Mills equations [26], in which any anti-self-dual Yang-Mills field can be reconstructed directly from a family of rational curves in the total space of a holomorphic vector bundle over a region in twistor space.[1] Similar constructions apply to a large family of integrable systems both by considering families of rational curves in larger complex manifolds, or by symmetry reduction of these two basic twistor correspondences, see [21] for a catalogue of such reductions and full details of the twistor correspondences.

In effect, the task of solving these equations reduces to that of the construction of such families of rational curves. A strategy proposed by one of us [9, 11, 12] is to consider *Admissible families* of

[1] This construction is usually viewed in the context of finding trivialisations of the holomorphic vector bundle over the lines in twistor space, but this, by Liouville's theorem, is equivalent to lifting the the lines in twistor space to the total space of the holomorphic vector bundle.

rational curves. An admissible family of curves is a local family of curves that admit an extension in some larger complex manifold to a family of compact rational curves that is complete (in the sense of Kodaira) family of rational curves. To construct such families, one can consider a simple space such as the total space of some bundle $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \oplus \cdots \oplus \mathcal{O}(k_r) \to \mathbb{CP}^1$, $k_1 \geq k_2 \cdots \geq k_r \geq 0$ that has too many rational curves; here the space of sections has dimension $\sum_{i=1}^{r}(k_i + 1)$. One can then restrict the family in such a way that the restricted family is an admissible family with the appropriate dimension and so can be regarded in its own right as a moduli space of rational curves in some (different) twistor space. There is a theorem due to Bernstein & Gindikin [10, 2] to the effect that admissible families can only be obtained from some large family by requiring intersection with or tangency to some family of submanifolds. This, in turn, corresponds to the operation of taking a blowup of or branched cover over the original twistor space at the chosen the submanifolds. These algebraic geometrical modifications of the twistor space can be thought of as finite non-linear deformations of the twistor space in Penrose's nonlinear graviton construction.

Such a procedure will not give rise to the general solution as these methods use data on subspaces of codimension at least 1 to that required for the general solution. Thus these solutions are special. In this paper, our aim is to show that many of the most important soliton-like solutions in mathematical physics arise from this construction. In particular, standard procedures for obtaining special solutions by introducing a hierarchy and imposing hidden symmetries fit naturally into this construction. We present this as evidence for a principal that all interesting soliton-like solutions arise in this way.

In Section 2 we outline the basic methodology associated to families of rational curves. In section 3 we explain how the imposition of restriction on a simple but large admissable family of rational curves arises naturally in situation in which hierarchies are considered and higher symmetries imposed. In the subsequent sections we show how various standard global solutions (solutions arising from the Ward ansatze such as instantons and monopoles in $\mathbb{R}^3$, Korteweg de Vries and non-linear Schrodinger solitons, and ALE gravitational instantons) fit into this framework. For the non-linear graviton construction, the construction of families of rational curves is connected only with the conformal part of the problem. In particular, the problem of constructing a self-dual 4-metric on $\mathcal{M}$ is equivalent to constructing a pair of 1-forms $\varphi(t), \psi(t)$ depending on a rational parameter t such that the 2-form $F(t) = \varphi(t) \wedge \psi(t)$ is quadratic in t [9]. The conformal part of this problem is the integrability of the kernel distribution if $\varphi(t)$ and $\psi(t)$ for all t. This weaker problem can be reformulated as a problem of finding an $\mathcal{O}(1) \oplus \mathcal{O}(1)$-family of rational curves. As a result we need to modify the tangency-intersection construction in such a way that we obtain solution to the complete problem, not just its conformal part. Thus, in Section 5.2 on the construction of ALE solutions, we work with a special type of tangency-intersection condition connected with the lifting of rational curves on certain branched coverings. In the subsequent sections, we demonstrate that classes of three-dimensional Einstein–Weyl spaces, and ODEs satisfying a set of over-determined Wünschmann–type constraints can be reconstructed from admissible families. The basic facts about bundles over $\mathbb{CP}^1$ and the deformation theory are summarised in the Appendix.

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2 Admissible families of rational curves.

In twistor theory, the data of a solution to an integrable equation is encoded into the complex structure of the twistor space $\mathcal{T}$. The reconstruction of the solution from the complex structure on $\mathcal{T}$ reduces to the construction of families of rational curves in $\mathcal{T}$. The parameter space $\mathcal{M}$ of the family is either a space-time or some related space (i.e., the total space of a Yang-Mills bundle over space-time) and the full solution to the integrable equation can be obtained directly from the correspondence with the rational curves in $\mathcal{T}$. In this article, we study such families of rational curves locally in $\mathcal{T}$, and the global condition that the curves extend to global rational curves in $\mathcal{T}$ is replaced by the condition that the family be admissible as defined below.

A family of rational curves in $\mathcal{T}$ is complete if it contains all small deformations in the family. We will give a *local* (in $\mathcal{T}$) characterization of complete families of rational curves. Since we are working locally, this characterization will be birationally invariant. Roughly speaking, this will mean that, perhaps after some extension and modification of $\mathcal{T}$, it will be the family of *all* rational curves of some fixed topological type in some manifold $\widetilde{\mathcal{T}}$. However, after another birational map, it may cease to be a maximal in the new manifold (and in general, it will not be maximal in $\mathcal{T}$). We will work in the complex holomorphic category throughout.

Initially, we will define families of local rational curves (which may not necessarily extend to become global). Let $\mathcal{T}$ be a complex manifold of dimension $r+1$ in which a family of curves are embedded, and let $\mathcal{M}$ be a manifold of dimension n, parametrising the curves with $m \in \mathcal{M}$ corresponding to the curve C_m in $\mathcal{T}$. Our considerations are local in the first instance, and we will assume that everything is in general position.

For $z \in \mathcal{T}$, let Z be the dual submanifold in $\mathcal{M}$, which parametrises the curves C that pass through z. Such a Z will be called an α-surface. Inside the tangent space $T_m\mathcal{M}$ at $m \in \mathcal{M}$, we define an α–plane to be a tangent plane $\alpha(z)$ to some Z, for which $z \in C_m$. Let v_m be the union of these α-planes in T_m.

Definition 2.1 *We will say that v_m satisfies the* locally rational *condition (LR) if it can be presented as the union of 2-dimensional flat cones, i.e., cones sitting inside 2-dimensional subspaces of $T_m\mathcal{M}$. These 2-planes, called β-planes, will be denoted $\beta(s)$ and we will assume that they intersect only at 0 and are transversal to the planes $\alpha(x)$. Let V_m be the union of these 2-planes, clearly $v_m \subset V_m$.*

Lemma 2.2 *If the curves do in fact admit an extension so that they form a complete family of global rational curves, then the LR condition is satisfied.*

Proof: In the case where the curves are global, the normal bundle NC_m of a rational curve C_m in $\mathcal{T}$ will be $\oplus_{i=1}^r \mathcal{O}(k_i)$ for some integers k_i and we will assume that these integers are constant in the family. (Here $\mathcal{O}(k)$ denotes the line bundle of Chern class k on $\mathbb{CP}^1$.) By Kodaira's theorem, $T_m\mathcal{M} = \Gamma(\mathbb{CP}^1_m, N)$. If we express $\mathbb{CP}^1$ as the projectivisation $\mathbb{P}S^*$ of a 2-dimensional complex vector space S^*, we obtain $T_m\mathcal{M} = \oplus_{i=1}^{n-1} \odot^{k_i} S$ (and so $n = \sum_i (k_i+1)$).

