Yang-Baxter maps and symmetries of integrable equations on quad-graphs

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Abstract

A connection between the Yang-Baxter relation for maps and the multi-dimensional consistency property of integrable equations on quad-graphs is investigated. The approach is based on the symmetry analysis of the corresponding equations. It is shown that the Yang-Baxter variables can be chosen as invariants of the multi-parameter symmetry groups of the equations. We use the classification results by Adler, Bobenko and Suris to demonstrate this method. Some new examples of Yang-Baxter maps are derived in this way from multi-field integrable equations.

1 Introduction

The quantum Yang-Baxter (YB) equation has its origins in the theory of solvable models in statistical mechanics [1, 2] and the quantum inverse scattering method [3]. The fact that this equation has also found many applications in representation theory, the construction of invariants in knot theory and that it lies at the foundation of quantum groups, gives to the quantum YB equation a prominent position among the basic equations in mathematical physics, see e.g. [4], [5] and references therein.

In its original form, the quantum YB equation is a relation for a linear operator $R : V \otimes V \to V \otimes V$, where V is a vector space. The relation has the form

$$R^{23}\, R^{13}\, R^{12} \, = R^{12}\, R^{13}\, R^{23} \,, \tag{1}$$

in $End(V \otimes V \otimes V)$, where R^{13} is meant as the identity in the second factor of the tensor product $V \otimes V \otimes V$ and as R in the first and third factors, and analogously for R^{12}, R^{23}. Supposing that X is any set, the maps R from the Cartesian product $X \times X$ into itself, which satisfy the relation (1) of composite maps are called set theoretic solutions of the quantum YB relation. The study of set theoretic solutions of the quantum YB equation was originally suggested by Drinfeld [6] (see also earlier work by Sklyanin [7], where the first interesting example of such solution was found) and since then they have attracted the interest of many researchers.

More recently, a general theory on the set theoretic solutions to the YB relation was developed in [8] and the notion of transfer maps, which can be considered as the dynamical analogues of the monodromy and transfer matrices in the theory of solvable models in statistical mechanics, was introduced in [9]. In many interesting examples of YB maps [1], such as maps arising from geometric crystals [10], the set X has the structure of an algebraic variety and R is a birational isomorphism. The case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ has been recently discussed in [11] in relation with the classification of the so-called *quadrirational maps*.

In this paper we investigate the relation between the YB property for maps and the *multi-dimensional consistency* condition for equations on quad-graphs, which is now commonly accepted as a definition of integrability for such equations (see [12, 13, 14]). Although the link between these two notions was known before (see e.g. concluding remarks in [14]) it was never explored systematically.

Our approach is based on the symmetry analysis of integrable equations on quad-graphs. The main idea is that the YB variables are suitable invariants of their symmetry groups. A good example is the *discrete potential Korteweg – de Vries* equation (dpKdV) [15, 16, 17]

$$(f_{1,2} - f)(f_1 - f_2) - \alpha_1 + \alpha_2 = 0 \,, \tag{2}$$

(see notation on Fig. 1). It is clearly invariant under the translation $f \to f + \text{const}$. The invariants

[1] Adopting the terminology in [9], set theoretic solutions to the YB equation will be referred in the following simply as YB maps.

$$x = f_1 - f, \quad y = f_{1,2} - f_1, \quad u = f_{1,2} - f_2, \quad v = f_2 - f, \tag{3}$$

satisfy the relation

$$x + y = u + v. \tag{4}$$

and equation (2) is written in terms of them as

$$(x + y)(x - v) = \alpha_1 - \alpha_2. \tag{5}$$

This allows to express u, v as functions of x, y, which leads to the following YB map

$$u = y + \frac{\alpha_1 - \alpha_2}{x + y}, \qquad v = x - \frac{\alpha_1 - \alpha_2}{x + y}\,, \tag{6}$$

known as the Adler map [18]. Note that the YB variables x, y, u, v, are attached to the edges of the lattice. The fact that the corresponding map satisfies the YB property follows directly from the 3D consistency property of dpKdV (see Fig. 5). This construction works for integrable equations on quad-graphs with one-parameter symmetry group.

One of the main findings of our paper is that this idea works for multi-parameter symmetry groups if one considers the extension of the equation on a multi-dimensional lattice. In that case the edges are replaced by higher dimensional faces. We show that in such a way one can derive from the same discrete potential KdV the following YB map

$$u = yQ, \quad v = xQ^{-1}, \quad Q = \frac{(1-\gamma_2) + (\gamma_2 - \gamma_1)\,x + \gamma_2(\gamma_1 - 1)\,x\,y}{(1-\gamma_1) + (\gamma_1 - \gamma_2)\,y + \gamma_1(\gamma_2 - 1)\,x\,y}. \tag{7}$$

We will call it *Harrison* map since it is closely related to the superposition formula of the Bäcklund transformation for the Ernst equation in general relativity introduced by Harrison [19]. After the change of variables $x \mapsto 1/x,\ v \mapsto 1/v,\ y \mapsto y/\gamma_2,\ u \mapsto u/\gamma_1$ it coincides with the F_I quadrirational map in [11], which corresponds to the most general case of two conics. Note that the most degenerate case F_V in the classification of [11] is simply related to the Adler map.

The plan of the paper is the following. We start in section 2 with the discussion of 3D consistency property for the equations on quad-graphs. As the examples we choose three equations from the classification list in [14]. By considering the invariants of their one-parameter symmetry groups we derive all five types of the quadrirational maps from [11]. Next, in section 3, we show how this symmetry method can be generalized in the case where the lattice equation admits a multi-parameter symmetry group. This is demonstrated on the example of the lattice KdV equation by extending it to a three dimensional cube and using the invariants of a two-parameter symmetry group as YB variables. Finally, we show how the Harrison map can be retrieved from the lattice KdV equation by exploiting its full three-parameter symmetry group and the consistency property on a four dimensional cube.

In section 4 we show that the same idea works equally well for multi-field integrable lattice equations and we derive certain examples of multi-component YB maps from lattice equations in the Boussinesq family, vector Calapso equation and its specialization to an integrable discrete version of the $O(n+2)$ nonlinear σ–model, introduced recently by Schief [20]. The paper concludes with perspectives where we address some questions for future study.

2 Multi-dimensional consistency and YB maps

2.1 Equations on quad-graphs and the 3D consistency property

Central to our considerations are integrable discrete equations on quad-graphs, which are specific equations associated to planar graphs with elementary quadrilaterals faces. In the simplest case one has complex fields $f : \mathbb{Z}^2 \to \mathbb{C}$ assigned on the vertices at sites (n_1, n_2) and two complex lattice parameters α_1, α_2 assigned on the edges of an elementary square being equal on opposite edges (see Fig. 1). The basic building block of such equations consists of a relation of the form

$$\mathcal{E}(f, f_1, f_2, f_{1,2}; \alpha_1, \alpha_2) = 0 \,, \tag{8}$$

between the values of four fields residing on the vertices of each elementary quadrilateral for which we use the shorthand notation:

$$f := f(n_1, n_2), \quad f_1 := f(n_1 + 1, n_2), \quad f_2 := f(n_1, n_2 + 1), \quad f_{1,2} := f(n_1 + 1, n_2 + 1) \,. \tag{9}$$

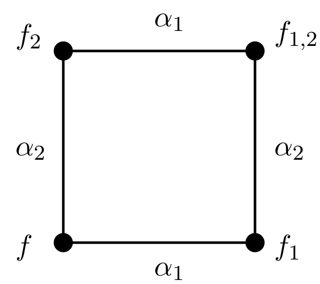

Figure 1: An elementary quadrilateral.

