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Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green's function

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Abstract. Two discretizations, linear and nonlinear, of basic notions of the complex analysis are considered. The underlying lattice is an arbitrary quasicrystallic rhombic tiling of a plane. The linear theory is based on the discrete Cauchy-Riemann equations, the nonlinear one is based on the notion of circle patterns. We clarify the role of the rhombic condition in both theories: under this condition the corresponding equations are integrable (in the sense of 3D consistency, which yields also the existense of zero curvature representations, Bäcklund transformations etc.). We demonstrate that in some precise sense the linear theory is a linearization of the nonlinear one: the tangent space to a set of integrable circle patterns at an isoradial point consists of discrete holomorphic functions which take real (imaginary) values on two sublattices. We extend solutions of the basic equations of both theories to $\mathbb{Z}^d$, where d is the number of different edge slopes of the quasicrystallic tiling. In the linear theory, we give an integral representation of an arbitrary discrete holomorphic function, thus proving the density of discrete exponential functions. We introduce the d-dimensional discrete logarithmic function which is a generalization of Kenyon's discrete Green's function, and uncover several new properties of this function. We prove that it is an isomonodromic solution of the discrete Cauchy-Riemann equations, and that it is a tangent vector to the space of integrable circle patterns along the family of isomonodromic discrete power functions.

1 Introduction

There is currently much interest in finding discrete counterparts of various structures of the classical (continuous, smooth) mathematics. In the present paper we are dealing with the discretization of the classical complex analysis.

There are two approaches to this problem. The first one, which we shall call the *linear theory*, is based on a discretization of the Cauchy-Riemann equations. Since the latter are linear, straightforward discretizations are linear as well. A discretization preserving apparently the most number of important structural features has been developed in [F, D1, D2, M1, K]. The first two references are dealing with *discrete holomorphic functions* $f : \mathbb{Z}^2 \to \mathbb{C}$ on the regular square lattice, satisfying the following *discrete Cauchy-Riemann equations*:

$$f_{m,n+1} - f_{m+1,n} = i(f_{m+1,n+1} - f_{m,n}). \qquad (1)$$

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A pioneering step was undertaken by Duffin [D2], where the combinatorics of $\mathbb{Z}^2$ was given up in favor of arbitrary planar graphs with rhombic faces. A far reaching generalization of these ideas is given in [M1], where the linear theory is extended to discrete Riemann surfaces. Planar graphs with rhombic faces are called *critical* in [M1]. Kenyon [K] developed a theory of the Dirac operator and constructed Green's function in the framework of the linear theory on critical graphs. See [CY, G] for combinatorial, resp. numerical aspects of Green's functions on graphs.

The second approach, which we will call the *nonlinear theory*, is based on the ideas by Thurston [T], and declares *circle patterns* to be natural discrete analogs of analytic functions [BeS, DS, Sch, S]. One of the most important achievements of this theory is the proof that the Riemann map can be (constructively) approximated by circle packings [RS, MR, HS]. The variational approach to circle patterns is discussed in detail in [BSp]. The word "nonlinear" refers to the basic feature of equations describing circle patterns. Often, the so-called *cross-ratio system* is used for this. For a function $f : \mathbb{Z}^2 \to \mathbb{C}$ on the regular square lattice, this system was introduced in [NC]:

$$\frac{(f_{m+1,n} - f_{m,n})(f_{m+1,n+1} - f_{m,n+1})}{(f_{m,n+1} - f_{m,n})(f_{m+1,n+1} - f_{m+1,n})} = -1. \tag{2}$$

For circle patterns with more sophisticated combinatorics, a generalization of this system to an arbitrary quad-graph (planar graph with quadrilateral faces) is required [BS].

It is not difficult to see in what sense solutions of equations like (1), (2) can be considered as discretized analytic functions. Indeed, assume that $\mathbb{Z}^2$ is embedded in the complex plane $\mathbb{C}$ with the grid size ε, i.e., the pair $(m, n) \in \mathbb{Z}^2$ corresponds to $(m+in)\varepsilon \in \mathbb{C}$. Then restrictions of analytic functions to this grid satisfy the corresponding equations up to $O(\varepsilon^2)$. More precisely, if $f : \mathbb{C} \to \mathbb{C}$ is analytic, then

$$\frac{f(z+i\varepsilon) - f(z+\varepsilon)}{f(z+\varepsilon+i\varepsilon) - f(z)} = i + O(\varepsilon^2),$$

and

$$\frac{\big(f(z+\varepsilon) - f(z)\big)\big(f(z+\varepsilon+i\varepsilon) - f(z+i\varepsilon)\big)}{\big(f(z+i\varepsilon) - f(z)\big)\big(f(z+\varepsilon+i\varepsilon) - f(z+\varepsilon)\big)} = -1 + O(\varepsilon^2).$$

Similar relations hold on more general graphs.

For a long time, the linear and the nonlinear theories of discrete analytic functions were considered separately. In the present paper, we show that in some precise sense the former is a *linearization* of the latter. We work in the set-up of rhombic tilings of a plane. The theory becomes especially rich for *quasicrystallic* tilings, – those with a finite number of different edge slopes. This class includes double periodic tilings (which are naturally considered on a torus), as well as non-periodic ones, like the Penrose tiling. We clarify the importance of rhombic embeddings of quad-graphs in both the linear and the nonlinear theories. Namely, we show that the rhombic property implies (actually, is almost synonymous with) *integrability.* Note that interrelations of circle patterns with the theory of integrable systems were already uncovered and studied in [BP, AB1, AB2, BHS, BH]. Note also that some of the ideas behind our unified treatment of integrability of linear and nonlinear systems, such as the use of zero curvature representations in both situations, are similar to the philosophy of Fokas's unified transform method for linear and nonlinear differential equations based on the Riemann-Hilbert boundary problem [Fo]. Our main results are the following.

- Discrete Cauchy-Riemann equations on a rhombically embedded quad-graph $\mathcal{D}$, with weights given by quotients of diagonals of the corresponding rhombi, are integrable. Integrability is understood here as 3D consistency [BS]. Therefore, discrete holomorphic functions on rhombic embeddings can (and should) be extended to multidimensional lattices. In particular, discrete holomorphic functions on a quasicrystallic rhombic embedding $\mathcal{D}$ with d different edge slopes can be considered as restrictions of discrete holomorphic functions on $\mathbb{Z}^d$ to a certain two-dimensional subcomplex $\Omega_{\mathcal{D}}$ in $\mathbb{Z}^d$.

- Cross-ratio equations on a rhombically embedded quad-graph $\mathcal{D}$, with cross-ratios read off the corresponding rhombi, are integrable as well. Therefore, solutions of the cross-ratio equations on a quasicrystallic rhombic embedding $\mathcal{D}$ are naturally extended to $\mathbb{Z}^d$.

- For a circle pattern, the centers and the intersection points of the circles yield a solution of cross-ratio equations, with the cross-ratios depending on the pairwise intersection angles of the circles. We say that a circle pattern is integrable, if the corresponding cross-ratio system is integrable. The combinatorics and intersection angles belong to an integrable circle pattern, if and only if they admit an isoradial realization. This latter realization gives a rhombic immersion of the corresponding quad-graph, and generates also a dual isoradial circle pattern. An integrable circle pattern can be alternatively described by the radii of the circles and the rotation angles of the configurations at the intersection points with respect to the isoradial realization. These data comprise a solution of an integrable *Hirota system.*

- The tangent space to the set of integrable circle patterns, at the point corresponding to an isoradial pattern, coincides with the space of discrete holomorphic functions on the corresponding rhombically embedded quad-graph, which take real (resp. pure imaginary) values on the white (resp. black) vertices. This holds in the description of circle patterns in terms of circle radii and rotation angles at the intersection points (Hirota equations). Discrete holomorphic functions obtained from these ones by discrete integration, comprise the tangent space to the set of integrable circle patterns, described in terms of circle centers and intersection points (cross-ratio equations).

- We define (in the linear theory) discrete exponential functions on $\mathbb{Z}^d$, and prove that they are dense in the space of discrete holomorphic functions, growing not faster than exponentially.

- We define (in the linear theory) a discrete logarithmic function on $\mathbb{Z}^d$, or, better, on a branched covering of certain d-dimensional octants [1] $S_m \subset \mathbb{Z}^d$, $m = 1, \dots, 2d$. On each such octant, the discrete logarithmic function is discrete holomorphic, with the distinctive property of being isomonodromic, in the sense of the integrable systems theory. We show that the real part of the discrete logarithmic function restricted to a surface $\Omega_{\mathcal{D}}$ in $\mathbb{Z}^d$ coming from a quasicrystallic quad-graph $\mathcal{D}$ is nothing but Green's function found in [K]. The integral representation of Green's function given in [K] is derived within the isomonodromic approach.

- We define (in the nonlinear theory) discrete power functions $w^{\gamma-1}$ (resp. z^γ) on the same branched covering of octants $S_m \subset \mathbb{Z}^d$, $m = 1, \dots, 2d$, where the discrete logarithmic function is defined. On each such sector, discrete $w^{\gamma-1}$ (resp. z^γ) is an isomonodromic solution of the Hirota (resp. cross-ratio) system. The tangent vector to the space of integrable circle patterns along the curve consisting of patterns $w^{\gamma-1}$, at the isoradial point corresponding to $\gamma = 1$, is shown to be the discrete logarithmic function.

In conclusion, we point out some generalizations of the concepts and results of this paper for the non-rhombic case.

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[1] We use this term for a subset of $\mathbb{Z}^d$ defined by fixing one of 2^d possible combinations of signs of the coordinates. An octant in the proper sense corresponds to $d = 3$, while by $d = 2$ this object is called quadrant.

2 Discrete harmonic and discrete holomorphic functions on graphs

We denote by $V(\mathcal{G})$, $E(\mathcal{G})$ and $\vec{E}(\mathcal{G})$ the sets of vertices, undirected and directed edges of a graph $\mathcal{G}$, respectively. Let there be given a complex-valued function $\nu : E(\mathcal{G}) \to \mathbb{C}$ on the edges. Then the *Laplacian* Δ corresponding to the weight function ν is the operator acting on functions $f : V(\mathcal{G}) \to \mathbb{C}$ by

$$(\Delta f)(x_0) = \sum_{x \sim x_0} \nu(x_0, x)(f(x) - f(x_0)). \tag{3}$$

Here the summation is extended over the set of all vertices x connected to x_0 by an edge. We will use the notation $\mathrm{star}(x_0) = \mathrm{star}(x_0; \mathcal{G})$ for the set of all edges of $\mathcal{G}$ incident to x_0, see Fig. 1.

Definition 1 *A function $f : V(\mathcal{G}) \to \mathbb{C}$ is called **discrete harmonic** (with respect to the weights ν), if $\Delta f = 0$.*

Of course, the most interesting case of these notions is that of real positive weights $\nu : E(\mathcal{G}) \to \mathbb{R}_+$.

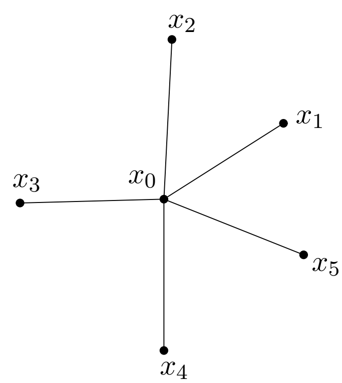

Figure 1: The star of the vertex x_0 in the graph $\mathcal{G}$.

In the continuous case, there is a canonical correspondence between harmonic and holomorphic functions on $\mathbb{C}$: the real and the imaginary parts of a holomorphic function are harmonic, and any real-valued harmonic function can be considered as a real part of a holomorphic function. This relation can be generalized for functions on graphs, but these two classes of functions live then on different graphs. Discrete holomorphic functions live on quad-graphs.

Definition 2 *A cell decomposition $\mathcal{D}$ of the plane $\mathbb{C}$ is called a **quad-graph**, if all its faces are quadrilaterals.*

A more general version of this definition deals with cell decompositions of an arbitrary oriented surface. So, quad-graphs are not just graphs, but are additionally assumed to be embedded in an oriented surface; we will deal with the case of $\mathbb{C}$ only.

To establish a relation with discrete harmonic functions, we consider the latter ones on graphs $\mathcal{G}$ with an additional structure, namely on those that come from general (not necessarily quadrilateral) cell decompositions of $\mathbb{C}$. We will denote by $F(\mathcal{G})$ the set of faces (2-cells) of $\mathcal{G}$. To any such $\mathcal{G}$ there corresponds canonically a combinatorial quad-graph called its double (or diamond [M1]), constructed from $\mathcal{G}$ and its dual $\mathcal{G}^*$. Recall that, in general, a *dual cell decomposition* $\mathcal{G}^*$ is only defined up to isotopy, but it can be fixed uniquely with the help of the Voronoi/Delaunay construction. The dual $\mathcal{G}^*$ is characterized as follows. Vertices of $\mathcal{G}^*$ are in a one-to-one correspondence to faces of $\mathcal{G}$, see Fig. 2. Each $\mathfrak{e} \in E(\mathcal{G})$ separates two faces of $\mathcal{G}$, which in turn correspond to two vertices of $\mathcal{G}^*$. It is declared that these two vertices are connected by the edge $\mathfrak{e}^* \in E(\mathcal{G}^*)$ dual to $\mathfrak{e}$. Finally, the faces of $\mathcal{G}^*$ are in a one-to-one correspondence with the vertices of $\mathcal{G}$: if $x_0 \in V(\mathcal{G})$, and $x_1, \ldots, x_n \in V(\mathcal{G})$ are its neighbors connected with x_0 by the edges $\mathfrak{e}_1 = (x_0, x_1), \ldots, \mathfrak{e}_n = (x_0, x_n) \in E(\mathcal{G})$, then the face of $\mathcal{G}^*$ corresponding to x_0 is defined by its boundary $\mathfrak{e}_1^* \cup \ldots \cup \mathfrak{e}_n^*$ (cf. Fig. 3). If one assigns a direction to an edge $\mathfrak{e} \in E(\mathcal{G})$, then it will be assumed that the dual edge $\mathfrak{e}^* \in E(\mathcal{G}^*)$ is also directed, in a way consistent with the orientation of the underlying surface, namely so that the pair $(\mathfrak{e}, \mathfrak{e}^*)$ is oriented directly at its crossing point. This orientation convention implies that $\mathfrak{e}^{**} = -\mathfrak{e}$.

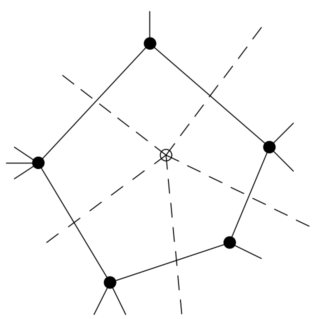

Figure 2: Vertex of $\mathcal{G}^*$ dual to a face of $\mathcal{G}$.

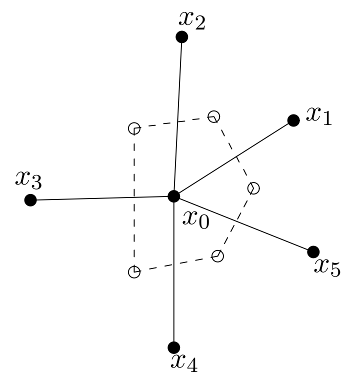

Figure 3: Face of $\mathcal{G}^*$ dual to a vertex of $\mathcal{G}$.

Now the *double* $\mathcal{D}$ is constructed from $\mathcal{G}$, $\mathcal{G}^*$ as follows. The set of vertices of the double $\mathcal{D}$ is $V(\mathcal{D}) = V(\mathcal{G}) \sqcup V(\mathcal{G}^*)$. Each pair of dual edges, say $\mathfrak{e} = (x_0, x_1) \in E(\mathcal{G})$ and $\mathfrak{e}^* = (y_0, y_1) \in E(\mathcal{G}^*)$, defines a quadrilateral (x_0, y_0, x_1, y_1). These quadrilaterals constitute the faces of the cell decomposition (quad-graph) $\mathcal{D}$, see Fig. 4. The edges of $\mathcal{D}$ belong neither to $E(\mathcal{G})$ nor to $E(\mathcal{G}^*)$. A star of a vertex $x_0 \in V(\mathcal{G})$ produces a flower of adjacent quadrilaterals from $F(\mathcal{D})$ around the common vertex x_0, see Fig. 5.

Observe that the double $\mathcal{D}$ is automatically *bipartite*, since its vertices $V(\mathcal{D})$ are decomposed into two complementary halves, $V(\mathcal{D}) = V(\mathcal{G}) \sqcup V(\mathcal{G}^*)$ ("black" and "white" vertices), such that the ends of each edge from $E(\mathcal{D})$ are of different colours. An arbitrary quad-graph embedded in $\mathbb{C}$ is automatically bipartite, and the above construction can be reversed for it, to produce a cell decomposition $\mathcal{G}$ along with its dual $\mathcal{G}^*$. The decomposition of $V(\mathcal{D})$ into $V(\mathcal{G})$ and $V(\mathcal{G}^*)$ is unique, up to interchanging the roles of $\mathcal{G}$ and $\mathcal{G}^*$. Edges of $\mathcal{G}$ (say) connect two "black" vertices along the diagonal of each face of $\mathcal{D}$.

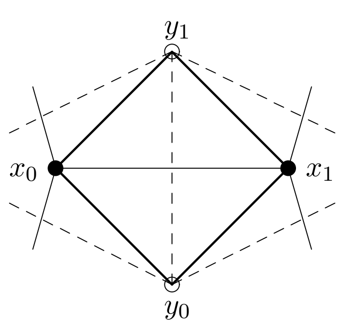

Figure 4: A face of the double $\mathcal{D}$

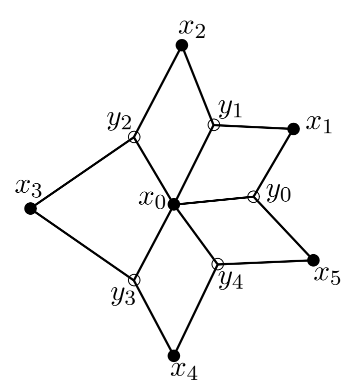

Figure 5: Faces of $\mathcal{D}$ around vertex x_0.

