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On Separation of Variables for Homogeneous SL(r) Gaudin Systems

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Abstract

By means of a recently introduced bihamiltonian structure for the homogeneous Gaudin models, we find a new set of Separation Coordinates for the $sl(r)$ case.

1 Introduction

In this paper we will discuss the Gaudin system with $sl(r)$-valued spins defined by the Hamiltonian

$$H_G = \sum_{i<j}^{n} \mathrm{Tr}(A_i \cdot A_j), \quad A_k \in sl(r) \tag{1.1}$$

on the manifold $M = sl(r)^n$ equipped with the standard product Lie–Poisson structure. We will refer to it, with a slight abuse of notation, as the *homogeneous* $sl(r)$ *XXX Gaudin system* (as in [9]).

The "conventional" approach to the integrability of this quite well studied problem is based on the Lax representation and the r-matrix theory (see, e.g., [18, 28]). Fixing n distinct parameters $a_i,\ i = 1, \ldots, n$ one introduces the matrix

$$\mathcal{L}(\lambda) = \sum_{i=1}^{n} \frac{A_i}{\lambda - a_i}, \tag{1.2}$$

to be considered as an element of the Loop algebra $sl(r)((\lambda))$. Along the Hamiltonian flow defined by (1.1), the Lax matrix $\mathcal{L}(\lambda)$ evolves according to a Lax equation

$$\frac{d\,\mathcal{L}(\lambda)}{dt} = [\mathcal{L}(\lambda), M].$$

Thanks to the existence of an r-matrix for the Lax matrix (1.2) the spectral invariants

$$I_i^{(\alpha)} = \mathrm{Res}\Big|_{\lambda=a_i} \mathrm{Tr}(\mathcal{L}(\lambda)^\alpha),\ i = 1, \dots, n,\ \alpha = 2, \dots, r, \tag{1.3}$$

are in involution. These integrals, together with the integrals of the motion associated with the invariance of the system under the global $SL(r)$ action given by

$$A_i \to G A_i G^{-1}, \tag{1.4}$$

to be referred to as *global gauge invariance*, provide a complete set of constants of the motion for H_G.

The separability of the Hamilton–Jacobi equations associated with the Gaudin Hamiltonian (1.1) was first studied [28, 29], for the low r cases, as a kind of byproduct of the solution of the Bethe Ansatz equations associated with the quantum Gaudin system. Separability was then proved for the general case in [14, 27]. and (implicitly) framed within the theory of Algebraically Complete Integrable Systems in [1, 5]. In this scheme, it turns out that one can find a set of "algebro-geometrical" Darboux coordinates (ζ_i, λ_i) as coordinates of a set of $d_n = r(r-1)(n-1)/2$ distinguished points on spectral curve

$$\Gamma(\zeta, \lambda) = \mathrm{Det}(\zeta - \mathcal{L}(\lambda)), \tag{1.5}$$

whose genus is $g = \dfrac{(r-1)}{2}\big((n-2)r + (r-2)\big)$.

In the recent paper [9] we have reconsidered the (homogeneous XXX) Gaudin model, and generalized to the case of an arbitrary Lie algebra $\mathfrak{g}$ an alternative set of integrals of the motion for H_G (see, e.g. [2]), introduced in the Hopf-algebraic approach to the integrability of the system. The distinguished feature of such integrals, which in the case of $\mathfrak{g} = sl(2)$ are given by the very simple expressions

$$K_l = \mathrm{Tr}\Big(\sum_{i=1}^{l} A_l\Big)^2, \tag{1.6}$$

is that they are independent of the (fake) parameters entering the definition of the Lax matrix (1.2).

These integrals were also introduced, in a different context [19], as Hamiltonians of a notable class of Hamiltonian flows on the moduli space of $n+3$-sided polygons in $\mathbb{R}^3$, and later generalized in [12] to the $\mathbb{R}^d$ case[1]. This moduli space turns out to be a suitable Marsden–Weinstein Hamiltonian quotient of the Poisson manifold $su(2)^{n+3}$ associated with the corresponding Gaudin model. The Hamiltonian flows associated with (1.6) were termed "bending flows" due to the following fact: if one draws, from a chosen vertex, the n possible diagonals of an $n+3$-sided polygon, the flow associated with the Hamiltonian K_k geometrically represents the bending of one side of the polygon along the k-th diagonal (the other side being kept fixed).

[1] We thank J. Harnad for drawing our attention to these references.

The key point for the analysis performed in [9] was the introduction, along with the standard Lie–Poisson structure P, of a particular second Poisson structure, hereinafter called R. In the $n = 3$ case, this structure is defined by its Hamiltonian vector fields as follows:

$$\begin{cases} \dot{A}_1 = [A_1, \frac{\partial F}{\partial A_2} + \frac{\partial F}{\partial A_3}] \\ \dot{A}_2 = [A_1, \frac{\partial F}{\partial A_1} - \frac{\partial F}{\partial A_2}] + [A_2, \frac{\partial F}{\partial A_2} + \frac{\partial F}{\partial A_3}] \\ \dot{A}_3 = [A_1, \frac{\partial F}{\partial A_1} - \frac{\partial F}{\partial A_3}] + [A_2, \frac{\partial F}{\partial A_2} - \frac{\partial F}{\partial A_3}] + 2[A_3, \frac{\partial F}{\partial A_3}], \end{cases} \tag{1.7}$$

where $\frac{\partial F}{\partial A_i}$ are elements of $sl(r)$ to be properly defined in Section 2.

The Poisson pencil $R - \lambda P$ and the integrals (1.6) fulfill standard Lenard–Magri relations, namely one can check that

$$\begin{aligned} & P d\mathrm{Tr}(A_i^2) = 0,\ i = 1, \ldots, n, \quad R d\mathrm{Tr}(A_1^2) = 0, \\ & R d\mathrm{Tr}\Big(\sum_{i=1}^{a} A_i^2\Big) = P dK_a, \quad a = 2, \ldots, n. \end{aligned} \tag{1.8}$$

For the general $sl(r)$ case one can show that it is possible to find a sufficient number of polynomial functions in involution that provide a set of integrals of the motion $K_l^{(\alpha)}$ alternative to the set defined by the Lax matrix (1.2). They share with the integrals (1.6) the property of being defined independently of the parameters entering the Lax matrix (1.2).

In the last Section of [9] we addressed the problem of separability of such flows, in the framework of the so–called bihamiltonian approach to the SoV problem (see, e.g., [24, 3, 20, 11], and the references quoted therein). In particular we solved it for the $sl(2)$ case by means of explicit computations, showing that the separation coordinates associated with the pencil $R - \lambda P$ are rational functions of the natural coordinates (h_i, e_i, f_i) in $sl(2)^n$, and the separation relations are quadratic equations in these coordinates.

In this paper we will solve the corresponding problem for the $sl(r)$ case, with r arbitrary. This task will be accomplished by means of a careful mixing of techniques of the theory of Lax equations with r–matrix structure, and the theory of bihamiltonian systems such as those exposed in a series of papers by Gel'fand and Zakharevich [15] and Magri and collaborators [8, 20]. In particular, we will make extensive use of (refinements of) the results presented in [10, 11] concerning the Separation of Variables of systems with an arbitrary number of Lenard–Magri chains. The key points for the analysis we are going to develop in this paper are:

1. It is possible to deform the Poisson tensor R into $\tilde{R}$ in such a way that
   a) $\tilde{R}$ is still compatible with P and restricts to the (generic) symplectic leaf $\mathcal{S}$ of P.
   b) The integrals $K_l^{(\alpha)}$ defined by the pencil $R - \lambda P$ are in involution also

w.r.t. $\tilde{R}$, although the recurrence relation they satisfy in relation with the new pencil $\tilde{R} - \lambda P$ are more complicated than the usual Lenard–Magri relations.

