

NON-INTERSECTING, SIMPLE, SYMMETRIC RANDOM WALKS AND THE EXTENDED HAHN KERNEL

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To Pierre van Moerbeke on his 60:th birthday

ABSTRACT. Consider a particles performing simple, symmetric, non-intersecting random walks, starting at points $2(j-1)$, $1 \leq j \leq a$ at time 0 and ending at $2(j-1)+c-b$ at time $b+c$. This can also be interpreted as a random rhombus tiling of an abc -hexagon, or as a random boxed planar partition confined to a rectangular box with side lengths a , b and c . The positions of the particles at all times gives a determinantal point process with a correlation kernel given in terms of the associated Hahn polynomials. In a suitable scaling limit we obtain non-intersecting Brownian motions which can be related to Dyson's Hermitian Brownian motion via a suitable transformation.

1. INTRODUCTION

We will consider a simple, symmetric random walks started at $2(j-1)$, $1 \leq j \leq a$, conditioned not to intersect in the time interval $[0, b+c]$, and end at $c-b+2(j-1)$ at time $b+c$. Here a, b, c , $c \geq b$, fixed positive integers. This model has several interpretations. One is as a uniform random rhombus tiling of an abc -hexagon, i.e. a hexagon with side lengths a, b, c, a, b, c , see [3]. This translates directly to a dimer or perfect matching representation, see e.g. [14], so it is a kind of two-dimensional statistical mechanics model. Another interpretation is as a boxed planar partition in a rectangular box with side lengths a , b and c , [18]. The number of possible configurations, the partition function of the model, was computed by MacMahon, and is given by

$$(1.1) \quad Z(a, b, c) = \prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i+j+k-1}{i+j+k-2},$$

see [18].

If we think of the random walks as the motion of particles, then at each time we have a certain particle configuration. By considering these particles at all times we get a discrete, finite point process. The purpose of this paper is to show that this is a determinantal point process and compute the correlation kernel in terms of the associated Hahn polynomials, [10], [2]. The derivation is based on the general framework of [11] and a variant of the orthogonal polynomial method. The main

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result is theorem 3.1 below. The proof of that theorem also gives a proof of MacMahon's formula. A certain continuous scaling limit of this model, namely a fixed and $b = c \rightarrow \infty$, converges to a model of non-intersecting Brownian motions all started at the origin and conditioned to end at the origin at time T . This Brownian motion model is a transformation of Dyson's Hermitian Brownian motion model. We will discuss these models in the next section and indicate how the correlation kernel can be computed in these models using Hermite polynomials and the orthogonal polynomial method. The result in this case is closely related to the work in [5], see also [7]. In the last section we will consider the discrete model where the orthogonal polynomial method is less obvious. At the end of that section we will give some remarks concerning asymptotics.

2. GENERAL FRAMEWORK AND DYSON'S BROWNIAN MOTION

2.1. General framework. Let X_r , $0 \leq r \leq m$ be subsets of \mathbb{R} , $\phi_{r,r+1} : X_r \rightarrow X_{r+1}$, $0 \leq r < m$, given functions and μ_r a measure on X_r , $1 \leq r \leq m$, e.g. Lebesgue or counting measure. An element $\underline{x} = (x^1, \dots, x^{m-1}) \in X_1^n \times X_2^n \times \dots \times X_{m-1}^n \doteq \mathcal{X}$ is called a configuration. We think of x_1^r, \dots, x_n^r , $x^r = (x_1^r, \dots, x_n^r)$, as the positions of particles in X_r , which we will call line r . Let $x^0 \in X_0^n$ and $x^m \in X_m^n$ be fixed configurations, the initial and final configurations respectively. Define $\phi_{r,s} : X_r \times X_s \rightarrow \mathbb{R}$ for $r < s$ by

$$(2.1) \quad \phi_{r,s}(x, y) = \int \phi_{r,r+1}(x, z_1) \dots \phi_{s-1,s}(z_{r-s-1}, y) d\mu_{r+1}(z_1) \dots d\mu_{s-1}(z_{r-s-1}),$$

and $\phi_{r,s} \equiv 0$ if $r \geq s$. We will consider probability measures on \mathcal{X} of the form

$$(2.2) \quad \frac{1}{Z_{n,m}} \prod_{r=0}^{m-1} \det(\phi_{r,r+1}(x_i^r, x_j^{r+1}))_{i,j=1}^n d\mu_1^n(x^1) \dots d\mu_{m-1}^n(x^{m-1}),$$

where $Z_{n,m}$ is a normalization constant. It is proved in [11] that the measure (2.2) has determinantal correlation functions, i.e. the probability density with respect to the reference measure $d\mu_{r_1}(y_1) \dots d\mu_{r_k}(y_k)$ of finding particles at $z_1 = (r_1, y_1), \dots, z_k = (r_k, y_k)$ is given by

$$(2.3) \quad \det(K_{n,m}(z_i; z_j))_{i,j=1}^k$$

where K is the so called correlation kernel. This kernel is given by

$$(2.4) \quad K_{n,m}(r, x; s, y) = -\phi_{r,s}(x, y) + \sum_{i,j=1}^n \phi_{r,m}(x, x_i^m) (A^{-1})_{i,j} \phi_{0,s}(x_j^0, y),$$

where $A = (\phi_{0,m}(x_i^0, x_j^m))_{i,j=1}^n$. Note that the kernel is not unique. We can multiply it by $\psi(r, x)/\psi(s, y)$ for an arbitrary function $\psi \neq 0$ and get the same correlation functions.