Let $\pi \in S$ then the general tangent vector to the α-plane corresponding to the point where π vanishes on $\mathbb{P}S^*$ will lie in the image of the vector space $W = \oplus_{i=1}^r \odot^{k_i-1} S$ embedded into $T_m\mathcal{M}$ by symmetrization of each summand with π. Hence, each element of $\mathbb{P}(W)$ corresponds to the two–plane in $T_m\mathcal{M}$ spanned by the image of symmetrizing each summand with S. $\Box$

The structure of the incidence cones $v_m \subset T_m\mathcal{M}$ supplies the curves C_m with the canonical projective structure of rational curves, induced by the projectivisation of one of the 2-planes $\beta(s)$. This structure is independent of the choice of s. If we consider curves C_m without parameterization, we can identify vectors in $T_m\mathcal{M}$ with sections of the normal bundles NC_m. For manifolds of parameterized curves we can interpret them as sections of tangent bundles TC_m.

Even in the general local case, the cones V_m must be linearly equivalent to cones for the case when $\mathcal{M}$ is the manifold of sections of a vector-bundle $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r)$, $k_1 \geq \cdots \geq k_r \geq 0$. These families of curves play the role of the flat model in this geometry. In the generic situation, V_m for different $m \in \mathcal{M}$ are linearly equivalent and correspond to some choice of $k_1 \geq \cdots \geq k_r \geq 0$.

This gives the manifold $\mathcal{M}$ a generalized conformal structure $\{V_m\}$. It is natural to investigate when this structure uniquely defines the manifold $\mathcal{T}$ with the curves C_m. The first step is to understand when the α and β planes can be uniquely reconstructed. It turns out that this is possible if $V_m \neq \mathbb{C}^p$ for some p, which is equivalent to the condition $k_2 > 0$. Then the α-surfaces, Z are defined as integral submanifolds whose tangent planes are α-planes at every point. It turns out that, with these conditions, the Z, if they exist, are unique (although existence is not automatic). Then we can construct $\mathcal{T}$ as the manifold of integral submanifolds of such Z, and the curves $C_m \subset \mathcal{T}$ defined as the sets of solutions passing through m.

For the definition of Z, in general it is necessary to produce an equation of second order, or to consider over $\mathcal{M}$ the fibering whose fiber over $m \in \mathcal{M}$ parameterizes the set of $\alpha(z) \subset T_m$ and a Frobenius distribution whose projection at an α-plane Z to $\mathcal{M}$ consists of the α-plane Z. Under the condition $k_2 > 0$, this Frobenius distribution can be uniquely reconstructed from the integrability conditions.

Proposition 2.3 (Generalized Desargues theorem [2, 12].) *If the V_m are subspaces of $T_m\mathcal{M}$ which are different from $\mathbb{C}^p$, $p \leq m$ (or equivalently $k_2 > 0$), then the field V_m determines at most one family of α-surfaces parametrized by a manifold $\widetilde{\mathcal{T}}$. If $V_m \subset T_m\mathcal{M}$ arose from a family of curves C_m in a manifold $\mathcal{T}$, then the manifold $\mathcal{T}$ with curves C_m can be reconstructed uniquely as a subset of $\widetilde{\mathcal{T}}$. All curves C_m admit a canonical extension up to global rational curves (after the appropriate extension of the manifold $\mathcal{T}$ to $\widetilde{\mathcal{T}}$).*

In the general case (when the condition of the above proposition is not satisfied) the extendibility of the curves C to global rational curves requires an algebraic condition on the family of α-planes. The simplest way to specify this is to use the (LR) condition as follows. We will say that a family of curves C satisfies the (R) condition if there exists an (LR)-family $\widetilde{C}$ parametrized by $\widetilde{\mathcal{M}}$ consisting of the family of all curves C together with projective parametrizations that are compatible with a local projective structures on them corresponding to a decomposition of V_m into β-planes. This gives an $n+3$-dimensional family $\widetilde{\mathcal{M}}$ of curves fibred over $\mathcal{M}$ with fibres $\mathrm{PSL}(2, \mathbb{C})$. An (LR)-structure on $\widetilde{\mathcal{M}}$ is what we define to be an (R)-structure on $\mathcal{M}$. This (LR)-structure on $\widetilde{\mathcal{M}}$ defines a canonical extension of curves C_m in $\widetilde{\mathcal{T}}$ to global rational curves in a larger manifold. The essential point here is the existence of a canonical decomposition of cones V_m into α-subspaces transversal to β-subspaces.

Let us remark that when $n = r + 1 = 2$ the condition (R) is equivalent to the Cartan condition: curves $U(m)$ are defined by a differential equation of 2-nd order which is a polynomial of degree 3 in the first derivative, i.e., they define a projective structure.

A principal result of the theory of families of rational curves concerns 'admissible' subfamilies:

Definition 2.4 *Given a (perhaps local) family of rational curves with the property (R), a subfamily will be said to be admissible if the property (R) is induced on it.*

It emerges that only very special subfamilies have this a property. Let us start from the generic case.

Proposition 2.5 *Let $\mathcal{M}$ be a family of (global) rational curves on $\mathcal{T}$. Let $T_1, ..., T_p$ be submanifolds in $\mathcal{T}$ of codimensions more then 1 and $S_1, ..., S_q$ have codimension 1 and $s_1, ..., s_q$ are natural numbers. Let $\mathcal{M}(T, S, s)$ be the subfamily of curves C_m which intersect all T_i and have at their intersection with each S_j tangency of order s_j. Then this subfamily is admissible and a generic admissible subfamily can be represent in such a form.*

To obtain all admissible subfamilies, not just the generic ones, we need to replace the conditions of intersection with the T_j by the condition that we take those curves that admit a lift to a tower $\tilde{\mathcal{T}}$ of blow ups of $\mathcal{T}$. Then we consider the hypersurfaces S_j to be in $\tilde{\mathcal{T}}$ and add the condition of tangency of order s_j.

There are two known proofs of this theorem: one use algebraic geometrical methods [2] and the other one uses geometrical methods of nonlinear differential equations [10]. The proof of this theorem splits into several steps: the condition of admissibility of a subfamily $\widetilde{\mathcal{M}} \subset \mathcal{M}$ can be written down as an explicit nonlinear differential equation. It turns out that this equation can be integrated by a generalized Hamilton-Jacobi method with multidimensional bicharacteristics.

3 Hierarchies for the anti-self-duality equations and invariance under hidden symmetries

A key feature of soliton solutions to, for example, the KdV equations is that they arise as solutions that are invariant under one or more hidden symmetries. A hidden symmetry is usually understood in the context of hierarchies associated to the equation.

A hierarchy associated to an integrable system is an overdetermined system of (completely integrable) partial differential equations on a higher-dimensional space, usually the cartesian product of the space-time for the original integrable system with a space of higher times. The hierarchy equations restrict to give the original completely integrable system on each leaf of the foliation on which the 'higher time' variables are constant. The flows along higher time variables are know as hidden symmetries since they evolve solutions to the original equations into different solutions to the original equations.

The connection with the above theory is that the solution to the hierarchy will arise from some admissible family of rational curves, and each solution to the original system at fixed values of the higher times arises from an admissible subfamily. Proposition 2.5 implies that such admissible subfamilies must be obtained by intersection and tangency.