Integrable discrete equations of the above type (8) are listed in a recent classification [14], where the 3D consistency property (see below) and some additional conditions were imposed. From that list we consider the following equations

$$\mathcal{E}_1 : \qquad (f_{1,2} - f)(f_1 - f_2) - \alpha_1 + \alpha_2 = 0 \,, \tag{10}$$

$$\mathcal{E}_2 : \qquad \alpha_1(f\, f_1 + f_2\, f_{1,2}) - \alpha_2(f\, f_2 + f_1\, f_{1,2}) + \delta({\alpha_1}^2 - {\alpha_2}^2) = 0 \,, \tag{11}$$

$$\mathcal{E}_3 : \qquad (1 - {\alpha_2}^2)(f_1 - \alpha_1 f)(f_2 - \alpha_1 f_{1,2}) - (1 - {\alpha_1}^2)(f_2 - \alpha_2 f)(f_1 - \alpha_2 f_{1,2}) = 0 \,. \tag{12}$$

Equation $\mathcal{E}_1$ is already mentioned as the dpKdV equation. Equation $\mathcal{E}_2$ with $\delta = 0$ is the modified discrete KdV or *Hirota equation* [15]. If $\delta \neq 0$ we may always assume that $\delta = 1$ using an appropriate gauge. Equation $\mathcal{E}_3$ corresponds to the equation labeled as $Q3_{\delta=0}$ in the classification of [14]. It is contained in the 4-parameter family of the equations derived earlier in [21], which contains also discrete versions of potential KdV, modified KdV and Schwarzian KdV (see [22] for a more recent discussion).

The integrability of such equations can be defined using the *three dimensional consistency* property. This means that the overdetermined system consisting of the difference equations

$$\mathcal{E}(f, f_i, f_j, f_{i,j}; \alpha_i, \alpha_j) = 0 \,, \qquad 1 \leq i < j \leq 3 \,, \tag{13}$$

and their shifted versions, is consistent on the three-dimensional lattice $\mathbb{Z}^3$. In practice, this property is verified as follows [12, 13, 14]. Consider an elementary initial value problem on the three-dimensional cube with initial data assigned on four vertices, not all of them lying on the same face. One such initial configuration is depicted in Fig. 2(a) with initial values f, f_i, $1 \leq i \leq 3$. Using equations (13) on the three faces adjacent to the vertex with value f, we determine uniquely the values $f_{i,j}$, $1 \leq i < j \leq 3$, in terms of the initial data. Then using shifted versions of (13) on each of the remaining three faces, we evaluate $f_{1,2,3}$ in three different ways. Consistency means that one obtains the same value for $f_{1,2,3}$ in terms of the initial data f, f_i, $1 \leq i \leq 3$ (independent of the way we choose to evaluate it). For the dpKdV equation (2) this value is

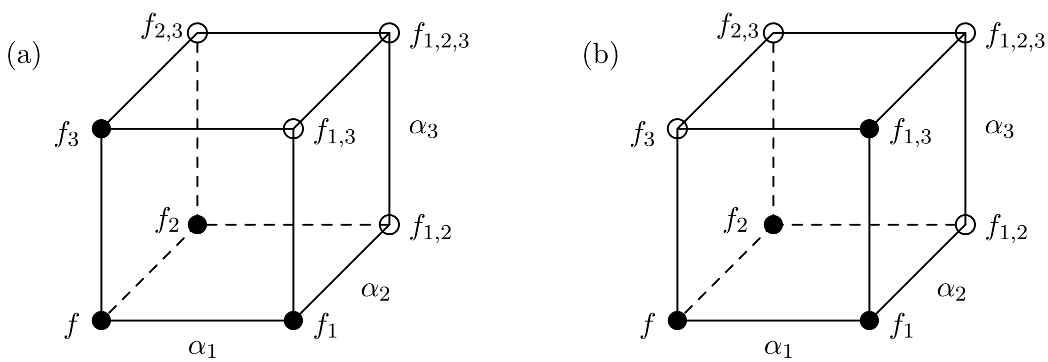

Figure 2: Elementary initial value problems on the cube

$$f_{1,2,3} = \frac{(\alpha_1 - \alpha_2) f_1 \, f_2 + (\alpha_3 - \alpha_1) f_1 \, f_3 + (\alpha_2 - \alpha_3) f_2 \, f_3}{(\alpha_2 - \alpha_1) f_3 + (\alpha_1 - \alpha_3) f_2 + (\alpha_3 - \alpha_2) f_1} \,. \tag{14}$$

Note that the right hand side of equation (14) is invariant under any permutation of the indices $(1, 2, 3)$ which label the field variables and the associated lattice parameters.

Another initial data configuration, which is best adapted to the YB property that we consider, is depicted in Fig. 2(b). A third possible initial configuration is to give the values $f, f_1, f_2, f_{1,2,3}$. The latter two configurations are equivalent to the first one by using the equation on one of the faces. For example, by using the front face equation we can exchange the value $f_{1,3}$ to f_3 in the set of initial data.

Using the fact that dpKdV equation possesses the 3D consistency property, one can show in a similar manner that it can be consistently imposed on each 2-dimensional face of a 4-cube. Since we are going to use this property later on we describe explicitly its derivation.

For given initial values f, f_i $i = 1, 2, 3, 4$, we determine the shifted values of the fields involving any two different directions, using the equations

$$(f_{i,j} - f)(f_i - f_j) = \alpha_i - \alpha_j \,, \tag{15}$$

$1 \leq i < j \leq 4$ (see Fig. 3). Successively, since dpKdV is 3-dimensional consistent, we determine the values f_{ijk}, $1 \leq i < j < k \leq 4$. Then the value $f_{1,2,3,4}$ can be found in six different ways, using the dpKdV equations on the six 2–dimensional facets containing the vertex where the value

$f_{1,2,3,4}$ is assigned. This vertex is contained also in four cubes, each one of them containing three of the six facets, and the incidence relations are such that taking into account the three dimensional consistency on each of the four cubes one proves that the value $f_{1,2,3,4}$ is uniquely determined in terms of initial data. By direct calculations also we find that this value is independent of the way that we used to calculate it, and equals

$$f_{1,i,j,k} = \frac{\underset{ijk}{\boldsymbol{\sigma}}(\alpha_1\alpha_i f_{i,j} + \alpha_j\alpha_k f_{j,k})(f_1 - f_i)(f_j - f_k)}{\underset{ijk}{\boldsymbol{\sigma}}(\alpha_1\alpha_i + \alpha_j\alpha_k)(f_1 - f_i)(f_j - f_k)}, \tag{16}$$

where $\underset{ijk}{\boldsymbol{\sigma}}$ denotes the cyclic sum over the subscripts $(i,j,k) = (2,3,4),(4,2,3),(3,4,2)$. It can be easily checked that $f_{1,2,3,4}$ given by (16), remains invariant under any permutation of the indices $(1,2,3,4)$, thus dpKdV is four dimensional consistent.

