Nevanlinna theory for the difference operator

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Abstract

Certain estimates involving the derivative $f \mapsto f'$ of a meromorphic function play key roles in the construction and applications of classical Nevanlinna theory. The purpose of this study is to extend the usual Nevanlinna theory to a theory for the exact difference $f \mapsto \Delta f = f(z+c) - f(z)$.

An a-point of a meromorphic function f is said to be *c-paired* at $z \in \mathbb{C}$ if $f(z) = a = f(z+c)$ for a fixed constant $c \in \mathbb{C}$. In this paper the distribution of paired points of finite-order meromorphic functions is studied. One of the main results is an analogue of the second main theorem of Nevanlinna theory, where the usual ramification term is replaced by a quantity expressed in terms of the number of paired points of f. Corollaries of the theorem include analogues of the Nevanlinna defect relation, Picard's theorem and Nevanlinna's five value theorem. Applications to difference equations are discussed, and a number of examples illustrating the use and sharpness of the results are given.

1. Introduction

Nevanlinna's theory of value distribution is concerned with the density of points where a meromorphic function takes a certain value in the complex plane. One of the early results in this area is a theorem by Picard [11] which states that a non-constant entire function can omit at most one value. Nevanlinna offered a deep generalization of Picard's theorem in the form of his *second main theorem* [8], which implies the defect relation:

$$\sum_{a} (\delta(a, f) + \theta(a, f)) \leq 2 \tag{1.1}$$

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where the sum is taken over all points in the extended complex plane, f is a non-constant meromorphic function and the quantities $\delta(a,f)$ and $\theta(a,f)$ are called the *deficiency* and the *index of multiplicity* of the value a, respectively (see Section 2.1). The defect relation (1.1) yields, for instance, Picard's theorem as an immediate corollary. It also implies that the maximum number of totally ramified values is at most four for any meromorphic function.

The appearance of the ramification index $\theta(a,f)$ in the defect relation (1.1) means that the density of high-multiplicity a-points is relatively low for most $a \in \mathbb{C}$. Similarly in this paper it is shown that a-points appearing in pairs with constant separation are rare for finite-order meromorphic functions, unless the function in question is periodic with the same period as the separation. For instance, if f is of finite order and not periodic with period c, then

$$\sum_a \left(\delta(a,f) + \pi_c(a,f)\right) \leq 2 \tag{1.2}$$

where the sum is taken over all points in the extended complex plane, and $\pi_c(a,f)$ is a measure of those a-points of f which appear in pairs separated by the constant $c \in \mathbb{C}$ (in other words, those points z_0 where $f(z_0) = a = f(z_0+c)$, see Section 2.1 for the exact definition.) The sharpness of inequality (1.2) is shown by giving an example of a finite-order meromorphic function, which is not periodic with period c, satisfying $\sum_a \pi_c(a,f) = 2$.

The defect relation (1.1) follows by an analysis of the behavior of the derivative $f \mapsto f'$ in the ramification term of the second main theorem. In what follows, (1.2) is obtained by proving a version of the second main theorem where the derivative of f is replaced by the exact difference $f \mapsto \Delta f = f(z+c) - f(z)$ of a meromorphic function. In the remainder of this paper difference analogues of Picard's theorem and Nevanlinna's theorem on functions sharing five values are given. In addition, the sharpness of the obtained results is discussed with the help of examples, and an application to difference equations is presented.

2. Nevanlinna theory for exact differences

Before going into details of value distribution of exact differences we must first give a precise answer to the following question: What is the difference analogue of a point with high multiplicity? By a formal discretisation of the derivative function $f'(z)$ we obtain

$$\frac{f(z+c)-f(z)}{c} =: \frac{\Delta_c f}{c}, \tag{2.1}$$

where $c \in \mathbb{C}$. As noted in the introduction, those a-points of f where the derivative vanishes, called *ramified* points, play a special role in Nevanlinna theory. The discretisation (2.1) of $f'(z)$ suggests that a-points appearing in pairs separated by a fixed constant c may have similar importance with respect to the operator Δ_c. This indeed turns out to be the case as seen in the following sections.

2.1. Lemma on the exact difference

We first briefly recall some of the basic definitions of Nevanlinna theory. We refer to [5] for a comprehensive description of the value distribution theory. The Nevanlinna *deficiency* is defined as

$$\delta(a,f) := \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)},$$

where a is in the extended complex plane, $m(r,a)$ is the Nevanlinna *proximity function* and $T(r,f)$ is the *characteristic function* of f. The *ramification index* is

$$\theta(a,f) := \liminf_{r\to\infty} \frac{N(r,a)-\overline{N}(r,a)}{T(r,f)},$$

where $N(r,a)$ is the *counting function* of the a-points of f, counting multiplicities, and $\overline{N}(r,a)$ the counting function ignoring multiplicities. The point $a \in \mathbb{C}$ is a *totally ramified value* of f if all a-points of f have multiplicity two or higher.