A standard strategy for obtaining soliton solutions is to require that a solution can be embedded into a solution to the hierarchy that is invariant under one or more higher flows. More generally, one can require that the solution to the hierarchy admits one or more symmetries that are not necessarily symmetries of the original system. Thus the solution to the hierarchy can be taken to arise from a simple admissible family, and the solitonic solution to the original system can be obtained by intersection and tangency as described above.

The twistor correspondences generalize straightforwardly to the hierarchy. In the case of the Bogomolny equations, the twistor space corresponding to a solution is the total space of a holomorphic vector bundle over 'minitwistor space', $\mathcal{O}(2)$ (the total space of the line bundle of Chern class 2 over the Riemann sphere, $\mathbb{CP}^1$). In [20] it was shown that the $\mathrm{SU}(N)$ Bogomolny equations embed into a hierarchy, referred to as the Bogomolny hierarchy, for which the twistor space is the total space of a holomorphic vector bundle $\mathcal{O}(n)$ for some (arbitrarily large) $n > 2$. The rational curves in this space have normal bundle that is a direct sum of the trivial $\mathbb{C}^N$ bundle with $\mathcal{O}(n)$. This yields the standard hierarchies for the KdV and the nonlinear Schrodinger equations under symmetry reduction. This was extended in [21] to a correspondence for hierarchies for the $\mathrm{SU}(N)$ ASDYM equations, in which a solution to the hierarchy corresponds to a holomomorphic vector bundle over the total space of $\mathcal{O}(n) \oplus \mathcal{O}(n) \to \mathbb{CP}^1$ so that the rational curves have normal bundle given by the direct sum of $\mathbb{C}^N$ with $\mathcal{O}(n) \oplus \mathcal{O}(n)$. In [6, 7] the twistor correspondences were extended to give a hierarchy for the hyper-Kähler equations in which the twistor space is as usual, a 3-dimensional complex manifold fibred over $\mathbb{CP}^1$ and admits a Poisson structure on the fibres, but now the family of rational curves has normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$.

It is natural, therefore, to regard the geometry arising on the moduli space of rational curves with some arbitrary, but fixed normal bundle, $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \oplus \cdots \oplus \mathcal{O}(k_r) \to \mathbb{CP}^1$, $k_1 \geq k_2 \cdots \geq k_r \geq 0$, as the most general hierarchy associated to equations that admit a twistor correspondence. This can be though of as a set of differential equations implied by the (LR)-condition on the generalised conformal structure formed by the family of incidence cones V_m in each $T_m\mathcal{M}$.

A solution to some version of the anti-self-duality equations will then extend to a hierarchy if it can be realised as arising from an admissible subfamily of the family of rational curves associated to that hierarchy. By proposition 2.5, if one is just given the solution to the hierarchy, admissible subfamilies are found by requiring intersection or tangency to submanifolds and all admissible subfamilies arise in this way.

A key application of hierarchies is to perform symmetry reduction, but with respect to a 'hidden' symmetry and in practice soliton solutions often arise in this way. This will mean that we will consider a solution that can be embedded into a hierarchy that admits at least one explicit symmetry. The hierarchy can admit many symmetries without the original solution admitting any at all.

If one wishes to look for such solutions, one can consider simple solutions to the hierarchy, perhaps even trivial ones corresponding to a constant conformal structure or flat connection, but then find non-trivial solutions by using intersection and tangency to find a non-trivial admissible subfamily.

4 Examples

We have seen then that solutions of many problems of mathematical physics that can be integrated by the inverse problem method require the construction of some family of rational curves. Using Proposition 2.5 it is possible to produce such families as (admissible) subfamilies of some simple families of rational curves depending of a larger number of parameters, for example, the families of sections of a vector bundle on the projective line. Let us remember that these families play the role of flat objects in this geometry and we are interested in solutions that admit embedding in flat solutions of a bigger dimension. Of course, not all local solutions can be produced such a way (such solutions depend on fewer functional parameters and will be partly algebraic) but we can expect that some 'good' global solutions can be included in this construction. They are in a sense quasi soliton solutions. We show here that it is indeed the case for several important problems: we will find intersection-tangency conditions in their solutions.

We believe that it is realistic to build Ansatze for solutions for a number of problems starting from these ideas.

4.1 The Ward ansatze and intersection conditions

In the case of the Ward construction for solutions to the ASD Yang-Mills equations, the twistor space $\mathcal{T}$ is the total space of a holomorphic vector bundle $E \to U$ where U is some region in $\mathbb{CP}^3$ and E is assumed to be trivial on each real line in U (i.e., on each line that is invariant under some anti-holomorphic conjugation $\sigma : U \to U$). Usually, the key step in the construction of a solution is the task of finding an explicit trivialisation of E over the (real) lines in U. This is equivalent to finding the rational curves in E that are sections of E over each line in U.

The process of finding a trivialisation of E over a line in U requires, in the Cech description of the bundle, the solution to a Riemann-Hilbert problem and this is difficult to find explicitly. However, it can be done explicitly in the case where the patching matrix is upper triangular and this is the Ward ansatze. When E has rank two with structure group $\mathrm{SL}(2, \mathbb{C})$ (which we will assume from hereon), this can be expressed as the requirement that E contains a line subbundle L^* and sits in a short exact sequence:

$$0 \longrightarrow L^* \longrightarrow E \longrightarrow L \longrightarrow 0\,.$$

The line bundle L must have non-negative degree $k \geq 0$ on each line if E is to be trivial on each line in U (a positive degree line subbundle L^* would contradict triviality, but not a negative degree one).

The assumption that E admits such a line bundle (or that the patching matrix can be expressed in upper triangular form) is known as the Ward ansatze and has been very fruitful in constructing solutions to the anti-self-dual Yang-Mills equations and its reductions, see Ward (1981). In particular, all instanton solutions can be obtained in this way [1], all monopoles [13], and solitons for the non-linear Schrodinger equations [20]. The following discussion is a paraphrase of Hitchin's discussion of monopoles, [13].

When the solution to the anti-self-dual Yang-Mills equations on space-time has gauge group SU(2) or SU(1,1) on a real slice, the reality conditions imply that the anti-holomorphic involution $\sigma : U \to U$ lifts to give give an isomorphism between $\overline{\sigma^* E}$ and E (see Atiyah 1979 or Mason & Woodhouse 1996 for a full discussion).

We assume that $\hat{L}^* := \overline{\sigma^* L^*}$ is generically a linearly independent line subbundle to L^*. In this case the intersection/tangency ideas of the previous section can be brought into play since we have a map

$$\rho : E \hookrightarrow L \oplus \hat{L}\,,$$

and this will be a fibrewise vector space isomorphism except on the codimension-1 set $T \subset U$ on which L^* and $\hat{L}^*$ coincide as line subbundles of E. Denote the image of ρ over T by $\widetilde{T}$. (Since on T, ρ is onto both L and $\hat{L}$ separately, $\widetilde{V}$ is the graph of an invertible map from L to $\hat{L}$, or alternatively a trivialisation e of $L \otimes \hat{L}^*$.)