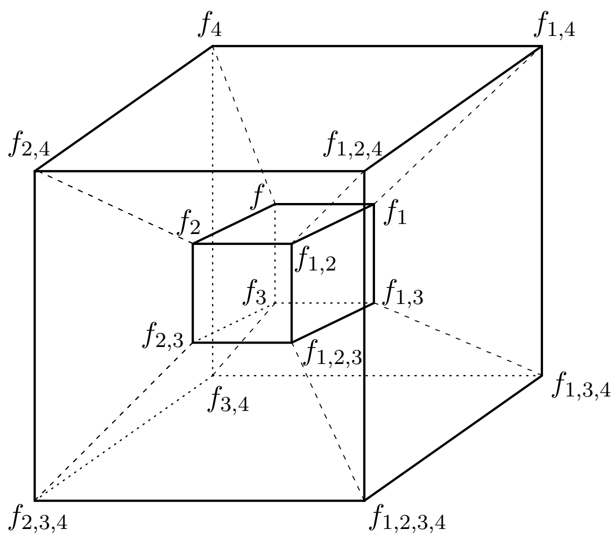

Figure 3: Discrete potential KdV in $\mathbb{Z}^4$

2.2 YB relation and 3D consistency property

Let $\mathbb{X}$ be any set and R a map of $\mathbb{X}\times\mathbb{X}$ into itself. Let $R^{ij} : \mathbb{X}^n \to \mathbb{X}^n$, where $\mathbb{X}^n = \mathbb{X}\times\mathbb{X}\times\ldots\times\mathbb{X}$, denotes the map which acts as R on the i and j factors and as the identity on the others. More explicitly, let us write $R(x,y)$, $x,y\in\mathbb{X}$, as

$$R(x,y) = \big(f(x,y), g(x,y)\big). \tag{17}$$

Then, for $n \geq 2$ and $1 \leq i,j \leq n$, $i \neq j$ we define

$$R^{ij}(x^1,x^2,\ldots x^n) = \begin{cases} (x^1,\ldots,x^{i-1},f(x^i,x^j),x^{i+1},\ldots,x^{j-1},g(x^i,x^j),x^{j+1},\ldots x^n) & i<j, \\ (x^1,\ldots,x^{j-1},g(x^i,x^j),x^{j+1},\ldots,x^{i-1},f(x^i,x^j),x^{i+1},\ldots x^n) & i>j\,. \end{cases} \tag{18}$$

In particular, for $n = 2$ we find that $R^{12} = R$ and $R^{21}(x, y) = \big(g(y, x), f(y, x)\big)$. The latter map can be written as a composition of maps as follows

$$R^{21} = P\,R\,P\,, \tag{19}$$

where P is the permutation map, i.e. $P(x, y) = (y, x)$.

A map R is called a *YB map* if it satisfies the YB relation (1), regarded as an equality of maps of $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ into itself. If in addition the relation $R^{21}\,R = \mathrm{Id}$ holds, then R is called reversible YB map.

In a more general setting we may consider a whole family of YB maps parametrized by continuous parameters α_i rather than a single map. The YB relation then takes the parameter-dependent form

$$R^{23}(\alpha_2, \alpha_3)\,R^{13}(\alpha_1, \alpha_3)\,R^{12}(\alpha_1, \alpha_2) = R^{12}(\alpha_1, \alpha_2)\,R^{13}(\alpha_1, \alpha_3)\,R^{23}(\alpha_2, \alpha_3)\,, \tag{20}$$

and the reversibility condition becomes

$$R^{21}(\alpha_2, \alpha_1)\,R(\alpha_1, \alpha_2) = \mathrm{Id}\,. \tag{21}$$

The relation between YB maps and integrable equations on quad-graphs can be demonstrated in the example of the discrete potential KdV equation $\mathcal{E}_1$. As we have already shown in the Introduction, by considering the differences of the values of the fields assigned on two adjacent vertices (3), we arrive at the Adler map (6). There is a different combination for the variables

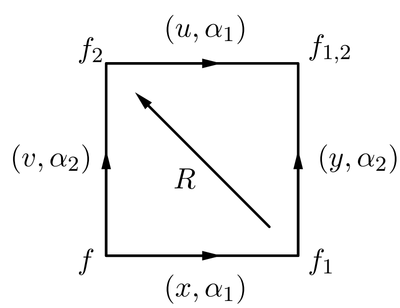

Figure 4: An oriented quadrilateral for the map $R(x, y) = (u, v)$.

assigned on the edges of the square, namely

$$x = f\,f_1, \quad y = f_1\,f_{1,2}, \quad u = f_2 f_{1,2}, \quad v = f\,f_2\,. \tag{22}$$

From the above relations (22) we deduce that

$$x\,u = y\,v\,. \tag{23}$$

Moreover, dpKdV can be also written in terms of the variables (22) as follows

$$y + v - x - u = \alpha_1 - \alpha_2\,. \tag{24}$$

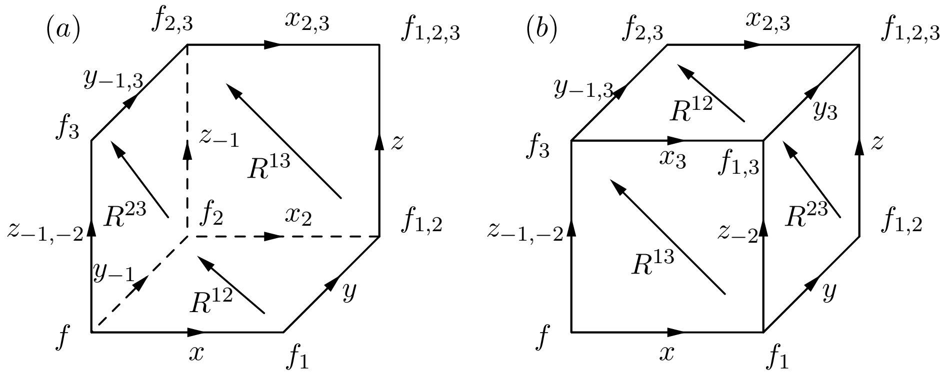

Figure 5: Three dimensional representation of the YB relation

Solving equations (23) and (24) for (u, v) we get the following map

$$u = y\left(1 + \frac{\alpha_1 - \alpha_2}{x - y}\right), \quad v = x\left(1 + \frac{\alpha_1 - \alpha_2}{x - y}\right). \tag{25}$$

The maps (6), (25), *automatically* satisfy the parameter dependent YB relation (20). Indeed, it is easily shown that the consistency property for a configuration of initial data on the vertices of a cube, as depicted in Fig. 2, is equivalent to that with initial values $f, f_1, f_{1,2}, f_{1,2,3}$. These initial data correspond to the values (x, y, z) on the edges (Fig. 5). The 3D consistency property guarantees that the composite maps

$$(a): \qquad (x,y,z) \xrightarrow{R^{12}} (x_2, y_{-1}, z) \xrightarrow{R^{13}} (x_{2,3}, y_{-1}, z_{-2}) \xrightarrow{R^{23}} (x_{2,3}, y_{-1,3}, z_{-1,-2}) \tag{26}$$

$$(b): \qquad (x,y,z) \xrightarrow{R^{23}} (x, y_3, z_{-2}) \xrightarrow{R^{13}} (x_3, y_3, z_{-1,-2}) \xrightarrow{R^{12}} (x_{2,3}, y_{-1,3}, z_{-1,-2}) \tag{27}$$

appearing in equation (20), applied on (x, y, z) give identical values for $(x_{2,3}, y_{-1,3}, z_{-1,-2})$.