The following theorem is a recently obtained difference analogue of the lemma on the logarithmic derivative [4].

Theorem 2.1. *Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) \tag{2.2}$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

In the original statement of Theorem 2.1 in [4] the error term on the right side of (2.2) has $T(r+|c|,f)$ instead of $T(r,f)$. But by the following lemma, [3, Lemma 2.1], we have $T(r+|c|,f)=(1+o(1))T(r,f)$ for all r outside of a set with finite logarithmic measure, whenever f is of finite order.

Lemma 2.2. *Let $T:(0,+\infty)\to(0,+\infty)$ be a non-decreasing continuous function, $s>0$, $\alpha<1$, and let $F\subset\mathbb{R}^+$ be the set of all r such that*

$$T(r)\le\alpha T(r+s). \tag{2.3}$$

If the logarithmic measure of is F infinite, that is, $\int_F \frac{dt}{t}=\infty$, then

$$\limsup_{r\to\infty}\frac{\log T(r)}{\log r}=\infty.$$

Let $f(z)$ be a non-constant meromorphic function of finite order, and let $a(z)$ be a finite-order periodic function with period c such that $f(z)\not\equiv a(z)$. Denote

$$\Delta_c f := f(z+c)-f(z),$$

and $\Delta_c^n f := \Delta_c^{n-1}(\Delta_c f)$ for all $n\in\mathbb{N}$, $n\ge 2$. Then by applying Theorem 2.1 with the function $f(z)-a(z)$, we have

$$\begin{aligned} m\left(r,\frac{\Delta_c f}{f-a}\right) &= m\left(r,\frac{f(z+c)-a(z+c)}{f(z)-a(z)}\right)+O(1)\\ &= o\left(\frac{T(r,f-a)}{r^\delta}\right)+O(1) \end{aligned} \tag{2.4}$$

outside of a possible exceptional set with finite logarithmic measure. We denote by $\mathcal{S}(f)$ the set of all meromorphic functions g such that $T(r,g)=o(T(r,f))$ for all r outside of a set with finite logarithmic measure. Functions in the set $\mathcal{S}(f)$ are called *small compared to* f, or *slowly moving* with respect to f. Also, if $g\in\mathcal{S}(w)$ we denote $T(r,g)=S(r,f)$.

Since by (2.4)

$$m\left(r,\frac{\Delta_c f}{f-a}\right)=S(r,f-a) \tag{2.5}$$

we arrive at the following lemma by induction and using the fact that

$$T(r,f(z+1))\le(1+\varepsilon)T(r+1,f(z))$$

for any $\varepsilon>0$ when r is large [15].

Lemma 2.3. *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let f be a meromorphic function of finite order. Then for all small periodic functions $a \in \mathcal{S}(f)$*

$$m\left(r, \frac{\Delta_c^n f}{f-a}\right) = S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Finally, an identity due to Valiron [13] and Mohon'ko [7] is needed in the following section. It states that if the function $R(z,f)$ is rational in f and has small meromorphic coefficients, then

$$T(r, R(z,f)) = \deg_f(R)T(r,f) + S(r,f). \tag{2.6}$$

For the proof see also [6].

2.2. Second main theorem

The lemma on the logarithmic derivative is one of the main components of the proof of the second main theorem of Nevanlinna theory. The following theorem is obtained by combining the standard method of proof for the second main theorem [8] together with Theorem 2.1. As a result a version of the second main theorem is obtained where, instead of the usual ramification term, there is a certain quantity expressed in terms of paired points of the considered function f. Since periodic functions are the analogues of constants for exact differences, it is natural to consider slowly moving periodic functions as target functions of f.

Theorem 2.4. *Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1(z), \ldots, a_q(z)$ be distinct meromorphic periodic functions with period c such that $a_k \in \mathcal{S}(f)$ for all $k = 1, \ldots, q$. Then*

$$m(r,f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f-a_k}\right) \leq 2T(r,f) - N_{pair}(r,f) + S(r,f)$$

where

$$N_{pair}(r,f) := 2N(r,f) - N(r,\Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right)$$

and the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Proof. By denoting

$$P(f) := \prod_{k=1}^{q} (f - a_k)\,,$$

we have

$$\frac{1}{P(f)} = \sum_{k=1}^{q} \frac{\alpha_k}{f - a_k},$$

where $\alpha_k \in \mathcal{S}(f)$ are certain periodic functions with period c. Hence, by (2.5), we obtain

$$m\left(r, \frac{\Delta_c f}{P(f)}\right) \leq \sum_{k=1}^{q} m\left(r, \frac{\Delta_c f}{f - a_k}\right) + S(r, f) = S(r, f),$$

and so

$$m\left(r, \frac{1}{P(f)}\right) = m\left(r, \frac{\Delta_c f}{P(f)} \frac{1}{\Delta_c f}\right) \leq m\left(r, \frac{1}{\Delta_c f}\right) + S(r, f). \tag{2.7}$$