The data of L, $\hat{L}$ and $\widetilde{V}$ is sufficient to reconstruct the original solution. Sections of E over lines in U correspond precisely to sections of $L \oplus \hat{L}$ that pass through $\widetilde{T}$. Since the line bundles L and $\hat{L}$ have degree k, there will be $2k+2$ sections of $L \oplus \hat{L}$ over each line in U. The submanifold T must therefore have degree $2k$ so that there are $2k$ conditions on the $2k+2$ sections, reducing the number of sections of E to 2 as required by triviality on lines.

We therefore have

Proposition 4.1 *Suppose $E \to U$ is a rank two holomorphic vector bundle such that $\overline{\sigma^* E} = E$ and there is a line subbundle $L^* \subset E$ such that $\hat{L}^* := \overline{\sigma^* L^*}$ is generically linearly independent from L^*. Then the admissible family of rational curves consisting of sections of E over lines in U is equivalent to the admissible subfamily of sections of $L \oplus \hat{L}$ over lines in U that intersect the codimension-2 subset $\widetilde{T}$ of $L \oplus \hat{L}$ as defined above.*

4.2 The monopole solutions

Monopoles are solutions to the ASD Yang-Mills equations on $\mathbb{R}^4$ with a single translation symmetry and with a finite energy condition. In [13] Hitchin showed that monopoles were, via the Ward correspondence, in a correspondence with holomorphic vector bundles on $T\mathbb{CP}^1$ that are constructed from a certain 'spectral curve' T in $T\mathbb{CP}^1$. The construction involves the use of homogeneous line bundles $L(n) \to T\mathbb{CP}^1$ that can be described as follows. Introduce affine coordinate λ on $\mathbb{CP}^1$ and corresponding fibre coordinate η on $T\mathbb{CP}^1$ so that $(\eta, \lambda) \leftrightarrow \eta\partial/\partial\lambda$. We can define $L(n)$ by the transition function $\lambda^{-n}\exp(\mu/\lambda)$ with respect to the open covering $U_0 = \{\lambda \neq \infty\}$ and $U_\infty = \{\lambda \neq 0\}$. ($\mathcal{O}(n)$ is the pullback of the $-n$th power of the tautological bundle on $\mathbb{CP}^1$ and $L(0)$ is the exponential of the class obtained by pulling back the 'unit element' of $H^1(\mathbb{CP}^1, T^*\mathbb{CP}^1)$ and contracting it with the vector $\eta\partial/\partial\zeta$, and $L(n) = L \otimes \mathcal{O}(n)$.)

Hitchin shows that if $E \to T\mathbb{CP}^1$ is the Ward transform of a monopole solution, then we can express E as an extension in two different ways

$$0 \to L(-k) \stackrel{\iota_+}{\to} E \stackrel{\sigma_+}{\to} L^*(k) \to 0\,, \qquad 0 \to L^*(-k) \stackrel{\iota_-}{\to} E \stackrel{\sigma_-}{\to} L(k) \to 0$$

where $L^*(k) = (L(0))^* \otimes \mathcal{O}(k)$. The map $\sigma_+\iota_-$ determines a section ψ of $\mathcal{O}(2k)$. The zero set of ψ is the spectral curve T. Such spectral curves determine E and are characterised by the conditions that (i) they are compact, (ii) invariant under the real structure $\tau : (\eta, \lambda) \to (-\bar{\eta}/\bar{\lambda}^2, -1/\bar{\lambda})$ and (iii) the line bundle $L(0)^2$ is trivial on restriction to T.

This construction fits into the previous discussion by virtue of the fact that, in order to reverse the Ward construction to reconstruct the solution to the Bogomoln'yi equations, one must first find a holomorphic trvialisation for E over each section $\sigma_x : \mathbb{CP}^1 \to T\mathbb{CP}^1$ that is invariant under the real structure τ, for $x \in \mathbb{R}^3$. This is equivalent to finding the appropriate family of holomorphically embedded rational curves in the total space of E that cover σ_x. By projection in each of the two short exact sequences above, these curves are a subset of those in $\tilde{E} = L(k) \oplus L^*(k)$ over σ_x. They can be characterised as the curves that intersect the codimension-2 subset of $\tilde{E}$ consisting of the the line subbundle $\tilde{T}$ of $\tilde{E}|_T$ defined by the the trivialisation of $L(0)^2$ over T.

It is worth noting that it is straightforward to find the trivialisation of $\tilde{E}$ over each σ_x. This is set out in [13] for $L(0)$ and one can simply multiply that section by a pair of polynomials of degree k in ζ leading to $2k+1$ sections. Given the spectral curve, T and the trivialisation e of $L^2(0)$ over T, the incidence condition with $\tilde{T}$ is $2k$ conditions on the $2k+1$ unknowns and leads to the desired 2-dimensional space of sections of E.

The implementation of this as a strategy for writing down exact solutions of monopole is hard. The principal difficulty is in choosing the spectral curve S (the simplest examples, the axisymmetric solutions, are known, but in general the triviality of $L(0)^2$ on T is a hard condition to impose explicitly). Even then, one must solve for the trivialisation of $L(0)^2$ on T and then find the intersection points of T with a generic σ_x in order to impose the incidence condition with $\tilde{T}$.

4.3 Korteweg de Vries and Non-linear Schrodinger solitons

In [20] it is shown that NLS solitons can be obtained from the Ward ansatze also, and so can be obtained from imposing intersection conditions on some simple admissible family. Here we can see that the construction can be made very explicit.

The KdV and NLS equations arise as symmetry reductions of the $\mathrm{SL}(2,\mathbb{C})$ Bogomolny equations in 2+1 signature under a null translation. The corresponding holomorphic vector bundles over regions in $\mathcal{O}(2)$ have rank two and admit a lift of the symmetry $K = \partial_\mu$ where $(\lambda\mu)$ are coordinates on $\mathcal{O}(2)$, with λ being an affine coordinate on the base $\mathbb{CP}^1$ and μ the fibre coordinate on $\mathcal{O}(2)$. Since $\mathcal{O}(2) = T\mathbb{CP}^1$, the choice of λ determines the trivialisation in which μ corresponds to the tangent vector $\mu\partial/\partial\lambda$.

Over a neighbourhood of $\lambda = \infty$, we use coordinates $(\lambda', \mu') = (1/\lambda, \mu/\lambda^2)$. In these coordinates $V = \lambda'^2 \partial/\partial\mu'$ and so K fixes the fibre at ∞ to 2nd order. The symmetry reductions to the KdV and NLS equations are distinguished by the action of the lift $\widetilde{K}$ of K to the bundle E at $\lambda = \infty$ to second order: we have $\widetilde{K} = \Lambda + O(1/\lambda^2)$ where

$$\text{For NLS} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{and for KdV} \qquad \Lambda = \begin{pmatrix} 0 & 1 \\ 1/\lambda & 0 \end{pmatrix}.$$

The trivial solutions correspond to bundles E_0 with $\widetilde{K}$ given as above on a neighbourhood of $\lambda = \infty$ and patching function $\exp(\mu\Lambda)$ to a trivialisation that extends over $\lambda \neq \infty$ in which $\widetilde{K}$ has trivial lift.