2.2. Dyson's Hermitian Brownian motion. Let $H(t)$ be an $n \times n$ Hermitian matrix whose elements evolve according to independent Ornstein-Uhlenbeck processes, see [4], [15]. We consider the stationary case. The probability measure for seeing the matrices H_1, \dots, H_{m-1} at times $t_1 < \dots < t_m$ is

$$(2.5) \quad \frac{1}{Z_{n,m}} e^{-\text{tr} H_1^2} \prod_{j=1}^{m-2} \exp\left(-\frac{\text{tr}(H_{j+1} - q_j H_j)^2}{1 - q_j^2}\right) dH_1 \dots dH_{m-1},$$

where dH_j is the Lebesgue measure on the space of Hermitian matrices, and $q_j = \exp(-(t_{j+1} - t_j))$, $1 \leq j \leq m-2$. Integrating out the angular variables using the HarishChandra/Itzykson-Zuber formula, [15], gives the eigenvalue measure

$$(2.6) \quad \frac{1}{Z'_{n,m}} \Delta_n(\lambda^1) \prod_{j=1}^n e^{-(\lambda_j^1)^2} \prod_{r=1}^{m-2} \det \left(\exp \left(-\frac{(\lambda_j^{r+1} - q_r \lambda_i^r)^2}{1 - q_r^2} \right) \right)_{i,j=1}^n \Delta_n(\lambda^{m-1}),$$

where $\Delta_n(\lambda) = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)$ is the Vandermonde determinant, and λ_j^r , $1 \leq j \leq n$, are the eigenvalues of H_r .

If we set $\phi_{0,1}(i, x) = p_i(x)e^{-x^2}$, $\phi_{m-1,m}(x, i) = p_i(x)$, where p_i is a polynomial of degree i , $\phi_{r,r+1}(x, y) = (\pi(1 - q_r^2))^{-1/2} \exp(-(y - q_r x)^2 / (1 - q_r^2))$, $X_0 = X_m = \{0, \dots, n-1\}$, $x_1^0 = x_1^m = i-1$, $X_r = \mathbb{R}$, $1 \leq r < m$ and μ_r the Lebesgue measure, we see that (2.6) is of the form (2.2). This is a basic example of a measure of the form (2.2). Here we have used the classical trick in the orthogonal polynomial method in random matrix theory to write the Vandermonde determinant as $\Delta_n(\lambda) = \det(p_i(\lambda_j))$. The polynomials p_i can be arbitrary but we choose them to be the normalized Hermite polynomials. This will lead to a formula for the kernel (2.4) in terms of the Hermite polynomials. The key is the expansion, see e.g. [1],

$$(2.7) \quad \frac{1}{\sqrt{\pi(1 - q^2)}} e^{-\frac{(qx-y)^2}{1-q^2}} = \sum_{k=0}^{\infty} p_k(x)p_k(y)q^k e^{-y^2},$$

$0 < q < 1$. Repeated use of this identity gives

$$(2.8) \quad \phi_{0,s}(j, y) = e^{-j(t_s - t_1)} p_j(y) e^{-y^2}.$$

Similarly,

$$(2.9) \quad \phi_{r,m}(x, j) = e^{-j(t_{m-r} - t_r)} p_j(x).$$

Using the orthonormality we obtain $\phi_{0,m}(i, j) = \exp(-j(t_{m-1} - t_1)) \delta_{ij}$ and hence $(A^{-1})_{ij} = \exp(j(t_{m-1} - t_1)) \delta_{ij}$. It also follows from (2.7) that if $1 \leq r < s < m$, then

$$(2.10) \quad \begin{aligned} \phi_{r,s}(x, y) &= \frac{1}{\sqrt{\pi(1 - e^{2(t_r - t_s)})}} \exp \left(-\frac{(e^{t_r - t_s} x - y)^2}{1 - e^{2(t_r - t_s)}} \right) \\ &= \sum_{k=0}^{\infty} p_k(x)p_k(y) e^{k(t_r - t_s)} e^{-y^2}. \end{aligned}$$

Set $\chi_{t,s} = 1$ if $t < s$ and $\chi_{t,s} = 0$ if $t \geq s$. From (2.4) we get the *extended Hermite kernel*,

$$(2.11) \quad \begin{aligned} K_{\text{ext.Herm.}}(t, x; s, y) &= -\frac{1}{\sqrt{\pi(1 - e^{2(t-s)})}} \exp \left(-\frac{(e^{t-s} x - y)^2}{1 - e^{2(t-s)}} \right) \chi_{t,s} \\ &\quad + \sum_{k=0}^{n-1} e^{k(t-s)} p_k(x)p_k(y) e^{-y^2} \end{aligned}$$

Using the second equality in (2.10) we obtain the alternative formula

$$(2.12) \quad K_{\text{ext.Herm.}}(t, x; s, y) = \begin{cases} \sum_{k=0}^{n-1} e^{k(t-s)} p_k(x)p_k(y) e^{-y^2} & , t \geq s \\ -\sum_{k=n}^{\infty} e^{k(t-s)} p_k(x)p_k(y) e^{-y^2} & , t < s. \end{cases}$$

Multiplying with $\exp(-x^2/2 + y^2/2)$ we get the ordinary Hermite kernel when $t = s$.