2. It is possible to define $n-1$ Lax matrices L_a linearly depending on a "spectral parameter" λ such that any formal vector field $X(\lambda)$ which is "Hamiltonian w.r.t. the pencil $R - \lambda P$ induces a Lax equation on each of the matrices L_a.

Thanks to the first property it will be possible to endow the generic symplectic leaf $\mathcal{S}$ of P with a special geometric structure, that is, a $(1,1)$ tensor with vanishing Nijenhuis torsion, whose "spectral" data will provide us with a set of separation coordinates for the H–J equations associated with H_G.

Thanks to the second property, as well as other specific features of the deformation $\tilde{R}$ of R, (to be fully discussed in the core of the paper) we will be able to show that the separation relations are provided by the spectral curves of the matrices L_a. The distinguished feature of such a SoV scheme is that the separation coordinates are defined iteratively in subsets of $d_r = r(r-1)$ coordinates, which are coordinates of a set of $r(r-1)/2$ points on a genus

$$g = (r-1)(r-2)/2$$

curve, irrespectively of the number n of "sites" of the Gaudin model.

A word of warning: the set of coordinates defined in this way on $\mathcal{S}$ must be completed by a set of $r(r-1)$ coordinates associated with the global $SL(r)$ invariance of the model, just like the set of integrals coming either from the Lax matrix (1.2) or from the construction discussed in [9] must be supplemented by the set of integrals associated with the global gauge invariance of the model. However, since these integrals are associated with a sort of "cyclic" coordinates, they will trivially enter the H-J equations and the problem of separability. So in the core of the paper, we will often "forget" about them.

The scheme of the paper is as follows: In Section 2 we will fix some conventions and notations to be used throughout the paper, review the results of [9] to be used in the sequel, and introduce the Lax matrices L_a. In Section 3 we will briefly recall the main points of the bihamiltonian scheme for SoV, and, in Section 3.1 we will discuss how to apply such a picture to the $sl(r)$ Gaudin models. Finally, in Section 4 we will give examples our constructions in the $sl(2)$ and $sl(3)$ case. In the last section we briefly summarize the content of the paper and add a few comments. In order to simplify the presentation, we collected the proofs of some important but somewhat technical points in three Appendices.

2 The bihamiltonian structures and the Lax matrices

Let $\mathfrak{g}$ be the Lie algebra $sl(r)$. It is known that it (as well as any simple Lie algebra) can be identified with its dual, e.g., via the dual pairing given by the trace in the fundamental representation. In this paper we will constantly use such an identification. The Lie Poisson structure on $M = \mathfrak{g}^n$ is the one defined, in the natural coordinates $\{A_1, \ldots, A_n\}$ by its Hamiltonian vector fields:

$$\dot{A}_i = [A_i, \frac{\partial F}{\partial A_i}], \tag{2.1}$$

where, if $X = (X_i, \ldots, X_n)$ represents a tangent vector to M, the elements $\dfrac{\partial F}{\partial A_i} \in sl(r)$, $i = 1, \ldots, n$ are those matrices defined by means of the expression of the Lie derivative of F w.r.t. X as

$$\text{Lie}_X(F) = \sum_{l=1}^{n} \text{Tr}\big(X_i \cdot \frac{\partial F}{\partial A_i}\big). \tag{2.2}$$

We will hereinafter denote the Poisson tensor associated with the Lie-Poisson natural bracket by P. From, e.g., [21, 26]) we know that we can endow M with a multi-parameter family of Poisson structures which are compatible with the natural one (2.1). In [9] a further linear Poisson structure, to be denoted by R, has been introduced. It can be described as follows.

We notice that relation (2.1) can be written as:

$$\dot{A}_i = \sum_{j,k=1}^{n} p_{ijk}[A_k, \frac{\partial F}{\partial A_j}], \quad \text{with } p_{ijk} = \delta_{ik}\delta_{ij}. \tag{2.3}$$

The new Poisson tensor R is analogously defined by the expression:

$$\dot{A}_i = \sum_{j,k=1}^{n} r_{ijk}[A_k, \frac{\partial F}{\partial A_j}], \tag{2.4}$$

with "structure constants" given by

$$r_{ijk} = (k-1)\delta_{ij}\delta_{jk} - \theta_{(i-k)}\delta_{ij} + \theta_{(j-i)}\delta_{ik} + \theta_{(i-j)}\delta_{jk} \tag{2.5}$$

where δ is the usual Kronecker symbol and $\theta_{(i)}$ equals 1 if $i > 0$, and vanishes for $i \leq 0$. Explicitly, the Hamiltonian vector field associated by R with a function F is given by:

$$\dot{A}_i = [A_i, (i-1)\frac{\partial F}{\partial A_i} + \sum_{k=i+1}^{N} \frac{\partial F}{\partial A_k}] + \sum_{k=1}^{i-1}[A_k, \frac{\partial F}{\partial A_k} - \frac{\partial F}{\partial A_i}]. \tag{2.6}$$

The following facts can be proven [9]:

Proposition 2.1 *1. The pencil of bivectors $R-\lambda P$ is a bihamiltonian structure on M, that is, R is a Poisson structure compatible with P.*

2. *the functions $H^{(1)}_{\alpha,1} = \frac{1}{\alpha+1} Tr(A_1^{\alpha+1})$, $\alpha = 1,,\ldots,r-1$ are* common *Casimirs for R and P. The Lenard-Magri chains starting at*

$$H^{(a)}_{\alpha,1} = \frac{1}{\alpha+1} Tr(A_a^{\alpha+1}),\ \ a = 2,\ldots,n$$

   provide us with further $d = (n-1)r(r-1)/2$ functionally independent integrals

$$H^{(a)}_{\alpha,p},\ a = 2,\ldots,n,\ p = 2,\ldots,\alpha+1,\ \alpha = 1,\ldots,r-1. \qquad (2.7)$$

3. *Taking into account the integrals associated with the global $SL(r)$ invariance of the model, that is the ring of functions generated by $F_\xi = Tr(\xi \cdot \sum_{i=1}^n A_i)$, those integrals insure complete Liouville integrability of the model.*

□

Remarks.

1. The Gaudin Hamiltonian (1.1) is expressed in terms of the integrals (2.7) as

$$H_G = \sum_{a=2}^n H^a_{2,2} = \sum_{a=2}^n \mathrm{Tr}\big(A_a \cdot (\sum_{b=1}^{a-1} A_b)\big).$$

2. A convenient choice of the integrals associated with the global $SL(r)$ invariance can be done as follows. We pick the $r-1$ independent elements $F_{h_1},\ldots,F_{h_{r-1}}$ associated with, say, the standard Cartan subalgebra of $sl(r)$, and the Gel'fand-Cetlyn invariants, that is, the Casimirs of the nested subalgebras

$$sl(2) \subset sl(3) \subset \cdots \subset sl(r), \qquad (2.8)$$

   under the map $sl(r)^n \to sl(r)$ sending the n-tuple $\{A_1,\ldots,A_n\}$ into the total sum, $A_{tot} = \sum_{i=1}^n A_i$.