Let there be given a function $\nu : E(\mathcal{G}) \to \mathbb{C}$ on undirected edges of $\mathcal{G}$. (It is assumed that both directed representatives $\pm\mathfrak{e}$ of any edge carry the same value $\nu(\mathfrak{e}) = \nu(-\mathfrak{e})$ as the underlying undirected one.) Extend the function ν to undirected edges of $\mathcal{G}^*$ according to the rule

$$\nu(\mathfrak{e}^*) = 1/\nu(\mathfrak{e}). \tag{4}$$

Definition 3 *A function $f : V(\mathcal{D}) \to \mathbb{C}$ is called* ***discrete holomorphic*** *(with respect to the weights ν), if for any positively oriented quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ there holds:*

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = i\nu(x_0, x_1) = -\frac{1}{i\nu(y_0, y_1)}. \tag{5}$$

These equations are called ***discrete Cauchy-Riemann equations****.*

Again, the most interesting case corresponds to the real positive weights $\nu : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{R}_+$. The theory of discrete holomorphic functions was developed in [M1]. In [K] a discrete Dirac operator was introduced, the kernel of which consists of discrete holomorphic functions. In the present paper, like in [M1], our attention belongs not to the discrete Dirac operator but to its kernel only. The next statement follows immediately.

Proposition 4 *a) If a function $f : V(\mathcal{D}) \to \mathbb{C}$ is discrete holomorphic, then its restrictions to $V(\mathcal{G})$ and to $V(\mathcal{G}^*)$ are discrete harmonic.*

b) Conversely, any discrete harmonic function $f : V(\mathcal{G}) \to \mathbb{C}$ admits a family of discrete holomorphic extensions to $V(\mathcal{D})$, differing by an additive constant on $V(\mathcal{G}^*)$. Such an extension is uniquely defined by a value at one arbitrary vertex $y \in V(\mathcal{G}^*)$.

3 Rhombic embeddings and labelings of a quad-graph

The paper [KS] studies *rhombic embeddings* of a quad-graph $\mathcal{D}$ in $\mathbb{C}$, i.e., embeddings with the property that each face of $\mathcal{D}$ is mapped to a rhombus. A combinatorial criterion for the existence of a rhombic embedding of a given quad-graph $\mathcal{D}$ found in [KS] is as follows.

Definition 5 *A* ***strip*** *$\mathcal{S}$ in $\mathcal{D}$ is a path $\{\mathfrak{a}_j^*\}_{j=-\infty}^{\infty}$ in $\mathcal{D}^*$ with the following property: for any two consecutive dual edges $\mathfrak{a}_j^*, \mathfrak{a}_{j+1}^* \in \mathcal{S} \subset E(\mathcal{D}^*)$ with the common point $\mathfrak{a}_j^* \cap \mathfrak{a}_{j+1}^* = q_j \in V(\mathcal{D}^*) \simeq F(\mathcal{D})$, the corresponding edges $\mathfrak{a}_j, \mathfrak{a}_{j+1} \in E(\mathcal{D})$ are two opposite sides of the quadrilateral q_j. The edges $\{\mathfrak{a}_j\}_{j=-\infty}^{\infty}$ are called the* ***traverse edges*** *of the strip $\mathcal{S}$.*

Theorem 6 [KS] *A planar quad-graph $\mathcal{D}$ admits a rhombic embedding in $\mathbb{C}$ if and only if the following two conditions are satisfied:*

- *No strip crosses itself or is periodic.*
- *Two distinct strips cross each other at most once.*

A rhombic embedding determines rhombus angles that are naturally assigned to the edges of $\mathcal{G}$ and $\mathcal{G}^*$, see Fig. 6. Such systems of rhombus angles $\phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to (0, \pi)$ are characterized, according to [KS], by the following two conditions: first,

$$\phi(\mathfrak{e}^*) = \pi - \phi(\mathfrak{e}), \qquad \forall \mathfrak{e} \in E(\mathcal{G}), \tag{6}$$

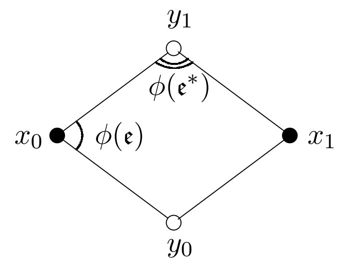

Figure 6: A rhombic embedding of a quadrilateral $(x_0, y_0, x_1, z_1) \in F(\mathcal{D})$, $\mathfrak{e} = (x_0, x_1)$, $\mathfrak{e}^* = (y_0, y_1)$

and, second,

$$\sum_{\mathfrak{e} \in \text{star}(x_0;\mathcal{G})} \phi(\mathfrak{e}) = 2\pi, \qquad \sum_{\mathfrak{e}^* \in \text{star}(y_0;\mathcal{G}^*)} \phi(\mathfrak{e}^*) = 2\pi, \qquad \forall x_0 \in V(\mathcal{G}),\ y_0 \in V(\mathcal{G}^*). \tag{7}$$

As mentioned in [KS], to each rhombic embedding of $\mathcal{D}$ there corresponds a set of parallelogram embeddings (wherein each face is mapped to a parallelogram), which are obtained by replacing each traverse edge of a strip with a real multiple (a different multiple for each strip).

Definition 7 *A* ***labeling*** *is a function $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ such that $\alpha(-\mathfrak{a}) = -\alpha(\mathfrak{a})$ for any $\mathfrak{a} \in \vec{E}(\mathcal{D})$, and the values of α on two opposite and equally directed edges of any quadrilateral from $F(\mathcal{D})$ are equal to one another.*

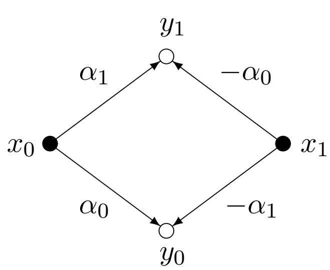

Figure 7: Labeling of directed edges

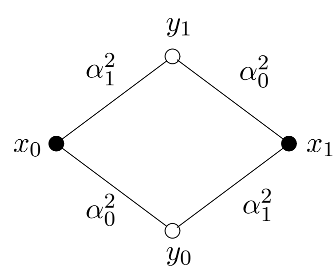

Figure 8: Labeling of undirected edges

This definition is illustrated in Fig. 7. Note that if edges of any given face of $\mathcal{D}$ are directed as on this figure (from black to white), then any two opposite edges carry opposite labels. For any labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ of directed edges, the function α^2 can be considered as a *labeling of undirected edges*, i.e., as a function $\alpha^2 : E(\mathcal{D}) \to \mathbb{C}$ such that its values on two opposite edges of any quadrilateral from $F(\mathcal{D})$ are equal to one another, see Fig. 8. Conversely, any labeling of undirected edges comes as a square of some labeling of directed edges.

The existence of a labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ is equivalent to the existence of a *parallelogram immersion* of the quad-graph $\mathcal{D}$. Indeed, given a parallelogram immersion $p : V(\mathcal{D}) \to \mathbb{C}$, one defines canonically a labeling by

$$\alpha(x, y) = p(y) - p(x), \quad \forall (x, y) \in \vec{E}(\mathcal{D}). \tag{8}$$

Conversely, given a labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$, the formula (8) correctly defines a function $p : V(\mathcal{D}) \to \mathbb{C}$ and assures that the p-image of any quadrilateral face of $\mathcal{D}$ is a parallelogram. If the labels α take values in $\mathbb{S}^1 = \{\theta \in \mathbb{C} : |\theta| = 1\}$, then the corresponding immersion is rhombic.

Definition 8 *A parallelogram immersion $p : V(\mathcal{D}) \to \mathbb{C}$ of a quad-graph $\mathcal{D}$ is called* ***quasicrystallic****, if the set of values of the corresponding labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$, defined by (8), is finite, say $A = \{\pm\alpha_1, \ldots, \pm\alpha_d\}$.*

It will be supposed that any two non-opposite elements of A are linearly independent over $\mathbb{R}$. This implies, in particular, that all parallelograms are non-degenerate.

It will be of a central importance for us that any quasicrystallic parallelogram immersion p can be seen as a sort of a projection of a certain two-dimensional subcomplex (combinatorial surface) $\Omega_{\mathcal{D}}$ of a multi-dimensional regular square lattice $\mathbb{Z}^d$. The vertices of $\Omega_{\mathcal{D}}$ are given by a map $P : V(\mathcal{D}) \to \mathbb{Z}^d$ constructed as follows. Fix some $x_0 \in V(\mathcal{D})$, and set $P(x_0) = \mathbf{0}$. For all other vertices of $\mathcal{D}$, their images in $\mathbb{Z}^d$ are defined recurrently by the property:

- For any two neighbors $x, y \in V(\mathcal{D})$, if $p(y) - p(x) = \pm\alpha_i \in A$, then $P(y) - P(x) = \pm\mathbf{e}_i$,

where $\mathbf{e}_i$ is the i-th coordinate vector of $\mathbb{Z}^d$. The edges and faces of $\Omega_{\mathcal{D}}$ correspond to edges and faces of $\mathcal{D}$. So, the combinatorics of $\Omega_{\mathcal{D}}$ is that of $\mathcal{D}$, and therefore Theorem 6 can be used to decide whether a given two-dimensional subcomplex of $\mathbb{Z}^d$ corresponds in this way to some rhombic embedding of a quad-graph in $\mathbb{C}$.

4 3D consistency

We now study a question about *integrability* of the discrete Cauchy-Riemann equations (5). These equations are just a specific linear issue of general equations on quad-graphs [BS]

$$\Phi(f(x_0), f(y_0), f(x_1), f(y_1)) = 0, \tag{9}$$

relating four fields f sitting on the four vertices of an arbitrary (oriented) face $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ of a quad-graph $\mathcal{D}$. Here the function Φ may depend on some parameters (in the case of discrete Cauchy-Riemann equations these are the weights ν), and it is supposed that equation (9) is uniquely solvable for any one of the fields in terms of other three (which is, of course, the case for discrete Cauchy-Riemann equations with non-vanishing weights ν).

The approach pushed forward in [BS] is based on the idea that integrability of such equations on quad-graphs is synonymous with their 3D consistency. To describe the latter notion, we extend the planar quad–graph $\mathcal{D}$ into the third dimension. Formally speaking, we consider the second copy $\widehat{\mathcal{D}}$ of $\mathcal{D}$ and add edges connecting each vertex $x \in V(\mathcal{D})$ with its copy $\widehat{x} \in V(\widehat{\mathcal{D}})$. On this way we obtain a "three-dimensional quad-graph" $\mathbf{D}$, whose set of vertices is

$$V(\mathbf{D}) = V(\mathcal{D}) \sqcup V(\widehat{\mathcal{D}}),$$

whose set of edges is

$$E(\mathbf{D}) = E(\mathcal{D}) \sqcup E(\widehat{\mathcal{D}}) \sqcup \{(x, \widehat{x}) : x \in V(\mathcal{D})\},$$

and whose set of faces is

$$F(\mathbf{D}) = F(\mathcal{D}) \sqcup F(\widehat{\mathcal{D}}) \sqcup \{(x, y, \widehat{y}, \widehat{x}) : (x, y) \in E(\mathcal{D})\}.$$

Elementary building blocks of $\mathbf{D}$ are cubes $(x_0, y_0, x_1, y_1, \widehat{x}_0, \widehat{y}_0, \widehat{x}_1, \widehat{y}_1)$, as shown on Fig. 9.

Clearly, if $\mathcal{D}$ is bipartite, then so is $\mathbf{D}$: each $\widehat{x} \in V(\widehat{\mathcal{D}})$ has the colour opposite to the colour of its counterpart $x \in V(\mathcal{D})$. Hence, we can extend the "black" graph $\mathcal{G}$ to a 3D object $\mathbf{G}$, with edges

$$E(\mathbf{G}) = E(\mathcal{G}) \sqcup E(\widehat{\mathcal{G}^*}) \sqcup \{(x, \widehat{y}) : x \in V(\mathcal{G}),\ y \in V(\mathcal{G}^*),\ (x, y) \in E(\mathcal{D})\}.$$

Edges of $E(\mathbf{G})$ within Fig. 9 are (x_0, x_1), $(\widehat{y}_0, \widehat{y}_1)$, $(x_0, \widehat{y}_0)$, $(x_0, \widehat{y}_1)$, $(x_1, \widehat{y}_0)$, and $(x_1, \widehat{y}_1)$, forming the black tetrahedron. Similarly, we have a 3D white graph $\mathbf{G}^*$.

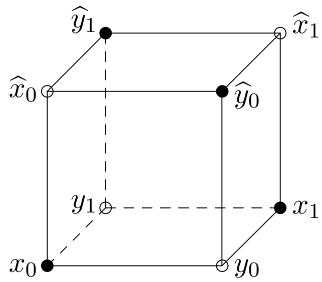

Figure 9: Elementary cube of $\mathbf{D}$

Definition 9 *Equation (9) is called* ***3D consistent*** *if it can be imposed on all faces of any elementary cube of* $\mathbf{D}$*, in such a manner that opposite faces carry one and the same equation (i.e., the same parameters).*

This should be understood as follows. Consider an elementary cube of $\mathbf{D}$, as on Fig. 9. Suppose that the values of the function f are given at the vertex x_0 and at its three neighbors y_0, y_1, and $\widehat{x}_0$. Then equation (9) uniquely determines the values of f at x_1, $\widehat{y}_0$, and $\widehat{y}_1$. After that equation (9) delivers three *a priori* different values for the value of the field f at the vertex $\widehat{x}_1$, coming from the faces $(y_0, x_1, \widehat{x}_1, \widehat{y}_0)$, $(x_1, y_1, \widehat{y}_1, \widehat{x}_1)$, and $(\widehat{x}_0, \widehat{y}_0, \widehat{x}_1, \widehat{y}_1)$, respectively. The 3D consistency means that these three values for $f(\widehat{x}_1)$ actually coincide, independently on the choice of initial conditions.

As discussed in detail in [BS], the 3D consistency of a given system (9) allows one to construct Bäcklund transformations and to find in an algorithmic way a zero curvature representation for it, which are traditionally considered as main attributes of integrability. Briefly, the constructions are as follows.

1) Given a solution $f : V(\mathcal{D}) \to \mathbb{C}$ to (9) and an arbitrary value $f(\widehat{x}_0) = \widehat{f}_0$ at some vertex $\widehat{x}_0 \in \widehat{\mathcal{D}}$, the 3D consistency allows one to extend the solution f to the whole of $V(\mathbf{D})$. Its restriction to $V(\widehat{\mathcal{D}})$ is thus a well-defined function $f : V(\widehat{\mathcal{D}}) \to \mathbb{C}$ which also solves the original equation (9). Setting $\widehat{f}(x) = f(\widehat{x})$ for all $x \in V(\mathcal{D})$, one can interpret this function as $\widehat{f} : V(\mathcal{D}) \to \mathbb{C}$, and this $\widehat{f}$ is called the ***Bäcklund transformation*** of f (defined by the value $\widehat{f}_0$ and the parameters sitting on the vertical faces).

2) Suppose that the function $\Phi(u_1, u_2, u_3, u_4)$ in (9) is affine-linear in all its arguments, so that this equation can be solved uniquely for an arbitrary argument u_i in terms of other three arguments, the solution being given by a fractional-linear function. For an arbitrary edge $\mathfrak{a} = (x, y) \in \vec{E}(\mathcal{D})$, consider the vertical face $(x, y, \widehat{y}, \widehat{x}) \in F(\mathbf{D})$ over this edge. The solution of equation $\Phi(f(x), f(y), f(\widehat{y}), f(\widehat{x})) = 0$ can be written as

$$f(\widehat{y}) = L(f(y), f(x)) \cdot f(\widehat{x}), \tag{10}$$

where $L(f(x), f(y)) \in PGL_2(\mathbb{C})$, and the standard notation for the action of $PGL_2(\mathbb{C})$ on $\mathbb{C}$ by Möbius transformations is used:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u = \frac{au + b}{cu + d}.$$

One assigns the matrix above to the edge $\mathfrak{a}$, so that $L(\mathfrak{a}) = L(f(y), f(x))$. Now it follows from the 3D consistency that for an arbitrary face $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ one has:

$$L(f(x_1), f(y_0))\, L(f(y_0), f(x_0)) = L(f(x_1), f(y_1))\, L(f(y_1), f(x_0)). \tag{11}$$

This expresses the flatness of the discrete connection L on $\mathcal{D}$ with values in $PGL_2(\mathbb{C})$, hence (11) is called the ***zero curvature representation*** of system (9). It is often possible to use suitable normalizations in order to lift this representation to the one with values in $GL_2(\mathbb{C})$.

5 3D consistent Cauchy-Riemann equations

To apply the notion of the 3D consistency to the discrete Cauchy-Riemann equation (5), one has to explain how to impose it on the vertical faces of $\mathbf{D}$. For this, we assume that the function ν is extended to $E(\mathbf{G}) \sqcup E(\mathbf{G}^*)$, still satisfying the condition $\nu(\mathfrak{e}^*) = 1/\nu(\mathfrak{e})$, and with an additional condition that *opposite edges carry the same values of* ν.

An interesting problem is, of course, to find functions ν on the "ground floor" $E(\mathcal{G}) \sqcup E(\mathcal{G}^*)$ which can be extended to the edges of $E(\mathbf{G}) \sqcup E(\mathbf{G}^*)$ lying in the "vertical" faces to give a 3D consistent system.