By combining the first main theorem, (2.7) and the Valiron-Mo'honko identity (2.6), we have

$$\begin{aligned}
T(r, \Delta_c f) &= m\left(r, \frac{1}{\Delta_c f}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + O(1) \\
&\geq m\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f) \\
&= qT(r, f) - \sum_{k=1}^{q} N\left(r, \frac{1}{f - a_k}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f) \\
&= \sum_{k=1}^{q} m\left(r, \frac{1}{f - a_k}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f).
\end{aligned}$$

Thus, by (2.5),

$$\begin{aligned}
m(r,f)+\sum_{k=1}^{q} m\left(r,\frac{1}{f-a_k}\right) &\leq T(r,f)+N(r,\Delta_c f)+m(r,\Delta_c f)\\
&\quad -N\left(r,\frac{1}{\Delta_c f}\right)-N(r,f)+S(r,f)\\
&\leq T(r,f)+N(r,\Delta_c f)+m(r,f)\\
&\quad -N\left(r,\frac{1}{\Delta_c f}\right)-N(r,f)+S(r,f)\\
&= 2T(r,f)+N(r,\Delta_c f)-N\left(r,\frac{1}{\Delta_c f}\right)\\
&\quad -2N(r,f)+S(r,f).
\end{aligned}$$

□

Let us now analyze the assertion of Theorem 2.4 more closely. By Lemma 2.2 $N(r+|c|,f)=(1+o(1))N(r,f)$ for all r outside of a set with finite logarithmic measure. Therefore,

$$\begin{aligned}
N_{pair}(r,f) &\geq N(r,f)-N(r+|c|,f)+N\left(r,\frac{1}{\Delta_c f}\right)\\
&= N\left(r,\frac{1}{\Delta_c f}\right)+S(r,f)
\end{aligned}$$

so clearly Theorem 2.4 is telling us something non-trivial about the value distribution of finite-order meromorphic functions. In order to better interpret the meaning of the *pair term* $N_{pair}(r,f)$ we introduce the counting function $n_c(r,a)$, $a\in\mathbb{C}$, which is the number of points z_0 where $f(z_0)=a$ and $f(z_0+c)=a$, counted according to the number of equal terms in the beginning of Taylor series expansions of $f(z)$ and $f(z+c)$ in a neighborhood of z_0. We call such points *c-separated a-pairs* of f in the disc $\{z:|z|\leq r\}$.

For instance, if $f(z)=a$ and $f(z+c)=a$ with multiplicities p and $q<p$, respectively, then the q first terms in the series expansions of $f(z)$ and $f(z+c)$ are identical, and so this point is counted q times in $n_c(r,a)$. Similarly, if in a neighborhood of z_0

$$f(z)=a+c_1(z-z_0)+c_2(z-z_0)^2+\alpha(z-z_0)^3+O\left((z-z_0)^4\right)$$

and

$$f(z+c) = a + c_1(z-z_0) + c_2(z-z_0)^2 + \beta(z-z_0)^3 + O\left((z-z_0)^4\right)$$

where $\alpha \neq \beta$, then the point z_0 is counted 3 times in $n_c(r,a)$.

The integrated counting function is defined as follows:

$$N_c(r,a) := \int_0^r \frac{n_c(t,a) - n_c(0,a)}{t}\, dt + n_c(0,a)\log r.$$

Similarly,

$$N_c(r,\infty) := \int_0^r \frac{n_c(t,\infty) - n_c(0,\infty)}{t}\, dt + n_c(0,\infty)\log r,$$

where $n_c(r,\infty)$ is the number of *c-separated pole pairs* of f, which are exactly the c-separated 0-pairs of $1/f$. This means that if f has a pole with multiplicity p at z_0 and another pole with multiplicity q at z_0+c then this pair is counted $\min\{p,q\}+m$ times in $n_c(r,\infty)$, where m is the number of equal terms in the beginning of the Laurent series expansions of $f(z)$ and $f(z+c)$ in a neighborhood of z_0. Of course, if $p \neq q$ then $m=0$.

Note that $n_c(r,a)$ is finite for any finite r, provided that the given function f is not periodic with period c. Otherwise there would be a point $z_0 \in \mathbb{C}$ in a neighborhood of which the series expansions of $f(z)$ and $f(z+c)$ would be identical. But this means that $f(z) \equiv f(z+c)$ in the whole complex plane, which contradicts the assumption. However, it is possible that $n_c(r,a)$ is strictly greater than the counting function $n(r,a)$.

A natural difference analogue of $\overline{N}(r,a)$ is

$$\widetilde{N}_c(r,a) := N(r,a) - N_c(r,a)$$

which counts the number of those a-points (or poles) of f which are *not* in c-separated pairs. We also use the notation $N_c(r, \frac{1}{f-a})$ instead of $N_c(r,a)$ and $N_c(r,f)$ instead of $N_c(r,\infty)$ when we want to emphasize the connection to the meromorphic function f. With this notation we may state the main result of this paper.