For the sequel of the paper the following construction is crucial. Let us introduce $n-1$ Lax matrices:

$$L_a = (\lambda - (a-2))A_a + \sum_{k=1}^{a-1} A_k \qquad a = 2,\ldots,n, \qquad (2.9)$$

It holds:

Proposition 2.2 *Let F be a smooth function on M and let us consider the pencil of vector fields*

$$X_F^\lambda = P_\lambda dF := (R - \lambda P)dF$$

(we say that X_F^λ is Hamiltonian w.r.t. the pencil P_λ). Then, along X_F^λ, the matrices L_i of eq. (2.9) evolve according to a Lax equation,

$$Lie_{X_F^\lambda}(L_a) = [L_a(\lambda), M_a(\lambda)] \tag{2.10}$$

with

$$M_a(\lambda) = (a - 1 - \lambda)\frac{\partial F}{\partial A_a} + \sum_{b=a+1}^{n} \frac{\partial F}{\partial A_b}$$

Proof: Let us denote $\alpha_i = \frac{\partial F}{\partial A_i}$, $i = 1, \ldots, n$. The vector field X_F^λ is explicitly given by:

$$\begin{aligned}
&\mathrm{Lie}_{X_F^\lambda}(A_i) = (P^\lambda dF)_i = \sum_{j,k} (r_{ijk} - \lambda p_{ijk}) \, [A_k, \alpha_k] = \\
&= \sum_{j,k} \left((k - \lambda - 1)\delta_{ij}\delta_{jk} - \theta(i - k)\delta_{ij} + \theta(j - i)\delta_{ik} + \theta(i - j)\delta_{jk}\right) [A_k, \alpha_k] = \\
&= \sum_{k=1}^{i-1} [A_k, \alpha_k - \alpha_i] + \left[A_i, (i - \lambda - 1)\alpha_i + \sum_{k=i+1}^{N} \alpha_k\right]
\end{aligned}$$

Substituting in $L_a(\lambda)$ we get:

$$\begin{aligned}
\mathrm{Lie}_{X_F^\lambda}(L_a) &= (\lambda - a + 2)\left(\sum_{k=1}^{a-1} [A_k, \alpha_k - \alpha_a] + \left[A_a, (a - 1 - \lambda)\alpha_a + \sum_{k=a+1}^{N} \alpha_k\right]\right) + \\
&+ \sum_{j=1}^{a-1} \left(\sum_{k=1}^{a-2} [A_k, \alpha_k - \alpha_j] + \left[A_j, (j - \lambda - 1)\alpha_j + \sum_{k=j+1}^{N} \alpha_k\right]\right) = \\
&= \left[\left((\lambda - a + 2)A_a + \sum_{j=1}^{a-1} A_j\right), \left((a - 1 - \lambda)\alpha_a + \sum_{k=a+1}^{N} \alpha_k\right)\right]
\end{aligned}$$

□

We can interpret this result by saying that we can associate with the homogeneous n-particle Gaudin system a set of $n-1$ matrices depending on a parameter λ, satisfying a Lax equation along the "formal" (i.e., depending on the parameter λ) flows of vector fields that are Hamiltonian with respect to the pencil P_λ.

Proposition 2.3 *The coefficients $K_\alpha^{(a)}(\lambda)$ of the expansion in powers of μ of the characteristic polynomial*

$$det(\mu - L_a(\lambda)) = \mu^r + \sum_{\alpha=1}^{r-1} K_\alpha^{(a)}(\lambda)\mu^{r-\alpha-1}$$

of every Lax matrix $L_a(\lambda)$ are polynomial Casimirs of the pencil $P_\lambda = R - \lambda P$. Moreover, along any vector field X associated with any of the non–trivial coefficient of such polynomial Casimir, all matrices $L_a(\lambda)$ evolve according to Lax equations

$$Lie_X(L_a(\lambda)) = [L_a(\lambda), M_a(X)]$$

for suitable matrices $M_a(X)$.

Proof. These assertion follow from the general theory of bihamiltonian pencils on loop algebras (see, e.g., [26] and [25]). We sketch the proof for completeness, considering the equivalent set of spectral invariants $H_m^{(\alpha)} = 1/(\alpha+1)\mathrm{tr}(L_m)^{\alpha+1}$. To prove the first statement, we must show that, for any one–form v, that we can assume to be exact, $v = dF$ we have

$$\langle v, P_\lambda dH_m^{(l)}\rangle = 0.$$

Now, switching the action of the Poisson pencil on $v = dF$ the LHS of this equation reads

$$\begin{aligned} L_{X_F^\lambda}(H_a^{(l)}) &= L_{X_F^\lambda}(1/(\alpha+1)\mathrm{Tr}(L_a(\lambda))^{\alpha+1}) \\ &= \frac{1}{\alpha+1}\sum_{p=0}^{\alpha-1}\mathrm{Tr}\left(L_a(\lambda)^p \cdot L_{X_F^\lambda}(L_a(\lambda)) \cdot L_a(\lambda)^{\alpha-p}\right) \qquad (2.11) \\ &= \mathrm{Tr}\left(L_a^\alpha \cdot L_{X_F^\lambda}(L_a)\right) = \mathrm{tr}\left(L_a^\alpha(\lambda)\cdot[L_a(\lambda), M_a^F(\lambda)]\right) = 0 \end{aligned}$$

This proves the first assertion of the proposition, and, in particular, shows that all the vector fields $X_{\alpha,p}^{(a)}$ associated (say, via P) with the coefficients of the expansion

$$K_\alpha^{(a)} = \sum_{p\geq 0} K_{\alpha,p}^{(a)}\lambda^p \qquad (2.12)$$

are indeed bihamiltonian vector fields.

To prove the second statement we notice, using a very simple trick well known to experts in the bihamiltonian theory of integrable system, that $X_{\alpha,p}^{(a)} = PK_{\alpha,p-1}^{(a)}$ can be written as a Hamiltonian vector field w.r.t. the pencil, considering the "truncated" polynomial $\left(\lambda^{-p}K_\alpha^{(a)}(\lambda)\right)_+$, where $(\cdot)_+$ denotes the nonnegative part of the expansion in λ. So we see that the bihamiltonian vector fields of the hierarchy are as well Hamiltonian vector fields w.r.t. the bihamiltonian pencil P_λ. The assertion then follows from Proposition 2.2.

□

Let us now focus our attention on a single Lax matrix, say $L_{\bar a}$; calling, for simplicity, $B_{\bar a} = \sum_{b=1}^{\bar a -1} A_b$, we have that the matrix

$$L_{\bar a} = (\lambda - \bar a + 2)A_{\bar a} + B_{\bar a}$$

is a Lax matrix with spectral parameter that evolve according to Lax equations along the vector fields of the hierarchy. Clearly, the Poisson brackets induced on $M^{(2)} = sl(r) \times sl(r)$ by the map $M \to M^{(2)}$ defined by $\{A_{\bar{a}}, B_{\bar{a}}\}$ are nothing but the Lie Poisson brackets on $M^{(2)}$. So, applying the formalism of [28, 1, 5],i.e., according to the Sklyanin "magic recipe" [30], we can get, *for every fixed* $\bar{a}$ a set of canonical coordinates $\{\xi^{\bar{a}}_{\alpha}, \lambda^{\bar{a}}_{\beta}\}$. Actually, we shall do this in Section 3.1.

The point is that, to get a set of canonical coordinates for the whole systems, we have to compare the different sets of coordinates coming from the different Lax matrices L_a, $a = 2, \ldots, n$ (and those coming from the global gauge invariance of the model).

To solve this problem, we shall make use of the bihamiltonian structure of the problem, and, namely, frame the Gaudin systems within the so–called bihamiltonian scheme for SoV. For the case of $sl(2)$, we were able to solve the problem by means of straightforward computations. For the general case, we have to use some slightly more sophisticated ideas and techniques of the bihamiltonian theory, to be discussed in the next Section.