Analysing these two examples one notices that the variables x, y, u, v, which we call YB variables, are *invariants* of certain *symmetry groups* of the relevant lattice equation. Now we are going to show that this symmetry method can be applied in more general situations as well.

2.3 Lattice invariants of symmetry groups and YB variables

Let us first recall the basic notions of Lie symmetry methods applied to lattice equations of the form (8). With minor modifications these are in accordance with the symmetry methods applied to algebraic or differential equations (see e.g. [23] for an extensive study on the subject).

Consider a lattice equation of the form (8) involving one field $f : \mathbb{Z}^2 \to \mathbb{C}\,(\mathbb{CP}^1)$. Let G be a one-parameter group of transformations acting on the domain of the dependent variables,

$$G : f \mapsto \Phi(n_1, n_2, f; \varepsilon), \quad \varepsilon \in \mathbb{C}. \tag{28}$$

The prolongation of the group action on the lattice jet space J with coordinates $(f, f_1, f_2, f_{1,2})$ is specified by

$$G : (f, f_1, f_2, f_{1,2}) \mapsto (\Phi(n_1, n_2, f; \varepsilon), \Phi_1(n_1{+}1, n_2, f_1; \varepsilon), \Phi(n_1, n_2{+}1, f_2; \varepsilon), \Phi(n_1{+}1, n_2{+}1, f_{1,2}; \varepsilon)). \tag{29}$$

The infinitesimal generator of the group action of G on f is the vector field

$$\mathbf{v} = Q(n_1,n_2,f)\,\partial_f\,, \qquad \text{where} \qquad Q(n_1,n_2,f) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi(n_1,n_2,f;\varepsilon)\Big|_{\varepsilon=0}\,. \tag{30}$$

There is a one-to-one correspondence between connected groups of transformations and their associated infinitesimal generators since the group action is reconstructed by the flow of the vector field $\mathbf{v}$ by exponentiation

$$\Phi(n_1,n_2,f;\varepsilon) = \exp(\varepsilon\,\mathbf{v})f\,. \tag{31}$$

The prolongation of the infinitesimal action of G given by (29), is generated by the prolonged vector field

$$\widehat{\mathbf{v}} = Q\,\partial_f + Q_1\,\partial_{f_1} + Q_2\,\partial_{f_2} + Q_{1,2}\,\partial_{f_{1,2}}\,, \tag{32}$$

where subscripts denote $Q_1 = Q(n_1+1,n_2,f_1)$, $Q_{1,2} = Q(n_1+1,n_2+1,f_{1,2})$, and so on.

The transformation G is a symmetry of the lattice equation (8), if it transforms any solution of (8) to another solution of the same equation. Equivalently, G is a symmetry of equation (8), if the equation is not affected by the transformation (29). The infinitesimal criterion for G to be a symmetry of equation (8) is

$$\widehat{\mathbf{v}}\big(\mathcal{E}(f,f_1,f_2,f_{1,2};\alpha_1,\alpha_2)\big) = 0\,, \tag{33}$$

whenever equation (8) holds.

A function $I : J \to \mathbb{C}$ is a *lattice invariant* of the transformation group G, if I is not affected under the action of G. The infinitesimal invariance condition for the lattice invariants is

$$\widehat{\mathbf{v}}(I) = 0\,. \tag{34}$$

Once we have determined a symmetry generator $\mathbf{v}$ of the lattice equation (8), the corresponding lattice invariants can be found from the solution of the first order partial differential equation (34), by using the method of characteristics. From the corresponding system of ordinary differential equations we may easily obtain the general solution, since it consists of equations with separated variables. We assign now to the edges of an elementary quadrilateral the following YB variables (Fig. 4)

$$x = I(f,f_1), \quad y = I(f_1,f_{1,2}), \quad u = I(f_2,f_{1,2}), \quad v = I(f,f_2)\,, \tag{35}$$

where I is an invariant depending on two neighboring values of f. Since G is a symmetry of the lattice equation, the latter can be written in terms of these variables:

$$\mathcal{D}(x,y,u,v;\alpha_1,\alpha_2) = 0\,. \tag{36}$$

This can be done in different ways since the variables (35) are not independent; there exists a relation among them

$$\mathcal{F}(x,y,u,v;\alpha_1,\alpha_2) = 0\,, \tag{37}$$

following from the fact that the space of G-orbits is three-dimensional.

Solving the system of equations (36), (37) for u,v in terms of x,y and assuming that the solution is unique, we obtain a map $R(x,y) = (u,v)$.

Proposition 2.1. *If the discrete equation $\mathcal{E}$ satisfies the 3D consistency property, then the map $R(x,y)=(u,v)$, which relates the lattice invariants (35), satisfies the YB relation.*

The proof follows from Fig. 5. Similar considerations hold for multi-field lattice equations, which give rise to multi-component YB maps (see section 4).

Example 2.2. Consider the dpKdV equation (2). Two infinitesimal symmetry generators of the latter equation are

$$\mathbf{v}_1=\partial_f\,,\qquad \mathbf{v}_2=(-1)^{n_1+n_2}\,f\,\partial_f\,,\tag{38}$$

(see [24]). They generate the symmetry transformations

$$G^1: f\mapsto f+\varepsilon_1\,,\qquad G^2: f\mapsto f\exp\left(\varepsilon_2(-1)^{n_1+n_2}\right)\,,\qquad \varepsilon_1,\varepsilon_2\in\mathbb{C}\,,\tag{39}$$

respectively. The lattice invariants assigned on the edges of a square for each one of the above symmetry transformations are the variables (3) and (22), respectively.

The consideration of the remaining two equations from the list (10)-(12), with the corresponding symmetry generators

$$\mathbf{v}_{\mathcal{E}_2}=(-1)^{n_1+n_2}f\,\partial_f,\qquad \mathbf{v}_{\mathcal{E}_3}=f\,\partial_f\,,\tag{40}$$

leads to the results summarized in the following Table 1.

Table 1: Yang-Baxter maps arising from equations $\mathcal{E}_2$ and $\mathcal{E}_3$.