The solitons can be obtained from the trivial solutions by considering first $E_0(k) = \mathcal{O}(k) \otimes E_0$. This bundle has $2k+2$ sections over each conic in $\mathcal{O}(2)$. We choose an admissible subfamily by choosing first k points $\{\lambda_1, \cdots, \lambda_k\}$ in the upper half plane in $\mathbb{C}$, and then an invariant section γ_i of $\mathbb{P}(E_0^*)$ over each fibre $\lambda = \lambda_i$ of $\mathcal{O}(2)$. We require that γ_i does not lie in an eigenspace for Λ. This data determines $2k$ codimension-2 submanifolds of $E_0(k)$, the kernel of γ_i over $\lambda = \lambda_i$ and the kernel of $\bar{\gamma}_i$ over $\lambda = \bar{\lambda}_i$. (Note that $\bar{\gamma}_i$ needs to be interpreted appropriately according to the reality condition that we wish the final solution to satisfy).

In order to see explicitly that this yields the standard formulae for the appropriate admissible subfamily of the space of sections of $E_0(k)$ over a given conic C in $\mathcal{O}(2)$, we first represent the sections of $E_0(k)$ over C as sections of E_0 that have simple poles at each λ_i. The residues must lie in the kernel of the corresponding γ_i and k of the remaining $k+2$ coefficients are fixed by the k conditions of lying in the kernels of the $\bar{\gamma}_i$ at $\bar{\lambda}_i$. See example 9.3.3 and section 12.4 from [21] for further details of such solutions.

4.4 The ALE solutions

In the previous subsections we have seen that see that we can express many of the most familiar soliton/instanton solutions in terms of subfamilies of some simple admissible family by imposing intersection conditions. In this subsection we see that the the ALE hyper-Kahler solutions (gravitational instantons) can be expressed as an admissible subfamily of a simple family by imposing tangency conditions.

Hyper-Kähler manifolds $(\mathcal{M}, g)$ that have the topology of $\mathbb{R}^4$ at infinity, and approach the flat Euclidean metric $\eta = \mathrm{d}{x_1}^2 + ... + \mathrm{d}{x_4}^2$ sufficiently fast, in the sense that

$$g_{ab} = \eta_{ab} + O(r^{-4}), \qquad (\partial_a)^p(g_{bc}) = O(r^{-4-p}), \qquad r^2 = x_1^2 + ... + x_4^2$$

have to be flat. A weaker asymptotic condition one can impose is that g should be asymptotically *locally* Euclidean (ALE).

The ALE spaces are non-compact, complete hyper-Kähler manifolds which satisfy the above condition only locally for $r \to \infty$. Globally the neighbourhood of infinity must look like $S^3/\Gamma \times \mathbb{R}$, where Γ is a finite group of isometries acting freely on S^3 (a Kleinian group). These manifolds belong to the class of *gravitational instantons* because their curvature is localised in a 'finite region' of a space-time.

Finite subgroups of $\Gamma \subset SU(2)$ correspond Platonic solids in $\mathbb{R}^3$. They are the cyclic groups, and the binary dihedral, tetrahedral, octahedral and icosahedral groups (one can think about the last three as Möbius transformations of $S^2 = \mathbb{CP}^1$ which leave the points corresponding to vertices of a given Platonic solid fixed). Each of them can be related to a Dynkin diagram of a simple Lie algebra. All Kleinian groups act on $\mathbb{C}^2$, and the 'infinity' $S^3 \subset \mathbb{C}^2$. Let $(z_1, z_2) \in \mathbb{C}^2$. For each Γ there exist three invariants x, y, z which are polynomials in (z_1, z_2) invariant under Γ. These invariants satisfy some algebraic relations which we list below:

Group	**Dynkin diagram**	**Relation** $F_\Gamma(x,y,z) = 0$
cyclic	A_k	$xy - z^k = 0$
dihedral	D_{k-1}	$x^2 + y^2 z + z^k = 0$
tetrahedral	E_6	$x^2 + y^3 + z^4 = 0$
octahedral	E_7	$x^2 + y^3 + yz^3 = 0$
icosahedral	E_8	$x^2 + y^3 + z^5 = 0$

In each case

$$\mathbb{C}^2/\Gamma \subset \mathbb{C}^3 = \{(x,y,z) \in \mathbb{C}^3, F_\Gamma(x,y,z) = 0\}.$$

The manifold $\mathcal{M}$ on which an ALE metric is defined is obtained by minimally resolving the singularity at the origin of $\mathbb{C}^2/\Gamma$. This desingularisation is achieved by taking $\mathcal{M}$ to be the zero set of

$$\widetilde{F}_\Gamma(x,y,z,\lambda) = F_\Gamma(x,y,z) + \sum_{i=1}^{r} a_i(\lambda) f_i(x,y,z),$$

where f_i span the ring of polynomials in (x, y, z) divided by the ideal generated by

$$< \partial_x F_\Gamma, \partial_y F_\Gamma, \partial_z F_\Gamma > .$$

The dimension r of this ring is equal to the number of non-trivial conjugacy classes of Γ which is $k-1, k+1, 6, 7$ and 8 respectively [3]. Kronheimer [17, 18] proved that for each Γ a unique hyper-Kähler metric exists on a minimal resolution $\mathcal{M}$, and that this metric is precisely the ALE metric with $\mathbb{R}^4/\Gamma$ as its infinity. His construction was a combination of the hyper-Kähler quotient [15] with the twistor theory.

The degrees p, q and r are such that $\widetilde{F}_\Gamma(x, y, z, \lambda)$ is a function homogeneous of some degree s. Therefore

$$\widetilde{F}_\Gamma : \mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \to \mathcal{O}(s).$$

To determine the integers p, q, r, s, we require that the normal bundle to a section of $\mathcal{PT} \longrightarrow \mathbb{CP}^1$ should have the Chern class 2. To impose this we restrict the Jacobian of the above map to the normal bundle of the curve, and notice that the Chern class is $p+q+r-s$ which should therefore be 2. This gives us the following

$$\begin{array}{rl}
A_k & \mathcal{T} = \{(x, y, z, \lambda) \in \mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2) \longrightarrow \mathbb{CP}^1, \\
 & xy - z^k - a_1 z^{k-2} - \ldots a_{k-1} = 0\}, \\
D_{k-1} & \mathcal{T} = \{(x, y, z, \lambda) \in \mathcal{O}(2k) \oplus \mathcal{O}(2k-2) \oplus \mathcal{O}(4) \longrightarrow \mathbb{CP}^1, \\
 & x^2 + y^2 z + z^k + a_1 y^2 + a_2 y + a_3 z^{k-2} + \ldots + a_k z + a_{k+1} = 0\}, \\
E_6 & \mathcal{T} = \{(x, y, z, \lambda) \in \mathcal{O}(12) \oplus \mathcal{O}(8) \oplus \mathcal{O}(6) \longrightarrow \mathbb{CP}^1, \\
 & x^2 + y^3 + z^4 + y(a_1 z^2 + a_2 z + a_3) + a_4 z^2 + a_5 z + a_6 = 0\} \\
E_7 & \mathcal{T} = \{(x, y, z, \lambda) \in \mathcal{O}(18) \oplus \mathcal{O}(12) \oplus \mathcal{O}(8) \longrightarrow \mathbb{CP}^1, \\
 & x^2 + y^3 + yz^3 + y^2(a_1 z + a_2) + y(a_3 z + a_4) + a_5 z^2 + a_6 z + a_7 = 0\} \\
E_8 & \mathcal{T} = \{(x, y, z, \lambda) \in \mathcal{O}(30) \oplus \mathcal{O}(20) \oplus \mathcal{O}(12) \longrightarrow \mathbb{CP}^1, \\
 & x^2 + y^3 + z^5 + y(a_1 z^3 + a_2 z^2 + a_3 z + a_4) + a_5 z^3 + a_6 z^2 + a_7 z + a_8 = 0\}
\end{array}$$