3 Bihamiltonian geometry and Separation of Variables

As we already remarked in the Introduction, a theory of Separation of variables based on the notions of bihamiltonian geometry has been quite recently introduced in the literature. The basic property of such a theoretical scheme which will enable us to solve the SoV problem of this paper can very simply stated as follows:

Proposition 3.1 *Let $(M, P_1 - \lambda P_0)$ be a bihamiltonian manifold and suppose that there exist functions $f, g, \lambda_f, \lambda_g$, with $\lambda_f \neq \lambda_g$, (eventually, λ_f and/or λ_g might be constant) satisfying*

$$P_1 df = \lambda_f P_0 df, \quad P_1 dg = \lambda_g P_0 dg. \tag{3.1}$$

Then $\{f, g\}_0 = \{f, g\}_1 = 0$.

Proof. The assertion easily follows from the equations

$$\begin{aligned} \{f, g\}_1 &= \langle df, P_1 dg \rangle = \lambda_g \{f, g\}_0 \\ \{g, f\}_1 &= \langle dg, P_1 df \rangle = -\lambda_f \{f, g\}_0, \end{aligned}$$

□

In words, calling (with a slight abuse of language) a function f satisfying (3.1) an "*eigenvector*" of the pair P_1, P_0 relative to the "*eigenvalue*" λ_f as in [17], this proposition simply says that eigenvectors belonging to different eigenspaces mutually commute.

If the Poisson tensors P_1 and P_0 do not share the same image and kernel, then a complete set of eigenvectors cannot be found. This is a typical instance in the Gel'fand-Zakharevich theory of bihamiltonian integrable systems, (and happens for the Poisson tensors R and P that we have considered so far).

In general, the bihamiltonian theory of SoV suggests to consider a suitable deformation $\widetilde{P}_1$ of P_1, such that it restricts to the (generic) symplectic leaves of P and it is still compatible with P. Upon restriction, the generic symplectic leaf $\mathcal{S}$ of P will be endowed with a regular bihamiltonian structure (that is, a bihamiltonian structure in which one element of the pencil is invertible). So, in the terminology of [11] the generic symplectic leaves of P are *ωN manifolds*, that is are symplectic manifolds (with symplectic form naturally induced by P), endowed with a compatible Nijenhuis (or hereditary) tensor N. In terms of the Poisson structures, the Nijenhuis tensor on the symplectic leaf $\mathcal{S}$ is defined by

$$N = \widetilde{P}_1|_{\mathcal{S}} \cdot P_0^{-1}|_{\mathcal{S}}.$$

To concoct out of P_1 the suitable deformation $\widetilde{P}_1$, one can adopt the following strategy:
First one fixes a complete set $C_1, \ldots, C_k$ of Casimirs of P_0, considers the first vector fields of the Lenard chains associated with C_a, i.e.,

$$X_a = P_1 dC_a, a = 1, \ldots, k,$$

and a a distribution $\mathcal{Z}$, transversal to the symplectic leaves of P_0. For any basis $W_1, \ldots, W_k$ in $\mathcal{Z}$, the matrix

$$[G_0]_{a,b} = \mathrm{Lie}_{W_b}(C_a) \tag{3.2}$$

is nonsingular (say on an open set $U \subset M$). So, the tensor defined by

$$\widetilde{P}_1 = P_1 - \sum_{a,b} X_a \wedge [G_0^{-1}]_{a,b} W_b \tag{3.3}$$

is well defined and restricts to the generic symplectic leaf $\mathcal{S}$ of P, since, by construction, $\widetilde{P}_1 dC_j = 0$, $j = 1, \ldots, k$. Notice that, if we define a new basis in $\mathcal{Z}$ by

$$Z_a = \sum_b [G_0^{-1}]^a_b W_b, \text{ so that } \mathrm{Lie}_{Z_a}(C_b) = \delta_{a,b}, \tag{3.4}$$

the expression of the deformed tensor $\widetilde{P}_1$ simplifies to

$$\widetilde{P}_1 = P_1 - \sum_a X_a \wedge Z_a. \tag{3.5}$$

We will call a basis of $\mathcal{Z}$ satisfying (3.4) a *normalized basis for the transversal distribution.*

The proof of the following Proposition can be found in [10, 4]

Proposition 3.2 *Let $(M, P_1 - \lambda P_0)$ be a $2n + k$ dimensional bihamiltonian manifold with corank$(P_0) = k$, and suppose that there exists an integrable distribution $\mathcal{Z} \subset TM$ of dimension k, s.t.:*

1. *$\mathcal{Z}$ intersect transversally the symplectic foliation of P_0.*

2. *the space of functions invariant under $\mathcal{Z}$ is a Poisson subalgebra for the whole pencil $(M, P_1 - \lambda P_0)$.*

Then, if $\widetilde{P}_1$ is the deformation of P_1 defined by (3.3), $\widetilde{P}_1 - \lambda P_0$ is still a Poisson pencil on M, and its restriction endows the generic symplectic leaves of P_0 with the structure of a ωN manifold. Furthermore, if $Z_1, \ldots, Z_k$ are a set of generators of $\mathcal{Z}$, normalized w.r.t. a given complete set $C_1, \ldots, C_k$ of Casimir functions of P_0,condition 2 above translates into the equations:

$$Lie_{Z_a} P_0 = 0, \quad Lie_{Z_a} P_1 = \sum_{b=1}^{k} [Z_a, X_b] \wedge Z_b, \quad \textit{where } X_a = P_1 dC_a, \quad a = 1, \ldots, k. \tag{3.6}$$

□

Definition 3.3 *We say that a bihamiltonian manifold $(M, P_1 - \lambda P_0)$, endowed with a transversal distribution $\mathcal{Z}$ satisfying the assumptions of Proposition 3.2 admits an affine structure if it is possible to choose a complete set of Casimir of P_0, and a corresponding basis of* normalized flat generators Z_b, $b = 1, \ldots, k$ *in $\mathcal{Z}$ such that, for every Casimir of the Poisson pencil $H^a(\lambda)$ and every $b, c = 1, \ldots k$ one has, in addition to Equation (3.6)*

$$Lie_{Z_b} Lie_{Z_c}(H^a(\lambda)) = 0. \tag{3.7}$$

The notion of affine structure for a bihamiltonian manifold was studied in [11] in connection with the problem of the Stäckel separability of a bihamiltonian system. For the purposes of the present paper, we remark that an affine Poisson pencil satisfies special properties, to be illustrated in the following.

Let $(M, P_1 - \lambda P_0)$ be a corank k affine bihamiltonian manifold, and let Z_a, $a = 1, \ldots, k$ be a set of normalized flat generators for the transversal distribution $\mathcal{Z}$. Let us consider the polynomial Casimirs

$$H^{(a)}(\lambda) = \lambda^{n_a} H_0^a + \cdots + H_{n_a}^a,$$

and their deformations along the flat generators, that is, the k^2 polynomials

$$D_b^a(\lambda) = \mathrm{Lie}_{Z_b} H^{(a)}(\lambda) = \lambda^{n_a} \delta_b^a - D_{b,1}^a \lambda^{n_a - 1} - \cdots - D_{b,n_a}^a. \tag{3.8}$$

The polynomials $D_b^a(\lambda)$ are invariant along $\mathcal{Z}$, so that they can be considered as functions on the generic symplectic leaves of P_0. They are the building blocks of the bihamiltonian set-up for SoV for GZ systems. Indeed it holds:

Proposition 3.4 *Let $\Delta(\lambda)$ be the determinant of the matrix D_b^a of (3.8). then*

1. *The roots λ_i of $\Delta(\lambda)$ satisfy*

$$\widetilde{P}_1 d\lambda_i = \lambda_i \, P_0 \, d\lambda_i,$$

2. *Let $\widetilde{D}(\lambda)$ denote the classical adjoint matrix of $D(\lambda)$, and let $[\widetilde{D}(\lambda)]_{a,c}$ be non identically vanishing. Then any ratio $\rho(\lambda) := [\widetilde{D}(\lambda)]_{a,b}/[\widetilde{D}(\lambda)]_{a,c}$ of elements belonging to the a-th row of $\widetilde{D}(\lambda)$, evaluated at the roots λ_i of $\Delta(\lambda)$ satisfy the equation*

$$\widetilde{P}_1 d\rho(\lambda_i) = \lambda_i P_0 d\rho(\lambda_i). \tag{3.9}$$

The proof of this Proposition is contained in Appendix A.