Let γ be a positively oriented circle around the origin with radius $r > 0$, and Γ the line $\mathbb{R} \ni \rightarrow L + it$ with $L > r$. Using the integral formulas, see [1],

$$H_n(x) = \frac{2^n}{i\sqrt{\pi}} e^{x^2} \int_{\Gamma} e^{w^2 - 2xw} w^n dw,$$

$$H_n(x) = \frac{n!}{2\pi i} \int_{\gamma} e^{-z^2 + 2xz} \frac{dz}{z^{n+1}},$$

and $p_n(x) = (\sqrt{\pi} 2^n n!)^{-1/2} H_n(x)$, it is not difficult to show that

$$(2.13) \quad K_{\text{ext.Herm.}}(t, x; s, y) = -\frac{1}{\sqrt{\pi(1 - e^{2(t-s)})}} \exp\left(-\frac{(e^{t-s}x - y)^2}{1 - e^{2(t-s)}}\right) \chi_{t,s}$$

$$+ \frac{2}{(2\pi i)^2} \int_{\Gamma} dw \int_{\gamma} dz \frac{w^n}{z^n} \frac{1}{w - z} e^{w^2 - 2yw - e^{2(t-s)}z^2 + 2e^{t-s}xz}.$$

This double contour integral can be useful for asymptotic computations, for example to show convergence to the extended Airy kernel when we have the edge scaling. To our knowledge the details for this has not been presented in the litterature, but using (2.13) and the integral formula for the extended Airy kernel it should be possible to do this similarly to what was done for the extended Airy kernel in [12].

2.3. Non-intersecting Brownian motions. A second closely related example is the following which involves non-intersecting Brownian motions. Consider n non-intersecting Brownian motions started at $x_i^0 = \epsilon(i - 1)$, $1 \leq i \leq n$, at time 0 and conditioned to end at the same points at time T . Let x_i^r , $1 \leq i \leq n$, denote the positions at time τ_r , $1 \leq r < m$, where $0 = \tau_0 < \tau_1 < \dots < \tau_{m-1} < \tau_m = T$. By the Karlin-McGregor theorem the probability density for $\underline{x} = (x^1, \dots, x^{m-1}) \in (\mathbb{R}^n)^{m-1}$ is given by

$$(2.14) \quad \frac{1}{Z_{n,m}^\epsilon} \prod_{r=0}^{m-1} \det(p_{\tau_{r+1}-\tau_r}(x_i^r, x_j^{r+1}))_{i,j=1}^n.$$

In the limit $\epsilon \rightarrow 0+$, corresponding to all particles starting at the origin at time 0 and ending at the origin at time T , we get the probability density

$$(2.15) \quad \frac{1}{Z_{n,m}} \Delta_n(y^1) \prod_{j=0}^{n-1} e^{-(y_j^1)^2/2\tau_1} \prod_{r=1}^{m-2} \det(e^{-(y_j^{r+1}-y_i^r)^2/2(\tau_{r+1}-\tau_r)})_{i,j=1}^n$$

$$\times \Delta_n(y^{m-1}) \prod_{j=0}^{n-1} e^{-(y_j^{m-1})^2/2(T-\tau_{m-1})}.$$

This has again the general form (2.2) with $\phi_{0,1}(i, y) = q_i(y) \exp(-y^2/2\tau_1)$, $\phi_{r,r+1}(x, y) = \exp(-(y-x)^2/2(\tau_{r+1}-\tau_r))$ and $\phi_{m-1,m}(x, i) = \tilde{q}_i(x) \exp(-x^2/2(T-\tau_{m-1}))$, where q_i and \tilde{q}_i are polynomials of degree i . The measure (2.15) is actually a transformation of the measure (2.6). Define

$$(2.16) \quad d_r = \sqrt{\frac{T}{2\tau_r(T-\tau_r)}},$$

$1 \leq r < m$, and $\tau_r = T(1 + e^{-2t_r})^{-1}$. If we set $\lambda_j^r = y_j^r d_r$, $1 \leq j \leq n$, $1 \leq r < m$, then a straightforward computation shows that (2.15) transforms into (2.6). In this

way we can also transform the extended Hermite kernel (2.11) into a correlation kernel for (2.15). However, let us indicate how we can obtain it directly.

Set

$$c_{r,j} = \pi^{1/2} \left(\frac{\tau_r(T - \tau_{r+1})}{\tau_{r+1}(T - \tau_r)} \right)^{j/2}.$$

Then

$$(2.17) \quad \int_{\mathbb{R}} e^{-\frac{(y-x)^2}{2(\tau_{r+1}-\tau_r)}} e^{-\frac{x^2}{2\tau_r}} p_j(xd_r) dx = c_{r,j} p_j(yd_{r+1}) e^{-\frac{y^2}{2\tau_{r+1}}},$$

where p_j is the j :th normalized Hermite polynomial. This can be deduced from the identity

$$(2.18) \quad \int_{\mathbb{R}} e^{-(x-y)^2} p_n(\alpha x) dx = \pi^{1/2} (1 - \alpha^2)^{n/2} p_n\left(\frac{\alpha y}{(1 - \alpha^2)^{1/2}}\right),$$

which in turn follows easily from the generating function for the Hermite polynomials. Choose $q_j(x) = p_j(xd_1)$ and $\tilde{q}_j(x) = p_j(xd_{m-1})$. It follows from (2.17) that

$$(2.19) \quad \phi_{0,s}(j, x) = \left(\frac{\tau_1}{\tau_s} \prod_{i=1}^{s-1} (\tau_{i+1} - \tau_i) \right)^{1/2} 2^{(s-1)/2} \prod_{i=1}^{s-1} c_{i,j} p_j(xd_s) e^{-x^2/2\tau_s}$$

and

$$(2.20) \quad \phi_{r,m}(x, j) = \left(\frac{T - \tau_{m-1}}{T - \tau_r} \prod_{i=r}^{m-2} (\tau_{i+1} - \tau_i) \right)^{1/2} 2^{(m-r-1)/2} \prod_{i=r}^{m-2} c_{i,j} p_j(xd_r) e^{-x^2/2(T-\tau_r)}.$$