Theorem 2.5. *Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1(z),\dots,a_q(z)$ be distinct meromorphic* *periodic functions with period c such that $a_k \in \mathcal{S}(f)$ for all $k = 1, \dots, q$. Then*

$$(q-1)T(r,f) \le \widetilde{N}_c(r,f) + \sum_{k=1}^{q} \widetilde{N}_c\left(r, \frac{1}{f-a_k}\right) + S(r,f)$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Before proving Theorem 2.5 we briefly discuss its implications. Analogously to the classical Nevanlinna theory, the counting function $\widetilde{N}_c(r,a)$ satisfies $\widetilde{N}_c(r,a) = T(r,f) + S(r,f)$ for all except at most countably many values a (see [5, pp. 43-44] for a proof of this). However, unlike $N(r,a)$, the counting function $\widetilde{N}_c(r,a)$ may, for some values a, be negative for all sufficiently large r. This fact has interesting consequences. By Theorem 2.5 any finite-order meromorphic function f is either periodic with period c, or it can have at most one non-deficient value a such that whenever $f(z) = a$ also $f(z+c) = a$ and the first two terms in the series expansions of $f(z)$ at z and $z+c$ are identical. For instance, consider the function $g(z) := \wp(z) + \exp(z)$ where $\wp(z)$ is a Weierstrass elliptic function with a period $c \neq 2\pi i$. Then $T(r,g) = N(r,g) + S(r,g)$ and each pole of g contributes 2 to $n(r,g)$ but -2 to $\widetilde{n}_c(r,g)$. Therefore $T(r,g) = -\widetilde{N}_c(r,g) + S(r,g)$ and so $\widetilde{N}_c(r,a) = T(r,g) + S(r,g)$ for all $a \in \mathbb{C}$ by Theorem 2.5.

Proof of Theorem 2.5. By Theorem 2.4 and the first main theorem, we obtain

$$\begin{aligned}(q-1)T(r,f) \le N(r,f) + \sum_{k=1}^{q} N\left(r, \frac{1}{f-a_k}\right) - N\left(r, \frac{1}{\Delta_c f}\right) \\ + N(r, \Delta_c f) - 2N(r,f) + S(r,f).\end{aligned} \tag{2.8}$$

We denote by $N_0(r,f)$ the counting function for those poles of f having Laurent series expansions at z_0 and z_0+c with identical principal parts, multiplicity counted according to the number of equal terms in the beginning of the analytic part of the series expansions. (For instance, if $f(z) = c/(z-z_0)^2 + b/(z-z_0) + a + \alpha(z-z_0) + O((z-z_0)^2)$ and $f(z+c) = c/(z-z_0)^2 + b/(z-z_0) + a + \beta(z-z_0) + O((z-z_0)^2)$ the pole at z_0 is counted once in $N_0(r,f)$ whenever $\alpha \neq \beta$.) Since $N(r,f) = N(r+|c|,f) + S(r,f)$ by Lemma 2.2, inequality (2.8) takes the form

$$\begin{aligned}(q-1)T(r,f) \le N(r,f) + N_0(r,f) + \sum_{k=1}^{q} N\left(r, \frac{1}{f-a_k}\right) - N\left(r, \frac{1}{\Delta_c f}\right) \\ + N(r, \Delta_c f) - 2N(r+|c|,f) - N_0(r,f) + S(r,f).\end{aligned} \tag{2.9}$$

The rest of the proof consists of estimates on different terms on the right side of (2.9). First, by the definition of a paired point, we have

$$N_0(r,f)+\sum_{k=1}^{q} N_c\left(r,\frac{1}{f-a_k}\right)\leq N\left(r,\frac{1}{\Delta_c f}\right)$$

for all r, and thus

$$N_0(r,f)+\sum_{k=1}^{q} N\left(r,\frac{1}{f-a_k}\right)-N\left(r,\frac{1}{\Delta_c f}\right)\leq \sum_{k=1}^{q} \widetilde{N}_c\left(r,\frac{1}{f-a_k}\right). \qquad (2.10)$$

Second, assume that $z_0\in\mathbb{C}$ is such that $f(z_0+kc)=\infty$ for all $k\in\mathbb{Z}$ with multiplicities $p_k\geq 0$. Here $p_k=0$ means that $f(z_0+kc)$ is finite. (Note that the case $p_k=0$ for all $k\neq 0$ is not ruled out.) Out of these points only finitely many are inside the disc $\{z\in\mathbb{C}:|z|\leq r+|c|\}$ for any $r>0$. By redefining z_0 if necessary, we may assume that these points are z_0+jc, $j=0,\ldots,K$, where $K\in\mathbb{N}$ is a constant depending only on r. Then $z_0+c,\ldots,z_0+(K-1)c$ are inside the disc with radius r centered at the origin, and $\Delta_c f$ has a pole with multiplicity $\max\{p_j,p_{j+1}\}-m_j'$ at z_0+jc, where $j=1,\ldots,K-1$ and m_j' is the number of equal terms in the beginning of the principal parts of the Laurent series expansions of $f(z)$ and $f(z+c)$ at z_0+jc. If principal parts are completely identical, the number of equal terms in the beginning of the analytic parts of the series at z_0+jc is denoted by m_j'', and moreover $m_j:=m_j'+m_j''$. Therefore the contribution to