□

3.1 Separation of Variables for the sl(r) Gaudin Systems

In this subsection we will specialize the results of Section 3 and show how the bihamiltonian structure $P_\lambda = R - \lambda P$ associated in Section 2 with the parameter independent integrals of the Gaudin model provides a set of separation coordinates and relations for the H-J equations associated with H_G.

The first step is to show that P_λ induces a ωN manifold structure on the generic symplectic leaf $\mathcal{S}$ of P, that is, that the tensor R can be suitably deformed. We consider in $M = sl(r)^n$ the $n(r-1)$ vector fields

$$W_i^\alpha := \frac{\partial}{\partial [A_i]_{r,\alpha}}, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, r-1, \tag{3.10}$$

that is, the vector fields defined by their action on the n-tuple of matrices $(A_1, \ldots, A_n)$ by

$$\mathrm{Lie}_{W_i^\alpha}(A_1, \ldots, A_n) = (0, 0, \cdots, \underbrace{e_{\alpha,r}}_{\text{i-th place}}, \cdots, 0), \tag{3.11}$$

where $e_{\alpha,r}$ is the elementary matrix $(e_{\alpha,r})_{ij} = \delta_{i,\alpha}\delta_{j,r}$.

Proposition 3.5 *The distribution $\mathcal{Z}$ spanned by the vector fields W_i^α satisfies the hypotheses of Proposition 3.2.*

The proof of this Proposition is contained in Appendix B.

□

To construct a set of flat generators Z_i^α for $\mathcal{Z}$, we can argue as follows. In the case of a single copy of $sl(r)$, we normalize the W^α with respect to the coefficients $C_1, \ldots, C_{r-1}$ of the characteristic polynomial of A. The normalization for the n site case is done site by site. Since the determinant of a matrix is a linear function of each of its entry, it is not difficult to realize that such normalized generators Z_i^α provide the GZ manifold $M, R - \lambda P$ with the structure of an affine GZ manifold, according to Definition 3.3.

Let us now consider the Lax matrices

$$L_a = (\lambda - a + 2)A_a + \sum_{b=1}^{a-1} A_b, \;\; a = 2, \dots, n, \text{ as well as } L_1 = A_1.$$

Define $M_i(\lambda, \xi) = \xi \mathbf{I} - L_i(\lambda)$, and denote their classical adjoint with $\widetilde{M_i}(\lambda, \xi)$. The determinants of the matrices M_i define, thanks to Proposition 2.3, polynomial Casimirs $K^{(i)}_\alpha(\lambda)$ for $R - \lambda P$, via:

$$\mathrm{Det}(M_i(\lambda, \xi)) = \xi^r + \sum_{\alpha=1}^{r-1} K^{(i)}_\alpha(\lambda)\xi^{r-\alpha-1}. \tag{3.12}$$

In particular, $K^{(1)}_\alpha$ are the common Casimirs of P and R, while $K^{(a)}(\lambda)$ are, for $a = 2, \dots, n$, the non trivial polynomial Casimirs of the pencil $R - \lambda P$.

Let us consider the n^2 matrices D_{ij} defined by

$$\big(D_{ij}\big)^\alpha_\beta = \mathrm{Lie}_{Z^\alpha_j}(K^{(i)}_\beta(\lambda)). \tag{3.13}$$

The proof of the following Proposition, which is based on a few elementary properties of the matrices $\big(D_{ij}\big)$ is contained in Appendix C.

Proposition 3.6 *The determinant $\mathcal{D}_a(\lambda)$ of the matrices $\big(D_{aa}\big), a = 2, \dots, n$, factors as*

$$\mathcal{D}_a(\lambda) = (\lambda - a + 2)^{r-1}\Delta_a(\lambda), \tag{3.14}$$

where $\Delta_a(\lambda)$ is a monic polynomial of degree $r(r-1)/2$. Let $\lambda^s_a, a = 2, \dots, n, s = 1, \dots, r(r-1)/2$, be the roots of $\Delta_a(\lambda)$, and let us consider a row (say, the first) $\delta_\alpha(\lambda) = \big(\widetilde{D_{aa}}(\lambda)\big)_{1,\alpha}$ of the adjoint matrix $\widetilde{D_{aa}}(\lambda)$ of $D_{aa}(\lambda)$. Finally, let ξ^s_a be the functions obtained by evaluating in $\lambda = \lambda^s_a$ the ratios $\delta_{r-2}(\lambda)/\delta_{r-1}(\lambda)$. Then, these $(n-1)r(r-1)/2$ pairs of functions $\{\xi^s_a, \lambda^s_a\}$ satisfy
1) the Jacobi separation relations

$$Det(M_a(\xi^s_a, \lambda^s_a)) = 0.$$

2) the differential relations

$$\widetilde{R}d\lambda^s_a = \lambda^s_a P d\lambda^s_a, \quad \widetilde{R}d\xi^s_a = \lambda^s_a P d\xi^s_a.$$

In particular, their brackets, (say, with respect to the Lie Poisson structure P), are of the separate form:

$$\{\lambda^s_b, \xi^t_a\}_P = \delta^{st}\delta_{ab}\varphi^s_a(\xi^s_a, \lambda^s_a), \tag{3.15}$$

where $\varphi^s_a(\xi^s_a, \lambda^s_a)$ are functions of the two variables (ξ^s_a, λ^s_a).

□

The meaning of this Proposition can be rephrased as follows. For every integer $a = 2, \ldots, n$ we can construct a Lax matrix, whose characteristic polynomial gives us a family of Casimirs of the bihamiltonian pencil $R - \lambda P$ defined on the manifold $M = sl(r)^n$. Separated coordinates are constructed, according to the bihamiltonian scheme, by deforming such Casimirs along normalized generators of a suitable distribution $\mathcal{Z}$ defined in M. In particular, for each $a = 2, \ldots, n$, we can algebraically construct a "cluster" of $(r(r-1))$ variables $\{\lambda_a^s, \xi_a^s\}_{s=1,\ldots r(r-1)/2}$ that are, in the terminology of [17], algebro-geometrical Nijenhuis coordinates, that is, satisfy properties 1) and 2) of Proposition 3.6.

To finish our job we have to:

i) Discuss about the coordinates associated with the global gauge invariance of the Gaudin Systems

ii) Explicitly construct, out of the coordinates found so far, a set of *canonical* separated coordinates (that is, a set of Darboux-Nijenhuis coordinates).