Using the orthogonality of the p_j :s and the general formula (2.4) we obtain the following expression for the correlation kernel

$$(2.21) \quad K_{\text{BM}}(\tau_r, x; \tau_s, y) = -\frac{1}{\sqrt{2\pi(\tau_r - \tau_s)}} e^{-\frac{(x-y)^2}{2(\tau_r - \tau_s)}} \chi_{\tau, \tau_s} \\ + \sum_{j=0}^{n-1} \left(\frac{\tau_r(T - \tau_s)}{\tau_s(T - \tau_r)} \right)^{j/2} \left(\frac{T}{2\tau_s(T - \tau_r)} \right)^{1/2} p_j(xd_r) p_j(yd_s) e^{-x^2/2(T-\tau_r) - y^2/2\tau_s}.$$

Here we have multiplied by the unimportant factor

$$(2\pi)^{\frac{r-s}{2}} \left(\frac{(\tau_s - \tau_{s-1}) \dots (\tau_2 - \tau_1)}{(\tau_r - \tau_{r-1}) \dots (\tau_2 - \tau_1)} \right)^{1/2}.$$

If we go back to the transformation discussed above we see that

$$(2.22) \quad \frac{1}{\sqrt{d_r d_s}} K_{\text{BM}}\left(\tau_r, \frac{x}{d_r}; \tau_s, \frac{y}{d_s}\right) e^{\frac{x^2 \tau_r}{T} - \frac{y^2 \tau_s}{T}} \left(\frac{T/\tau_r - 1}{T/\tau_s - 1} \right)^{1/4} = K_{\text{ext.Herm.}}(t_r, x; t_s, y),$$

with $\tau_r = T(1 + \exp(-2t_r))^{-1}$ and $K_{\text{ext.Herm.}}$ given by (2.11).

3. THE EXTENDED HAHN KERNEL

3.1. Derivation of the kernel. Consider a symmetric, simple random walks with initial points $(0, 2j)$ and final points $(b+c, c-b+2j)$, $0 \leq j \leq a-1$, conditioned not to intersect in the whole time interval $[0, b+c]$. The single step transition kernel for one particle is

$$(3.1) \quad \frac{1}{2}\phi(x, y) = \frac{1}{2}\delta_{x-1, y} + \frac{1}{2}\delta_{x+1, y}.$$

The configuration at time $t = r$, which we also call the configuration on the r :th line, is given by points z_j^r , $0 \leq j < a$, $z_0^r < \dots < z_{a-1}^r$, where $z_j^0 = 2j$, $z_j^{b+c} = c-b+2j$. We think of these points as the positions of particles. By the Lindström-Gessel-Viennot method, [19], our probability measure on the set of configurations $\underline{z} = (z_j^r)$ in $(\mathbb{Z}^a)^{b+c-1}$ is

$$(3.2) \quad p(\underline{z}) = \frac{1}{Z(a, b, c)} \prod_{r=0}^{b+c-1} \det(\phi(z_j^r, z_k^{r+1}))_{j, k=0}^{a-1}.$$

Here $Z(a, b, c)$ is the total number of configurations and is given by MacMahon's formula (1.1).

The measure (3.2) has exactly the general form (2.2) (with μ counting measure on \mathbb{Z}), and we want to compute the correlation kernel (2.4). To do this we will use the orthogonal polynomial method in a similar way that was used for the non-intersecting Brownian motions in the last section. How this should be done is not obvious from (3.2). It is shown in [9], that the induced probability ensemble on a single line is an orthogonal polynomial ensemble, where the relevant polynomials are the associated Hahn polynomials. This indicates that we should modify the first and the last factors in (3.2) by doing row operations so that we get a situation where the matrix A in (2.4) is diagonal.

The normalized associated Hahn polynomials, [16], [10], [2], can be defined using a hypergeometric function by

$$(3.3) \quad \begin{aligned} q_{n, N}^{(\alpha, \beta)}(x) &= \frac{(-N-\beta)_n (-N)_n}{d_{n, N}^{(\alpha, \beta)} n!} {}_3F_2 \left(\begin{matrix} -n, n-2N-\alpha-\beta-1, -x \\ -N-\beta, -N \end{matrix}; 1 \right) \\ &= \frac{(-N-\beta)_n (-N)_n}{d_{n, N}^{(\alpha, \beta)} n!} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(-x)_j (n-2N-\alpha-\beta-1)_j}{(-N-\beta)_j (-N)_j}, \end{aligned}$$

where

$$(3.4) \quad \left(d_{n, N}^{(\alpha, \beta)} \right)^2 = \frac{(\alpha + \beta + N + 1 - n)_{N+1}}{(\alpha + \beta + 2N + 1 - 2n)n!(\beta + N - n)!(\alpha + N - n)!(N - n)!},$$

and we use the standard notation $(a)_n = a(a+1)\dots(a+n-1)$. These polynomials are orthogonal with respect to the weight

$$(3.5) \quad w_N^{(\alpha, \beta)}(x) = \frac{1}{x!(x+\alpha)!(N+\beta-x)!(N-x)!},$$

on $\{0, 1, \dots, N\}$, i.e.

$$(3.6) \quad \sum_{x=0}^N q_{n, N}^{(\alpha, \beta)}(x) q_{m, N}^{(\alpha, \beta)}(x) w_N^{(\alpha, \beta)}(x) = \delta_{n, m},$$

for $0 \leq n, m \leq N$. Below we will sometimes use the convention that $1/n! = 0$ if $n < 0$, so that the summation in (3.6) for example could be extended to $x \in \mathbb{Z}$.