$$n(r,\Delta_c f)-2n(r+|c|,f)-n_0(r,f)$$

from the points z_0+jc, $j=0,\ldots,K$, is

$$\begin{aligned}
&\sum_{j=1}^{K-1}\left(\max\{p_j,p_{j+1}\}-m_j'\right)-2\sum_{j=0}^{K}p_j-\sum_{j=1}^{K-1}m_j''\\
&\quad=\sum_{j=1}^{K-1}\left(\max\{p_j,p_{j+1}\}-m_j'-m_j''\right)\\
&\qquad-\left(p_0+\sum_{j=0}^{K-1}\left(\max\{p_j,p_{j+1}\}+\min\{p_j,p_{j+1}\}\right)+p_K\right)\\
&\quad\leq-\sum_{j=1}^{K-1}\left(\min\{p_j,p_{j+1}\}+m_j\right).
\end{aligned} \qquad (2.11)$$

The quantity on the right side of (2.11) is by definition exactly the same as the contribution to $-n_c(r, f)$ from the points $z_0 + jc$, $j = 0, \ldots, K$. Therefore, by summing over all poles of f, we obtain

$$N(r, f) + N(r, \Delta_c f) - 2N(r + |c|, f) - N_0(r, f) \leq \widetilde{N}_c(r, f). \tag{2.12}$$

The assertion follows by combining (2.9), (2.10) and (2.12). □

2.3. Defect relation and Picard's theorem

Nevanlinna's second main theorem is a deep generalization of Picard's theorem, and as such it has many important consequences for the value distribution of meromorphic functions. In this section we present difference analogues of a number of these results, including Picard's theorem and Nevanlinna's theorems on the total deficiency sum and completely ramified values of a meromorphic function. All of the results in this section follow from Theorem 2.5.

A difference analogue of the index of multiplicity $\theta(a, f)$ is called the *c-separated pair index*, and it is defined as follows:

$$\pi_c(a, f) := \liminf_{r\to\infty} \frac{N_c(r, a)}{T(r, f)},$$

where a is either a slowly moving periodic function with period c, or $a = \infty$. Similarly, we define

$$\Pi_c(a, f) := 1 - \limsup_{r\to\infty} \frac{\widetilde{N}_c(r, a)}{T(r, f)},$$

which is an analogue of

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a)}{T(r, f)}$$

in the usual value distribution theory.

The following corollary says that a non-periodic meromorphic function of finite order cannot have too many a-points which appear in pairs. It is a difference analogue of Nevanlinna's theorem on deficient values.

Corollary 2.6. *Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Then $\Pi_c(a, f) = 0$ except for at most countably many meromorphic periodic functions a with period c such that $a \in \mathcal{S}(f)$, and*

$$\sum_a (\delta(a, f) + \pi_c(a, f)) \leq \sum_a \Pi_c(a, f) \leq 2. \tag{2.13}$$

By the second main theorem it follows that $\Theta(a, f) = 0$ for all except at most countably many values a, see, for instance, [5, pp. 43–44]. The same reasoning can be applied to prove that Theorem 2.5 implies Corollary 2.6.

Probably the most distinct difference between the classical Nevanlinna theory and its difference analogue is that, although $0 \leq \Theta(a, f) \leq 1$ for all meromorphic functions f and for all a in the extended complex plane, the maximal deficiency sum

$$\sum_a \Pi_c(a, f) = 2$$

may be attained by a single value a. For instance, the function $g(z) = \wp(z) + \exp(z)$, where $\wp(z)$ is a Weierstrass elliptic function with a period $c \neq 2\pi i$, satisfies $\Pi_c(\infty, g) = 2$. In fact, by the definition of $\Pi_c(a, f)$ alone, it is not even clear that $\Pi_c(a, f)$ has an upper bound what so ever. The fact that $\Pi_c(a, f) \leq 2$ for all a follows by Corollary 2.6.

We say that a is an *exceptional paired value* of f *with the separation* c if the following property holds for all except at most finitely many a-points of f: *Whenever* $f(z) = a$ *then also* $f(z + c) = a$ *with the same or higher multiplicity.* Clearly $N(r, a) \leq N_c(r, a)$ for all exceptional paired values a of f. Note also that by this definition all Picard exceptional values of f are also exceptional paired values. The following corollary is an analogue of Picard's theorem.

Corollary 2.7. *If a finite-order meromorphic function f has three exceptional paired values with the separation c, then f is a periodic function with period c.*

Corollary 2.7 implies that if a finite-order meromorphic function w has two groups of three exceptional paired values with two different separations, say c_1 and c_2 independent over the reals, then either w is a constant or w is an elliptic function with periods c_1 and c_2 and therefore exactly of order 2.