Theorem 10 *The function $\nu : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ can be extended to $E(\mathbf{G}) \sqcup E(\mathbf{G}^*)$ giving a 3D consistent system of discrete Cauchy-Riemann equations, if and only if the following condition is satisfied:*

$$\prod_{\mathfrak{e}\in\text{star}(x_0;\mathcal{G})} \frac{1+i\nu(\mathfrak{e})}{1-i\nu(\mathfrak{e})} = 1, \qquad \prod_{\mathfrak{e}^*\in\text{star}(y_0;\mathcal{G}^*)} \frac{1+i\nu(\mathfrak{e}^*)}{1-i\nu(\mathfrak{e}^*)} = 1, \qquad \forall x_0 \in V(\mathcal{G}),\ y_0 \in V(\mathcal{G}^*). \tag{12}$$

Proof. Consider a flower of quadrilaterals around x_0, with $\mathfrak{e}_k = (x_0, x_k)$, $\mathfrak{e}_k^* = (y_{k-1}, y_k)$ (in notations of Fig. 5). Build the extension of this flower to the third dimension (for one of its petals corresponding to $k = 1$ this extension is shown on Fig. 9). Denote

$$\nu(y_{k-1}, y_k) = \nu(\mathfrak{e}_k^*) = \nu_k, \quad \nu(y_k, \widehat{x}_0) = \mu_k. \tag{13}$$

Lemma 11 *Discrete Cauchy-Riemann equations are 3D consistent on the cube over the k-th petal, if and only if*

$$1 + \nu_k\mu_{k-1} - \nu_k\mu_k + \mu_{k-1}\mu_k = 0. \tag{14}$$

Proof of Lemma 11. Consider the elementary cube on Fig. 9, corresponding to $k = 1$. On the first step of checking the 3D consistency we find:

$$\begin{aligned} f(x_1) &= f(x_0) + i\nu_1(f(y_0) - f(y_1)), \\ f(\widehat{y}_0) &= f(x_0) + i\mu_0(f(y_0) - f(\widehat{x}_0)), \\ f(\widehat{y}_1) &= f(x_0) + i\mu_1(f(y_1) - f(\widehat{x}_0)). \end{aligned}$$

On the second step we find (from the condition that opposite faces support the same equations):

$$\begin{aligned} f(\widehat{x}_1) &= f(\widehat{x}_0) + i\nu_1(f(\widehat{y}_0) - f(\widehat{y}_1)), \\ &= f(y_1) + i\mu_0(f(x_1) - f(\widehat{y}_1)), \\ &= f(y_0) + i\mu_1(f(x_1) - f(\widehat{y}_0)). \end{aligned}$$

After simple computations we find:

$$\begin{aligned} f(\widehat{x}_1) &= (1 + \nu_1\mu_0 - \nu_1\mu_1)f(\widehat{x}_0) - \nu_1\mu_0 f(y_0) + \nu_1\mu_1 f(y_1), \\ &= -\mu_0\mu_1 f(\widehat{x}_0) - \nu_1\mu_0 f(y_0) + (1 + \nu_1\mu_0 + \mu_0\mu_1)f(y_1), \\ &= -\mu_0\mu_1 f(\widehat{x}_0) + (1 - \nu_1\mu_1 + \mu_0\mu_1)f(y_0) + \nu_1\mu_1 f(y_1). \end{aligned}$$

Comparison of these expressions leads to $1 + \nu_1\mu_0 - \nu_1\mu_1 + \mu_0\mu_1 = 0$, which proves the lemma. ∎

Continuing the proof of Theorem 10, we derive from (14):

$$\mu_k = \frac{\nu_k \mu_{k-1} + 1}{\nu_k - \mu_{k-1}} = \begin{pmatrix} \nu_k & 1 \\ -1 & \nu_k \end{pmatrix} \cdot \mu_{k-1},$$

where the standard notation for the action of $PGL_2(\mathbb{C})$ on $\mathbb{C}$ by Möbius transformations is used. Starting with an arbitrary μ_0, we can define all μ_k's consecutively. This procedure is consistent, if running around x_0 returns the value of μ_0 we started with. This holds for any μ_0, if and only if the matrix product $\overset{\curvearrowleft}{\prod_k} \begin{pmatrix} \nu_k & 1 \\ -1 & \nu_k \end{pmatrix}$ is a scalar matrix. It is easy to see by induction that the above matrix product may be presented as

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \text{with} \quad A = \frac{1}{2}\Big(\prod_k (\nu_k + i) + \prod_k (\nu_k - i)\Big), \quad B = \frac{1}{2i}\Big(\prod_k (\nu_k + i) - \prod_k (\nu_k - i)\Big).$$

Therefore, a necessary and sufficient condition for this matrix to be scalar is

$$B = 0 \quad \Leftrightarrow \quad \prod_k \frac{\nu_k + i}{\nu_k - i} = 1,$$

which is equivalent to the first equality in (12), because of $\nu_k = \nu(\mathfrak{e}_k^*) = 1/\nu(\mathfrak{e}_k)$. The second condition in (12) is proved similarly, by considering a flower of quadrilaterals around $y_0 \in V(\mathcal{G}^*)$. ■

As pointed out above, the most interesting case is when ν takes values in $\mathbb{R}_+$. In this case we will use the notation

$$\nu(\mathfrak{e}) = \tan \frac{\phi(\mathfrak{e})}{2}, \quad \phi(\mathfrak{e}) \in (0, \pi). \tag{15}$$

The condition $\nu(\mathfrak{e}^*) = 1/\nu(\mathfrak{e})$ is translated in this case into (6). The integrability condition (12) takes in this case the form

$$\prod_{\mathfrak{e} \in \mathrm{star}(x_0;\mathcal{G})} \exp(i\phi(\mathfrak{e})) = 1, \qquad \prod_{\mathfrak{e}^* \in \mathrm{star}(y_0;\mathcal{G}^*)} \exp(i\phi(\mathfrak{e}^*)) = 1, \qquad \forall x_0 \in V(\mathcal{G}),\ y_0 \in V(\mathcal{G}^*). \tag{16}$$

The latter condition is a generalization of (7), and is equivalent to saying that the system of angles $\phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to (0, \pi)$ comes from a realization of the quad-graph $\mathcal{D}$ by a *rhombic ramified embedding* in $\mathbb{C}$. Flowers of such an embedding can wind around its vertices more than once.

Lemma 12 *Let a quad-graph $\mathcal{D}$ be a double for a pair of dual cell decompositions $\mathcal{G}$, $\mathcal{G}^*$. Let $\Phi : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ be a function satisfying*

$$\Phi(\mathfrak{e}^*) = -1/\Phi(\mathfrak{e}), \qquad \forall \mathfrak{e} \in E(\mathcal{G}). \tag{17}$$

Then the necessary and sufficient condition for the existence of a labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ such that, in the notations of Fig. 7,

$$\Phi(\mathfrak{e}) = \Phi(x_0, x_1) = \frac{\alpha_1}{\alpha_0} \quad \Leftrightarrow \quad \Phi(\mathfrak{e}^*) = \Phi(y_0, y_1) = -\frac{\alpha_0}{\alpha_1}, \tag{18}$$

is given by the equations

$$\prod_{\mathfrak{e} \in \mathrm{star}(x_0;\mathcal{G})} \Phi(\mathfrak{e}) = 1, \qquad \prod_{\mathfrak{e}^* \in \mathrm{star}(y_0;\mathcal{G}^*)} \Phi(\mathfrak{e}^*) = 1, \qquad \forall x_0 \in V(\mathcal{G}),\ y_0 \in V(\mathcal{G}^*). \tag{19}$$

Proof. The necessity is obvious. To prove sufficiency, we construct α by assigning an arbitrary value (say, $\alpha = 1$) to some edge of $\mathcal{D}$, and then extending it successively using either of the equations (18) and the definition of labeling. Conditions (19) assure the consistency of this procedure. ■

Corollary 13 *Integrability condition (12) for the function $\nu : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ is equivalent to the following one: there exists a labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ of directed edges of $\mathcal{D}$, such that, in notations of Fig. 7,*

$$\nu(y_0, y_1) = \frac{1}{\nu(x_0, x_1)} = i\,\frac{\alpha_1 + \alpha_0}{\alpha_1 - \alpha_0}. \tag{20}$$

Under this condition, the 3D consistency of the discrete Cauchy-Riemann equations is assured by the following values of the weights ν on the edges of $E(\mathbf{G}) \sqcup E(\mathbf{G}^*)$ lying in the vertical faces:

$$\nu(y, \widehat{x}) = \frac{1}{\nu(x, \widehat{y})} = i\,\frac{\lambda + \alpha}{\lambda - \alpha}, \tag{21}$$

where $\alpha = \alpha(x, y)$, and $\lambda \in \mathbb{C}$ is an arbitrary number having the interpretation of the label carried by all vertical edges of $\mathbf{D}$: $\lambda = \alpha(x, \widehat{x}) = \alpha(y, \widehat{y})$.

Proof. Apply Lemma 12 with the function

$$\Phi(\mathfrak{e}) = \frac{1 + i\nu(\mathfrak{e})}{1 - i\nu(\mathfrak{e})}, \tag{22}$$

which satisfies (17) due to the property (4) of the weights ν. Note that in the case $\nu(\mathfrak{e}) \in \mathbb{R}_+$ the notation (15) implies that $\Phi(\mathfrak{e}) = \exp(i\phi(\mathfrak{e}))$. The formula (18) with the function (22) is clearly equivalent to (20). To prove the second statement, we use notations of Lemma 11, in particular the formula $1 + \nu_1\mu_0 - \nu_1\mu_1 + \mu_0\mu_1 = 0$. According to (20), we have: $\nu_1 = i\frac{\alpha_1+\alpha_0}{\alpha_1-\alpha_0}$. Parametrize the (arbitrary) value of μ_0 as $\mu_0 = i\frac{\lambda+\alpha_0}{\lambda-\alpha_0}$. Then it follows from the above formula that $\mu_1 = i\frac{\lambda+\alpha_1}{\lambda-\alpha_1}$. An easy induction proves (21) for all edges in the vertical faces. ■

So, integrability of the discrete Cauchy-Riemann equations is equivalent to the existence of a labeling α of directed edges satisfying (20). Let $p : V(\mathcal{D}) \to \mathbb{C}$ be a parallelogram realization of $\mathcal{D}$ defined by $p(y) - p(x) = \alpha(x, y)$. Then discrete holomorphic functions are characterized by

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = \frac{\alpha_1 - \alpha_0}{\alpha_1 + \alpha_0} = \frac{p(y_1) - p(y_0)}{p(x_1) - p(x_0)}. \tag{23}$$

In other words, the quotient of diagonals of the f-image of any quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ is equal to the quotient of diagonals of the corresponding parallelogram. In the case of positive weights $\nu \in \mathbb{R}_+$, the labels α take values in $\mathbb{S}^1$, and have a geometric interpretation of edges of a rhombic realization of $\mathcal{D}$.

Proposition 14 *The discrete Cauchy-Riemann equations (23) admit a zero curvature representation (11) in $GL_2(\mathbb{C})[\lambda]$, with transition matrices along $(x, y) \in \vec{E}(\mathcal{D})$ given by*

$$L(y, x, \alpha; \lambda) = \begin{pmatrix} \lambda + \alpha & -2\alpha(f(x) + f(y)) \\ 0 & \lambda - \alpha \end{pmatrix}, \quad \textit{where} \quad \alpha = p(y) - p(x). \tag{24}$$

Proof. This result is easy to check. It can be also systematically derived using the procedure outlined at the end of Sect. 4. Indeed, setting $\lambda = p(\widehat{x}) - p(x)$, one writes the equation (23) on the vertical face $(x, y, \widehat{y}, \widehat{x})$ as

$$\frac{f(\widehat{x}) - f(y)}{f(\widehat{y}) - f(x)} = \frac{\lambda - \alpha}{\lambda + \alpha} \quad \Leftrightarrow \quad f(\widehat{y}) = \frac{\lambda + \alpha}{\lambda - \alpha} f(\widehat{x}) + \Big(f(x) - \frac{\lambda + \alpha}{\lambda - \alpha} f(y) \Big) = M(y, x, \alpha; \lambda) \cdot f(\widehat{x}),$$

where

$$M(y, x, \alpha; \lambda) = \begin{pmatrix} \lambda + \alpha & (\lambda - \alpha) f(x) - (\lambda + \alpha) f(y) \\ 0 & \lambda - \alpha \end{pmatrix}.$$

One easily shows that these matrices form a zero curvature representation with values in $GL_2(\mathbb{C})[\lambda]$, i.e., that (11) holds literally, and not only projectively (up to a scalar factor). Finally, observe that the matrices L in (24) are gauge equivalent to the matrices M:

$$L(y, x, \alpha; \lambda) = \begin{pmatrix} 1 & -f(y) \\ 0 & 1 \end{pmatrix} M(y, x, \alpha; \lambda) \begin{pmatrix} 1 & f(x) \\ 0 & 1 \end{pmatrix}.$$

This finishes the proof. ■

The main result of the present section can be formulated as follows. *Discrete Cauchy-Riemann equations on a quad-graph $\mathcal{D}$ are integrable if and only if they come from a parallelogram immersion of $\mathcal{D}$ in $\mathbb{C}$, weights $i\nu$ being the quotients of diagonals of the corresponding parallelograms. In the case of real positive weights ν the parallelograms are actually rhombi.*

6 Extension of discrete holomorphic functions to $\mathbb{Z}^d$

To exploit analytic possibilities provided by 3D consistency of the discrete Cauchy-Riemann equations, we restrict our considerations to quasicrystallic rhombic embeddings $\mathcal{D}$, with the set of labels $A = \{\pm\alpha_1, \ldots, \pm\alpha_d\}$. Construct the two-dimensional subcomplex $\Omega_{\mathcal{D}}$ in $\mathbb{Z}^d$ corresponding to $\mathcal{D}$, as explained at the end of Sect. 3. Extend the labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ to all edges of $\mathbb{Z}^d$, assuming that all edges parallel to (and directed as) $\mathbf{e}_k$ carry the label α_k. Now, 3D consistency of the discrete Cauchy-Riemann equations allows us to impose them not only on $\Omega_{\mathcal{D}}$, but on the whole of $\mathbb{Z}^d$.

Definition 15 *A function $f : \mathbb{Z}^d \to \mathbb{C}$ is called **discrete holomorphic**, if it satisfies, on each elementary square of $\mathbb{Z}^d$, the equation*

$$\frac{f(\mathbf{n} + \mathbf{e}_j + \mathbf{e}_k) - f(\mathbf{n})}{f(\mathbf{n} + \mathbf{e}_j) - f(\mathbf{n} + \mathbf{e}_k)} = \frac{\alpha_j + \alpha_k}{\alpha_j - \alpha_k}. \tag{25}$$

Obviously, for any discrete holomorphic function $f : \mathbb{Z}^d \to \mathbb{C}$, its restriction to $V(\Omega_{\mathcal{D}}) \sim V(\mathcal{D})$ is a discrete holomorphic function on $\mathcal{D}$. To justify the reverse procedure, i.e., the extension of an arbitrary discrete holomorphic function on $\mathcal{D}$ to $\mathbb{Z}^d$, keeping the property of being discrete holomorphic, more thorough considerations are necessary.

Definition 16 *For a given set $V \subset \mathbb{Z}^d$, its **hull** $\mathcal{H}(V)$ is the minimal set $\mathcal{H} \subset \mathbb{Z}^d$ containing V and satisfying the condition: if three vertices of an elementary square belong to $\mathcal{H}$, then so does the fourth vertex.*

This notion is tailored for arbitrary 3D consistent four-point equations of the type (9), including the discrete Cauchy-Riemann equations. It is not difficult to show by induction that the hull of an arbitrary connected subcomplex of $\mathbb{Z}^d$ is a *brick*, i.e., a set of the type

$$\Pi_{\mathbf{a},\mathbf{b}} = \left\{ \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d : \ a_k \le n_k \le b_k, \ \ k = 1, \ldots, d \right\}, \qquad (26)$$

where $\mathbf{a} = (a_1, \ldots, a_d)$, $\mathbf{b} = (b_1, \ldots, b_d)$ are some integer vectors, with infinite values $a_k = -\infty$, $b_k = \infty$ allowed. (Observe that Definition 15 is equally well applicable to functions on bricks.) However, there exist combinatorial surfaces Ω (two-dimensional subcomplexes of $\mathbb{Z}^d$), like the one shown on Fig. 10, that support discrete holomorphic function which cannot be extended to $\mathbb{Z}^d$: the recursive process of extending an arbitrary discrete holomorphic function from $V(\Omega)$ to its hull $\mathcal{H}(V(\Omega))$ will lead to contradictions. The reason for this is a non-monotonicity of Ω: it contains pairs of points which cannot be connected by a path in Ω with all edges lying in one octant. However, such surfaces do not come from rhombic embeddings. We will prove the absence of contradictions in the case of $\Omega_{\mathcal{D}}$.

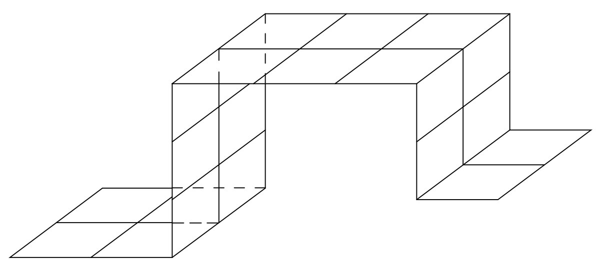

Figure 10: A non-monotone surface in $\mathbb{Z}^3$

Proposition 17 *For a combinatorial surface $\Omega_{\mathcal{D}}$ in $\mathbb{Z}^d$ coming from a rhombic embedding of a quad-graph $\mathcal{D}$, set*

$$a_k = a_k(\Omega_{\mathcal{D}}) = \min_{\mathbf{n} \in V(\Omega_{\mathcal{D}})} n_k, \qquad b_k = b_k(\Omega_{\mathcal{D}}) = \max_{\mathbf{n} \in V(\Omega_{\mathcal{D}})} n_k, \qquad k = 1, \ldots, d. \qquad (27)$$

(In case that n_k are unbounded from below or from above on $V(\Omega_{\mathcal{D}})$, set $a_k(\Omega_{\mathcal{D}}) = -\infty$, resp. $b_k(\Omega_{\mathcal{D}}) = \infty$.) Then $\mathcal{H}(V(\Omega_{\mathcal{D}})) = \Pi_{\mathbf{a},\mathbf{b}}$, and an arbitrary discrete holomorphic function on $\Omega_{\mathcal{D}}$ can be extended to a discrete holomorphic function on $\Pi_{\mathbf{a},\mathbf{b}}$ in a unique and unambiguous way.

For a proof of this proposition, a more detailed study of the surface $\Omega_{\mathcal{D}}$ will be necessary. In order to fix the ideas, we will assume, without loss of generality, that the circular order of the points $\pm\alpha_k$ on the positively oriented unit circle $\mathbb{S}^1$ is the following: $\alpha_1, \ldots, \alpha_d, -\alpha_1, \ldots, -\alpha_d$. We set $\alpha_{k+d} = -\alpha_k$ for $k = 1, \ldots, d$, and then define α_m for all $m \in \mathbb{Z}$ by $2d$-periodicity.

Consider the set $A_m = \{\alpha_m, \ldots, \alpha_{m+d-1}\}$ of d consecutive edge slopes. The opening angle of the sector spanned by α_m and α_{m+d-1} is in $(0, \pi)$. The set A_m contains exactly one member $\epsilon_k \alpha_k$ of each pair $\pm\alpha_k$, $k = 1, \ldots, d$. This associates to any $m \in \mathbb{Z}$ the set of signs $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$, $\epsilon_k = \pm 1$, which will be denoted by $\boldsymbol{\epsilon}(m)$. The sets of signs $\boldsymbol{\epsilon}(m)$ repeat $2d$-periodically, therefore not all possible sets of signs appear among them, but only the following $2d$ different ones. If $m \in [1, d]$, then the corresponding $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(m)$ is given by

$$\epsilon_k(m) = \begin{cases} -1, & 1 \le k < m, \\ +1, & m \le k \le d, \end{cases}$$

and if $m \in [d+1, 2d]$, then

$$\epsilon_k(m) = \begin{cases} +1, & 1 \le k < m-d, \\ -1, & m-d \le k \le d. \end{cases}$$

Fix an arbitrary $x_0 \in V(\mathcal{D})$, and define the "sector" U_m on the embedding plane $\mathbb{C}$ of the quad-graph $\mathcal{D}$ as the set of all points of $V(\mathcal{D})$ which can be reached from x_0 along paths with all edges from $A_m = \{\alpha_m, \ldots, \alpha_{m+d-1}\}$.