	Yang-Baxter variables	Functional relation and lattice equation	Yang-Baxter map
$\mathcal{E}_2$	$x=f\,f_1/\alpha_1$ $y=f_1\,f_{1,2}/\alpha_2$ $u=f_2\,f_{1,2}/\alpha_1$ $v=f\,f_2/\alpha_2$	$\gamma_1\,x\,u=\gamma_2\,y\,v$ $\gamma_1\,(x+u+\delta)=\gamma_2\,(y+v+\delta)$ where $\gamma_i={\alpha_i}^2$	$u=\dfrac{y}{\gamma_1}\,\dfrac{\gamma_1(x+\delta)-\gamma_2(y+\delta)}{x-y}$ $v=\dfrac{x}{\gamma_2}\,\dfrac{\gamma_1(x+\delta)-\gamma_2(y+\delta)}{x-y}$
$\mathcal{E}_3$	$x=f_1/(\alpha_1\,f)$ $y=f_{1,2}/(\alpha_2\,f_1)$ $u=f_{1,2}/(\alpha_1 f_2)$ $v=f_2/(\alpha_2 f)$	$x\,y=u\,v$ $\dfrac{1-x^{-1}}{1-v^{-1}}=\dfrac{1-\gamma_1}{1-\gamma_2}\,\dfrac{1-\gamma_2\,y}{1-\gamma_1\,u}$ where $\gamma_i={\alpha_i}^2$	$u=y\,Q,\qquad v=x\,Q^{-1}$ $Q=\frac{(1-\gamma_2)+(\gamma_2-\gamma_1)\,x+\gamma_2(\gamma_1-1)\,x\,y}{(1-\gamma_1)+(\gamma_1-\gamma_2)\,y+\gamma_1(\gamma_2-1)\,x\,y}$

These maps are simply related to quadrilateral maps F_I-F_{III} from the Adler-Bobenko-Suris list of [11]. Namely, setting $\delta=0$ in the YB map constructed from $\mathcal{E}_2$, we retrieve the map labeled as F_{III} map in [11]. The case $\delta=-1$ corresponds to the F_{II} map. Finally, the YB map constructed from $\mathcal{E}_3$ under the transformation $x \mapsto 1/x,\ v \mapsto 1/v,\ y \mapsto y/\gamma_2,\ u \mapsto u/\gamma_1$ turns into

$$u = \gamma_1\, y\, \widetilde{Q}, \qquad v = \gamma_2\, x\, \widetilde{Q}, \qquad \widetilde{Q} = \frac{(1-\gamma_2)x + \gamma_2 - \gamma_1 + (\gamma_1 - 1)y}{\gamma_2(1-\gamma_1)x + (\gamma_1 - \gamma_2)x\, y + \gamma_1(\gamma_2 - 1)y}\,, \tag{41}$$

which corresponds to the F_I map in the classification in [11]. As we have already mentioned it is closely related to the superposition formula of the Bäcklund transformation for the Ernst equation in general relativity introduced by Harrison [19], c.f. [25].

The remaining maps F_{IV} and F_V from [11] are related in a simple way to the maps (6) and (25) derived from dpKdV already in section 2.2.

Thus all 5 types of quadrilateral maps from the Adler-Bobenko-Suris classification [11] are equivalent to the YB maps coming from the integrable equations on quad-graphs.

Remark 2.3. We should mention that the equivalence of the quadrilateral maps considered in [11] allows independent change of variables x, y, u, v and therefore does not respect the YB property, which is preserved in general only under the diagonal action of the Möbius group. In particular, quadrilateral maps in general do not satisfy the YB relation (contrary to what one might conclude from [11]). However sometimes two YB maps are related by non-diagonal action (see the example of Harrison map and F_V above). The question how many such pairs exist needs further investigation (see the discussion of this in [26]).

3 Multi-parameter symmetry groups and multi-dimensional consistency

The purpose of this section is to show that the symmetry method described in the preceding section works equally well for multi-parameter symmetry groups. The idea is to consider the extension of the equation into many dimensions using the 3D-consistency property and then prescribe the YB variables not to the edges but, for example, to higher dimensional faces.

We demonstrate how the method works in the example of dpKdV equation, which is invariant under the three-parameter symmetry group G with infinitesimal generators

$$\mathbf{v}_1 = \partial_f, \qquad \mathbf{v}_2 = (-1)^{n_1+n_2}\, f\, \partial_f\,, \qquad \mathbf{v}_3 = (-1)^{n_1+n_2}\, \partial_f\,. \tag{42}$$

Their commutators are

$$[\mathbf{v}_1, \mathbf{v}_2] = \mathbf{v}_3\,, \quad [\mathbf{v}_2, \mathbf{v}_3] = -\mathbf{v}_1\,, \quad [\mathbf{v}_1, \mathbf{v}_3] = 0\,, \tag{43}$$

from which it is immediately seen that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span a solvable Lie algebra. This algebra is actually isomorphic to the Lie algebra of the group $Iso(\mathbb{R}^{1,1})$ of isometries of Minkowski plane, so G can be considered as the connected component of identity of this group.

3.1 Consistency of dpKdV around a 3-cube and the F_{III} map

Consider the dpKdV equation imposed on each face of an elementary cube (see Fig. 2) and the abelian subgroup H of the full symmetry group G generated by $\{\mathbf{v}_1, \mathbf{v}_3\}$ (translations of the

Minkowski plane). These two symmetries can be extended to the corresponding system on $\mathbb{Z}^3$, e.g. $\mathbf{v}_3 = (-1)^{n_1+n_2+n_3}\partial_f$.

We consider now the following invariants of the subgroup H

$$x = f_1 - f_3\,, \quad y = f_{1,2} - f_{1,3}\,, \quad u = f_{1,2} - f_{2,3}\,, \quad v = f_2 - f_3\,, \tag{44}$$

assigned on four faces of the 3-cube. Using the dpKdV equations

$$f_{1,2} - f = \frac{\alpha_1 - \alpha_2}{f_1 - f_2}\,, \quad f_{1,3} - f = \frac{\alpha_1 - \alpha_3}{f_1 - f_3}\,, \quad f_{2,3} - f = \frac{\alpha_2 - \alpha_3}{f_2 - f_3}\,, \tag{45}$$

we easily find that the invariants (44) are related by

$$u\,v = x\,y\,, \qquad u - \frac{\beta_1}{x} = y - \frac{\beta_2}{v}\,, \tag{46}$$

where $\beta_1 = \alpha_1 - \alpha_3$, $\beta_2 = \alpha_2 - \alpha_3$. Solving the system (46) for (u, v) in terms of (x, y) we obtain the map

$$u = yP\,, \qquad v = xP^{-1}\,, \qquad P = \frac{\beta_1 + x\,y}{\beta_2 + x\,y}\,, \tag{47}$$

which satisfies the YB relation as it can be checked by direct calculations. This fact is also related to the higher dimensional consistency of dpKdV on $\mathbb{Z}^4$ as it is explained in the following.

Considering the five initial values $f_3, f_{1,3}, f_1, f_{1,2}, f_{1,2,4}$ on the vertices of the 4-cube (see Fig. 3) one can find the values on all other vertices in a unique way using the dpKdV equation on each 2-dimensional face because of its 4-dimensional consistency. From these five initial values we form the differences

$$x = f_1 - f_3\,, \quad y = f_{1,2} - f_{1,3} \quad z = f_{1,2,4} - f_{1,2,3}\,, \tag{48}$$

which are assigned on (the diagonals of) the 2-dimensional faces of the 4-cube (Fig. 3). We note that the value $f_{1,2,3}$ can be expressed already in terms of $f_3, f_{1,3}, f_1, f_{1,2}$ through the 3-dimensional consistency of dpKdV on the "inner" cube. Next we apply successively the map $R : (x, y) \mapsto (u, v)$ given by (47) on the (a) "inner", "front", "left" and (b) "right", "back", "outer" 3-dimensional cubes to obtain the following composite maps

$$(a): \qquad (x, y, z) \xrightarrow{R^{12}} (x_2, y_{-1}, z) \xrightarrow{R^{13}} (x_{2,4}, y_{-1}, z_{-1}) \xrightarrow{R^{23}} (x_{2,4}, y_{-1,4}, z_{-1,-2})\,, \tag{49}$$

$$(b): \qquad (x, y, z) \xrightarrow{R^{23}} (x, y_4, z_{-2}) \xrightarrow{R^{13}} (x_4, y_4, z_{-1,-2}) \xrightarrow{R^{12}} (x_{2,4}, y_{-1,4}, z_{-1,-2})\,. \tag{50}$$

The fact that the two ways of obtaining the values for $(x_{2,4}, y_{-1,4}, z_{-1,-2})$ lead to identical results, and thus to the YB property of the map (47), is guaranteed by the 4D consistency. One can notice that the evolution of the Yang Baxter variables takes place on two parallel layers of the $\mathbb{Z}^3$ lattice i.e. $\{(n_1, n_2, 0)\,, (n_1, n_2, 1)\,, n_1, n_2 \in \mathbb{Z}\}$. This is reflected to the fact that there are six out of the eight 3-dimensional faces (cubes) of the 4-cube involved in the compatibility.