There is no hope of extending Corollary 2.7 (or Corollary 2.6) to include all infinite order meromorphic functions, since the function $\exp(\exp(z))$ has three exceptional paired values with the separation $\log 2$: In addition to the Picard exceptional zeros and poles, the value 1 is exceptionally paired, although non-deficient.

An example of a finite-order meromorphic function which has exactly two exceptional paired values with the separation $2K$ is given by the elliptic function $\mathrm{sn}(z, k)$, where $k \in (0, 1)$ is the elliptic modulus and K is the complete elliptic integral. The function $\mathrm{sn}(z, k)$ is periodic with the periods $4K$ and $2iK'$, and it attains the value zero at points $2nK + 2miK'$ and has its poles at $2nK + (2m +$

$1)iK'$, where $n, m \in \mathbb{Z}$. The function $\operatorname{sn}(z,k)$ has no deficient values, but it has the maximal four completely ramified values at ± 1 and $\pm 1/k$. Therefore, the function $g(z) = \operatorname{sn}(z,k)$ satisfies

$$\sum_a \pi_{2K}(a, g) = 2$$

and, moreover,

$$\sum_a (\theta(a, g) + \pi_{2K}(a, g)) = 4.$$

Analogously to complete ramification, we say that a point a is *completely paired with the separation* c if whenever $f(z) = a$ then either $f(z+c) = a_j$ or $f(z-c) = a_j$, with the same multiplicity. Then a non-periodic meromorphic function of finite order can have at most four values which only appear in pairs.

Corollary 2.8. *Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Then f has at most four completely paired points with separation c.*

Similarly, a non-periodic finite-order function f can have at most three values a which only appear such that for some $z_0 \in \mathbb{C}$, $f(z_0) \neq a$, $f(z_0 + jc) = a$ with the same multiplicity for each $j = 1, 2, 3$, and $f(z_0 + 4c) \neq a$. We say that such values appear in lines of three. Similarly, a finite-order meromorphic function can have a maximum of two values which appear only in lines of four or more.

2.4. Functions sharing values

Another consequence of Nevanlinna's second main theorem is the five value theorem, which says that if two non-constant meromorphic functions share five values ignoring multiplicity then these functions must be identical. By considering periodic functions instead of constants, and by ignoring paired points instead of multiplicity, we obtain a difference analogue of the five value theorem.

We say that two meromorphic functions f and g share a point a, *ignoring c-separated pairs*, when $f(z) = a$ if and only if $g(z) = a$ with the same multiplicity, unless a is a c-separated pair of f or g. In short, all paired points are ignored when determining whether or not f and g share a. This also means that if f has a paired a-point at z_0 and g has a single a-point at the same location, this point is not shared by f and g.

Theorem 2.9. *Let $c \in \mathbb{C}$, and let f and g be meromorphic functions of finite order. If there are five distinct periodic functions $a_k \in \mathcal{S}(f)$ such that f and g share a_k, ignoring c-separated pairs, for all $k = 1, \ldots, 5$ then either $f(z) \equiv g(z)$ or both f and g are periodic with period c.*

Proof. We follow the reasoning of the proof of the five value theorem [5]. Suppose first that f is periodic with period c. Then by definition all a-points of f are paired. Since f and g share five points, ignoring pairs, g has at least five exceptional paired values, and therefore it must also be periodic with period c by Corollary 2.7.

Assume now that neither f nor g is periodic with period c, and that $f \not\equiv g$. Then by Theorem 2.5, for any $\varepsilon > 0$,

$$(4+\varepsilon)T(r,f) \le \widetilde{N}_c(r,f) + \sum_{k=1}^{5} \widetilde{N}_c\left(r, \frac{1}{f-a_k}\right) \tag{2.14}$$

and

$$(4+\varepsilon)T(r,g) \le \widetilde{N}_c(r,g) + \sum_{k=1}^{5} \widetilde{N}_c\left(r, \frac{1}{g-a_k}\right) \tag{2.15}$$

outside a set with finite logarithmic measure. Since

$$\widetilde{N}_c\left(r, \frac{1}{g-a_k}\right) = \widetilde{N}_c\left(r, \frac{1}{f-a_k}\right)$$

for all $k = 1, \ldots, 5$, inequalities (2.14) and (2.15) imply

$$\begin{aligned} T\left(r, \frac{1}{f-g}\right) &\le T(r,f) + T(r,g) + O(1) \\ &\le \frac{2}{3+\varepsilon} \sum_{k=1}^{5} \widetilde{N}_c\left(r, \frac{1}{f-a_k}\right) \\ &\le \frac{2}{3+\varepsilon} N\left(r, \frac{1}{f-g}\right) \\ &\le \frac{2}{3+\varepsilon} T\left(r, \frac{1}{f-g}\right). \end{aligned}$$

This is only possible when $f-g$ is a constant, say $g(z) = f(z) + k$. But now, since $f(z)$ and $f(z)+k$ share five points out of which at most two can be either exceptionally paired or Picard exceptional, $k = 0$, and the assertion follows. □

The elliptic functions $\operatorname{sn} z$ and $1/\operatorname{sn} z$ show that the number five cannot be replaced by four in Theorem 2.9. Namely, for both functions zero and infinity are exceptional paired values, and they share the points 1 and -1, counting multiplicities. Therefore, $\operatorname{sn} z$ and $1/\operatorname{sn} z$ share the points -1, 0, 1 and ∞, ignoring pairs.