This can be re-formulated in terms of $\Omega_{\mathcal{D}}$ as follows. Recall that the map P which identifies $\Omega_{\mathcal{D}}$ with $\mathcal{D}$ depends on the choice of the point $x_0 \in V(\mathcal{D})$ corresponding to $\mathbf{0} \in V(\Omega_{\mathcal{D}})$. The map P sends U_m to the set of the points of $V(\Omega_{\mathcal{D}})$ which can be reached from $\mathbf{0}$ along paths in $V(\Omega_{\mathcal{D}})$ with all edges from $\{\epsilon_1\mathbf{e}_1, \ldots, \epsilon_d\mathbf{e}_d\}$, where $\epsilon_k = \epsilon_k(m)$ for $k = 1, \ldots, d$. To formulate it in a still another way, put into a correspondence to any set of signs $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$ the d-dimensional octant

$$S_{\boldsymbol{\epsilon}} = (\epsilon_1\mathbb{Z}_+) \times \ldots \times (\epsilon_d\mathbb{Z}_+) \subset \mathbb{Z}^d. \tag{28}$$

In case of $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(m)$, use the notation $S_{\boldsymbol{\epsilon}(m)} = S_m$. Then the definition of U_m is equivalent to saying that $U_m = P^{-1}(V(\Omega_{\mathcal{D}}) \cap S_m)$. The following statement will be of a key importance.

Lemma 18 *The union $\bigcup_{m=1}^{2d} U_m$ covers the whole of the quad-graph $\mathcal{D}$. Equivalently, the combinatorial surface $\Omega_{\mathcal{D}}$ coming from a rhombic embedding of $\mathcal{D}$ lies entirely in $\bigcup_{m=1}^{2d} S_m$.*

Proof. Clearly, U_m lies within the sector of the embedding plane with the tip at x_0, spanned by the directions α_m and α_{m+d-1}. The set A_m can be ordered: $\alpha_m \prec \ldots \prec \alpha_{m+d-1}$. The lower boundary U_m^- (upper boundary U_m^+) of U_m can be described as the path in $\mathcal{D}$ from the point x_0 obtained by following, at each vertex of the path, the edge with the least (resp. the largest) slope from A_m available at this vertex, with respect to the above mentioned ordering in A_m. The fact that $\mathcal{D}$ is embedded implies that all vertices of $\mathcal{D}$ between U_m^- and U_m^+ belong to U_m. Indeed, suppose that there are vertices between U_m^- and U_m^+ which cannot be reached from x_0 along a path with all edges from A_m. Take such a vertex x, combinatorially nearest to x_0. It cannot be reached from x_0 along a path with the *last* edge from A_m. Then one of the corners of one of the faces adjacent to x is free from edges from A_m and therefore has an internal angle larger than π, in a contradiction with embeddedness. Thus, U_m can be described as a set of vertices between U_m^- and U_m^+. Further, observe that the boundaries of the sectors U_m are interlaced: U_m contains all U_r^- with $r \in [m+1, m+d-2]$, and all U_r^+ with $r \in [m-d+2, m-1]$. This yields that the union of all U_m's covers the whole of $\mathcal{D}$. ■

See Fig. 11 for an illustration.

We say that a subset $\mathcal{I} \subset \Pi_{\mathbf{a},\mathbf{b}}$ is an *initial values locus*, if, prescribing arbitrarily values of f on $\mathcal{I}$, one can extend f in virtue of the 3D consistent Cauchy-Riemann equations in a unique and unambiguous way from $\mathcal{I}$ to the whole of $\Pi_{\mathbf{a},\mathbf{b}}$ (cf. [AV]). We will use two types of initial values loci.

- Any monotone path from $\mathbf{a}$ to $\mathbf{b}$, with all edges directed positively:

$$\mathcal{I}_1 = \{\mathbf{n}_r\}_{r=0}^N \quad \text{with} \quad \mathbf{n}_0 = \mathbf{a},\ \ \mathbf{n}_N = \mathbf{b}, \quad \text{and} \quad \mathbf{n}_{r+1} - \mathbf{n}_r \in \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}. \tag{29}$$

- The intersection of $\Pi_{\mathbf{a},\mathbf{b}}$ with all coordinate axes:

$$\mathcal{I}_2 = \bigcup_{k=1}^{d} \{\mathbf{n} = n\mathbf{e}_k :\ a_k \le n \le b_k\}. \tag{30}$$

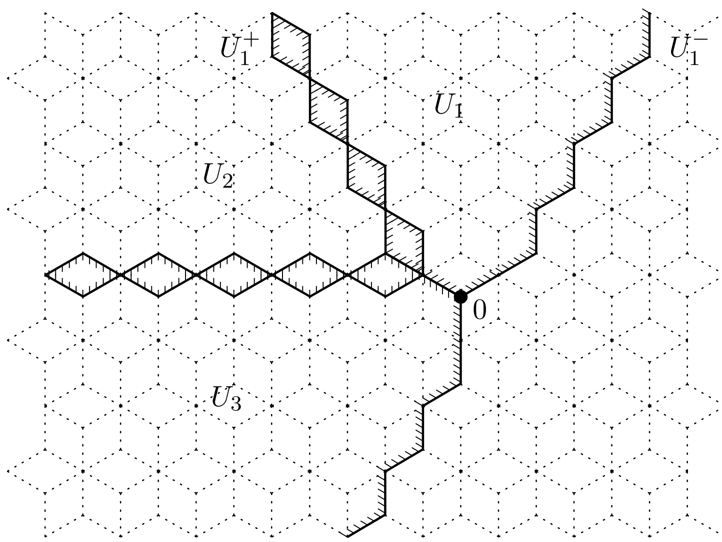

Figure 11: Sectors of the dual kagome lattice, $d = 3$, $\alpha_k = \exp((2k-1)\pi i/6)$.

Proof of Proposition 17. We have to show that, for any set of signs $\boldsymbol{\epsilon}$, the values of f on $V(\Omega_{\mathcal{D}}) \cap S_{\boldsymbol{\epsilon}}$ determine f uniquely on the hull $\Pi_{\mathbf{a},\mathbf{b}} \cap S_{\boldsymbol{\epsilon}}$.

First, we prove this for $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(m)$, so that $S_{\boldsymbol{\epsilon}} = S_m$. For the sake of notational simplicity, we do this for $m = 1$ only, i.e., for the hull $\mathcal{H}(V(\Omega_{\mathcal{D}}) \cap S_1) = \Pi_{\mathbf{0},\mathbf{b}}$. Indeed, an arbitrary point $\mathbf{n} \in V(\Omega_{\mathcal{D}}) \cap S_1$ can be reached from $\mathbf{0}$ along a path in $V(\Omega_{\mathcal{D}})$ with all edges from $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$. This is a path of the type $\mathcal{I}_1$, as in (29), hence it is an initial value locus for the brick $\Pi_{\mathbf{0},\mathbf{n}}$. Since the union of the bricks $\Pi_{\mathbf{0},\mathbf{n}}$ over all $\mathbf{n} \in V(\Omega_{\mathcal{D}}) \cap S_1$ exhausts the brick $\Pi_{\mathbf{0},\mathbf{b}}$, our claim is proved.

The bricks $\Pi_{\mathbf{a},\mathbf{b}} \cap S_{\boldsymbol{\epsilon}}$ with $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}(m)$ do not contain points of $V(\Omega_{\mathcal{D}})$ in their interior. However, on the first step of the proof, we obtain values of f on all the coordinate axes. This gives an initial values locus of the type $\mathcal{I}_2$, as in (30), for any brick of this type. ∎

Note that intersections of $\Omega_{\mathcal{D}}$ with bricks correspond to *combinatorially convex subsets* of $\mathcal{D}$, as defined in [M2].

7 Discrete exponential functions

A particularly important discrete holomorphic function on $\mathbb{Z}^d$ is the *discrete exponential function*, defined as

$$e(\mathbf{n}; z) = \prod_{k=1}^{d} \Big(\frac{z + \alpha_k}{z - \alpha_k}\Big)^{n_k}. \tag{31}$$

For $d = 2$, this function was considered in [F, D1]. The discrete Cauchy-Riemann equations for the discrete exponential function are easily checked: they are equivalent to a simple identity

$$\Big(\frac{z+\alpha_j}{z-\alpha_j} \cdot \frac{z+\alpha_k}{z-\alpha_k} - 1\Big) \Big/ \Big(\frac{z+\alpha_j}{z-\alpha_j} - \frac{z+\alpha_k}{z-\alpha_k}\Big) = \frac{\alpha_j + \alpha_k}{\alpha_j - \alpha_k}.$$

At a given $\mathbf{n} \in \mathbb{Z}^d$, the discrete exponential function is rational with respect to the parameter z, with poles at the points $\epsilon_1\alpha_1, \ldots, \epsilon_d\alpha_d$, where $\epsilon_k = \operatorname{sign} n_k$.

Equivalently, one can identify the discrete exponential function by its initial values on the axes:

$$e(n\mathbf{e}_k;z)=\Big(\frac{z+\alpha_k}{z-\alpha_k}\Big)^n. \tag{32}$$

A still another characterization says that $e(\cdot;z)$ is the Bäcklund transformation of the zero solution of discrete Cauchy-Riemann equations on $\mathbb{Z}^d$, with the "vertical" parameter z.

Restriction of the function $e(\cdot;z)$ to $V(\Omega_{\mathcal{D}})\sim V(\mathcal{D})$ is a discrete exponential function on $\mathcal{D}$ defined and studied in [M1, M2, K]. Note that the latter depends on the choice of the point $x_0\in V(\mathcal{D})$. A question posed in [K] asks whether discrete exponential functions are dense in the space of discrete holomorphic functions on $\mathcal{D}$. We now show that the answer to this question is in affirmative, in some natural class of functions (growing not faster than exponentially).

Theorem 19 *Let f be a discrete holomorphic function on $V(\mathcal{D})\sim V(\Omega_{\mathcal{D}})$, satisfying*

$$|f(\mathbf{n})|\le\exp(C(|n_1|+\ldots+|n_d|)),\qquad\forall\mathbf{n}\in V(\Omega_{\mathcal{D}}), \tag{33}$$

with some $C\in\mathbb{R}$. Extend it to a discrete holomorphic function on $\mathcal{H}(V(\Omega_{\mathcal{D}}))$. Then inequality (33) holds for all $\mathbf{n}\in\mathcal{H}(V(\Omega_{\mathcal{D}}))$, possibly with some larger constant C. There exists a function g defined on the disjoint union of small neighborhoods around the points $\pm\alpha_k\in\mathbb{C}$ and holomorphic on each one of these neighborhoods, such that

$$f(\mathbf{n})-f(\mathbf{0})=\frac{1}{2\pi i}\int_\Gamma g(\lambda)e(\mathbf{n};\lambda)d\lambda,\qquad\forall\mathbf{n}\in\mathcal{H}(V(\Omega_{\mathcal{D}})), \tag{34}$$

where Γ is a collection of $2d$ small loops, each one running counterclockwise around one of the points $\pm\alpha_k$.

Proof. In order to extend f from $V(\Omega_{\mathcal{D}})$ to $\mathcal{H}(V(\Omega_{\mathcal{D}}))$, one makes elementary steps based on eq. (15). For instance, within the octant S_1 these elementary steps consist of calculating the left-hand side of the following equation through the quantities on the right-hand side:

$$f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)=f(\mathbf{n})+\frac{\alpha_j+\alpha_k}{\alpha_j-\alpha_k}\big(f(\mathbf{n}+\mathbf{e}_j)-f(\mathbf{n}+\mathbf{e}_k)\big).$$

(In other octants everything is similar, but notations become slightly more complicated.) A simple induction shows that if the constant C in (33) satisfies the inequality

$$1+2\max_{j\neq k}\left|\frac{\alpha_j+\alpha_k}{\alpha_j-\alpha_k}\right|\exp(C)\le\exp(2C),$$

then (33) propagates in the extension process. This proves the first statement of the theorem.

To prove the second one, it is enough to find $g(\lambda)$ such that (34) holds on the coordinate axes, that is,

$$f_n^{(k)}-f(\mathbf{0})=\operatorname*{Res}_{\lambda=\alpha_k}g(\lambda)\left(\frac{\lambda+\alpha_k}{\lambda-\alpha_k}\right)^n,\quad f_{-n}^{(k)}-f(\mathbf{0})=\operatorname*{Res}_{\lambda=-\alpha_k}g(\lambda)\left(\frac{\lambda-\alpha_k}{\lambda+\alpha_k}\right)^n,\quad\forall n>0, \tag{35}$$

where $f_n^{(k)}$ are the restrictions of $f:\mathcal{H}(V(\Omega_{\mathcal{D}}))\to\mathbb{C}$ to the coordinate axes:

$$f_n^{(k)}=f(n\mathbf{e}_k),\qquad a_k(\Omega_{\mathcal{D}})\le n\le b_k(\Omega_{\mathcal{D}}).$$

Set $g(\lambda) = \sum_{k=1}^d (g_k(\lambda) + g_{-k}(\lambda))$, where the functions $g_{\pm k}(\lambda)$ vanish everywhere except in small neighborhoods of the points $\pm\alpha_k$, respectively, and are given there by convergent series

$$g_k(\lambda) = \frac{1}{2\lambda}\left(f_1^{(k)} - f(\mathbf{0}) + \sum_{n=1}^{\infty}\Big(\frac{\lambda-\alpha_k}{\lambda+\alpha_k}\Big)^n\big(f_{n+1}^{(k)} - f_{n-1}^{(k)}\big)\right), \tag{36}$$

and a similar formula for $g_{-k}(\lambda)$. (Convergence of these series is assured by the exponential growth of $f_n^{(k)}$.) The easy-to-check formula

$$\operatorname*{Res}_{\lambda=\alpha_k} \frac{1}{\lambda}\Big(\frac{\lambda+\alpha_k}{\lambda-\alpha_k}\Big)^n = 1-(-1)^n, \quad n \geq 0,$$

shows that the so defined function g satisfies (35). ■

8 Isomonodromic discrete logarithmic function

We first give a construction of the discrete logarithmic function on $\mathcal{D}$ which is equivalent to Kenyon's one [K]. This function is defined, after fixing some point $x_0 \in V(\mathcal{D})$, by the formula

$$f(x) = \frac{1}{2\pi i}\int_\Gamma \frac{\log(\lambda)}{2\lambda}\, e(x;\lambda)d\lambda, \qquad \forall x \in V(\mathcal{D}). \tag{37}$$

Here the integration path Γ is the same as in Theorem 19, and fixing x_0 is necessary for the definition of the discrete exponential function on $\mathcal{D}$. To make (37) a valid definition, one has to specify which branch of $\log(\lambda)$ is chosen around each point $\pm\alpha_k$. This choice depends on x, and is done as follows.

For each $m \in \mathbb{Z}$, assign to $\alpha_m = \exp(i\theta_m) \in \mathbb{S}^1$ a certain value of argument $\theta_m \in \mathbb{R}$: choose a value θ_1 of the argument of α_1 arbitrarily, and then extend it according to the rule

$$\theta_{m+1} - \theta_m \in (0,\pi), \qquad \forall m \in \mathbb{Z}.$$

Clearly, there holds $\theta_{m+d} = \theta_m + \pi$, and therefore also $\theta_{m+2d} = \theta_m + 2\pi$. It will be convenient to consider the points α_m, supplied with the arguments θ_m, as belonging to the Riemann surface $\widetilde{\Lambda}$ of the logarithmic function (a branched covering of the complex λ-plane).

The definition domain of the discrete logarithmic function is a branched covering

$$\widetilde{U} = \bigcup_{m=-\infty}^{\infty} \widetilde{U}_m$$

of the quad-graph $\mathcal{D}$. Here $\widetilde{U}_m$ is the sector U_m equipped with additional data – the interval

$$\log(\alpha_r) \in [i\theta_m, i\theta_{m+d-1}], \qquad r = m, \ldots, m+d-1 \tag{38}$$

of length less than π for the logarithms of the slopes of edges $\alpha_m, \ldots, \alpha_{m+d-1}$. If m increases by $2d$, the interval on the right-hand side of (38) is shifted by $2\pi i$. Two sectors $\widetilde{U}_{m_1}$ and $\widetilde{U}_{m_2}$ have a non-empty intersection, if and only if $|m_1 - m_2| < d$. It follows from Lemma 18 that $\widetilde{U}$ is, indeed, a branched covering of $\mathcal{D}$. Definition (37) should be read as follows: for $x \in \widetilde{U}_m$, the poles of $e(x;\lambda)$ are exactly the points $\alpha_m, \ldots, \alpha_{m+d-1} \in \widetilde{\Lambda}$. Therefore, one can assume that the integration path Γ consists of d small loops around these points, and the values of $\log(\lambda)$ at these points satisfy (38).

Proposition 20 [K] *The discrete logarithmic function on $\mathcal{D}$, restricted to $V(\mathcal{G})$, coincides with* ***discrete Green's function*** *on $\mathcal{G}$, up to a constant factor 2π.*

Proof. It is not difficult to see that the restriction of the discrete logarithmic function to black points does not branch: it is a well-defined real-valued function on $V(\mathcal{G})$. Clearly, this function is harmonic everywhere except the origin. At the origin, its Laplacian equals to the increment of f upon running once around the origin through its white neighbors. The values of f at the vertices neighboring to the origin are nothing but the arguments of the corresponding edges. Therefore, the above mentioned increment is equal to 2π. In order to obtain asymptotic results for the discrete logarithmic function, one can deform the integration path Γ into a connected contour lying on a single leaf of the Riemann surface of the logarithm, and then use standard methods of the complex analysis [K]. This possibility is due to the fact that functions g_k in integral representation (34) of an arbitrary discrete holomorphic function, defined originally in disjoint neighborhoods of the points α_r, in the case of the discrete logarithmic function are actually restrictions of a single analytic function $\log(\lambda)/(2\lambda)$ to these neighborhoods. ■

Now we extend the discrete logarithmic function to $\mathbb{Z}^d$. To this end, recall that the sector U_m of $\mathcal{D}$ is nothing but the preimage w.r.t. P of the part of $\Omega_{\mathcal{D}}$ lying in the octant $S_m \subset \mathbb{Z}^d$. Therefore, it is natural to introduce a branched covering

$$\widetilde{S} = \bigcup_{m=-\infty}^{\infty} \widetilde{S}_m$$

of the set $\bigcup_{m=1}^{2d} S_m \subset \mathbb{Z}^d$. Here $\widetilde{S}_m$ is the octant S_m equipped with the set of values of $\log(\epsilon_k \alpha_k)$ satisfying (38). Recall that $\epsilon_k = \epsilon_k(m)$, $k = 1, \ldots, d$, are the signs of the coordinate semi-axes of S_m, defined in Sect. 6. By definition, $\widetilde{S}_{m_1}$ and $\widetilde{S}_{m_2}$ intersect, if the underlying octants S_{m_1} and S_{m_2} have a non-empty intersection spanned by the common coordinate semi-axes, and the data $\log(\epsilon_k \alpha_k)$ for these common semi-axes match. It is easy to see that $\widetilde{S}_{m_1}$ and $\widetilde{S}_{m_2}$ intersect, if and only if $|m_1 - m_2| < d$.