Our goal is to give a formula for the correlation kernel in terms of the associated Hahn polynomials. First, we need some notation. Let $a, b, c \in \mathbb{Z}^+$, $b \leq c$. Set $a_r = |c - r|$, $b_r = |b - r|$,

$$(3.7) \quad \alpha_r = \begin{cases} -r & , 0 \leq r \leq b \\ r - 2b & , b \leq r \leq b + c \end{cases}$$

and

$$(3.8) \quad \gamma_r = \begin{cases} r + a - 1 & , 0 \leq r \leq b \\ b + a - 1 & , b \leq r \leq c \\ a + b + c - 1 - r & , c \leq r \leq b + c. \end{cases}$$

Define

$$(3.9) \quad \omega_r(x) = \begin{cases} ((b_r + x)!(\gamma_r + a_r - x))^{-1} & , 0 \leq r \leq b \\ (x!(\gamma_r + a_r - x))^{-1} & , b \leq r \leq c \\ (x!(\gamma_r - x))^{-1} & , c \leq r \leq b + c \end{cases}$$

and

$$(3.10) \quad \tilde{\omega}_s(x) = \begin{cases} (y!(\gamma_s - y))^{-1} & , 0 \leq s \leq b \\ ((b_s + y)!(\gamma_s - y))^{-1} & , b \leq s \leq c \\ ((b_s + y)!(\gamma_s + a_s - y))^{-1} & , c \leq s \leq b + c. \end{cases}$$

Theorem 3.1. *The point process on $(\mathbb{Z}^a)^{b+c-1}$ defined by (3.21) has determinantal correlation functions with kernel given by*

$$(3.11) \quad \begin{aligned} K_H(r, \alpha_r + 2x; s, \alpha_s + 2y) &= -\phi_{r,s}(\alpha_r + 2x, \alpha_s + 2y) \\ &+ \sum_{n=0}^{a-1} \sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-1-n)!}} q_{n,\gamma_r}^{(b_r, a_r)}(x) q_{n,\gamma_s}^{(b_s, a_s)}(y) \omega_r(x) \tilde{\omega}_s(y), \end{aligned}$$

for $0 < r, s < b + c$, $x, y \in \mathbb{Z}$. Here $\phi_{r,s} \equiv 0$ if $r \geq s$ and

$$(3.12) \quad \phi_{r,s}(x, y) = \binom{s-r}{\frac{y-x+s-r}{2}}$$

if $r < s$.

Proof. Set

$$c_{j,k} = \frac{1}{(a-k)(j-k)!(a-1-j)!},$$

for $0 \leq j, j < a$,

$$f_{n,k} = \binom{n}{k} \frac{(n-2a-b-c+1)_k}{(-a-c+1)_k(-a)_k}$$

and

$$f_{n,k}^* = \binom{n}{k} \frac{(n-2a-b-c+1)_k}{(-a-b+1)_k(-a)_k}.$$

for $0 \leq k \leq n$. Define

$$(3.13) \quad \begin{aligned} \psi(n, z) &= \sum_{m=0}^n f_{n,m} \sum_{j=m}^{a-1} c_{j,m} \phi(2j, z), \\ \psi^*(n, z) &= \sum_{m=0}^n f_{n,m}^* \sum_{j=m}^{a-1} c_{j,m} \phi(c-b+2j, z) \end{aligned}$$

$0 \leq n < a$, $z \in \mathbb{Z}$.

We will now do row operations to modify the first and the last factor in (3.2).

$$(3.14) \quad \begin{aligned} \det(\phi(x_j^0, x_k^1))_{j,k=0}^{a-1} &= \det(\phi(2m, x_k^1))_{m,k=0}^{a-1} \\ &= \det\left(\frac{1}{c_{m,m}} \sum_{j=m}^{a-1} c_{j,m} \phi(2j, x_k^1)\right)_{m,k=0}^{a-1} \\ &= \prod_{m=0}^{a-1} \frac{1}{c_{m,m}} \det\left(\sum_{j=n}^{a-1} c_{j,n} \phi(2j, x_k^1)\right)_{n,k=0}^{a-1} \\ &= \prod_{m=0}^{a-1} \frac{1}{c_{m,m}} \det\left(\frac{1}{f_{n,n}} \sum_{m=0}^n f_{n,m} \sum_{j=m}^{a-1} c_{j,m} \phi(2j, x_k^1)\right)_{n,k=0}^{a-1} \\ &= \prod_{m=0}^{a-1} \frac{1}{c_{m,m} f_{m,m}} \det(\psi(n, x_k^1))_{n,k=0}^{a-1}. \end{aligned}$$

In the same way we obtain

$$(3.15) \quad \det(\phi(x_j^{b+c-1}, x_k^{b+c}))_{j,k=0}^{a-1} = \prod_{m=0}^{a-1} \frac{1}{c_{m,m} f_{m,m}^*} \det(\psi(n, x_k^{b+c-1}))_{n,k=0}^{a-1}.$$

If we now set $\phi_{0,1}(n, y) = \psi(n, y)$, $\phi_{b+c-1, b+c}(y, n) = \psi^*(n, y)$ and $\phi_{r, r+1}(x, y) = \phi(x, y)$, $1 \leq r < b+c-1$, the probability measure (3.2) can be written

$$(3.16) \quad p(\underline{y}) = \frac{1}{Z(a, b, c)} \prod_{m=0}^{a-1} \frac{1}{c_{m,m}^2 f_{m,m} f_{m,m}^*} \prod_{r=0}^{b+c-1} \det(\phi_{r, r+1}(y_j^r, y_k^{r+1}))_{j,k=0}^{a-1},$$

where $y_j^0 = y_j^{b+c} = y$, $0 \leq j < a$.