2.5. An application to difference equations

In this section we give an example of how to apply the obtained results to study meromorphic solutions of difference equations. We consider the equation

$$w(z+1)+w(z-1)=\frac{a_2 w(z)^2+a_0}{1-w(z)^2} \tag{2.16}$$

where the right side is irreducible in w and the coefficients a_j are constants. Equation (2.16) is a subcase of a more general equation studied in [3] where it was shown that the existence of one finite-order meromorphic solution is sufficient to reduce a large class of difference equations into a difference Painlevé equation or into a linear difference equation, provided that the solution does not simultaneously satisfy a difference Riccati equation. Suppose that (2.16) has a finite-order meromorphic solution $w(z)$ and consider a Laurent series expansion of w in a neighborhood of a point z_0 such that $w(z_0)=\delta$ with the multiplicity $k\geq 1$, where $\delta:=\pm 1$. Then w has a pole of order at least k at z_0-1 or z_0+1.

Consider first the case where $w(z_0+1)=\infty$ with the multiplicity k and $w(z_0-1)$ is either finite or a pole with multiplicity strictly less than k. Then by iterating (2.16), we have

$$\begin{aligned}
w(z+4n) &= \delta+\alpha(z-z_0)^k+O\left((z-z_0)^{k+1}\right)\\
w(z+2n+1) &= \frac{\left((-1)^n(\frac{1}{4}n+\frac{1}{8})-\frac{1}{8}\right)(a_0+a_2)}{\alpha\delta}(z-z_0)^{-k}+O\left((z-z_0)^{1-k}\right)\\
w(z+4n+2) &= -a_2-\delta+O\left((z-z_0)\right)
\end{aligned} \tag{2.17}$$

for all $n\in\mathbb{N}\cup\{0\}$ and for all z in a suitably small neighborhood of z_0, provided that $a_2\neq 0$. Since we assumed the right side of (2.16) to be irreducible $a_0+a_2\neq 0$ and so $w(z+2n+1)=\infty$ for all $n\in\mathbb{N}\cup\{0\}$. The iteration in the case where $w(z_0+1)$ is finite, or a pole with low order, and $w(z_0-1)=\infty$ is symmetric with (2.17).

Suppose now that $w(z_0) = \delta$ and $w(z_0 \pm 1) = \infty$ all with the same multiplicity k. Then, assuming $c_1 \in \mathbb{C}$ and $c_{-1} \in \mathbb{C}$ such that $c_1 c_{-1} \neq 0$, we have

$$\begin{aligned} w(z+4n) &= \delta + \alpha(z-z_0)^k + O\left((z-z_0)^{k+1}\right) \\ w(z+4n+2) &= -a_2 - \delta + O\left((z-z_0)\right) \\ w(z+2n+1) &= c_{2n+1}(z-z_0)^{-k} + O\left((z-z_0)^{1-k}\right) \end{aligned} \tag{2.18}$$

for all $n \in \mathbb{Z}$ as long as none of the constants c_{2n+1} vanish. But if $c_{2n_0+1} = 0$ for some $n_0 \in \mathbb{Z}$ then we are back in the situation (2.17) with the starting point $z_0 + 2n_0 + 1$ instead of $z_0 - 1$. Note also that a closer inspection of the iteration in (2.18) shows that

$$c_{k\pm 4} = c_k + \frac{a_2 + a_0}{2\alpha} \tag{2.19}$$

for all $k \in \mathbb{Z}$.

The final possibility is that $w(z_0) = \delta$ with the multiplicity k and $w(z_0 \pm 1) = \infty$ for both choices of the sign with the multiplicity strictly greater than k. But in this case it is immediately seen that $w(z)$ has a pole with the same order in $z_0 + 2n + 1$ for all $n \in \mathbb{Z}$.

We conclude that all poles, 1-points and -1-points of w appear in lines where each point is separated from its neighbors by the constant 4, with the possible exception of the endpoints of sequences of points appearing as a part of (2.17). In fact for our purposes it is sufficient to know that all poles and δ-points of w appear in groups of four or more, with 4-separation. Assume that w is not periodic with period four. Then by Theorem 2.5,

$$\begin{aligned} T(r,w) &\leq \widetilde{N}_4(r,\infty) + \widetilde{N}_4(r,1) + \widetilde{N}_4(r,-1) + S(r,w) \\ &\leq \frac{1}{4}N(r,\infty) + \frac{1}{4}N(r,1) + \frac{1}{4}N(r,-1) + S(r,w) \\ &\leq \frac{3}{4}T(r,w) + S(r,w), \end{aligned}$$

which is a contradiction. Therefore, either $a_2 = 0$, or w is periodic with period 4 or of infinite order.