Definition 21 *The* ***discrete logarithmic function*** *on $\widetilde{S}$ is given by the formula*

$$f(\mathbf{n}) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log \lambda}{2\lambda} e(\mathbf{n}; \lambda) d\lambda, \qquad \forall \mathbf{n} \in \widetilde{S}, \tag{39}$$

where the integration path Γ consists, for $\mathbf{n} \in \widetilde{S}_m$, of d loops around $\alpha_m, \ldots, \alpha_{m+d-1}$, and the branch of the logarithm on Γ is defined by inequality (38).

The discrete logarithmic function on $\mathcal{D}$ can be described as the restriction of the discrete logarithmic function on $\widetilde{S}$ to a branched covering of $\Omega_{\mathcal{D}} \sim \mathcal{D}$. This holds for an *arbitrary* quasicrystallic quad-graph with the set of edge slopes A.

Now we are in a position to give an alternative definition of the discrete logarithmic function. Clearly, it is completely characterized by its values $f(n\epsilon_k \mathbf{e}_k)$ on the coordinate semi-axes of an arbitrary octant $\widetilde{S}_m$.

Proposition 22 *For the discrete logarithmic function on $\widetilde{S}$, each of d sequences $f_n^{(k)} = f(n\epsilon_k \mathbf{e}_k)$, $k = 1, \ldots, d$, solves the difference equation*

$$n(f_{n+1} - f_{n-1}) = 1 - (-1)^n, \tag{40}$$

with the initial conditions

$$f_0^{(k)} = f(\mathbf{0}) = 0, \qquad f_1^{(k)} = f(\epsilon_k \mathbf{e}_k) = \log(\epsilon_k \alpha_k). \tag{41}$$

Explicitly,

$$f_{2n}^{(k)} = \sum_{\ell=1}^{n} \frac{2}{2\ell - 1}, \qquad f_{2n+1}^{(k)} = \log(\epsilon_k \alpha_k), \qquad k = 1, \dots, d, \qquad n \geq 0. \tag{42}$$

Here $\epsilon_k = \epsilon_k(m)$, and the values $\log(\epsilon_k \alpha_k)$ are chosen in the interval (38).

Proof. According to eq. (36), the values $f_n^{(k)}$, with $f_0^{(k)} = 0$, are defined by the expansion near $\lambda = \epsilon_k \alpha_k$,

$$\log(\lambda) = \log(\epsilon_k \alpha_k) + \log\Big(\frac{\lambda}{\epsilon_k \alpha_k}\Big) = f_1^{(k)} + \sum_{n=1}^{\infty} \Big(\frac{\lambda - \epsilon_k \alpha_k}{\lambda + \epsilon_k \alpha_k}\Big)^n (f_{n+1}^{(k)} - f_{n-1}^{(k)}). \tag{43}$$

This is equivalent to

$$f_1^{(k)} = \log(\epsilon_k \alpha_k), \qquad f_{n+1}^{(k)} - f_{n-1}^{(k)} = \frac{1 - (-1)^n}{n}. \tag{44}$$

The solution to these recurrent relations is given by (42). ■

Observe that values (42) at even (resp. odd) points imitate the behaviour of the real (resp. imaginary) part of the function $\log(\lambda)$ along the semi-lines $\arg(\lambda) = \arg(\epsilon_k \alpha_k)$. This can be easily extended to the whole of $\widetilde{S}$. Restricted to black points $\mathbf{n} \in \widetilde{S}$ (those with $n_1 + \ldots + n_d$ even), the discrete logarithmic function models the real part of the logarithm. In particular, this restricted function is real-valued and does not branch: its values on $\widetilde{S}_m$ depend on $m \pmod{2d}$ only. In other words, it is a well defined function on S_m. On the contrary, the discrete logarithmic function restricted to white points $\mathbf{n} \in \widetilde{S}$ (those with $n_1 + \ldots + n_d$ odd) takes purely imaginary values, and increases by $2\pi i$, as m increases by $2d$. Hence, this restricted function models the imaginary part of the logarithm.

It turns out that recurrent relations (40) are characteristic for an important class of solutions of the discrete Cauchy-Riemann equations, namely for the isomonodromic ones. Recall the definition of this class. For a discrete holomorphic function $f : \mathbb{Z}^d \to \mathbb{C}$, the *transition matrices* are (cf. (24)),

$$L_k(\mathbf{n}; \lambda) = \begin{pmatrix} \lambda + \alpha_k & -2\alpha_k(f(\mathbf{n} + \mathbf{e}_k) + f(\mathbf{n})) \\ 0 & \lambda - \alpha_k \end{pmatrix}. \tag{45}$$

The *moving frame* $\Psi(\cdot, \lambda) : \mathbb{Z}^d \to GL_2(\mathbb{C})[\lambda]$ is defined by prescribing some $\Psi(\mathbf{0}; \lambda)$, and by extending it recurrently according to the formula

$$\Psi(\mathbf{n} + \mathbf{e}_k; \lambda) = L_k(\mathbf{n}; \lambda)\Psi(\mathbf{n}; \lambda). \tag{46}$$

Finally, define the matrices $A(\cdot; \lambda) : \mathbb{Z}^d \to GL_2(\mathbb{C})[\lambda]$ by

$$A(\mathbf{n}; \lambda) = \frac{d\Psi(\mathbf{n}; \lambda)}{d\lambda} \Psi^{-1}(\mathbf{n}; \lambda). \tag{47}$$

These matrices are defined uniquely after fixing some $A(\mathbf{0}; \lambda)$.

Definition 23 *A discrete holomorphic function $f : \mathbb{Z}^d \to \mathbb{C}$ is called* ***isomonodromic*** [2]*, if, for some choice of $A(\mathbf{0}; \lambda)$, the matrices $A(\mathbf{n}; \lambda)$ are meromorphic in λ, with poles whose positions and orders do not depend on $\mathbf{n} \in \mathbb{Z}^d$.*

[2] This term originates in the theory of integrable nonlinear differential equations, where it is used for solutions with a similar analytic characterization [IN].

It is clear how to extend this definition to functions on the covering $\widetilde{S}$.

Theorem 24 *The discrete logarithmic function is isomonodromic.*

This is an immediate consequence of the following statement, which we formulate for functions on $S_1 = (\mathbb{Z}_+)^d$ for notational simplicity, but which holds actually for any octant $S_{\boldsymbol{\epsilon}}$.

Proposition 25 *Let*

$$A(\mathbf{0};\lambda) = \frac{1}{\lambda}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{48}$$

and let there be d sequences $\{f_n^{(k)}\}_{n=0}^{\infty}$ satisfying, for all $k = 1, \ldots, d$, the recurrent relation (40). Then the discrete holomorphic function $f : (\mathbb{Z}_+)^d \to \mathbb{C}$, defined by the values $f(n\mathbf{e}_k) = f_n^{(k)}$ on the coordinate semi-axes, is isomonodromic. At any point $\mathbf{n} \in (\mathbb{Z}_+)^d$ there holds:

$$A(\mathbf{n};\lambda) = \frac{A^{(0)}(\mathbf{n})}{\lambda} + \sum_{l=1}^{d}\Big(\frac{B^{(l)}(\mathbf{n})}{\lambda+\alpha_l} + \frac{C^{(l)}(\mathbf{n})}{\lambda-\alpha_l}\Big), \tag{49}$$

with

$$A^{(0)}(\mathbf{n}) = \begin{pmatrix} 0 & (-1)^{n_1+\ldots+n_d} \\ 0 & 0 \end{pmatrix}, \tag{50}$$

$$B^{(l)}(\mathbf{n}) = n_l\begin{pmatrix} 1 & -(f(\mathbf{n}) + f(\mathbf{n}-\mathbf{e}_l)) \\ 0 & 0 \end{pmatrix}, \qquad C^{(l)}(\mathbf{n}) = n_l\begin{pmatrix} 0 & f(\mathbf{n}+\mathbf{e}_l) + f(\mathbf{n}) \\ 0 & 1 \end{pmatrix}. \tag{51}$$

Moreover, at any point $\mathbf{n} \in (\mathbb{Z}_+)^d$ there holds an isomonodromic constraint,

$$\sum_{l=1}^{d} n_l\Big(f(\mathbf{n}+\mathbf{e}_l) - f(\mathbf{n}-\mathbf{e}_l)\Big) = 1 - (-1)^{n_1+\ldots+n_d}. \tag{52}$$

Proof. The matrices A satisfy a recurrent relation, which results by differentiating (46),

$$A(\mathbf{n}+\mathbf{e}_k;\lambda) = \frac{dL_k(\mathbf{n};\lambda)}{d\lambda}L_k^{-1}(\mathbf{n};\lambda) + L_k(\mathbf{n};\lambda)A(\mathbf{n};\lambda)L_k^{-1}(\mathbf{n};\lambda). \tag{53}$$

Fix some $k = 1, \ldots, d$, and consider the matrices $A(n\mathbf{e}_k;\lambda)$ along the kth coordinate semi-axis. Formula (53) shows that singularities of $A(n\mathbf{e}_k;\lambda)$ are poles at $\lambda = 0$ and at $\lambda = \pm\alpha_k$. It is easy to see that the pole $\lambda = 0$ remains simple for all $n > 0$. As one can show (see Lemma 43 in Appendix A), the recurrent relation (40) for $f_n = f(n\mathbf{e}_k)$ assures that the poles $\lambda = \pm\alpha_k$ are simple for all $n > 0$. So, under condition (40) there holds:

$$A(n\mathbf{e}_k;\lambda) = \frac{A^{(0)}(n\mathbf{e}_k)}{\lambda} + \frac{B^{(k)}(n\mathbf{e}_k)}{\lambda+\alpha_k} + \frac{C^{(k)}(n\mathbf{e}_k)}{\lambda-\alpha_k}, \tag{54}$$

i.e., eq. (49) is valid on the kth coordinate semi-axis, with $B^{(l)}(n\mathbf{e}_k) = C^{(l)}(n\mathbf{e}_k) = 0$ for $l \neq k$. The proof continues by induction, whose scheme follows filling out the hull of the coordinate semi-axes: each new point is of the form $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$, $j \neq k$, with three points $\mathbf{n}$, $\mathbf{n}+\mathbf{e}_j$ and $\mathbf{n}+\mathbf{e}_k$ known from the previous steps, where the statements of the proposition are assumed to hold. So, suppose that (49) holds at $\mathbf{n}+\mathbf{e}_j$, $\mathbf{n}+\mathbf{e}_k$. The new matrix $A(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k;\lambda)$ is obtained by two alternative formulas,

$$A(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k;\lambda) = \left(\frac{dL_k(\mathbf{n}+\mathbf{e}_j;\lambda)}{d\lambda} + L_k(\mathbf{n}+\mathbf{e}_j;\lambda)A(\mathbf{n}+\mathbf{e}_j;\lambda)\right)L_k^{-1}(\mathbf{n}+\mathbf{e}_j;\lambda), \qquad (55)$$

and the one with interchanged roles of k and j. Eq. (55) shows that all poles of $A(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k;\lambda)$ remain simple, with the possible exception of $\lambda = \pm\alpha_k$, whose orders might increase by 1. The same statement holds with k replaced by j. Therefore, all poles remain simple, and (49) holds at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$. The proof of formulas (50), (51) and of the constraint (52) is based on computations presented in Appendix A. ■

The reason for considering isomonodromic solutions on octants like $(\mathbb{Z}_+)^d$ is clear from (40): indeed, this second-order ordinary difference equation has a special form, enforcing that its solution on the semi-axis $n \geq 0$ is completely defined by the values at $n = 0, 1$, and does not depend on f_{-1}. Thus, a discrete holomorphic function from Proposition 25 is uniquely defined by its initial values $f(\mathbf{0}) = f_0$ and $f(\mathbf{e}_k) = f_1^{(k)}$ for $k = 1, \ldots, d$.

Remark. The isomonodromic constraint (52) was found in [NRGO], without any relation to the discrete logarithmic function. Observe that our formulation allows us to avoid a major computational problem arising in [NRGO] in this context, namely that of compatibility of the constraint with the discrete Cauchy-Riemann equations.

Summing up: *discrete Green's function on a quasicrystallic quad-graph is the real part (i.e., restriction to $V(\mathcal{G})$) of the discrete logarithmic function. The latter can be extended to a function on a branched covering of certain octants $S_m \subset \mathbb{Z}^d$, $m = 1, \ldots, 2d$. On each such octant, the discrete logarithmic function is discrete holomorphic, with the distinctive property of being isomonodromic. This function is uniquely defined either by the integral representation (39), or by the values on the coordinate semi-axes (42), or else by the initial values (41) and the constraint (52).*

9 3D consistent cross-ratio equations

The cross-ratio system is one of the simplest and at the same time one of the most fundamental and important *nonlinear* integrable systems on quad-graphs. Recall the definition of the cross-ratio of four complex numbers:

$$q(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_0)}, \qquad (56)$$

which yields the property

$$q(z_0, z_1, z_2, z_3) = 1/q(z_1, z_2, z_3, z_0). \qquad (57)$$

Let there be given a function $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ satisfying the condition

$$Q(\mathfrak{e}^*) = 1/Q(\mathfrak{e}), \qquad \forall \mathfrak{e} \in E(\mathcal{G}). \qquad (58)$$

Definition 26 *A function $z : V(\mathcal{D}) \to \mathbb{C}$ is said to solve the* ***cross–ratio equations*** *on $\mathcal{D}$ corresponding to the function Q, if for any positively oriented face (x_0, y_0, x_1, y_1) of $\mathcal{D}$ there holds:*

$$q(z(x_0), z(y_0), z(x_1), z(y_1)) = Q(x_0, x_1) = 1/Q(y_0, y_1). \qquad (59)$$

Like in Sect. 5, an interesting question is on the 3D consistency of the system of cross-ratio equations corresponding to a given function Q.

Theorem 27 *The function $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ can be extended to $E(\mathbf{G}) \sqcup E(\mathbf{G}^*)$ giving a 3D consistent system of cross-ratio equations, if and only if the following condition is satisfied:*

$$\prod_{\mathfrak{e}\in\mathrm{star}(x_0;\mathcal{G})} Q(\mathfrak{e}) = 1, \qquad \prod_{\mathfrak{e}^*\in\mathrm{star}(y_0;\mathcal{G}^*)} Q(\mathfrak{e}^*) = 1, \qquad \forall x_0 \in V(\mathcal{G}),\ y_0 \in V(\mathcal{G}^*). \tag{60}$$

Proof. We proceed as in the proof of Theorem 10. Consider a flower of quadrilaterals around x_0, with $\mathfrak{e}_k = (x_0, x_k)$, $\mathfrak{e}_k^* = (y_{k-1}, y_k)$. Build the extension of this flower to the third dimension, as in Sect. 4. Denote

$$Q(x_0, x_k) = Q(\mathfrak{e}_k) = Q_k, \quad Q(x_0, \widehat{y}_k) = \mu_k. \tag{61}$$

Then there holds a statement analogous to Lemma 11: the cross-ratio equations are 3D consistent on the cube over the k-th petal, if and only if

$$\mu_{k-1} = Q_k \mu_k. \tag{62}$$

This is donstraightforward, as in the proof of Lemma 11. For the cube over the petal with $k = 1$, one determines on the first step the values of z at x_1, $\widehat{y}_0$ and $\widehat{y}_1$ from

$$\begin{aligned} q(z(x_0), z(y_0), z(x_1), z(y_1)) &= Q_1, \\ q(z(x_0), z(y_0), z(\widehat{y}_0), z(\widehat{x}_0)) &= \mu_0, \\ q(z(x_0), z(y_1), z(\widehat{y}_1), z(\widehat{x}_0)) &= \mu_1. \end{aligned}$$

On the second step one has three alternative values for $z(\widehat{x}_1)$ from

$$\begin{aligned} q(z(\widehat{x}_0), z(\widehat{y}_0), z(\widehat{x}_1), z(\widehat{y}_1)) &= Q_1, \\ q(z(y_1), z(x_1), z(\widehat{x}_1), z(\widehat{y}_1)) &= \mu_0, \\ q(z(y_0), z(x_1), z(\widehat{x}_1), z(\widehat{y}_0)) &= \mu_1. \end{aligned}$$

A direct computation shows that these three values for $z(\widehat{x}_1)$ coincide if and only if $\mu_0 = Q_1 \mu_1$, and then

$$\widehat{x}_1 = \frac{\mu_0 y_0(y_1 - \widehat{x}_0) + \mu_1 y_1(\widehat{x}_0 - y_0) + \widehat{x}_0(y_0 - y_1)}{\mu_0(y_1 - \widehat{x}_0) + \mu_1(\widehat{x}_0 - y_0) + (y_0 - y_1)}.$$

Thus, (62) is proved. This relation yields immediately that running around x_0 returns back an (arbitrary) initial μ_0, if and only if the first condition in (60) holds. The second one follows similarly. ■

Corollary 28 *The integrability condition (60) for the function $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{C}$ is equivalent to the existence of a labeling $\alpha^2 : E(\mathcal{D}) \to \mathbb{C}$ of undirected edges of $\mathcal{D}$, such that, in notations of Fig. 8,*

$$Q(x_0, x_1) = \frac{1}{Q(y_0, y_1)} = \frac{\alpha_0^2}{\alpha_1^2}. \tag{63}$$

This formula assures the 3D consistency of the cross-ratio equations, if one assumes that all vertical edges of $\mathbf{D}$ carry one and the same label $\lambda^2 \in \mathbb{C}$.

Let the labeling α^2 come from a labeling $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ of directed edges. Let $p : V(\mathcal{D}) \to \mathbb{C}$ be a parallelogram realization of $\mathcal{D}$ defined by $p(y) - p(x) = \alpha(x, y)$. Then the cross-ratio equations are written as

$$q(z(x_0), z(y_0), z(x_1), z(y_1)) = \frac{\alpha_0^2}{\alpha_1^2} = q(p(x_0), p(y_0), p(x_1), p(y_1)). \tag{64}$$

In other words, the cross-ratio of the vertices of the f-image of any quadrilateral $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ is equal to the cross-ratio of the vertices of the corresponding parallelogram.