A final comment about the map (47) is that under the transformation $x \mapsto -x^{-1}$, $v \mapsto -v^{-1}$, $u \mapsto \beta_1 u$, $y \mapsto \beta_2 y$ it becomes the first YB map in Table 1 for $\delta = 0$. Thus the F_{III} map in the classification of [11] is also retrieved from dpKdV in $\mathbb{Z}^3$, by using the invariants of the symmetry subgroup H.

3.2 Consistency of dpKdV around a 4-cube and the Harrison map

We are going to show that the Harrison map (7) appears in a similar manner as previously using now the invariants of the *full* symmetry group G of dpKdV and extending both the equation and its symmetry group in $\mathbb{Z}^4$.

The symmetry group G can be naturally extended to the corresponding system in $\mathbb{Z}^4$. Now let us consider the following invariants of this group:

$$x = \frac{f_1 - f_3}{f_2 - f_3}, \qquad v = \frac{f_1 - f_4}{f_2 - f_4}, \qquad y = \left(\frac{f_1 - f_4}{f_2 - f_4}\right)_3, \qquad u = \left(\frac{f_1 - f_3}{f_2 - f_3}\right)_4, \tag{51}$$

where the subscript i means the shift in the i-th direction. The natural place for them to live in are the corresponding 2-dimensional faces. Next we derive the relations between these variables.

First of all equations (15) and their forward shifts with respect to the lattice directions 3 and 4 imply that the following relations

$$(f_1 - f_3)(f_1 - f_4)_3 = (f_1 - f_4)(f_1 - f_3)_4\,, \tag{52a}$$

$$(f_2 - f_3)(f_2 - f_4)_3 = (f_2 - f_4)(f_2 - f_3)_4\,, \tag{52b}$$

$$(f_{1,3} - f)(f_{1,4} - f)_3 = (f_{1,4} - f)(f_{1,3} - f)_4\,, \tag{52c}$$

$$(f_{2,3} - f)(f_{2,4} - f)_3 = (f_{2,4} - f)(f_{2,3} - f)_4\,, \tag{52d}$$

hold on the “back” and “left” 3-cubes of the 4-cube depicted in Fig. 3. Dividing memberwise equations (52a), (52b) and rearranging terms we get

$$\frac{f_1 - f_3}{f_2 - f_3}\left(\frac{f_1 - f_4}{f_2 - f_4}\right)_3 = \frac{f_1 - f_4}{f_2 - f_4}\left(\frac{f_1 - f_3}{f_2 - f_3}\right)_4. \tag{53}$$

In terms of the variables (51) equation (53) reads

$$x\,y = u\,v\,. \tag{54}$$

On the other hand, we can rewrite equation (52c) in the equivalent form

$$(f_{1,3} - f)\left(1 - \frac{(f_{2,3} - f)_4}{(f_{1,3} - f)_4}\right) = (f_{1,4} - f)\left(1 - \frac{(f_{2,4} - f)_3}{(f_{1,4} - f)_3}\right), \tag{55}$$

which by using equations (15) reads

$$\frac{\alpha_1 - \alpha_3}{f_1 - f_3}\left(1 - \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\left(\frac{f_1 - f_3}{f_2 - f_3}\right)_4\right) = \frac{\alpha_1 - \alpha_4}{f_1 - f_4}\left(1 - \frac{\alpha_2 - \alpha_4}{\alpha_1 - \alpha_4}\left(\frac{f_1 - f_4}{f_2 - f_4}\right)_3\right). \tag{56}$$

Multiplying both terms of equation (56) with $(\alpha_1 - \alpha_2)(f_1 - f_2)$ and rearranging terms, the latter takes the form

$$\begin{aligned}&\left(1 - \frac{\alpha_2 - \alpha_4}{\alpha_1 - \alpha_4}\right)\left(1 - \frac{f_2 - f_3}{f_1 - f_3}\right)\left(1 - \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\left(\frac{f_1 - f_3}{f_2 - f_3}\right)_4\right) = \\ &\left(1 - \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\right)\left(1 - \frac{f_2 - f_4}{f_1 - f_4}\right)\left(1 - \frac{\alpha_2 - \alpha_4}{\alpha_1 - \alpha_4}\left(\frac{f_1 - f_4}{f_2 - f_4}\right)_3\right).\end{aligned} \tag{57}$$

Finally, recalling the defining relations of the variables x, y, u, v, equation (57) becomes

$$(1-\gamma_2)(1-x^{-1})(1-\gamma_1\, u) = (1-\gamma_1)(1-v^{-1})(1-\gamma_2\, y)\,, \tag{58}$$

where

$$\gamma_1 = \frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}, \qquad \gamma_2 = \frac{\alpha_2-\alpha_4}{\alpha_1-\alpha_4}\,. \tag{59}$$

A similar calculation starting with (52d) and using (54), delivers the same relation (58).

Proposition 3.1. *The invariants (51) of the symmetry group G of the dpKdV equation extended to $\mathbb{Z}^4$ are related by the YB map, which coincides with the Harrison map (7).*

Indeed, comparing (54) and (58) with the relations from which we obtain the last map in Table 1, we deduce that x, y and u, v are related by the Harrison map. The fact that this map satisfies the YB relation can be also derived from the consistency property and geometry of the 5-dimensional lattice.

4 Multi-component YB maps

In this section we show that YB maps can be constructed equally well from $3D$ consistent multi-field discrete equations, for which no classification scheme exploiting the multi-dimensional consistency property has been obtained yet.