Suppose finally that w is periodic with period 4. Then 1 and -1 are Picard exceptional values of w by (2.17), (2.18) and (2.19). Therefore all poles of w appear in lines where each pole is separated from its neighbors by the constant 2, and so w is periodic with period 2. But then, by periodicity, $w(z+1)$, $w(z-1)$ and $w(z+1) + w(z-1)$ are infinite simultaneously. On the other hand, the right side of (2.16) is never infinite since the values ± 1 are Picard exceptional. Hence also the value infinity is Picard exceptional for w, and therefore w is a constant by Picard's theorem. We conclude that if (2.16) has a non-constant meromorphic solution of finite order then $a_2 = 0$.

The existence of finite-order meromorphic solutions of (2.16) is guaranteed in the case $a_2 = 0$, $a_0 \neq 0$. Then (2.16) has solutions of the form

$$w(z) = \frac{\alpha \operatorname{sn}(\Omega z + C) + \beta}{\gamma \operatorname{sn}(\Omega z + C) + \delta} \tag{2.20}$$

where $C \in \mathbb{C}$ is arbitrary, and $\alpha, \beta, \gamma, \delta, \Omega$ are certain constants depending on another free parameter. The meromorphic solutions (2.20) are of order 2 and periodic, but not of period 4.

3. Discussion

Nevanlinna's second main theorem implies that a non-constant meromorphic function cannot have too many points with high multiplicity. In this study a difference analogue of the second main theorem of Nevanlinna theory was given, which shows that a non-periodic finite-order meromorphic function cannot have many values which only appear in pairs, separated by a fixed constant. Then a number of results on the value distribution of finite-order meromorphic functions were derived by combining existing proof techniques from Nevanlinna theory together with the difference analogue of the second main theorem. These include analogues of Picard's theorem, the theorem on the deficiency sum and the theorem on meromorphic functions sharing five values. Sharpness of these results was discussed with the help of examples. Also, an example of how to apply some of these results to study complex difference equations was given.

All concepts of Nevanlinna theory related to ramification have a natural difference analogue. For instance, constant functions are analogous to periodic functions, and a pole with multiplicity $n > 1$ is analogous to a line of n poles with the same multiplicity, each separated from its neighbors by a fixed constant. Similarly as a pole is counted only once in the counting function $\overline{N}(r, f)$ regardless of its multiplicity, only one pole from the above line of poles contributes to $\widetilde{N}_c(r, f)$. However, some notions in the difference Nevanlinna theory seem to go, in a sense, further than their classical counterparts. If a line of points consists of poles with different multiplicities, the contribution from these poles to $\widetilde{N}_c(r, f)$ is nevertheless strictly less than the contribution to $N(r, f)$. Therefore this situation is still exceptional in the sense of the difference deficiency relation (1.2). On the other hand, if all poles in the line have similar enough Laurent series expansions, then the contribution to $\widetilde{N}_c(r, f)$ from these poles may be *negative*. This implies that the maximal value two in the difference deficiency relation (1.2) may be attained by one value a, which is impossible for the classical deficiencies (1.1).

4. Open problems

In addition to his ground-breaking results in the field of value distribution theory, Nevanlinna proposed a number of problems many of which have remained open until recently. In this section we briefly discuss two of them.

4.1. Inverse problem

The *inverse problem* for the deficiency relation is to find a meromorphic function f which at prescribed points has certain non-zero deficiencies and ramification indices. This problem was proposed and partially solved by Nevanlinna himself, see [9], but the complete solution had to wait until 1977 when Drasin [1] settled the issue by a clever use of quasi-conformal mappings. Later on Drasin [2] established a related corollary by F. Nevanlinna, which states that if a meromorphic function f has finite order λ and $\sum_a \delta(a, f) = 2$ then 2λ is a natural number greater or equal to two. In the view of Corollary 2.6 it is natural to ask under what conditions it is possible to find a meromorphic function of finite order for which the pair index $\pi(a, f)$ and the deficiency $\delta(a, f)$ have certain non-zero values at prescribed points a?

4.2. Slowly moving targets

Another question proposed by Nevanlinna is whether or not the relation (1.1) remains valid if the sum is taken over all small functions with respect to f. Partial answer was given by Steinmetz [12] and Osgood [10] who showed that

$$\sum_a \delta(a, f) \leq 2$$

where the sum is taken over distinct small functions with respect to f. A complete solution to this problem was given recently by Yamanoi [14] who showed that (1.1) indeed remains valid if the sum is taken over the larger field small functions, rather than just constants. Similarly we propose that the (2.13) remains valid even if the sum is taken over the field $\mathcal{S}(f)$. It can be immediately seen, by a modification of the reasoning in [5, p. 47], that the assertion holds for at most three functions.

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