Proposition 29 *The cross-ratio equations (64) admit a zero curvature representation with the values in $GL_2(\mathbb{C})[\lambda]$, with transition matrices along $(x, y) \in \vec{E}(\mathcal{D})$ given by*

$$L(y, x, \alpha; \lambda) = \begin{pmatrix} 1 & z(x) - z(y) \\ \lambda\alpha^2/(z(x) - z(y)) & 1 \end{pmatrix}, \quad \textit{where} \quad \alpha = p(y) - p(x). \tag{65}$$

Proof. This result is easy to check. To derive it systematically using 3D consistency and the procedure outlined at the end of Sect. 4, observe that eq. (64) on the vertical face $(x, y, \widehat{y}, \widehat{x})$ can be written as

$$q(z(x), z(y), z(\widehat{y}), z(\widehat{x})) = \lambda\alpha^2 \quad \Leftrightarrow \quad z(\widehat{y}) - z(y) = L(y, x, \alpha; \lambda) \cdot (z(\widehat{x}) - z(x)),$$

where $\lambda = (p(\widehat{x}) - p(x))^{-2}$, and the matrices $L(y, x, \alpha; \lambda)$ are as in (65). One easily shows that these matrices L form a zero curvature representation with values in $GL_2(\mathbb{C})[\lambda]$, i.e., that (11) holds as it stands, and not only up to a scalar factor. ∎

The main result of the present section can be formulated as follows. *Integrable cross-ratio equations on a quad-graph $\mathcal{D}$ come from parallelogram immersions of $\mathcal{D}$ in $\mathbb{C}$, the coefficients Q being the cross-ratios of the corresponding parallelograms. In the case of unitary values $Q \in \mathbb{S}^1$ the parallelograms are actually rhombi.*

10 Circle patterns and the cross-ratio system

Definition 30 *A **Delaunay decomposition** of $\mathbb{C}$ is a cell decomposition $\mathcal{G}$ such that the boundary of each face is a polygon inscribed in a circle, and these circles have no vertices in their interior. These circles build a **circle pattern with the combinatorics of** $\mathcal{G}$.*

The vertices $z : V(\mathcal{G}) \to \mathbb{C}$ of a Delaunay decomposition are the intersection points of the circles of the corresponding pattern. The circle of the pattern corresponding to a face $y \in F(\mathcal{G})$ will be denoted by $C(y)$. If two faces $y_0, y_1 \in F(\mathcal{G})$ have a common edge (x_0, x_1), then the circles $C(y_0)$ and $C(y_1)$ intersect in the points $z(x_0), z(x_1)$. In other words, the edges of $\mathcal{G}$ correspond to pairs of neighboring (intersecting) circles of the pattern. Similarly, if several faces $y_1, y_2, \dots, y_m$ of $\mathcal{G}$ meet at one vertex $x_0 \in V(\mathcal{G})$, then the corresponding circles $C(y_1), C(y_2), \dots, C(y_m)$ also have a common intersection point $z(x_0)$.

Given a circle pattern with the combinatorics of $\mathcal{G}$, we can extend the function z to the vertices of the dual graph $\mathcal{G}^*$, setting

$$z(y) = \text{center of the circle } C(y), \quad y \in F(\mathcal{G}) \simeq V(\mathcal{G}^*).$$

After this extension, the map z is defined on all of $V(\mathcal{D}) = V(\mathcal{G}) \sqcup V(\mathcal{G}^*)$, where $\mathcal{D}$ is the double of $\mathcal{G}$. Consider a face of the double. Its vertices x_0, x_1, y_0, y_1 correspond to the intersection points and to the centers of two neighboring circles C_0, C_1 of the pattern. The following statement is obtained by a simple computation.

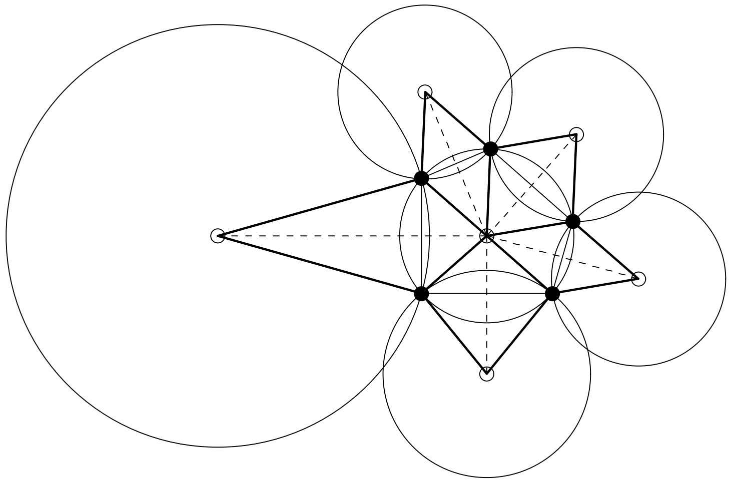

Figure 12: Circle pattern

Lemma 31 *If ϕ is the intersection angle of C_0, C_1, as on Fig. 13, then*

$$q(z(x_0), z(y_0), z(x_1), z(y_1)) = \exp(2i\phi). \tag{66}$$

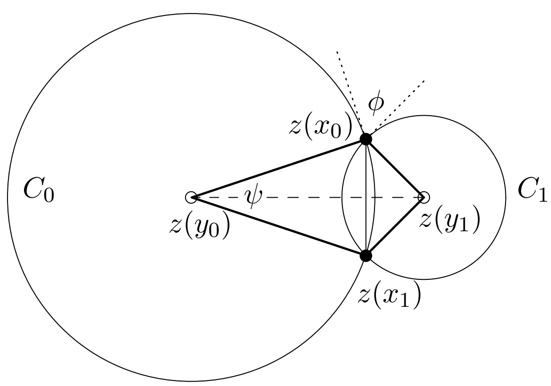

Figure 13: Two intersecting circles

It will be convenient to assign the intersection angle ϕ of $C(y_0), C(y_1)$ to the edge $(y_0, y_1) \in E(\mathcal{G}^*)$. Extend the function $\phi : E(\mathcal{G}^*) \to (0, \pi)$ to $E(\mathcal{G})$ by setting $\phi(\mathfrak{e}) = \pi - \phi(\mathfrak{e}^*)$.

Proposition 32 *Let $\mathcal{G}$ be Delaunay decomposition of a plane, and consider a circle pattern with the combinatorics of $\mathcal{G}$ and with the intersection angles $\phi : E(\mathcal{G}^*) \to (0, \pi)$. Let $\{z(x) : x \in V(\mathcal{G})\}$ and $\{z(y) : y \in V(\mathcal{G}^*)\}$ consist of intersection points of the circles, resp. of their centers. Then $z : V(\mathcal{D}) \to \mathbb{C}$ satisfies a system of cross-ratio equations with the function $Q : E(\mathcal{G}) \sqcup E(\mathcal{G}^*) \to \mathbb{S}^1$ defined as $Q(\mathfrak{e}) = \exp(2i\phi(\mathfrak{e}))$. There holds:*

$$\prod_{\mathfrak{e} \in \mathrm{star}(x_0;\mathcal{G})} \exp(2i\phi(\mathfrak{e})) = 1, \quad \forall x_0 \in V(\mathcal{G}). \tag{67}$$

The following condition is necessary and sufficient for the integrability of the system of cross-ratio equations:

$$\prod_{\mathfrak{e}^* \in \mathrm{star}(y_0;\mathcal{G}^*)} \exp(2i\phi(\mathfrak{e}^*)) = 1, \quad \forall y_0 \in V(\mathcal{G}^*), \tag{68}$$

i.e., for each circle of the pattern the sum of its intersection angles with all neighboring circles of the pattern vanishes $\pmod \pi$.

Proof. The relation (67) is obvious for geometrical reasons: for an arbitrary common intersection point of circles of the pattern, the sum of their consecutive pairwise intersection angles vanishes $\pmod \pi$. Now the claim follows from Theorem 27. ■

We can formulate the main result of this section as follows. *Combinatorial data $\mathcal{G}$ and intersection angles $\phi : E(\mathcal{G}^*) \to (0, \pi)$ belong to an integrable circle pattern, if and only if they admit an isoradial realization. This latter realization gives a rhombic immersion of the double $\mathcal{D}$, and generates also a dual isoradial circle pattern with the combinatorial data $\mathcal{G}^*$ and intersection angles $\phi : E(\mathcal{G}) \to (0, \pi)$.*

11 Hirota system

We have seen that integrable circle patterns deliver solutions $z : V(\mathcal{D}) \to \mathbb{C}$ to integrable cross-ratio systems. These solutions are characterized as follows: the z-image of any quadrilateral (x_0, y_0, x_1, y_1) from $F(\mathcal{D})$ is a *kite* with two pairs of sides of equal length (incident to the white vertices $z(y_0)$, $z(y_1)$), and with the prescribed angle $\pi - \phi$ at the black vertices $z(x_0)$, $z(x_1)$. The following transformation of the cross-ratio system is useful in order to single out this class of kite solutions.

Definition 33 *Let $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ be a labeling, and let $p : V(\mathcal{D}) \to \mathbb{C}$ be the corresponding parallelogram realization of $\mathcal{D}$ defined by $p(y) - p(x) = \alpha(x, y)$. A function $w : V(\mathcal{D}) \to \mathbb{C}$ is said to solve the corresponding* ***Hirota system****, if for any positively oriented face $(x_0, y_0, x_1, y_1) \in F(\mathcal{D})$ there holds, in the notations of Fig. 7:*

$$\alpha_0 w(x_0)w(y_0) + \alpha_1 w(y_0)w(x_1) - \alpha_0 w(x_1)w(y_1) - \alpha_1 w(y_1)w(x_0) = 0, \tag{69}$$

or, in a more invariant fashion,

$$\begin{aligned} w(x_0)w(y_0)\big(p(y_0) - p(x_0)\big) + w(y_0)w(x_1)\big(p(x_1) - p(y_0)\big) + \\ w(x_1)w(y_1)\big(p(y_1) - p(x_1)\big) + w(y_1)w(x_0)\big(p(x_0) - p(y_1)\big) = 0. \end{aligned} \tag{70}$$

Obviously, a black-white scaling, i.e., a transformation $w \to cw$ on $V(\mathcal{G})$ and $w \to c^{-1}w$ on $V(\mathcal{G}^*)$ with a constant c, maps solutions of the Hirota system into solutions. We will identify solutions related by such a transformation.

Proposition 34 *Let $w : V(\mathcal{D}) \to \mathbb{C}$ be a solution of the Hirota system. Then the relation*

$$z(y) - z(x) = \alpha(x, y)w(x)w(y) = w(x)w(y)\big(p(y) - p(x)\big), \qquad \forall (x, y) \in \vec{E}(\mathcal{D}), \tag{71}$$

correctly defines a unique (up to an additive constant) function $z : V(\mathcal{D}) \to \mathbb{C}$ which is a solution of the cross-ratio system (64). Conversely, for any solution z of the cross-ratio system (64), relation (71) defines a function w correctly and uniquely (up to a black-white scaling); this function w solves the Hirota system (69).

Proof. Simple calculation based on closing conditions around the quadrilateral (x_0, y_0, x_1, y_1). ∎

Proposition 35 *Let all $\alpha \in \mathbb{S}^1$, so that $p : V(\mathcal{D}) \to \mathbb{C}$ is a rhombic realization of $\mathcal{D}$. Let $z : V(\mathcal{D}) \to \mathbb{C}$ be a solution of the corresponding cross-ratio system (64). It corresponds to a circle pattern, if and only if the corresponding function w satisfies the condition*

$$w(x) \in \mathbb{S}^1, \quad w(y) \in \mathbb{R}_+, \quad \forall x \in V(\mathcal{G}),\ y \in V(\mathcal{G}^*). \tag{72}$$

The values $w : V(\mathcal{G}^*) \to \mathbb{R}_+$ have then the interpretation of the radii of the circles.

Proof. The function z corresponds to a circle pattern, if and only if all elementary quadrilaterals $(z(x_0), z(y_0), z(x_1), z(y_1))$ are of the kite form with the properties:
– the pairs of edges incident with white vertices have equal length,
– the angles at black vertices are equal to the corresponding angles of the underlying rhombi.
As easily seen, these conditions are equivalent to:

$$\frac{|w(x_0)|}{|w(x_1)|} = 1 \quad \text{and} \quad \frac{w(y_0)}{w(y_1)} \in \mathbb{R}_+,$$

respectively. This yields (72), possibly upon a black-white scaling. ∎

Remark. The conditions (72) form an *admissible reduction* of the Hirota system corresponding to a rhombic realization, in the following sense: if any three of the four points $w(x_0)$, $w(y_0)$, $w(x_1)$, $w(y_1)$ satisfy the condition (72), then so does the fourth one. This is immediately seen, if one rewrites the Hirota equation (69) in one of the equivalent forms:

$$\frac{w(x_1)}{w(x_0)} = \frac{\alpha_1 w(y_1) - \alpha_0 w(y_0)}{\alpha_1 w(y_0) - \alpha_0 w(y_1)} \quad \Leftrightarrow \quad \frac{w(y_1)}{w(y_0)} = \frac{\alpha_0 w(x_0) + \alpha_1 w(x_1)}{\alpha_0 w(x_1) + \alpha_1 w(x_0)}. \tag{73}$$

Proposition 36 *a) Let $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ be a labeling, and let $p : V(\mathcal{D}) \to \mathbb{C}$ be the corresponding parallelogram realization of $\mathcal{D}$ defined by $p(y) - p(x) = \alpha(x, y)$. Then the corresponding Hirota system is 3D consistent.*

b) Let all $\alpha \in \mathbb{S}^1$, so that $p : V(\mathcal{D}) \to \mathbb{C}$ is a rhombic realization of $\mathcal{D}$. Consider a solution $w : V(\mathcal{D}) \to \mathbb{C}$ corresponding to a circle pattern with the combinatorics of $\mathcal{G}$, i.e., satisfying (72). Consider its Bäcklund transformation $\widehat{w} : V(\mathcal{D}) \to \mathbb{C}$ with an arbitrary parameter $\lambda \in \mathbb{S}^1$ and with an arbitrary initial value $\widehat{w}(x_0) \in \mathbb{R}_+$ or $\widehat{w}(y_0) \in \mathbb{S}^1$. Then there holds:

$$\widehat{w}(x) \in \mathbb{R}_+, \quad \widehat{w}(y) \in \mathbb{S}^1, \quad \forall x \in V(\mathcal{G}),\ y \in V(\mathcal{G}^*), \tag{74}$$

so that $\widehat{w}$ corresponds to a circle pattern with the combinatorics of $\mathcal{G}^*$.

Proof. Statement a) is a matter of a direct computation. In the notations of Fig. 9, suppose that the Hirota equation (69) holds on all faces of the cube, wherein the vertical edges carry the (arbitrary) label λ. One finds that all three alternative ways to compute $w(\widehat{x}_1)$ lead to one and the same result, namely

$$w(\widehat{x}_1) = \frac{\lambda(\alpha_0^2 - \alpha_1^2) w(y_0) w(y_1) + \alpha_1(\lambda^2 - \alpha_0^2) w(y_0) w(\widehat{x}_0) + \alpha_0(\alpha_1^2 - \lambda^2) w(y_1) w(\widehat{x}_0)}{\lambda(\alpha_0^2 - \alpha_1^2) w(\widehat{x}_0) + \alpha_1(\lambda^2 - \alpha_0^2) w(y_1) + \alpha_0(\alpha_1^2 - \lambda^2) w(y_0)}.$$

Statement b) follows from the Remark above. ∎

Proposition 37 *The Hirota system (69) admits a zero curvature representation with the values in $GL_2(\mathbb{C})[\lambda]$, with transition matrices along $(x,y) \in \vec{E}(\mathcal{D})$ given by*

$$L(y,x,\alpha;\lambda) = \begin{pmatrix} 1 & -\alpha w(y) \\ -\lambda\alpha/w(x) & w(y)/w(x) \end{pmatrix}, \quad where \quad \alpha = p(y) - p(x). \tag{75}$$

Proof. The matrix (75) is obtained also directly from (65) by the substitution (71), followed by a simple gauge transformation

$$L \mapsto \begin{pmatrix} 1 & 0 \\ 0 & w(y) \end{pmatrix} L \begin{pmatrix} 1 & 0 \\ 0 & 1/w(x) \end{pmatrix}.$$

Alternatively, a systematic derivation of this result is based on the 3D consistency and the procedure outlined at the end of Sect. 4. ■

The main result of the present section is as follows. *Integrable circle patterns can be alternatively described by solutions of the Hirota system with a special property of being real-valued on $V(\mathcal{G})$ and unimodular on $V(\mathcal{G}^*)$.*

12 Linearization

Let $\alpha : \vec{E}(\mathcal{D}) \to \mathbb{C}$ be a labeling, and let $p : V(\mathcal{D}) \to \mathbb{C}$ be the corresponding parallelogram realization of $\mathcal{D}$ defined by $p(y) - p(x) = \alpha(x,y)$. Then the formula

$$z_0(x) = p(x), \qquad w_0(x) = 1, \qquad \forall x \in V(\mathcal{D}),$$

gives a (trivial) solution of the cross-ratio system (64) and the corresponding (trivial) solution of the Hirota system. Suppose that $z_0 : V(\mathcal{D}) \to \mathbb{C}$ belongs to a differentiable one-parameter family of solutions $z_\epsilon : V(\mathcal{D}) \to \mathbb{C}$, $\epsilon \in (-\epsilon_0, \epsilon_0)$, of the same cross-ratio system, and denote by $w_\epsilon : V(\mathcal{D}) \to \mathbb{C}$ the corresponding solutions of the Hirota system. Denote

$$g = \left.\frac{dz_\epsilon}{d\epsilon}\right|_{\epsilon=0}, \qquad f = \left(w_\epsilon^{-1}\frac{dw_\epsilon}{d\epsilon}\right)_{\epsilon=0}. \tag{76}$$

Proposition 38 *Both functions $f, g : V(\mathcal{D}) \to \mathbb{C}$ solve discrete Cauchy-Riemann equations (23).*

Proof. By differentiating (71), we obtain a relation between the functions $f, g : V(\mathcal{D}) \to \mathbb{C}$:

$$g(y) - g(x) = \big(f(x) + f(y)\big)\big(p(y) - p(x)\big), \qquad \forall (x,y) \in \vec{E}(\mathcal{D}). \tag{77}$$

The proof of proposition is based on this relation solely. Indeed, the closeness condition for the form on the right-hand side reads:

$$\big(f(x_0) + f(y_0)\big)\big(p(y_0) - p(x_0)\big) + \big(f(y_0) + f(x_1)\big)\big(p(x_1) - p(y_0)\big) + \\ \big(f(x_1) + f(y_1)\big)\big(p(y_1) - p(x_1)\big) + \big(f(y_1) + f(x_0)\big)\big(p(x_0) - p(y_1)\big) = 0,$$

which is equivalent to (23) for the function f. Similarly, the closeness condition for f, that is,

$$\big(f(x_0) + f(y_0)\big) - \big(f(y_0) + f(x_1)\big) + \big(f(x_1) + f(y_1)\big) - \big(f(y_1) + f(x_0)\big) = 0,$$

yields:

$$\frac{g(y_0)-g(x_0)}{p(y_0)-p(x_0)} - \frac{g(x_1)-g(y_0)}{p(x_1)-p(y_0)} + \frac{g(y_1)-g(x_1)}{p(y_1)-p(x_1)} - \frac{g(x_0)-g(y_1)}{p(x_0)-p(y_1)} = 0.$$

Under the condition $p(y_0)-p(x_0)=p(x_1)-p(y_1)$, this is equivalent to (23) for g. ■

Remark. This proof shows that, given a discrete holomorphic function $f: V(\mathcal{D})\to\mathbb{C}$, relation (77) correctly defines a unique, up to an additive constant, function $g: V(\mathcal{D})\to\mathbb{C}$, which is also discrete holomorphic. Conversely, for any g satisfying the discrete Cauchy-Riemann equations (23), relation (77) defines a function f correctly and uniquely (up to an additive black-white constant); this function f also solves the discrete Cauchy-Riemann equations (23). Actually, formula (77) expresses that the discrete holomorphic function f is the *discrete derivative* of g, so that g is obtained from f by *discrete integration.* This operation was considered in [D1, D2, M1].