Point i) can be solved as follows. One notices that any function φ depending only on the "global" matrix variable $A_T = \sum_{i=1}^n A_i$, which is invariant along the distribution $\mathcal{Z}$ satisfies the differential relation

$$\widetilde{R} d\varphi = (n-1) P d\varphi. \tag{3.16}$$

In particular, this family includes the mutually commuting Hamiltonians of Gel'fand-Cetlyn type discussed in the Remark after Proposition 2.1. The property (3.16) follows from the fact that they trivially satisfy the relation $Rd\varphi = (n-1)Pd\varphi$ w.r.t. the undeformed pencil, and from the property that φ commutes with all the Hamiltonians of the hierarchy. Inside this ring of functions one can find a set of $r(r-1)/2$ canonical coordinates that complement the Gel'fand-Cetlyn Hamiltonians. Thanks to (3.16) they will have vanishing Poisson brackets with the Nijenhuis coordinates of Proposition 3.6.

The solution to point ii) above can be simply done by means of a direct computation of the Poisson brackets between ξ_a^s and λ_a^s. In particular, this computation will implicitly prove that these quantities are functionally independent.

Proposition 3.7 *The Poisson brackets, w.r.t. the Lie Poisson pencil P of the coordinates λ_a^s, ξ_a^s defined above are given by*

$$\{\lambda_a^s, \xi_a^s\} = (\lambda_a^s - a + 2)(\lambda_a^s - a + 1) \tag{3.17}$$

Proof. The proof of this proposition follows verbatim that of Theorem 1.3 in [1], to which we refer for full details. Indeed, the coordinates λ_a^s, ξ_a^s can be seen as common zeroes of the first row of the matrix $\widetilde{M}_a(\lambda, \xi)$. So we can apply all the considerations of [1], the only difference being that the Poisson brackets of the entries of the matrix $M_a(\lambda,\xi)$ are given by:

$$\{M_a^{ij}(\lambda,\xi), M_a^{kl}(\sigma,\eta)\} =$$
$$= \mathrm{tr}\left[\left((\lambda-a+2)(\sigma-a+2)A_a + \sum_{r=1}^{a-1} A_r\right)(e_{kj}\delta_{li} - e_{li}\delta_{kj})\right] =$$
$$= \frac{1}{\lambda-\sigma}\left[(\lambda-a+1)(\sigma-a+2)(M_a^{jk}(\lambda,\xi)\delta_{il} - M_a^{il}(\lambda,\xi)\delta_{jk}) + \right.$$
$$\left. + (\lambda-a+2)(\sigma-a+1)(M_a^{il}(\sigma,\eta)\delta_{jk} - M_a^{jk}(\sigma,\eta)\delta_{il})\right]. \quad (3.18)$$

The presence of the factors $(\lambda-a+2)(\sigma-a+1)$ and $(\lambda-a+2)(\sigma-a+1)$ is responsible for the factor $(\lambda_a^s-a+2)(\lambda_a^s-a+1)$ in Eq. (3.17).

□

4 Examples

In this Section we will specialize the constructions presented in the paper for the cases of $sl(2)$ and $sl(3)$.

4.1 The sl(2) case

Here we briefly reframe the explicit computations of the last Section of [9] within the formalism exposed in this paper. We consider the manifold $M = sl(2)^n$, endowed with the Poisson pencil $R - \lambda P$ of Section 2. It is explicitly parametrized in terms of the n matrices

$$A_i = \begin{bmatrix} h_i & e_i \\ f_i & -h_i \end{bmatrix}. \quad (4.1)$$

The generic symplectic leaf S of P is a $2n$ dimensional symplectic manifold, defined by the equations

$$C_i = \frac{1}{2}\mathrm{Tr}A_i^2 = h_i^2 + e_i f_i, \quad i = 1, \ldots, n,$$

and can be (generically) endowed with the $2n$ coordinates (h_i, f_i), $i = 1, \ldots, n$.

A set of normalized transverse vector fields are given in this case by

$$Z_i = \frac{1}{f_i}\frac{\partial}{\partial e_i}, \quad (4.2)$$

The matrices L_a are explicitly given by

$$L_a(\lambda) = \begin{pmatrix} (\lambda-a+2)h_a + \sum_{b=1}^{a-1} h_b & (\lambda-a+2)e_a + \sum_{b=1}^{a-1} e_b \\ (\lambda-a+2)f_a + \sum_{b=1}^{a-1} f_b & -((\lambda-a+2)h_a + \sum_{b=1}^{a-1} h_b) \end{pmatrix}. \quad (4.3)$$

As canonical coordinates associated with the global $SL(2)$ invariance one can choose the two functions

$$\lambda_1 = \sum_{i=1}^{n} f_i, \quad \phi_1 = \frac{\sum_{i=1}^{n} h_i}{\sum_{i=1}^{n} f_i}.$$

The non trivial separation coordinates are gotten simply considering the zeroes z_a of the elements $[L_a]_{2,1}$, and the values μ_a on these zeroes of the elements $[L_a]_{2,2}$, normalized as in the previous Section. One sees that

$$z_a = -\frac{\sum_{k=1}^{a-1} f_k}{f_a} + (a-2), \quad a = 2, \ldots, n$$

Shifting these values by the unessential term $a-2$, we find that the separation coordinates are given, for $a = 2, \ldots, n$, by

$$\lambda_a = -\frac{\sum_{k=1}^{a-1} f_k}{f_a}, \quad \mu_a = -\frac{\lambda_a h_a + \sum_{k=1}^{a-1} h_k}{\lambda_a(\lambda_a - 1)} \tag{4.4}$$

They fulfill the separation relations

$$\mu_a^2 = \frac{1}{2(\lambda_a(\lambda_a-1))^2}\big(C_a^2\lambda_a^2 + \mathrm{Tr}(A_a(\sum_{b=1}^{a-1} A_b))\lambda_a + \mathrm{Tr}((\sum_{b=1}^{a-1} A_b)^2)\big). \tag{4.5}$$

In other words, the separation coordinates are coordinates of suitable points on the rational curves (4.5). The corresponding Hamilton-Jacobi equations can be explicitly solved by means of algebraic functions.

4.2 The sl(3) case

We consider the Poisson manifold $M = sl(3)^n$, endowed with the Poisson pencil $R - \lambda P$ and parametrized by the n matrices

$$A_i = \begin{pmatrix} h_{1,i} & e_{1,i} & e_{3,i} \\ f_{1,i} & h_{2,i} - h_{1,i} & e_{2,i} \\ f_{3,i} & f_{2,i} & -h_{2,i} \end{pmatrix} \quad i = 1, \ldots, n. \tag{4.6}$$

On this manifold the Poisson tensor P has $2n$ Casimirs:

$$C_i^2 = \frac{1}{2}\mathrm{Tr}\big((A_i^2)\big), \quad C_i^3 = \frac{1}{3}\mathrm{Tr}\big((A_i)^3\big) \quad i = 1, \ldots, n. \tag{4.7}$$

The characteristic polynomials of the Lax matrices $L_a = (\lambda - a + 2)A_a + \sum_{b=1}^{a-1} A_b$ are expressed as

$$\Gamma^a(\mu, \lambda) = \mu^3 - \mu H_2^{(a)}(\lambda) - H_3^{(a)}(\lambda). \tag{4.8}$$

The transversal distribution $\mathcal{Z}$ is generated by the $2n$ flat generators:

$$
\begin{aligned}
Z_i^2 &= \frac{1}{d}\left[(f_{3,i}(h_{1,i}-h_{2,i})+f_{2,i}f_{1,i})\frac{\partial}{\partial e_{2,i}}+(f_{2,i}h_{1,i}-f_{3,i}e_{1,i})\frac{\partial}{\partial e_{3,i}}\right] \\
Z_i^3 &= \frac{1}{d}\left[f_{2,i}\frac{\partial}{\partial e_{3,i}}-f_{3,i}\frac{\partial}{\partial e_{2,i}}\right] \\
d &= f_{2,i}f_{3,i}(2h_{1,i}-h_{2,i})+f_{2,i}^2f_{1,i}-f_{3,i}^2e_{1,i}
\end{aligned}
$$

The symplectic leaves of P are generically parametrized by matrices A_i of the form:

$$
\begin{pmatrix} h_{1,i} & e_{1,i} & \Phi_{3,i} \\ f_{1,i} & h_{2,i}-h_{1,i} & \Phi_{2,i} \\ f_{3,i} & f_{2,i} & -h_{2,i} \end{pmatrix}
$$

where $\Phi_{2,i}$ and $\Phi_{3,i}$ are suitable functions of the coordinates $h_{1,i}, h_{2,i}, f_{1,i}, f_{2,i}, f_{3,i}, e_{1,i}$, parametrically depending on the Casimirs (4.7).