Write $\phi^{*n}(x, y) = \phi * \dots * \phi(x, y)$ (n factors) if $n \geq 2$, $\phi^{*1}(x, y) = \phi(x, y)$ and $\phi^{*0}(x, y) = \delta_{x,y}$. We want to compute $\phi_{0,s}$, $\phi_{r, b+c}$ and $\phi_{0, b+c}$ for $1 \leq r, s < b+c$. By definition

$$(3.17) \quad \phi_{0,r}(n, y) = \sum_{z \in \mathbb{Z}} \psi(n, z) \phi^{*(r-1)}(z, y),$$

and

$$(3.18) \quad \phi_{r, b+c}(y, n) = \sum_{z \in \mathbb{Z}} \psi^*(n, z) \phi^{*(b+c-r-1)}(z, y),$$

since $\phi(x, y) = \phi(y, x)$.

Claim 3.2. *If $z \in 2\mathbb{Z} + 1$, then*

$$(3.19) \quad \psi(n, z) = \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-c+1)_j (-a)_j \left(\frac{z+1}{2} - j\right)! \left(a - \frac{z+1}{2}\right)!},$$

and if $z \in 2\mathbb{Z}$, then $\psi(n, z) = 0$.

Proof. By definition

$$\psi(n, z) = \sum_{m=0}^n \binom{n}{m} \frac{(n-2a-b-c+1)_j}{(-a-c-1)_m (-a)_m} \sum_{j=m}^{a-1} \frac{\phi(2j, z)}{(a-m)(j-m)!(a-1-j)!}.$$

Now, with $z = 2\zeta - 1$,

$$\begin{aligned} & \sum_{j=m}^{a-1} \frac{\delta_{2j-1, z} + \delta_{2j+1, z}}{(a-m)(j-m)!(a-1-j)!} \\ &= \frac{1}{(a-m)(\zeta-m)!(a-1-\zeta)!} + \frac{1}{(a-m)(\zeta-1-m)!(a-\zeta)!} \\ &= \frac{1}{\left(\frac{z+1}{2}-m\right)!(a-\frac{z+1}{2})!}, \end{aligned}$$

and (3.19) follows. \square

If $z \in 2\mathbb{Z} - 1$, then similarly

$$(3.20) \quad \psi^*(n, c-b+z) = \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-b+1)_j (-a)_j \left(\frac{z+1}{2}-j\right)!(a-\frac{z+1}{2})!}.$$

Claim 3.3.

(3.21)

$$\phi_{0,r}(n, y) = (a+1)_{r-1} \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-c+1)_j (-a-r+1)_j} \frac{1}{\left(\frac{y+r}{2}-j\right)!(a-1-\frac{y-r}{2})!}.$$

Proof. By induction on r . The statement is true for $r = 1$ by (3.19). We have

$$\begin{aligned} \phi_{0,r+1}(n, y) &= \sum_{x \in \mathbb{Z}} \phi_{0,r}(n, x) \phi(x, y) \\ &= \sum_{x \in \mathbb{Z}} \phi_{0,r}(n, x) (\delta_{x, y+1} + \delta_{x, y-1}) = \phi_{0,r}(n, y+1) + \phi_{0,r}(n, y-1) \\ &= (a+1)_{r-1} \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-c+1)_j (-a-r+1)_j} \frac{1}{\left(\frac{y+r+1}{2}-j\right)!(a-1-\frac{y-r-1}{2})!} \\ &\quad \times \left[a - \frac{y-r+1}{2} + \frac{y+r+1}{2} - j \right]. \end{aligned}$$

Now,

$$\frac{(a+1)_{r-1}}{(-a-r+1)_j} (a-r-j) = \frac{(a+1)_r}{(-a-r)_j},$$

and the claim is proved. \square

Also,

$$\begin{aligned} \phi_{r,b+c}(c-b+x, n) &= \sum_{z \in \mathbb{Z}} \psi^*(n, z) \phi^{*(b+c-r-1)}(z, c-b+x) \\ &= \sum_{z \in \mathbb{Z}} \psi^*(n, c-b+z) \phi^{*(b+c-r-1)}(c-b+z, c-b+x) \\ &= \sum_{z \in \mathbb{Z}} \psi^*(n, c-b+z) \phi^{*(b+c-r-1)}(z, x). \end{aligned}$$

We can now proceed exactly as in the proof of claim 3.3 and show that

$$(3.22) \quad \begin{aligned} \phi_{r,b+c}(y, n) &= (a+1)_{b+c-r-1} \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-b+1)_j(-a-b-c+r+1)_j} \\ &\times \frac{1}{\left(\frac{y-r}{2} + b - j\right)! (a+c-1 - \frac{y+r}{2})!}. \end{aligned}$$

Introduce new coordinates, which we will call the *Hahn coordinates* on line r by

$$x_k^r = \frac{y_k^r - \alpha_r}{2}.$$

Then, $0 \leq x_k^r \leq \gamma_r$. One motivation to use these coordinates is that it is easier to recognize the Hahn polynomials when using them. Since $\phi_{0,r}(i, z)$ is zero unless $z+r$ is even, i.e. unless $z - \alpha_r$ is even, we obtain

$$(3.23) \quad A_{nm} = \sum_{z \in \mathbb{Z}} \phi_{0,r}(n, \alpha_r + 2z) \phi_{r,b+c}(\alpha_r + 2z, m).$$