Summarizing, we have the following statement.

Theorem 39 *a) A tangent space to the set of solutions of an integrable cross-ratio system, at a point corresponding to a rhombic embedding of a quad-graph, consists of discrete holomorphic functions on this embedding. This holds in both descriptions of the above set: in terms of variables z satisfying the cross-ratio equations, and in terms of variables w satisfying the Hirota equations. The corresponding two descriptions of the tangent space are related by taking the discrete derivative (resp. anti-derivative) of discrete holomorphic functions.*

b) A tangent space to the set of integrable circle patterns of a given combinatorics, at a point corresponding to an isoradial pattern, consists of discrete holomorphic functions on the rhombic embedding of the corresponding quad-graph, which take real values at white vertices and purely imaginary values at black ones. This holds in the description of circle patterns in terms of circle radii and rotation angles at intersection points (Hirota equations).

13 Isomonodromic discrete power function

Like in Sect. 6, one can consider functions $z:\mathbb{Z}^d\to\mathbb{C}$ and $w:\mathbb{Z}^d\to\mathbb{C}$, satisfying, on each elementary square of $\mathbb{Z}^d$, the cross-ratio and the Hirota equation, respectively, and ask about isomonodromic solutions. As shown in [AB1, AB2, BH], this leads to a discrete analog of the power function. Since the latter references contain a detailed presentation of these results in terms of the cross-ratio variables z, we restrict ourselves here to similar results in terms of the Hirota variables w. (Recall that transition matrices in these two formulations actually coincide, up to a simple gauge transformation by diagonal matrices which do not depend on λ.)

Transition matrices for the Hirota system on $\mathbb{Z}^d$ are:

$$L_k(\mathbf{n};\lambda) = \begin{pmatrix} 1 & -\alpha_k w(\mathbf{n}+\mathbf{e}_k) \\ -\lambda\alpha_k/w(\mathbf{n}) & w(\mathbf{n}+\mathbf{e}_k)/w(\mathbf{n}) \end{pmatrix}. \tag{78}$$

With these transition matrices, isomonodromic solutions are defined in exactly the same manner as in Sect. 8.

Proposition 40 *Let*

$$A(\mathbf{0};\lambda) = \frac{1}{\lambda}\begin{pmatrix} -\gamma/2 & 0 \\ 0 & \gamma/2 \end{pmatrix}, \tag{79}$$

and let there be d sequences $\{w_n^{(k)}\}_{n=0}^\infty$ satisfying, for all $k=1,\ldots,d$, the recurrent relation

$$n\,\frac{w_{n+1}-w_{n-1}}{w_{n+1}+w_{n-1}} = \left(\gamma-\tfrac{1}{2}\right)\left(1-(-1)^n\right). \tag{80}$$

Then the solution $w : (\mathbb{Z}_+)^d \to \mathbb{C}$ of the Hirota system, defined by the values $w(n\mathbf{e}_k) = w_n^{(k)}$ on the coordinate semi-axes, is isomonodromic. At any point $\mathbf{n} \in (\mathbb{Z}_+)^d$ there holds:

$$A(\mathbf{n};\lambda) = \frac{A^{(0)}(\mathbf{n})}{\lambda} + \sum_{l=1}^{d} \frac{B^{(l)}(\mathbf{n})}{\lambda - \alpha_l^{-2}} \, , \tag{81}$$

with

$$A^{(0)}(\mathbf{n}) = \begin{pmatrix} -\gamma/2 & * \\ 0 & \gamma/2 \end{pmatrix}, \tag{82}$$

$$B^{(l)}(\mathbf{n}) = \frac{n_l}{w(\mathbf{n}+\mathbf{e}_l) + w(\mathbf{n}-\mathbf{e}_l)} \begin{pmatrix} w(\mathbf{n}+\mathbf{e}_l) & \alpha_l w(\mathbf{n}+\mathbf{e}_l) w(\mathbf{n}-\mathbf{e}_l) \\ 1/\alpha_l & w(\mathbf{n}-\mathbf{e}_l) \end{pmatrix}. \tag{83}$$

The upper right entry of the matrix $A^{(0)}(\mathbf{n})$, denoted by the asterisk in (82), is actually given by

$$A_{12}^{(0)}(\mathbf{n}) = -\sum_{l=1}^{d} B_{12}^{(l)}(\mathbf{n}). \tag{84}$$

Moreover, at any point $\mathbf{n} \in (\mathbb{Z}_+)^d$ there holds an isomonodromic constraint,

$$\sum_{l=1}^{d} n_l \, \frac{w(\mathbf{n}+\mathbf{e}_l) - w(\mathbf{n}-\mathbf{e}_l)}{w(\mathbf{n}+\mathbf{e}_l) + w(\mathbf{n}-\mathbf{e}_l)} = \big(\gamma - \tfrac{1}{2}\big)\big(1 - (-1)^{n_1+\ldots+n_d}\big). \tag{85}$$

Proof. The scheme of the proof is the same as for Proposition 25. Fix some $k = 1, \ldots, d$, and consider the matrices $A(n\mathbf{e}_k;\lambda)$ along the kth coordinate semi-axis. It follows from formula (53) that singularities of $A(n\mathbf{e}_k;\lambda)$ are poles at $\lambda = 0$ and at $\lambda = \alpha_k^{-2}$. While the pole $\lambda = 0$ automatically remains simple for all $n > 0$, this is not necessarily so for the pole $\lambda = \alpha_k^{-2}$. As one can show (see Lemma 44 in Appendix B), the recurrent relation (80) for $w_n = f(n\mathbf{e}_k)$ assures that the pole $\lambda = \alpha_k^{-2}$ is simple for all $n > 0$, and

$$A(n\mathbf{e}_k;\lambda) = \frac{A^{(0)}(n\mathbf{e}_k)}{\lambda} + \frac{B^{(k)}(n\mathbf{e}_k)}{\lambda - \alpha_k^{-2}} \, , \tag{86}$$

i.e., eq. (81) is valid on the kth coordinate semi-axis, with $B^{(l)}(n\mathbf{e}_k) = 0$ for $l \neq k$. To prove that eq. (81) is valid also elsewhere, one argues by induction: suppose that (81) holds at $\mathbf{n}+\mathbf{e}_j$, $\mathbf{n}+\mathbf{e}_k$. Then eq. (55) shows that all poles of $A(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k;\lambda)$ remain simple, with the possible exception of $\lambda = \alpha_k^{-2}$, whose order might increase by 1. The same statement holds with k replaced by j. Hence, all poles remain simple. Therefore, (81) holds at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$, possibly up to a term which does not vanish with $\lambda \to \infty$. Such a term is absent, if the right-hand side of (53) vanishes with $\lambda \to \infty$, that is, if

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \left(A^{(0)}(\mathbf{n}) + \sum_{l=1}^{d} B^{(l)}(\mathbf{n}) \right) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0. \tag{87}$$

Clearly, the latter equation is equivalent to (84). Computations towards the proof of (84), as well as of (82), (83) and of constraint (85), are presented in Appendix B. ■

Remark. Again, the isomonodromic constraint (85) was found in [NRGO]. In the approach of that paper, consistency of the constraint with the Hirota equations (called lattice MKdV there) is a difficult problem, only manageable with the help of a computer system for symbolic manipulations. In our formulation, this comes for free, as a natural consequence of the construction based on the 3D consistency of the Hirota system.

A solution of the Hirota system given in Proposition 40 is completely defined by its initial values $w(0) = w_0$ and $w(\mathbf{e}_k) = w_1^{(k)}$ for $k = 1, \dots, d$. The choice

$$w_0 = 1, \qquad w_1^{(k)} = \exp(i\rho_k)\,, \quad k = 1, \dots, d, \tag{88}$$

with arbitrary constants ρ_k, leads to the following solution on the semi-axes:

$$w_{2n}^{(k)} = \prod_{\ell=1}^{n} \frac{\ell - 1 + \gamma}{\ell - \gamma}\,, \qquad w_{2n+1}^{(k)} = \exp(i\rho_k). \tag{89}$$

Observe the asymptotics at $n \to \infty$,

$$w_{2n}^{(k)} = c(\gamma) n^{2\gamma-1}\big(1 + O(n^{-1})\big). \tag{90}$$

The following special choice of ρ_k defines the *discrete analog of the function* $w \mapsto w^{2\gamma-1}$ *on* $(\mathbb{Z}_+)^d$:

$$i\rho_k = (2\gamma - 1)\log\alpha_k, \quad \text{so} \quad \text{that} \quad w(\mathbf{e}_k) = \alpha_k^{2\gamma-1}. \tag{91}$$

Remark. In the variables z the initial values $\{z_n^{(k)}\}_{n=0}^{\infty}$ on the semi-axes are given by the following analog (and consequence) of (80):

$$n\,\frac{(z_{n+1} - z_n)(z_n - z_{n-1})}{z_{n+1} - z_{n-1}} = \gamma z_n, \tag{92}$$

and then the corresponding solution of the cross-ratio equation satisfies an analog of (85):

$$\sum_{j=1}^{d} n_j\,\frac{(z(\mathbf{n} + \mathbf{e}_j) - z(\mathbf{n}))(z(\mathbf{n}) - z(\mathbf{n} - \mathbf{e}_j))}{z(\mathbf{n} + \mathbf{e}_j) - z(\mathbf{n} - \mathbf{e}_j)} = \gamma z(\mathbf{n}). \tag{93}$$

Recall also [AB1, AB2, BH] that the discrete analog of the function $z \mapsto z^{2\gamma}$ on $(\mathbb{Z}_+)^d$ is characterized by the constraint (92) and the following choice of the initial conditions:

$$z(0) = 0, \qquad z(\mathbf{e}_k) = \alpha_k^{2\gamma}, \quad k = 1, \dots, d. \tag{94}$$

Clearly, this choice of initial conditions is equivalent to (88), if one takes into account the basic relation (71) between the variables z and w.

Like in Sect. 8, isomonodromic solutions similar to those of Proposition 40 can be defined not only on $(\mathbb{Z}_+)^d$ but on any octant $S_{\boldsymbol{\epsilon}}$. They are characterized by the initial data

$$w(0) = 0, \qquad w(\epsilon_k \mathbf{e}_k) = (\epsilon_k \alpha_k)^{2\gamma-1}, \quad k = 1, \dots, d, \tag{95}$$

and give discrete analogs of the function $w \mapsto w^{2\gamma-1}$ on $S_{\boldsymbol{\epsilon}}$. Such a solution is fixed by an independent choice of branches of the function $w^{2\gamma-1}$ at the points $w = \epsilon_k \alpha_k$. This is equivalent to choosing the branches of the function $\log w$, because of $w^{2\gamma-1} = \exp((2\gamma - 1)\log w)$.

Definition 41 *The* ***discrete power function*** $w^{2\gamma-1}$ *on* $\widetilde{S}$ *is a complex-valued function whose restriction to* $\widetilde{S}_m$ *is defined as the unique isomonodromic solution* $w : S_m \to \mathbb{C}$ *of the Hirota system on the corresponding* S_m *with the initial data (95) fixed by the condition (38).*

Clearly, the discrete power function takes real values at the white points and unimodular values at the black points, so that it corresponds to a circle pattern.

Proposition 42 *The tangent vector to the space of integrable circle patterns along the curve consisting of patterns $w^{2\gamma-1}$, at the point corresponding to $\gamma = 1/2$, is the discrete logarithmic function.*

Proof. We have to prove that the discrete logarithm f and the discrete power function $w^{2\gamma-1}$ are related by

$$f(\mathbf{n}) = \Big(\frac{1}{2}\frac{d}{d\gamma}\, w^{2\gamma-1}(\mathbf{n})\Big)_{\gamma=1/2}.$$

It is enough to prove this on the coordinate semi-axes of each octant S_m. But this follows by differentiating with respect to γ constraint (80) and initial conditions (95) at the point $\gamma = 1/2$, where all $w = 1$: the results coincide with (40) and (41), respectively. ∎

14 Concluding remarks

Results of Sect. 12 can be generalized to the case of linearization at an arbitrary (not necessarily parallelogram) solution z_0 of the cross-ratio system and the corresponding solution w_0 of the Hirota system. In this case the relation between derivatives (76), coming to replace eq. (77), reads:

$$g(y) - g(x) = \big(f(x) + f(y)\big)\big(z_0(y) - z_0(x)\big), \qquad \forall (x, y) \in \vec{E}(\mathcal{D}). \tag{96}$$

Arguments similar to those of the proof of Proposition 38 show that in this case the function f is discrete holomorphic with respect to z_0, i.e.,

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = \frac{z_0(y_1) - z_0(y_0)}{z_0(x_1) - z_0(x_0)}, \qquad \forall (x_0, y_0, x_1, y_1) \in F(\mathcal{D}).$$

The function g, in general, is no longer discrete holomorphic.

Thus, a tangent space to the set of integrable circle patterns of a given combinatorics, at an arbitrary point, consists of functions, discrete holomorphic with respect to the kite-form embedding z_0 of the corresponding quad-graph. This holds for the description of circle patterns in terms of circle radii and rotation angles at intersection points (Hirota equations). The elements of the tangent space are characterized by the property of being real at white vertices and purely imaginary at black ones.

A number of constructions of the present paper can be generalized to the case of kite-form (rather than rhombic) embeddings coming from an integrable circle pattern. In particular, differentiating the discrete $w^{2\gamma-1}$ with respect to γ at a point $\gamma \neq 1/2$, one obtains a sort of the discrete logarithmic (and Green's) functions on the kite-form quad-graph corresponding to $z^{2\gamma}$.

A Appendix: proof of Proposition 25

Lemma 43 *Let the matrix $A(\mathbf{0}; \lambda)$ be as in (48). Fix some $k = 1, \ldots, d$. Then singularities of the matrices $A(n\mathbf{e}_k; \lambda)$ are poles at $\lambda = 0$, $\lambda = \pm\alpha_k$. For $n > 0$, the poles $\lambda = 0$ and $\lambda = -\alpha_k$ are simple. The pole $\lambda = \alpha_k$ is simple for all $n > 0$, if and only if recurrent relation (40) holds for*

$f_n = f(n\mathbf{e}_k)$. In this case there holds (54) with

$$A^{(0)}(n\mathbf{e}_k) = \begin{pmatrix} 0 & (-1)^n \\ 0 & 0 \end{pmatrix}, \tag{97}$$

$$B^{(k)}(n\mathbf{e}_k) = n \begin{pmatrix} 1 & -(f_n + f_{n-1}) \\ 0 & 0 \end{pmatrix}, \qquad C^{(k)}(n\mathbf{e}_k) = n \begin{pmatrix} 0 & f_{n+1} + f_n \\ 0 & 1 \end{pmatrix}. \tag{98}$$

Proof proceeds by induction. Putting matrices (45) into the recurrent definition (53), one finds immediately:

$$A_{11}(n\mathbf{e}_k; \lambda) = \frac{n}{\lambda + \alpha_k}, \qquad A_{22}(n\mathbf{e}_k; \lambda) = \frac{n}{\lambda - \alpha_k},$$

and the following recurrent relation for the upper right entry of the matrix $A(\cdot; \lambda)$,

$$A_{12}((n+1)\mathbf{e}_k; \lambda) = \frac{\lambda + \alpha_k}{\lambda - \alpha_k} A_{12}(n\mathbf{e}_k; \lambda) + \frac{2\alpha_k(f_{n+1} + f_n)}{\lambda - \alpha_k} \left(\frac{n+1}{\lambda + \alpha_k} - \frac{n}{\lambda - \alpha_k} \right).$$

Assume that there holds eq. (54). Then it holds also with $n \mapsto n+1$, if and only if no higher order pole appears at $\lambda = \alpha_k$ by this transition, what is equivalent to

$$C_{12}^{(k)}(n\mathbf{e}_k) = n(f_{n+1} + f_n). \tag{99}$$

This has to be considered as a recursive definition of the sequence $\{f_n\}$ (isomonodromic constraint). Under this condition, we find the recurrent relations for the upper right entries of the matrices $A^{(0)}$, $B^{(k)}$, $C^{(k)}$:

$$\begin{aligned} A_{12}^{(0)}((n+1)\mathbf{e}_k) &= -A_{12}^{(0)}(n\mathbf{e}_k), & (100) \\ B_{12}^{(k)}((n+1)\mathbf{e}_k) &= -(n+1)(f_{n+1} + f_n), & (101) \\ C_{12}^{(k)}((n+1)\mathbf{e}_k) &= (n+1)(f_{n+1} + f_n) + B_{12}^{(k)}(n\mathbf{e}_k) + C_{12}^{(k)}(n\mathbf{e}_k) + 2A_{12}^{(0)}(n\mathbf{e}_k). & (102) \end{aligned}$$

Now (100), (101) yield:

$$A_{12}^{(0)}(n\mathbf{e}_k) = (-1)^n, \qquad B_{12}^{(k)}(n\mathbf{e}_k) = -n(f_n + f_{n-1}). \tag{103}$$

Adding all three equations (100)–(102), we find that

$$A_{12}^{(0)}(n\mathbf{e}_k) + B_{12}^{(k)}(n\mathbf{e}_k) + C_{12}^{(k)}(n\mathbf{e}_k) = 1, \tag{104}$$

and this together with (99), (103) implies explicit form (40) of the isomonodromic constraint. ■

Proof of Proposition 25, continued. As shown in the main text, the induction from $\mathbf{n} + \mathbf{e}_j$, $\mathbf{n} + \mathbf{e}_k$ to $\mathbf{n} + \mathbf{e}_j + \mathbf{e}_k$ proves formula (49). From the diagonal part of eq. (55) one easily derives that for all $\mathbf{n} \in \mathbb{Z}^d$,

$$A_{11}(\mathbf{n}; \lambda) = \sum_{l=1}^{d} \frac{n_l}{\lambda + \alpha_l}, \qquad A_{22}(\mathbf{n}; \lambda) = \sum_{l=1}^{d} \frac{n_l}{\lambda - \alpha_l}.$$