In each case the twistor space is the three dimensional hyper-surface $\widetilde{F}_\Gamma(x, y, z) = 0$ in the rank-three bundle $\mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \to \mathbb{CP}^1$ where now $x(\lambda), y(\lambda), z(\lambda)$ are coordinates up the fibres of $\mathcal{O}(p), \mathcal{O}(q), \mathcal{O}(r)$ respectively, $f_i = f_i(x, y, z)$, and $a_i = a_i(\lambda)$ is a global section of the appropriate power of $\mathcal{O}(1)$ to make $\widetilde{F}$ homogeneous. Therefore we have projections

$$f_p : \mathcal{PT} \longrightarrow \mathcal{O}(p), \qquad f_q : \mathcal{PT} \longrightarrow \mathcal{O}(q), \qquad f_r : \mathcal{PT} \longrightarrow \mathcal{O}(r),$$

and we can, for example, express $\mathcal{PT}$ as a branched cover of $\mathcal{O}(p) \oplus \mathcal{O}(q)$ branched over the singular locus of $f_p \oplus f_q$. Rational curves in $\mathcal{PT}$ project to give rational curves in $\mathcal{O}(p) \oplus \mathcal{O}(q)$ tangent to the singular locus of $f_p \oplus f_q$ and the condition that a rational curve (i.e., a section) in $\mathcal{O}(p) \oplus \mathcal{O}(q)$ admits a lift to $\mathcal{PT}$ is that it should be tangent to the singular locus of $f_p \oplus f_q$. This is an admissible subfamily by 2.5. Thus, in particular, we see that the ALE spaces can be realized as admissible subfamilies of the spaces of sections of $\mathcal{O}(p) \oplus \mathcal{O}(q)$, in fact in three different ways.

4.5 Three–dimensional Einstein–Weyl structures

Here we consider classes of Einstein–Weyl spaces of dimension three. Such spaces arise as the generic geometry on the space of rational curves lying in a surface with normal bundle $\mathcal{O}(2)$. In this section we see how examples of 3–dimensional Einstein–Weyl spaces can be constructed by considering admissible subfamilies of relatively simple higher dimensional families of rational curves on surfaces by imposing intersection and tangency.

Let $\mathcal{M}$ be an 3-dimensional manifold with a torsion-free connection D, and a conformal structure $[h]$ which is compatible with D in a sense that $Dh = \omega \otimes h$ for some one-form ω. Here $h \in [h]$ is a representative metric in a conformal class. If we change this representative by $h \rightarrow \psi^2 h$, then $\omega \rightarrow \omega + 2\mathrm{d}\ln\psi$, where ψ is a non-vanishing function on $\mathcal{M}$. The space of oriented D–geodesics in $\mathcal{M}$ is a manifold $\mathcal{T}$ of dimension 4. There exists a fixed point free map $\tau : \mathcal{T} \longrightarrow \mathcal{T}$ which reverses an orientation of each geodesics. Let γ be an oriented geodesic in $\mathcal{M}$, and let U be a vector field tangent to γ.

The almost-complex structure on $\mathcal{T}$ defined by

$$J(V) = \frac{U \times V}{\sqrt{h(U,U)}},$$

is integrable if for any choice of $h \in [h]$ the symmetrised Ricci tensor of D is proportional to h. This is the conformally invariant Einstein–Weyl condition. Hitchin [13] has demonstrated the one-to-one correspondence between local solutions solutions to the Einstein–Weyl equations, and complex surfaces (twistor spaces) $\mathcal{T}$ equipped with a fixed-point free anti-holomorphic involution τ, and a τ-invariant rational curve with a normal bundle $\mathcal{O}(2)$.

The EW space can be completely reconstructed form the twistor data; Since $H^0(\mathbb{CP}^1, \mathcal{O}(2)) = \mathbb{C}^3$, and $H^1(\mathbb{CP}^1, \mathcal{O}(2)) = 0$ we can use Kodaira's theorem A.1. The EW space is a space of those $\mathcal{O}(2)$ curves which are τ-invariant. The family of such curves passing through a given point (and its conjugate) is a geodesic of a Weyl connection of D. To construct a conformal structure $[h]$ consider a point on a τ- invariant $\mathcal{O}(2)$ curve C_m. This point represents a point in a sphere of directions $(T_m\mathcal{M} - 0)/\mathbb{R}^+$, and the conformal structure on C_m induces a quadratic conformal structure in $\mathcal{M}$.

One class of solutions can constructed by taking an n–fold covering of a neighbourhood of a $(1, n)$ curve $\zeta = P(\lambda)/Q(\lambda)$ in the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$. Here ζ and λ are affine coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and P, Q are polynomials of degree n in λ. The curve has a normal bundle $\mathcal{O}(2n)$, and the space of such curves is parametrised by $\mathbb{CP}^{2n+1}$ minus the hypersurface where the resultant of P and Q vanishes. In [22] Pedersen considered an n-fold cover $\mathcal{T}$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a fixed curve $\zeta = \lambda^n$. The $(1, n)$ curves which meet the fixed curve to the nth order give rise to curves with a normal bundle $\mathcal{O}(2)$ in $\mathcal{T}$ satify the condition

$$\frac{P(\lambda)}{Q(\lambda)} = \lambda^n - \frac{(a\lambda^2 + b\lambda + c)^n}{Q(\lambda)}.$$

Here a, b, c are complex coordinates on the resulting EW space. More work is required to impose Euclidean or Lorentzian reality conditions.

Another class of EW spaces could be constructed by blowing up a point on the quadric, and considering all $(1, n)$ curves passing through this point. The resulting curves in the blown up surfaces have normal bundle $\mathcal{O}(2n - 1)$. This process may also be combined with the taking the branched covering.

The explicit forms of resulting EW structures were determined only for $n = 2$ and $n = 3$ [22, 23]. They are quite complicated but components of h and ω are algebraic expressions in local coordinates (a, b, c). According to our proposal these solutions should be regarded as 'solitons' of the EW geometry. Further analysis of the corresponding conformal invariants is required to justify this claim.

Yet another class of examples arises form a rational curve in $\mathbb{CP}^2$ of degree $d > 1$ whose singularities are $D = (d - 1)(d - 2)/2$ distinct nodes. Let $\mathcal{T}$ be a surface obtained from $\mathbb{CP}^2$ by blowing up points on $S \cup N$, where S is a set of s non-singular points and N is the set of n nodes on the curve. The resulting curve in the blown up space will have a normal bundle $\mathcal{O}(k)$ where k depends on n and s.

4.6 Generalised Wünschmann conditions

As a final application of the above ideas, related to the previous subsection, we consider an n-dimensional family of curves in a surface Σ. Such curves have a natural lift to the bundle of $(n-2)$–jets of such curves $\mathcal{T} = J^{n-2}\Sigma$ over Σ. This family of curves will be expressed as a family of solutions to an ordinary differential equation of order n below and will be considered as a family of curves in $\mathcal{T}$. We then ask whether we can characterise those differential equations that give rise to an admissible family of rational curves in $\mathcal{T}$. This turns out to be given by what we will refer to as the *Wünschmann condition*, Wünschmann (1905), and its generalisations on the coefficients of the equations. One can ask whether the projection of this family to a families of curves in $J^r\Sigma$, $r < n-2$, are admissible, and these will lead to further generalisations of the Wünschmann conditions.