The correlation kernel is given by

$$(3.24) \quad \begin{aligned} K(r, 2x + \alpha_r; s, 2y + \alpha_s) &= -\phi_{r,s}(2x + \alpha_r, 2y + \alpha_s) \\ &+ \sum_{i,j=0}^{a-1} \phi_{r,b+c}(2x + \alpha_r, i) (A^{-1})_{ij} \phi_{0,s}(j, 2y + \alpha_s) \end{aligned}$$

according to (2.4). We want to express $\phi_{0,r}(j, 2y + \alpha_r)$ and $\phi_{r,b+c}(2x + \alpha_r, i)$ in terms of the associated Hahn polynomials. In order to do so we have to distinguish three cases, $1 \leq r \leq b$, $b \leq r \leq c$ and $c \leq r \leq b+c$.

Set $a_r = |c-r|$ and $b_r = |b-r|$.

(i) Consider first the case $1 \leq r \leq b$. By (3.3) and (3.21)

$$(3.25) \quad \begin{aligned} \phi_{0,r}(n, \alpha_r + 2z) &= (a+1)_{r-1} \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-c+1)_j(-a-r+1)_j} \\ &\times \frac{1}{(z-j)!(a+r-1-z)!} = \frac{(a+1)_{r-1} d_{n,\gamma_r}^{(b_r, a_r)} n!}{(-a-c+1)_n (-a-r+1)_n} q_{n,\gamma_r}^{(b_r, a_r)}(z) \frac{1}{z!(\gamma_r - z)!}. \end{aligned}$$

Also, by (3.22),

$$\begin{aligned} \phi_{r,b+c}(\alpha_r + 2z, n) &= (a+1)_{b+c-r-1} \sum_{j=0}^n \binom{n}{j} \frac{(n-2a-b-c+1)_j}{(-a-b+1)_j(-a-b-c+r+1)_j} \\ &\times \frac{1}{(b-r+z-j)!(a+c-1-z)!} \\ &= \frac{(a+1)_{b+c-r-1}}{(b-r+z)!(a+c-1-z)!} \sum_{j=0}^n \frac{(-n)_j (n-2a-b-c)_j (-b+r-z)_j}{j! (-a-b+1)_j (-a-b-c+r+1)_j} \\ &= \frac{(a+1)_{b+c-r-1}}{(b_r+z)!(\gamma_r + \alpha_r - z)!} {}_3F_2 \left(\begin{matrix} -n, n-2a-b-c+1, -b+r-z \\ -a-b+1, -a-b-c+r+1 \end{matrix}; 1 \right). \end{aligned}$$

We can rewrite this using the following hypergeometric identity, [1] p. 141,

$$(3.26) \quad {}_3F_2 \left(\begin{matrix} -n, a, b \\ d, e \end{matrix}; 1 \right) = \frac{(d-a)_n (e-a)_n}{(d)_n (e)_n} {}_3F_2 \left(\begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix}; 1 \right).$$

This gives

$$\begin{aligned}
\phi_{r,b+c}(\alpha_r + 2z) &= \frac{(a+1)_{b+c-r-1}(a+c-n)_n(a+r-n)_n}{(-a+b+1)_n(-a-b-c+r+1)_n(b_r+z)!(\gamma_r+a_r-z)!} \\
&\times {}_3F_2\left(\begin{matrix} -n, n-2\gamma_r-a_r-b_r-1, -z \\ -\gamma_r-a_r, -\gamma_r \end{matrix}; 1\right). \\
&= \frac{(a+1)_{b+c-r-1}(a+c-n)_n(a+r-n)_n d_{n,\gamma_r}^{(b_r,a_r)} n!}{(-a+b+1)_n(-a-b-c+r+1)_n(-a-c+1)_n(-a-r+1)_n} \\
(3.27) \quad &\times q_{n,\gamma_r}^{(b_r,a_r)}(z) \frac{1}{(b_r+z)!(\gamma_r+a_r-z)!}
\end{aligned}$$

We can now compute A_{nm} given by (3.23) by picking r between 1 and b , the choice does not matter. Using (3.6), (3.25) and (3.27) we obtain, after some simplification

$$(3.28) \quad A_{nm} = C_n(a, b, c)^{-1} \delta_{n,m},$$

where

$$(3.29) \quad C_n(a, b, c) = \frac{(a+b-1)!(a+c-1)!(2a+b+c-2n-1)a!^2}{n!(2a+b+c-n-1)!}.$$

(ii) Next we consider the case $b \leq r \leq c$. The computations are similar to those in the previous case. We find

$$\begin{aligned}
\phi_{0,r}(n, \alpha_r + 2z) &= \frac{(a+1)_{r-1}(a+b-n)_n(a+b+c-r-n)_n d_{n,\gamma_r}^{(b_r,a_r)} n!}{(-a-c+1)_n(-a-r+1)_n(-a-b+1)_n(-a-b-c+1+r)_n} \\
(3.30) \quad &\times q_{n,\gamma_r}^{(b_r,a_r)}(z) \frac{1}{(b_r+z)!(\gamma_r-z)!}.
\end{aligned}$$