4.1 YB map from the discrete modified Boussinesq system

The discrete modified Boussinesq (dmBSQ) equations [27] involve two fields $f, g : \mathbb{Z}^2 \to \mathbb{CP}^1$ and are given by the system

$$f_{1,2} = g\,\frac{\alpha_1\, f_2 - \alpha_2\, f_1}{\alpha_1\, g_1 - \alpha_2\, g_2}\,, \qquad g_{1,2} = \frac{g}{f}\,\frac{\alpha_1\, f_1\, g_2 - \alpha_2\, f_2\, g_1}{\alpha_1\, g_1 - \alpha_2\, g_2}\,. \tag{60}$$

Its $3D$ consistency is provided by a lengthy but straightforward calculation that delivers the symmetric values

$$f_{1,2,3} = f\,\frac{\underset{ijk}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j f_k(\alpha_i g_i - \alpha_j g_j)}{\underset{ij}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j(\alpha_i f_i g_j - \alpha_j f_j g_i)}\,, \qquad g_{1,2,3} = g\,\frac{\underset{ijk}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j g_k(\alpha_i f_j - \alpha_j f_i)}{\underset{ij}{\boldsymbol{\sigma}}\, \alpha_i\alpha_j(\alpha_i f_i g_j - \alpha_j f_j g_i)}\,, \tag{61}$$

with respect to any permutation of the indices $(1,2,3)$. Here the cyclic sum $\underset{ijk}{\boldsymbol{\sigma}}$ is over the subscripts $(i,j,k) = (1,2,3), (3,1,2), (2,3,1)$, and similarly the cyclic sum $\underset{ij}{\boldsymbol{\sigma}}$ is over $(i,j) = (1,2), (2,3), (3,1)$. The explicit dependence of $f_{1,2,3}$, $g_{1,2,3}$ on the values f, g implies that dmBSQ does not satisfy the so-called *tetrahedron* property, which is an additional assumpion in the classification scheme in [14] for one-field discrete equations.

Using the symmetry generators

$$\mathbf{v}_1 = f\,\partial_f\,, \qquad \mathbf{v}_2 = g\,\partial_g\,, \tag{62}$$

of the dmBSQ equations, we define as YB variables the following joint lattice invariants

$$x^1 = \frac{f_1}{f}, \quad y^1 = \frac{f_{1,2}}{f_1}, \quad u^1 = \frac{f_{1,2}}{f_2}, \quad v^1 = \frac{f_2}{f}, \tag{63}$$

$$x^2 = \frac{g_1}{g}, \quad y^2 = \frac{g_{1,2}}{g_1}, \quad u^2 = \frac{g_{1,2}}{g_2}, \quad v^2 = \frac{g_2}{g}. \tag{64}$$

It is immediately seen that the above equations imply that

$$x^1 y^1 = u^1 v^1, \qquad x^2 y^2 = u^2 v^2. \tag{65}$$

Moreover, the lattice equations (60) can be expressed in terms of the above invariants as follows

$$u^1 v^1 = \frac{\alpha_1 v^1 - \alpha_2 x^1}{\alpha_1 x^2 - \alpha_2 v^2}, \qquad u^2 v^2 = \frac{\alpha_1 x^1 v^2 - \alpha_2 v^1 x^2}{\alpha_1 x^2 - \alpha_2 v^2}. \tag{66}$$

Finally, solving equations (65), (66) for (u^i, v^i) we obtain the reversible YB map

$$u^1 = y^1\, A, \qquad v^1 = x^1\, A^{-1} \qquad A = \frac{{\alpha_1}^2\, x^1 + {\alpha_2}^2\, x^1\, x^2\, y^1 + \alpha_1\, \alpha_2\, x^2\, y^2}{\alpha_1\, \alpha_2\, x^1 + {\alpha_1}^2\, x^1\, x^2\, y^1 + {\alpha_2}^2\, x^2\, y^2}, \tag{67a}$$

$$u^2 = y^2\, B, \qquad v^2 = x^2\, B^{-1} \qquad B = \frac{{\alpha_1}^2\, x^1 + {\alpha_2}^2\, x^1\, x^2\, y^1 + \alpha_1\, \alpha_2\, x^2\, y^2}{{\alpha_2}^2\, x^1 + \alpha_1\, \alpha_2\, x^1\, x^2\, y^1 + {\alpha_1}^2\, x^2\, y^2}. \tag{67b}$$

4.2 YB map from the discrete potential Boussinesq system

Discrete potential Boussinesq (dpBSQ) equations is the second member in the so-called lattice Gel'fand-Dikii hierarchy [28]. The dpBSQ equations, in the form they were studied recently in [29], involve three fields $f, g, h : \mathbb{Z}^2 \to \mathbb{CP}^1$, and they are given by the following system

$$h_1 = f\, f_1 - g\,, \tag{68a}$$

$$h_2 = f\, f_2 - g\,, \tag{68b}$$

$$h = f\, f_{1,2} - g_{1,2} - \frac{\alpha_1 - \alpha_2}{f_1 - f_2}\,. \tag{68c}$$

For the purposes of the present discussion, equations (68) exhibit the interesting feature that the joint invariants of two symmetry generators are enough to construct a YB map. In connection with this issue we note that for an elementary Cauchy problem on a staircase, we should impose initial values (f, g, h), (f_1, g_1), (f_2, g_2), only. From these data the values (h_1, h_2) and $(f_{1,2}, g_{1,2}, h_{1,2})$ are determined uniquely. In particular, equations (68a), (68b) imply that

$$f_{1,2} = \frac{g_1 - g_2}{f_1 - f_2}\,, \tag{69}$$

and subsequently the values $h_{1,2}$ and $g_{1,2}$ are determined from equation (68a) (or equivalently (68b)) and (68c), respectively.

Using the infinitesimal invariance criterion (33) to determine the symmetries of equations (68), we find that two particular symmetry generators are given by the following vector fields

$$\mathbf{v}_1 = \partial_f + f\, \partial_g + f\, \partial_h\,, \quad \mathbf{v}_2 = \partial_g - \partial_h\,. \tag{70}$$

They generate the symmetry transformations

$$G^1 : (f, g, h) \mapsto (f + \varepsilon_1, g + \varepsilon_1 f + \frac{{\varepsilon_1}^2}{2}, h + \varepsilon_1 f + \frac{{\varepsilon_1}^2}{2}), \tag{71}$$

$$G^2 : (f, g, h) \mapsto (f, g + \varepsilon_2, h - \varepsilon_2), \tag{72}$$

respectively. We define now as YB variables the following invariants

$$\begin{aligned}
x^1 &= f_1 - f, & y^1 &= f_{1,2} - f_1, \\
x^2 &= g_1 - g - f(f_1 - f), & y^2 &= g_{1,2} - g_1 - f_1(f_{1,2} - f_1), \\
x^3 &= h_1 - h - f(f_1 - f), & y^3 &= h_{1,2} - g_1 - f_1(f_{1,2} - f_1), \\
\\
u^1 &= f_{1,2} - f_2, & v^1 &= f_2 - f, \\
u^2 &= g_{1,2} - g_2 - f_2(f_{1,2} - f_2), & v^2 &= g_2 - g - f(f_2 - f), \\
u^3 &= h_{1,2} - h_2 - f_2(f_{1,2} - f_2), & v^3 &= h_2 - h - f(f_2 - f).
\end{aligned} \tag{73}$$