The following formula is an easy consequence of eq. (55) under the limit $\lambda \to \infty$:

$$A_{12}^{(0)}(\mathbf{n}) + \sum_{l=1}^{d} \left(B_{12}^{(l)}(\mathbf{n}) + C_{12}^{(l)}(\mathbf{n}) \right) = 1. \tag{105}$$

It remains to show that the following relations propagate in the evolution defined by the recurrent relation (55):

$$A_{12}^{(0)}(\mathbf{n}) = (-1)^{n_1+\ldots+n_d}\,, \tag{106}$$

$$B_{12}^{(l)}(\mathbf{n}) = -n_l(f(\mathbf{n})+f(\mathbf{n}-\mathbf{e}_l)), \qquad C_{12}^{(l)}(\mathbf{n}) = n_l(f(\mathbf{n}+\mathbf{e}_l)+f(\mathbf{n})). \tag{107}$$

Indeed, constraint (52) follows then immediately, because it coincides with (105), if one takes (106)–(107) into account. Writing now the upper right entry of eq. (55) in length, we find the following recurrent relations:

$$A_{12}^{(0)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) = -A_{12}^{(0)}(\mathbf{n}+\mathbf{e}_j), \tag{108}$$

$$B_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) = \frac{\alpha_l-\alpha_k}{\alpha_l+\alpha_k}\,B_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j) - \frac{2\alpha_k}{\alpha_l+\alpha_k}(n_l+\delta_{lj}+\delta_{lk})(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)+f(\mathbf{n}+\mathbf{e}_j)), \tag{109}$$

$$C_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) = \frac{\alpha_l+\alpha_k}{\alpha_l-\alpha_k}\,C_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j) - \frac{2\alpha_k}{\alpha_l-\alpha_k}(n_l+\delta_{lj})(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)+f(\mathbf{n}+\mathbf{e}_j)), \tag{110}$$

the latter formula being valid for $l\neq k$ only. For $C_{12}^{(k)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)$ there holds a similar but much longer formula, which we actually will not need. Now, eq. (108) readily yields (106). By the induction hypothesis, eqs. (109) and (110) with $l\neq k$ can be rewritten as

$$\begin{aligned} & B_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) = \\ & \quad -(n_l+\delta_{lj})\bigg(\frac{\alpha_l-\alpha_k}{\alpha_l+\alpha_k}(f(\mathbf{n}+\mathbf{e}_j)+f(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_l)) + \frac{2\alpha_k}{\alpha_l+\alpha_k}(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)+f(\mathbf{n}+\mathbf{e}_j))\bigg), \\ & C_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) = \\ & \quad (n_l+\delta_{lj})\bigg(\frac{\alpha_l+\alpha_k}{\alpha_l-\alpha_k}(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_l)+f(\mathbf{n}+\mathbf{e}_j)) - \frac{2\alpha_k}{\alpha_l-\alpha_k}(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)+f(\mathbf{n}+\mathbf{e}_j))\bigg). \end{aligned}$$

But the discrete Cauchy-Riemann equation for the corresponding elementary squares imply that the latter two equations are equivalent to

$$\begin{aligned} B_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) &= -(n_l+\delta_{lj})(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)+f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k-\mathbf{e}_l)), \\ C_{12}^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) &= (n_l+\delta_{lj})(f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_l)+f(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)), \end{aligned}$$

which coincide with (107) at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$ for $l\neq k$. By interchanging the roles of k and j, we see that (107) at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$ holds also for $l\neq j$, and thus for all $l=1,\ldots,d$. ■

B Appendix: proof of Proposition 40

Lemma 44 *Let the matrix $A(\mathbf{0};\lambda)$ be as in (79). Fix some $k=1,\ldots,d$. Then singularities of the matrices $A(n\mathbf{e}_k;\lambda)$ are poles at $\lambda=0$, $\lambda=\alpha_k^{-2}$. The pole $\lambda=0$ is simple. The pole $\lambda=\alpha_k^{-2}$ is simple for all $n>0$, if recurrent relation (92) holds for $w_n=w(n\mathbf{e}_k)$. In this case there holds (86)* *with*

$$A^{(0)}(n\mathbf{e}_k)=\begin{pmatrix}-\gamma/2 & * \\ 0 & \gamma/2\end{pmatrix}, \tag{111}$$

$$B^{(k)}(n\mathbf{e}_k)=\frac{n}{w_{n+1}+w_{n-1}}\begin{pmatrix}w_{n+1} & \alpha_k w_{n+1}w_{n-1} \\ 1/\alpha_k & w_{n-1}\end{pmatrix}. \tag{112}$$

The upper right entry of the matrix $A^{(0)}(n\mathbf{e}_k)$, denoted in (111) by the asterisk, is given by

$$A^{(0)}_{12}(n\mathbf{e}_k)=-B^{(k)}_{12}(n\mathbf{e}_k). \tag{113}$$

Proof. Assume that eq. (86) holds. Put matrices (78) into recurrent definition (53). It is easy to see that no higher order pole appears at $\lambda=\alpha_k^{-2}$ by the transition $n\mapsto n+1$, if and only if

$$\begin{pmatrix}1 & -\alpha_k w_{n+1} \\ -1/(\alpha_k w_n) & w_{n+1}/w_n\end{pmatrix}B^{(k)}(n\mathbf{e}_k)\begin{pmatrix}1 & \alpha_k w_n \\ 1/(\alpha_k w_{n+1}) & w_n/w_{n+1}\end{pmatrix}=0.$$

This is equivalent to

$$\begin{pmatrix}1 & -\alpha_k w_{n+1}\end{pmatrix}B^{(k)}(n\mathbf{e}_k)\begin{pmatrix}1 \\ 1/(\alpha_k w_{n+1})\end{pmatrix}=0, \tag{114}$$

or, written in length, to

$$B^{(k)}_{22}(n\mathbf{e}_k)-B^{(k)}_{11}(n\mathbf{e}_k)+\alpha_k w_{n+1}B^{(k)}_{21}(n\mathbf{e}_k)-\frac{1}{\alpha_k w_{n+1}}B^{(k)}_{12}(n\mathbf{e}_k)=0. \tag{115}$$

This is a recursive definition of the sequence $\{w_n\}$ (an isomonodromic constraint). Notice that this is a *quadratic* equation for w_{n+1}, unlike (99), which was a *linear* equation for f_{n+1}.

Under condition (114), or (115), eq. (86) holds also by $n\mapsto n+1$, possibly with an additional λ-independent term on the right-hand side, which vanishes if and only if (113) holds. We will show in a moment that this is indeed the case. One readily finds recurrent relations for the matrices $A^{(0)}(n\mathbf{e}_k)$ and $B^{(k)}(n\mathbf{e}_k)$. For the matrix $A^{(0)}(n\mathbf{e}_k)$ they read:

$$A^{(0)}((n+1)\mathbf{e}_k)=\begin{pmatrix}1 & -\alpha_k w_{n+1} \\ 0 & w_{n+1}/w_n\end{pmatrix}A^{(0)}(n\mathbf{e}_k)\begin{pmatrix}1 & \alpha_k w_n \\ 0 & w_n/w_{n+1}\end{pmatrix}. \tag{116}$$

This proves formula (111), with a recurrent relation for the upper right entry:

$$A^{(0)}_{12}((n+1)\mathbf{e}_k)=-\gamma\alpha_k w_n+\frac{w_n}{w_{n+1}}A^{(0)}_{12}(n\mathbf{e}_k). \tag{117}$$

For the matrix $B^{(k)}(n\mathbf{e}_k)$ the recurrent relations read, in components:

$$B^{(k)}_{11}((n+1)\mathbf{e}_k) = \gamma+B^{(k)}_{22}(n\mathbf{e}_k)-\frac{1}{\alpha_k w_{n+1}}\big(A^{(0)}_{12}(n\mathbf{e}_k)+B^{(k)}_{12}(n\mathbf{e}_k)\big), \tag{118}$$

$$B^{(k)}_{22}((n+1)\mathbf{e}_k) = 1-\gamma+B^{(k)}_{11}(n\mathbf{e}_k)+\frac{1}{\alpha_k w_{n+1}}\big(A^{(0)}_{12}(n\mathbf{e}_k)+B^{(k)}_{12}(n\mathbf{e}_k)\big), \tag{119}$$

$$B^{(k)}_{12}((n+1)\mathbf{e}_k) = \gamma\alpha_k w_n-\frac{w_n}{w_{n+1}}A^{(0)}_{12}(n\mathbf{e}_k), \tag{120}$$

$$B^{(k)}_{21}((n+1)\mathbf{e}_k) = \frac{1-\gamma}{\alpha_k w_n}+\frac{1}{\alpha_k^2 w_n w_{n+1}}\big(A^{(0)}_{12}(n\mathbf{e}_k)+2B^{(k)}_{12}(n\mathbf{e}_k)\big) -\frac{1}{\alpha_k w_n}\big(B^{(k)}_{22}(n\mathbf{e}_k)-B^{(k)}_{11}(n\mathbf{e}_k)\big). \tag{121}$$

Comparing now (120) with (117), we see that (113) holds for all n, as claimed above. Upon using this fact and constraint (115), we can rewrite formulas (118)–(121) as follows:

$$\begin{aligned} B_{11}^{(k)}((n+1)\mathbf{e}_k) &= \gamma + B_{22}^{(k)}(n\mathbf{e}_k), && (122)\\ B_{22}^{(k)}((n+1)\mathbf{e}_k) &= 1-\gamma + B_{11}^{(k)}(n\mathbf{e}_k), && (123)\\ B_{12}^{(k)}((n+1)\mathbf{e}_k) &= \gamma\alpha_k w_n + \frac{w_n}{w_{n+1}}B_{12}^{(k)}(n\mathbf{e}_k), && (124)\\ B_{21}^{(k)}((n+1)\mathbf{e}_k) &= \frac{1-\gamma}{\alpha_k w_n} + \frac{w_{n+1}}{w_n}B_{21}^{(k)}(n\mathbf{e}_k). && (125) \end{aligned}$$

These relations together with constraint (115) define the sequence $\{w_n\}$ and the matrices $B^{(k)}(n\mathbf{e}_k)$ completely. First of all, there follows from (122), (123):

$$B_{11}^{(k)}(n\mathbf{e}_k) + B_{22}^{(k)}(n\mathbf{e}_k) = n, \qquad B_{11}^{(k)}(n\mathbf{e}_k) - B_{22}^{(k)}(n\mathbf{e}_k) = \big(\gamma - \tfrac{1}{2}\big)\big(1-(-1)^n\big). \tag{126}$$

Further, there follows from (122)–(125):

$$\begin{aligned} B_{11}^{(k)}((n+1)\mathbf{e}_k) - \frac{1}{\alpha_k w_n}B_{12}^{(k)}((n+1)\mathbf{e}_k) &= B_{22}^{(k)}(n\mathbf{e}_k) - \frac{1}{\alpha_k w_{n+1}}B_{12}^{(k)}(n\mathbf{e}_k), && (127)\\ B_{22}^{(k)}((n+1)\mathbf{e}_k) - \alpha_k w_n B_{21}^{(k)}((n+1)\mathbf{e}_k) &= B_{11}^{(k)}(n\mathbf{e}_k) - \alpha_k w_{n+1}B_{21}^{(k)}(n\mathbf{e}_k). && (128) \end{aligned}$$

Subtracting these two equations and taking (115) into account, we find, upon downshifting n:

$$B_{11}^{(k)}(n\mathbf{e}_k) - B_{22}^{(k)}(n\mathbf{e}_k) + \alpha_k w_{n-1}B_{21}^{(k)}(n\mathbf{e}_k) - \frac{1}{\alpha_k w_{n-1}}B_{12}^{(k)}(n\mathbf{e}_k) = 0. \tag{129}$$

This yields that one of the solutions of eq. (115), considered as a quadratic equation for w_{n+1}, is $w_{n+1} = -w_{n-1}$. We will be interested in the second one. To find it, add eqs. (115) and (129) and derive, under the condition $w_{n+1} + w_{n-1} \neq 0$:

$$B_{12}^{(k)}(n\mathbf{e}_k) = \alpha_k^2 w_{n+1}w_{n-1}B_{21}^{(k)}(n\mathbf{e}_k). \tag{130}$$

Due to (115), the right–hand sides of (127), (128) are equal to one another. Using there (130), we finally come to

$$B_{11}^{(k)}(n\mathbf{e}_k) = \alpha_k w_{n+1}B_{21}^{(k)}(n\mathbf{e}_k), \qquad B_{22}^{(k)}(n\mathbf{e}_k) = \alpha_k w_{n-1}B_{21}^{(k)}(n\mathbf{e}_k). \tag{131}$$

This together with (126) yields

$$B_{21}^{(k)}(n\mathbf{e}_k) = \frac{n}{\alpha_k(w_{n+1}+w_{n-1})}, \tag{132}$$

and now both the expression (112) and explicit form (80) of the constraint (115) follow readily. ■

Proof of Proposition 40, continued. As demonstrated in the main text, if eq. (81) holds at $\mathbf{n}+\mathbf{e}_j$, $\mathbf{n}+\mathbf{e}_k$, then it holds also at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$, provided eq. (84) is valid. To prove eq. (84), put expression (78) into eq. (55), and find recurrent relations for the matrices $A^{(0)}(\mathbf{n})$ and $B^{(l)}(\mathbf{n})$. Upon use of the abbreviation $L_{j,k}(\lambda) = L_k(\mathbf{n}+\mathbf{e}_j;\lambda)$, we have:

$$\begin{aligned} A^{(0)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) &= L_{j,k}(0)A^{(0)}(\mathbf{n}+\mathbf{e}_j)L_{j+k,-k}(0), && (133)\\ B^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) &= L_{j,k}(\alpha_l^{-2})\,\frac{B^{(l)}(\mathbf{n}+\mathbf{e}_j)}{1-\alpha_k^2\alpha_l^{-2}}\,L_{j+k,-k}(\alpha_l^{-2}), \quad l\neq k, && (134)\\ B^{(k)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k) &= -L_{j,k}(\alpha_k^{-2})\bigg(A^{(0)}(n+\mathbf{e}_j) + \sum_{l\neq k}\frac{B^{(l)}(\mathbf{n}+\mathbf{e}_j)}{1-\alpha_k^2\alpha_l^{-2}}\bigg)L_{j+k,-k}(\alpha_k^{-2}) \\ &\quad + \text{lower triangular matrix.} && (135) \end{aligned}$$

Here we used the fact that $L_{j,k}^{-1}(\lambda)=L_{j+k,-k}(\lambda)/(1-\lambda\alpha_k^{-2})$. Taking into account that the upper triangular part of the matrix $L_{j,k}(\lambda)$ does not depend on λ (and coincides with $L_{j,k}(0)$), we see that eqs. (133)–(135) imply the desired property (84), which proves formula (81) at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$.

After eq. (81) has been proved, it is instructive to rewrite eq. (53) as

$$A(\mathbf{n}+\mathbf{e}_k;\lambda)L_k(\mathbf{n};\lambda)-L_k(\mathbf{n};\lambda)A(\mathbf{n};\lambda)=\frac{dL_k(\mathbf{n};\lambda)}{d\lambda},$$

and consider the limit $\lambda\to\infty$ of *this* formula. Due to (78) and (81), this limit reads:

$$\Big(A^{(0)}(\mathbf{n}+\mathbf{e}_k)+\sum_{l=1}^{d}B^{(l)}(\mathbf{n}+\mathbf{e}_k)\Big)\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\Big(A^{(0)}(\mathbf{n})+\sum_{l=1}^{d}B^{(l)}(\mathbf{n})\Big)=\begin{pmatrix}0&0\\1&0\end{pmatrix}. \quad (136)$$

Clearly, this equation contains more information than (87). More precisely, the diagonal terms of (136) are equivalent to (87), while the lower left entry gives an additional identity.

Further, eq. (133) yields immediately that the matrix $A^{(0)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)$ retains the upper triangular form (82). Turning to the matrices $B^{(l)}(\mathbf{n})$, observe first of all that formula (83) is equivalent to

$$B^{(l)}(\mathbf{n})\sim\begin{pmatrix}\alpha_l w(\mathbf{n}+\mathbf{e}_l)\\ 1\end{pmatrix}\begin{pmatrix}1 & \alpha_l w(\mathbf{n}-\mathbf{e}_l)\end{pmatrix},\qquad \operatorname{tr}B^{(l)}(\mathbf{n})=n_l, \quad (137)$$

where the sign $\sim$ means "equal up to a scalar factor". Suppose that this holds at the points $\mathbf{n}+\mathbf{e}_j$ and $\mathbf{n}+\mathbf{e}_k$ for all $l=1,\dots,d$. Then it follows from (134) that for all $l\neq k$ the matrix $B^{(l)}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)$ is also of rank 1, and its trace is equal to $\operatorname{tr}B^{(l)}(\mathbf{n}+\mathbf{e}_j)=n_l+\delta_{jl}=n_l+\delta_{jl}+\delta_{kl}$. It remains to prove that

$$L_{j,k}(\alpha_l^{-2})\begin{pmatrix}\alpha_l w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_l)\\ 1\end{pmatrix}\sim\begin{pmatrix}\alpha_l w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_l)\\ 1\end{pmatrix}, \quad (138)$$

$$\begin{pmatrix}1 & \alpha_l w(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_l)\end{pmatrix}L_{j+k,-k}(\alpha_l^{-2})\sim\begin{pmatrix}1 & \alpha_l w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k-\mathbf{e}_l)\end{pmatrix}. \quad (139)$$

These equations, written in length, read:

$$w(\mathbf{n}+\mathbf{e}_j)\,\frac{\alpha_k w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)-\alpha_l w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_l)}{\alpha_k w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_l)-\alpha_l w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)} = w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_l), \quad (140)$$

$$w(\mathbf{n}+\mathbf{e}_j)\,\frac{\alpha_k w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)+\alpha_l w(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_l)}{\alpha_k w(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_l)+\alpha_l w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k)} = w(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k-\mathbf{e}_l), \quad (141)$$

and are nothing but the Hirota equations on the corresponding elementary squares. Thus, recurrent relation (134) implies that formula (137) holds at $\mathbf{n}+\mathbf{e}_j+\mathbf{e}_k$ for $l\neq k$. By interchanging the roles of j and k, formula (137) holds for $l\neq j$, and thus for all $l=1,\dots,d$. It remains to prove the isomonodromic constraint (85). But it is not difficult to see that it is a direct consequence of the lower left entry of eq. (136), if one takes into account expressions (82) and (83). ■

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