The coordinates $\{\lambda_a^s, \xi_a^s\}$ can quite explicitly be found by means of the following steps:

We consider the matrix $M_a(\lambda,\xi)=\xi-L_a(\lambda)$ and its adjoint $\widetilde{M_a}(\lambda,\xi)$. We have to look for the common zeroes of the elements $\widetilde{M_a}(\lambda,\xi)_{3,1}$ and $\widetilde{M_a}(\lambda,\xi)_{3,2}$, that is, for the common zeroes of

$$
\mathrm{Det}\begin{pmatrix} -L_a(\lambda)_{2,1} & \xi-L_a(\lambda)_{2,2} \\ -L_a(\lambda)_{3,1} & -L_a(\lambda)_{3,2} \end{pmatrix}, \qquad \mathrm{Det}\begin{pmatrix} \xi-L_a(\lambda)_{1,1} & -L_a(\lambda)_{1,2} \\ -L_a(\lambda)_{3,1} & -L_a(\lambda)_{3,2} \end{pmatrix}. \tag{4.9}
$$

Taking into account the form of the vector fields Z_a^α and of the characteristic polynomial (4.8), we can identify the system (4.9) with

$$
\begin{cases} \xi \mathrm{Lie}_{Z_a^2} H_2^{(a)} + \mathrm{Lie}_{Z_a^2} H_3^{(a)} = 0 \\ \xi \mathrm{Lie}_{Z_a^3} H_2^{(a)} + \mathrm{Lie}_{Z_a^3} H_3^{(a)} = 0 \end{cases}, \tag{4.10}
$$

where

$$
\mathrm{Det}(M_a(\xi,\lambda)) = \xi^3 - H_2^{(a)}\xi - H_3^{(a)}. \tag{4.11}
$$

As we have noticed in Section 3.1 , we can factor out $(\lambda-a+2)$ from each line of this system, and consider, in matrix form the equivalent system:

$$
\langle \xi, 1| \begin{pmatrix} G^a_{2,2} & G^a_{2,3} \\ G^a_{3,2} & G^a_{3,2} \end{pmatrix}, \quad \mathrm{with} G^a_{\alpha,\beta} = \mathrm{Lie}_{Z_a^\alpha} H_\beta^{(a)}/(\lambda-a+2). \tag{4.12}
$$

We notice that $G^a_{\alpha,\beta}$ are polynomials in λ of degree $\alpha - 1$, so that the three zeroes $\lambda^1_a, \lambda^2_a, \lambda^3_a$ of the equation

$$\Delta_a(\lambda) = \mathrm{Det}\begin{pmatrix} G^a_{2,2} & G^a_{2,3} \\ G^a_{3,2} & G^a_{3,2} \end{pmatrix} = 0 \tag{4.13}$$

are the compatibility condition for the system (4.10); the corresponding coordinates $\xi^1_a, \xi^2_a, \xi^3_a$ are thus given by, e.g.,

$$\xi^s_a = -G^a_{3,3}/G^a_{2,3}\big|_{\lambda=\lambda^s_a} \tag{4.14}$$

We remark that our procedure for finding separation coordinates exactly matches the one introduced, in the framework of r-matrix theory, in [5].

Finally, defining $\zeta^s_a = \lambda^s_a - a + 2,\ a = n, \ldots, n$ and considering the pairs

$$\{\zeta^s_a,\ \rho^s_a\} = \xi^s_a/\zeta^s_a(\zeta^s_a - 1), s = 1, 2, 3,\ a = 2, \ldots, n,$$

we see that the solution W of the (stationary) Hamilton-Jacobi equations of the $sl(3)$ Gaudin model can be expressed as:

$$W = \sum_{a=2}^{n}\left(\sum_{s=1}^{3}\int^{P^s_a}\frac{\xi d\zeta}{\zeta(\zeta-1)}\right) + \sum_{\alpha=s}^{3} H^s_T q^s_T, \tag{4.15}$$

where P^α_a denotes the point $(\xi^\alpha_a, \zeta^\alpha_a)$ on the genus $g = 1$ algebraic curve defined by equation (4.11), and H^s_T and q^S_T denote, respectively, a suitable complete family of Gel'fand-Cetlyn Hamiltonians associated with the global $SL(3)$ invariance of the model, and their canonically conjugated variables.

Standard arguments show that the linearization of the flows associated with the mutually commuting Hamiltonians we have considered in this paper, and hence also the flow associated with the "physical" Hamiltonian of the $sl(3)$ Gaudin system can be achieved by means of the Abel maps associated with the differentials (the last four being of the third kind)

$$\frac{d\zeta}{\Gamma^a_\xi}, \frac{\zeta d\zeta}{(\zeta-1)\Gamma^a_\xi}, \frac{d\zeta}{(\zeta-1)\Gamma^a_\xi}, \frac{d\zeta}{\zeta(\zeta-1)\Gamma^a_\xi}, \frac{\xi d\zeta}{\zeta(\zeta-1)\Gamma^a_\xi}, \tag{4.16}$$

where $\Gamma^a_\xi = \dfrac{\partial \mathrm{Det}(M_a(\xi, \lambda))}{\partial \xi}$. The case of $sl(r), r > 3$ can be treated analogously.

5 Conclusion and discussion

In this paper we have reconsidered the SoV problem for the (homogeneous XXX) Gaudin systems based on the Lie algebras $sl(r)$, from a particular standpoint. Namely, we considered the n-site system as a Gel'fand-Zakharevich system defined on the manifold $M = sl(r)^n$, with respect to the Poisson pencil

$P_\lambda = R - \lambda P$, where P is the usual Lie Poisson bracket on M, while R, given by equations (2.4) and (2.5) is a further linear Poisson structure on M that was introduced in [9].

We showed that the system admits a set of $n-1$ "Lax" matrices, linear in the spectral parameter λ that evolve according to Lax equations along any vector field that is Hamiltonian with respect to the Poisson pencil P_λ.

Thanks to this property, by using the bihamiltonian set-up for SoV, we managed to define a set of separating coordinates, quite explicitly given in equations (4.4) for the $sl(2)$ case and by equations (4.13) and (4.14) for the $sl(3)$ case.