They are functionally related by

$$u^1 + v^1 = x^1 + y^1, \tag{74a}$$

$$u^2 + v^2 = x^2 + y^2 + x^1 y^1 - u^1 v^1, \tag{74b}$$

$$u^3 + v^3 = x^3 + y^3 + x^1 y^1 - u^1 v^1. \tag{74c}$$

Moreover, the system of equations formed by (68a)-(68c), (68b)-(68c) and (69) can be written in terms of the above invariants as follows

$$x^1 = -y^1 + \frac{x^2}{x^1 - v^1} - \frac{v^2}{x^1 - v^1}, \tag{75a}$$

$$x^3 = x^2 + y^2 + x^1 y^1 + \frac{\alpha_1 - \alpha_2}{x^1 - v^1}, \tag{75b}$$

$$v^3 = u^2 + v^2 + u^1 v^1 + \frac{\alpha_1 - \alpha_2}{x^1 - v^1}, \tag{75c}$$

Solving the system of equations (74), (75) for (u^i, v^i) we obtain the following YB map

$$\begin{aligned}
u^1 &= y^1 - (\alpha_1 - \alpha_2)\Gamma^{-1}, & v^1 &= x^1 + (\alpha_1 - \alpha_2)\Gamma^{-1}, \\
u^2 &= y^2 + (\alpha_1 - \alpha_2)\big(\alpha_1 - \alpha_2 - 2y^1\Gamma\big)\Gamma^{-2}, & v^2 &= x^2 + (\alpha_1 - \alpha_2)(x^1 + y^1)\Gamma^{-1}, \\
u^3 &= y^3 + (\alpha_1 - \alpha_2)\big(\alpha_1 - \alpha_2 + (x^1 - y^1)\Gamma\big)\Gamma^{-2}, & v^3 &= x^3,
\end{aligned} \tag{76}$$

where $\Gamma = x^2 - x^3 + x^1 y^1 + y^2$.

4.3 YB map from discrete Calapso equation and nonlinear σ–model.

In a recent study on discrete isothermic surfaces, Schief [20] introduced the following vector generalization of the dpKdV

$$(\boldsymbol{f}_{1,2} - \boldsymbol{f}) = \frac{\alpha_1 - \alpha_2}{|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2}(\boldsymbol{f}_1 - \boldsymbol{f}_2), \tag{77}$$

$\boldsymbol{f} : \mathbb{Z}^2 \mapsto \mathbb{C}^n$, under the name *discrete Calapso equation.* Equation (77) is three dimensional consistent since for an initial value configuration $\boldsymbol{f}, \boldsymbol{f}_i$ as in Fig. 2(a), one finds that the value $\boldsymbol{f}_{1,2,3}$ is given by

$$\boldsymbol{f}_{1,2,3} = \frac{\lambda\,|\boldsymbol{f}_2 - \boldsymbol{f}_3|^2 \boldsymbol{f}_1 - \mu\,|\boldsymbol{f}_1 - \boldsymbol{f}_3|^2 \boldsymbol{f}_2 + \nu\,|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2 \boldsymbol{f}_3}{\lambda\,|\boldsymbol{f}_2 - \boldsymbol{f}_3|^2 - \mu\,|\boldsymbol{f}_1 - \boldsymbol{f}_3|^2 + \nu\,|\boldsymbol{f}_1 - \boldsymbol{f}_2|^2}\,, \tag{78}$$

where

$$\lambda = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\,, \quad \mu = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)\,, \quad \nu = (\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)\,. \tag{79}$$

The consistency property is readily checked since $\boldsymbol{f}_{1,2,3}$ is invariant under any permutation of the indices $(1,2,3)$ labeling the field variables and the corresponding lattice parameters.

The aim now is to construct a YB map from the equation (77). Using the translational invariance of equation (77), we define the following YB variables

$$\boldsymbol{x} = \boldsymbol{f}_1 - \boldsymbol{f}, \quad \boldsymbol{y} = \boldsymbol{f}_{1,2} - \boldsymbol{f}_1, \quad \boldsymbol{u} = \boldsymbol{f}_{1,2} - \boldsymbol{f}_2, \quad \boldsymbol{v} = \boldsymbol{f}_2 - \boldsymbol{f}, \tag{80}$$

on the edges of a square, which are related by

$$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{u} + \boldsymbol{v}. \tag{81}$$

On the other hand, equation (10) can be written in terms of the variables (80) in the form

$$(\boldsymbol{u} - \boldsymbol{y}) = \frac{\alpha_1 - \alpha_2}{|\boldsymbol{x} + \boldsymbol{y}|^2}(\boldsymbol{x} + \boldsymbol{y}). \tag{82}$$

Hence, equations (81), (82) deliver the following reversible YB map

$$\boldsymbol{u} = \boldsymbol{y} + \frac{\alpha_1 - \alpha_2}{|\boldsymbol{x} + \boldsymbol{y}|^2}\,(\boldsymbol{x} + \boldsymbol{y}), \qquad \boldsymbol{v} = \boldsymbol{x} - \frac{\alpha_1 - \alpha_2}{|\boldsymbol{x} + \boldsymbol{y}|^2}\,(\boldsymbol{x} + \boldsymbol{y})\,. \tag{83}$$

Moreover, in [20] it was shown that discrete Calapso equation (77) can be specialized to an integrable discrete version of the $O(n+2)$ nonlinear σ- model. This reduction is accomplished by imposing the constraint

$$|\boldsymbol{f}|^2 = 1\,, \tag{84}$$

on the discrete Calapso equation (77). Since the shifted values of $\boldsymbol{f}$ with respect to any lattice directions should also satisfy constraint (84), equation (77) is compatible with this constraint whenever

$$2\boldsymbol{f} \cdot \boldsymbol{f}_2 - 2\boldsymbol{f} \cdot \boldsymbol{f}_1 = \alpha_1 - \alpha_2\,. \tag{85}$$

This requirement can be satisfied by taking

$$-2\boldsymbol{f} \cdot \boldsymbol{f}_1 = \alpha_1\,, \qquad -2\boldsymbol{f} \cdot \boldsymbol{f}_2 = \alpha_2\,. \tag{86}$$

In terms of the variables (80), the above constraints translate to

$$|\boldsymbol{x}|^2 = 2 + \alpha_1\,, \qquad |\boldsymbol{y}|^2 = 2 + \alpha_2\,. \tag{87}$$

In view of the previous relations the map (83) obtains the form

$$\boldsymbol{u} = \boldsymbol{y} + \frac{|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2}{|\boldsymbol{x}+\boldsymbol{y}|^2}\,(\boldsymbol{x}+\boldsymbol{y}), \qquad \boldsymbol{v} = \boldsymbol{x} - \frac{|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2}{|\boldsymbol{x}+\boldsymbol{y}|^2}\,(\boldsymbol{x}+\boldsymbol{y})\,. \tag{88}$$

By straightforward calculations one finds that

$$|\boldsymbol{u}|^2 = |\boldsymbol{x}|^2\,, \qquad |\boldsymbol{v}|^2 = |\boldsymbol{y}|^2\,. \tag{89}$$

Using the above identity it is easily established that the map $R : (\boldsymbol{x},\boldsymbol{y}) \mapsto (\boldsymbol{u},\boldsymbol{v})$ given by (88) is a reversible YB map. Up to a permutation this map was first considered by Adler [30] in the geometric problem about recuttings of the polygons.

5 Perspectives

We have shown how the symmetry analysis of integrable equations on quad-graphs can be used in order to construct YB maps. In particular, we derived the Harrison map from the consistently extended discrete potential Korteweg – de Vries equation to the 4-dimensional lattice. The main question now is how far this example can be generalized. In particular, for a given multi-parametric symmetry group, is there a general relation between the structure of the invariants and the geometry of the objects which the YB variables are assigned to ? What is the dimension of the lattice in which the discrete equation should be extended to ? The analysis of other equations from [14] may clarify these issues.

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