Here we have used the hypergeometric identity (3.26). Also, we find

$$\begin{aligned}
\phi_{r,b+c}(\alpha_r + 2z, n) &= \frac{(a+1)_{b+c-r-1} d_{n,\gamma_r}^{(b_r,a_r)} n!}{(-a-b+1)_n(-a-b-c+1+r)_n} \\
(3.31) \quad &\times q_{n,\gamma_r}^{(b_r,a_r)}(z) \frac{1}{z!(\gamma_r+a_r-z)!}.
\end{aligned}$$

(iii) Finally we come to the case $c \leq r \leq b+c$, and again the computations are similar. We obtain

$$\begin{aligned}
\phi_{0,r}(n, \alpha_r + 2z) &= \frac{(a+1)_{r-1}(a+b-n)_n(a+b+c-r-n)_n d_{n,\gamma_r}^{(b_r,a_r)} n!}{(-a-c+1)_n(-a-r+1)_n(-a-b+1)_n(-a-b-c+1+r)_n} \\
(3.32) \quad &\times q_{n,\gamma_r}^{(b_r,a_r)}(z) \frac{1}{(b_r+z)!(\gamma_r+a_r-z)!},
\end{aligned}$$

where we have used the identity (3.26). Also,

$$\begin{aligned}
\phi_{r,b+c}(\alpha_r + 2z, n) &= \frac{(a+1)_{b+c-r-1} d_{n,\gamma_r}^{(b_r,a_r)} n!}{(-a-b+1)_n(-a-b-c+1+r)_n} \\
(3.33) \quad &\times q_{n,\gamma_r}^{(b_r,a_r)}(z) \frac{1}{z!(\gamma_r-z)!}.
\end{aligned}$$

We now have all the ingredients in (2.4). It follows from (3.28) that

$$(A^{-1})_{ij} = C_i(a, b, c) \delta_{ij},$$

and some computation now gives (3.11). Note that $\phi_{r,s}(x,y)$ is the number of random walk paths from x to y in $s-r$ steps and hence is given by (3.12).

The computations in the proof of the theorem also gives a proof of MacMahons formula. We have

$$(3.34) \quad Z(a,b,c) = \prod_{n=0}^{a-1} \frac{1}{c_{n,n}^2 d_{n,n} d_{n,n}^*} \det A.$$

A computation gives

$$\prod_{n=0}^{a-1} \frac{1}{c_{n,n}^2 d_{n,n} d_{n,n}^*} = \prod_{n=0}^{a-1} \frac{(2a+b+c-2n-1)!^2 (a+b-1)! (a+c-1)!}{(a+b-1-n)! (a+c-1-n)! (2a+b+c-n-1)!^2}.$$

It follows from (3.28) and (3.29) that

$$\det A = \prod_{n=0}^{a-1} \frac{n! (2a+b+c-n-1)!}{(a+b-1)! (a+c-1)! (2a+b+c-2n-1)! a!^2}.$$

Hence, by (3.34) and after some simplification

$$Z(a,b,c) = \prod_{n=0}^{a-1} \frac{n! (b+c+n)!}{(b+n)! (c+n)!},$$

which is the same as (1.1). \square

3.2. Some remarks about asymptotics. As discussed above the non-intersecting Brownian motion model (2.15) is a kind of continuum version of the random walk model. In fact it can be obtained as a scaling limit of the random walk model. For the associated Hahn polynomials we have the asymptotics

$$(3.35) \quad \lim_{N \rightarrow \infty} d_{n,N}^{(\alpha,\alpha)} n! \left(-\frac{2}{N^{3/2} \sqrt{(2t+1)(t+1)}} \right)^n p_{n,N}^{(\alpha,\alpha)} \left(\frac{N}{2} + 2z \sqrt{\frac{2t+1}{t+1}} N \right) = H_n(z),$$

where $\alpha/N \rightarrow t \geq 0$, uniformly for z in a compact subset of \mathbb{C} . Here $H_n(z)$ is the ordinary Hermite polynomial of degree n . This can be proved by a slight modification of the argument in [8] based on the recurrence relation. Using (3.35) and standard asymptotics for the binomial coefficient it follows that

$$(3.36) \quad 2^{r-s} \sqrt{\frac{k}{2T}} K_H(r,x;s,y) \rightarrow K_{\text{BM}}(\tau,\xi;\sigma,y)$$

as $k \rightarrow \infty$ if $r/k \rightarrow 2\tau/T$, $s/k \rightarrow 2\sigma/T$, $x/\sqrt{k} \rightarrow \xi\sqrt{2/T}$, $y/\sqrt{k} \rightarrow \eta\sqrt{2/T}$, where K_{BM} is given by (2.21). So in this sense we have convergence to the Brownian motion model. It should also be possible to prove this directly, i.e. that the measure (3.2) converges, when rescaled as above, to the measure (2.15), compare the arguments in [13].

A more interesting, and also much more difficult limit is to consider the case when, a , b and c go to infinity with the same rate, say $a = b = c \rightarrow \infty$. In particular it is interesting to consider the fluctuations of the top (and bottom) curves which bound the so called frozen regions, [3], in the tiling. If we restrict to a single line, this has been done recently by [2] using very precise asymptotics for Hahn polynomials derived using Riemann-Hilbert techniques. This shows for

example that if $n = a = b = c$ then the last (first) particle fluctuates like $n^{1/3}$ in the appropriate region and that the fluctuations are given by the Tracy-Widom distribution, see [2] for details. If these asymptotic results could be extended to the extended (associated) Hahn kernel, (3.11), it should be possible to prove the convergence of the boundary curve of the frozen region to the Airy process, [17], [11], as has been done for some other tiling problems in [6] and [12].

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