Consider a relation of the form

$$\Psi(x, y, m) = 0$$

between the complex variables $m = (m_1, m_2, ...m_n)$ (local coordinates on an n-dimensional manifold $\mathcal{M}$), and (x, y) (complex local coordinates on a two-dimensional manifold Σ). For each fixed choice of (x, y) the relation defines an α-surface in $\mathcal{M}$. Conversely each choice of m defines a curve C_m in Σ. We can apply the implicit function theorem to $\Psi = 0$, and regard C_m as a graph $x \longrightarrow (x, y = Z(x, m))$. Consider a system of equations consisting of $y = Z(x, m)$, and the first $(n-1)$ derivatives with respect to x. Solving this system for m, and differentiating once more with respect to x yields the ODE

$$y^{(n)} := \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = F(x, y, y', ..., y^{(n-1)}),$$

where the explicit form of F is completely determined by Ψ.

Asking that the α-sufaces in $\mathcal{M}$ arise from specific geometric structures on $\mathcal{M}$ (which from now on will be identified with the space of solutions to the ODE) imposes additional constraints on F. This idea goes back to Cartan [5], and his program of ‘geometrising’ ODEs.

A different approach based on twistor theory was suggested by Hitchin [14] and LeBrun [19]. In this approach the relation $\Psi = 0$ represents part of a rational curve in $\Sigma = \mathcal{T}$ with a prescribed normal bundle. The local differential geometry of $\mathcal{M}$ is encoded in the global complex structure of $\mathcal{T}$, and the globality of the curve implies that α-curves are the geodesics of a projective structure. The ODE does not explicitly appear in the correspondence between $\mathcal{M}$ and $\mathcal{T}$. The details of the Hitchin-LeBrun construction and its connection with the ODE approach have only been worked out fully for $n = 2$. In this case there exists an embedding of rational curve with a normal bundle $\mathcal{O}(1)$ in $\mathcal{T}$ if and only if

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{11} - 4\frac{\mathrm{d}}{\mathrm{d}x}F_{01} - F_1\frac{\mathrm{d}}{\mathrm{d}x}F_{11} + 4F_1F_{01} - 3F_0F_{11} + 6F_{00} = 0,$$

where $F_0 = \partial F/\partial y, F_1 = \partial F/\partial y'$ and $\mathrm{d}/\mathrm{d}x = \partial/\partial x + y'\partial/\partial y + F\partial/\partial y'$. The two-dimensional moduli space $\mathcal{M}$ of $\mathcal{O}(1)$ curves (the space of solutions to the ODE is in this case equipped with a projective structure, in the sense that the α-surfaces (here curves) of constant (x, y) are the geodesics of a torsion-free projective connection. Conversely, given a projective structure on $\mathcal{M}$ one defines $\mathcal{T}$ as a quotient space of the foliation of $P(T\mathcal{M})$ by the orbits of the geodesic flow. Each projective tangent space $P(T_m\mathcal{M})$ maps to a rational curve with self-intersection number one in $\mathcal{T}$.

The case $n = 3$ which goes back to Cartan [5] and was recently revisited by Tod [25]. The conformal structure on $\mathcal{M}$ is defined by demanding that hyper-surfaces $z \subset \mathcal{M}$ corresponding to points in $\mathcal{T}$ are null (it could also be defined by declaring the curves in $\mathcal{M}$ that correspond to points of $\mathcal{T}J^1\Sigma$ to be null geodesics). This conformal structure doesn't depend on $(x,y)\in\Sigma$ if $F(x,y,y',y'')$ satisfies a second-order differential constraint

$$\frac{1}{3}F_2\frac{\mathrm{d}}{\mathrm{d}x}F_2-\frac{1}{6}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}F_1-\frac{2}{27}\left(F_2\right)^3-\frac{1}{3}F_2F_1-F_0=0.$$

This constraint has already appeared in a work Wünschmann [28].

The only other case which has attracted some attention is $n=4$. Bryant [4] has shown that there exist a correspondence between a class of fourth order ODEs, and exotic non-metric holonomies in dimension four. The conditions on F are only implicit in Bryant's work.

We shall say that F satisfies the generalised Wünschmann conditions if there exists $SL(2,\mathbb{C})$ invariant paraconformal structure

$$T\mathcal{M}\cong\mathbb{C}^2\odot\mathbb{C}^2\odot...\odot\mathbb{C}^2=\mathrm{S}^{n-1}\mathbb{C}^2.$$

The explicit form of the generalised Wünschmann conditions has been worked out recursively in [8]. For example if $n=4$ one gets

$$\begin{aligned}
&\frac{11}{1600}\left(F_3\right)^4-\frac{9}{50}\left(F_3\right)^2\frac{\mathrm{d}}{\mathrm{d}x}F_3-\frac{1}{200}\left(F_3\right)^2F_2+\frac{21}{100}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_3\right)^2+\frac{1}{50}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_3\right)F_2\\
&-\frac{9}{100}\left(F_2\right)^2+\frac{7}{20}F_3(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_3-\frac{1}{5}\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_3+\frac{3}{10}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_2-\frac{1}{4}F_3\frac{\mathrm{d}}{\mathrm{d}x}F_2-F_0=0,\\
&\frac{9}{4}F_3\frac{\mathrm{d}}{\mathrm{d}x}F_3-\frac{3}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_3+3\frac{\mathrm{d}}{\mathrm{d}x}F_2-\frac{3}{8}\left(F_3\right)^3-\frac{3}{2}F_2F_3-3F_1=0.
\end{aligned}$$

If F satisfies these conditions, then the space $\mathcal{M}$ of solutions to the corresponding ODE is equipped with a torsion–free connection with holonomy G_3 in the terminology of [4].

In general F has to satisfy an over-determined system of $n-2$ PDEs, and a priori it is not clear that any solutions exist. It can however be verified that the method of admissible curves provides (some) solutions to all the constraints. For example one can consider the blow-ups applied to described in the last section to find a simple solutions $F=(4/3)(y''')^2/y''$, or $F=(ay'''+b)^{4/3}$ when $n=4$.

A Appendix: Rational curves and their embedings

Let $\mathbb{C}^2$ be a vector space with coordinates $\pi=(\pi_0,\pi_1)$. Remove $\pi=(0,0)$ and use $[\pi]$ as homogeneous coordinates on $\mathbb{CP}^1$. We shall also use the affine coordinate $\lambda=\pi_0/\pi_1$. Holomorphic functions on $\mathbb{C}^2-0$ extend to holomorphic functions on $\mathbb{C}^2$ (Hartog's Theorem). Therefore homogeneous functions on $\mathbb{CP}^1$ are polynomials. In particular, holomorphic functions homogeneous of degree 0 are constant (Liouville theorem). Let us summarize some facts about holomolrphic line bundles over $\mathbb{CP}^1$. First define a tautological line bundle

$$\mathcal{O}(-1)=\{(\lambda,(\pi_0,\pi_1))\in\mathbb{CP}^1\times\mathbb{C}^2|\lambda=\pi_0/\pi_1\}.$$

Other line bundles can be obtained from $\mathcal{O}(-1)$ by algebraic operations:

$$\mathcal{O}(-n)=\mathcal{O}(-1)^{\otimes n},\qquad\mathcal{O}(n)=\mathcal{O}(-n)^*,\qquad\mathcal{O}=\mathcal{O}(-1)\otimes\mathcal{O}(1),\qquad n\in\mathbb{N}.$$

Equivalently $\mathcal{O}(n)$ denotes the line bundle over $\mathbb{CP}^1$ with transition functions λ^{-n} from the set $\lambda\neq\infty$ to $\lambda\neq 0$ (i.e. Chern class n). Its sections are given by functions homogeneous of degree n in a sense that $f(\xi\pi)=\xi^nf(\pi)$. These are polynomials in λ of degree n with complex coefficients. The theorem of Grothendick states that all holmorphic line bundles over a rational curve are equivalent to $\mathcal{O}(n)$ for some n. The spaces of global sections, and the first cohomology groups are

$$H^0(\mathbb{CP}^1, \mathcal{O}(n)) = \begin{cases} 0 & \text{for } n < 0 \\ \mathbb{C}^{n+1} & \text{for } n \geq 0. \end{cases} \qquad H^1(\mathbb{CP}^1, \mathcal{O}(-n)) = \begin{cases} 0 & \text{for } n < 2 \\ \mathbb{C}^{n-1} & \text{for } n \geq 2. \end{cases}$$

The following result of Kodaira underlies the twistor approach to curved geometries. Let $\mathcal{T}$ be a complex manifold of dimension $d + r$. A pair $(E, \mathcal{M})$ is called *a complete analytic family of compact sub-manifolds of $\mathcal{T}$ of dimension d* if

- E is a complex analytic sub-manifold of $\mathcal{T} \times \mathcal{M}$ of codimension r with the property that for each $m \in \mathcal{M}$ the intersection $C_m := E \cap (\mathcal{T} \times m)$ is a compact sub-manifold of $\mathcal{T} \times m$ of dimension d.
- There exists an isomorphism
$$T_m\mathcal{M} \simeq H^0(C_m, NC_m)$$
where $NC_m \longrightarrow C_m$ is the normal bundle of C_m in $\mathcal{T}$.

Theorem A.1 (Kodaira[16]) *Let E be a complex compact sub-manifold of $\mathcal{T}$ of dimension d, and let NE be the normal bundle of E in $\mathcal{T}$. If $H^1(E, NE) = 0$ then there exists a complete analytic family $(E, \mathcal{M})$ such that $E = E(m_0)$ for some $m_0 \in \mathcal{M}$.*

We will apply the above theorem to the situation when $\mathcal{T}$ is a twistor space and $E = \mathbb{CP}^1$. Roughly speaking, the moduli space $\mathcal{M}$ is the 'arena' of differential geometry and integrable systems.

References

[1] Atiyah, M.F. & Ward, R.S. (1977) Instantons and algebraic geometry, *Comm. Math. Phys.*, **55**, 111-124.

[2] Bernstein, J.& Gindikin S.(2003) Notes on Integral geometry for Manifolds of Curves, Amer.Math Soc.Transl.(2) **vol 210**, 57–80

[3] Brieskorn, E. (1970) Singular elements of semisimple algebraic groups, Actes Congres Intern. Math. **vol 2**, 279-284.

[4] Bryant, R.L. (1991) Two exotic holonomies in dimension four, path geometries, and twistor theory. Proc. Symp. Pure. Maths. Vol. 53, 33–88

[5] Cartan, E.(1941) La Geometria de las Ecuaciones Diferenciales de Tercer Orden. Rev. Mat. Hispano-Amer. 4, 1–31

[6] Dunajski, M.& Mason, L.J. (2000) Hyper-Kähler Hierarchies and their twistor theory Comm. Math. Phys **213** 641-672

[7] Dunajski, M. & Mason, L.J. (2003) *Twistor theory of hyper-Kähler metrics with hidden symmetries*, J. Math. Phys. **44**, 3430-3454.

[8] Dunajski, M. & Tod K.P. (2004) A geometry of nth order ODEs. Preprint.

[9] Gindikin, S. (1982) Bundles of differential forms and the Einstein equation, Nuclear Phys. **36**, 2(8), 537-548.

[10] Gindikin S.(1984) Reduction of manifolds of rational curves and related problems of differential equations, Funktionsal'nyi Analiz Ego Prilozheniya, **18** No.4, 14–39 (Russian). English translation.Functional Anal.Appl.**18**,4,278–298

[11] Gindikin, S. (1985) Some solutions of the self-duality Einsteinequations, Funktionsal'nyi Analiz Ego Prilozheniya, Vol. 19, No.3, pp58-60 (Russian). English translation English translation.Functional Anal.Appl.**19**,3,210–213

[12] Gindikin, S. (1990) Generalized conformal structures, in *Twistors in Mathematics and Physics*, eds Bailey, T.N. & Baston, R.J., LMS lecture notes series 156, CUP.

[13] Hitchin, N. (1982) Monopoles and geodesics, Commun. Math. Phys. **83**, 589-602.

[14] Hitchin, N. (1982) Complex manifolds and Einstein's equations, in *Twistor Geometry and Non-Linear systems*, Springer LNM 970, ed. Doebner, H.D. & Palev, T.D..

[15] Hitchin, N.J., Karlhede, A., Lindstrom, U.& Rocek, M. (1987) Comm. Math. Phys **108** 21

[16] Kodaira, K. (1963) On stability of compact submanifolds of complex manifolds, Am. J. Math. **85**, 79-94.

[17] Kronheimer, P. (1989) The Construction Of ALE Spaces As Hyperkahler Quotient, J.Diff.Geom. **29** 665

[18] Kronheimer, P. (1989) A Torelli Type Theorem For Gravitational Instantons, J.Diff.Geom. **29** 685

[19] LeBrun, C.R. (1980) Spaces of complex geodesics and related structures, D.Phil. thesis, Oxford University,

[20] Mason, L.J. & Sparling, G.A.J. (1992) Twistor correspondences for the soliton hierarchies, J. Geom. Phys., **8**, 243-271.

[21] Mason, L.J., and Woodhouse, N.M.J. (1996) Integrability, Self-duality and Twistor theory, LMS Monograph, CUP.

[22] Pedersen, H. (1986) Einstein–Weyl spaces and $(1, n)$ curves in the quadric surface, Ann.Global Anal.Geom. 4, No 1, 89–120.

[23] Pedersen, H. & Tod, K.P. (1993) Three-Dimensional Einstein–Weyl Geometry, Adv. in Math. **97** 74-109.

[24] Penrose, R. (1976) Nonlinear gravitons and curved twistor theory, Gen. Rel. Grav. **7**, 31-52.

[25] Tod, K.P. (2000) Einstein-Weyl Spaces and Third Order Differential Equations, J. Math, Phys. 41, 5572.

[26] Ward, R.S. (1977) On self-dual gauge fields, Phys. Lett. **61A**, 81-82.

[27] Ward, R.S. (1981) Ansatze for self-dual Yang-Mills fields, Comm. Math. Phys., **80**, 563-74.

[28] Wünschmann, K.W. (1905) Über Beruhrungsbedingungen bei Dierentialgleichungen, Dissertation, Greifswald.