We notice that this set of coordinates provide an alternative set of separation coordinates for the Hamilton–Jacobi equations associated with the Gaudin Hamiltonian w.r.t. the coordinates that can be found by means of the conventional approach based on the rational Lax matrix (1.2), i,e,

$$\mathcal{L}(\lambda) = \sum_{i=1}^{n} \frac{A_i}{\lambda - a_i}. \tag{5.1}$$

In this respect, a few remarks are in order:

First we notice that the existence of different sets of separation coordinates is to be ascribed to the *super-integrability* of the system. This is particularly clear in the bihamiltonian setting, where the separation coordinates are obtained, by means of the procedure outlined in Section 3, from the polynomial Casimirs of a Gel'fand-Zakharevich Poisson pencil. The general problem of the connections between super-integrability and "multi-separability" is, to the best of our knowledge, still an open problem (see, [23] and the references quoted therein). In particular, the classification problem for these systems have been solved only for systems with a small number of degrees of freedom. The homogeneous XXX Gaudin models are systems with an arbitrarily high number of degrees of freedom where the connection between super integrability and multi-separability happens, and their study might shed light on the structural properties of this phenomena.

The second remark is the following. The SoV scheme based on the Lax matrix 5.1 leads to the definition of a divisor of degree

$$d_R = r(r-1)(n-1)/2$$

on the spectral curve $R(\lambda, \mu) = \mathrm{Det}(\mu - \mathcal{L}(\lambda))$ (see, e.g., [16]). It is not difficult to ascertain that the genus of this spectral curve is, for $\mathfrak{g} = sl(r)$

$$g_{\mathcal{L}} = \frac{(r-1)}{2}\big((n-2)r + (r-2)\big),$$

that is, it grows linearly with the number n of sites of the model.

As we have shown in Section 3.1 and exemplified in Section 4 for $r = 2, 3$, the SoV scheme herewith outlined leads to consider the set of $n-1$ Lax matrices

$L_a(\lambda)$ given by

$$L_a(\lambda) = (\lambda - a + 2)A_a + B_a, \quad B_a = \sum_{b=1}^{a-1} A_b. \tag{5.2}$$

The separation coordinates parametrize sets of degree $r(r-1)/2$ divisors on the spectral curves $\Gamma^a(\lambda,\mu) = \mathrm{Det}(\mu - L_a(\lambda))$. We see that the genus of such curves is

$$g_{\Gamma^a} = (r-1)(r-2)/2,$$

that is depends only on the rank r of the algebra, and not on the number n of sites, showing that the equations of motions can be explicitly solved by means of θ functions of genus g_{Γ^a} for all n's.

This also implies the existence of canonical transformations between (suitable open sets of) the degree $d_{\mathcal{L}}$ Jacobian of the spectral curve $R(\mu,\lambda)$ associated with $\mathcal{L}(\lambda)$ and the (corresponding open subsets) of the Cartesian product of the degree $r(r-1)/2$ Jacobians associated with the curves $\Gamma^a(\lambda,\mu)$[2]. This simply follows from the fact that both the algebro-geometrical coordinates found from $\mathcal{L}(\lambda)$ and those we discovered in this paper are *canonical coordinates* for the standard Lie-Poisson bracket P_0 on $sl(r)^n$, and, in particular, (together with those coordinates associated with the global $SL(r)$ invariance) are Darboux coordinates for the restriction of P_0 to its generic symplectic leaves.

It is outside the size of this paper to fully discuss this issue here. However we think it is appropriate to display this transformation in the simple case of $sl(2)$. This goes as follows.

We recall that the n matrices A_i can be explicitly parametrized by means of $3n$ coordinates $h_i, e_i, f_i,\ i = 1,\ldots,n$:

$$A_i = \begin{bmatrix} h_i & e_i \\ f_i & -h_i \end{bmatrix},$$

with Lie-Poisson brackets given by

$$\{h_i, e_j\}_P = \delta_{ij} e_j, \quad \{h_i, f_j\}_P = -\delta_{ij} f_j, \quad \{e_i, f_j\}_P = 2\delta_{ij} h_j$$

Let us denote with λ_i, μ_i the set of separation variables associated with the Lax matrix $\mathcal{L}(\lambda)$ (see eq. (5.1)) of the n-particles $sl(2)$-Gaudin model and with ζ_i, ρ_i those associated with the bihamiltonian picture discussed in this paper.

We can assume that the coordinates associated with the global $SL(2)$ invariance of the problem are the same; in particular, on the symplectic leaves of the Lie Poisson tensor P_0 we have to consider the pair

$$\lambda_1 = \zeta_1 = \sum_{j=1}^{n} f_j, \quad \mu_1 = \rho_1 = \frac{\sum_{j=1}^{n} h_j}{\sum_{j=1}^{n} f_j}$$

[2]This observation is due to B. Dubrovin.

According to the "Sklyanin" recipe, the $\lambda_a,\ a=2,\dots,n$ are the zeroes of the $(1,2)$ entry of the rational Lax matrix (1.2), i.e. the roots of the polynomial:

$$\Delta_{\mathcal{L}}(\lambda)\equiv\sum_{k=0}^{n-1}\lambda^k C_k=\frac{1}{\sum_{i=1}^n f_i}\sum_{k=0}^{n-1}\lambda^k\sum_{j=1}^n(-1)^{n-k-1}s_{n-k-1}(a_1,\dots,\widehat{a_j},\dots,a_n)f_j. \tag{5.3}$$

Here the polynomials $s_{n-k-1}(a_1,\dots,\hat{a}_i,\dots,a_n)$ are the elementary symmetric polynomials in $n-1$ letters $b_1,\dots,b_{n-1}$, defined by

$$\prod_{j=1}^{n-1}(\lambda-b_j)=\sum_{j=0}^{n-1}(-1)^{n-j-1}s_{n-j-1}(b_1,\dots,b_{n-1})\lambda^j,\qquad \text{where } s_0\equiv 1, \tag{5.4}$$

evaluated for $b_1=a_1,\dots,b_{j-1}=a_{j-1},b_j=a_{j+1},\dots,b_{n-1}=a_n$.

We can express the "physical" coordinates f_i in terms of the λ_i as follows:

$$\frac{f_i}{\sum_{j=1}^n f_j}=\frac{1}{\prod_{k\neq i}(a_i-a_k)}\sum_{j=0}^{n-1}a_i^j C_j,\qquad \sum_{j=1}^n f_j=\lambda_1. \tag{5.5}$$

Since the coordinates ζ_i are rational functions of the f_i alone, and namely,

$$\zeta_a=-\frac{\sum_{k=1}^{a-1}f_k}{f_a},\qquad a=2,\dots,n,$$

we can explicitly find the transformation yielding the ζ_i in terms of the λ_i as:

$$\zeta_b=-\sum_{k=1}^{b-1}\left(\frac{\Pi_{l\neq b}(a_b-a_l)}{\Pi_{l\neq k}(a_k-a_l)}\cdot\frac{\sum_{j=0}^{n-1}a_k^j(-1)^{n-j-1}s_{n-j-1}(\lambda_2,\dots,\lambda_n)}{\sum_{j=0}^{n-1}a_b^j(-1)^{n-j-1}s_{n-j-1}(\lambda_2,\dots,\lambda_n)}\right), \tag{5.6}$$

for $b=2,\dots,n$, with, obviously

$$\zeta_1=\lambda_1. \tag{5.7}$$

The variables $\mu_a, a=2,\dots,n$ are the values of the $(1,1)$ entry of the rational Lax matrix (1.2) for $\lambda=\lambda_a$, while the ρ_a are be given by the values of the $(1,1)$ entry of the Lax matrix L_a in $\lambda=\zeta_a$, divided by the normalizing factor $\zeta_a(\zeta_a-1)$. Explicitly:

$$\mu_a=\sum_{j=1}^n\frac{h_j}{\lambda_a-a_j}\quad a=2,\dots,n,\qquad \mu_1=\frac{\sum_{j=1}^n h_j}{\lambda_1}$$
$$\rho_a=\frac{\zeta_a h_a+\sum_{j=1}^{a-1}h_j}{\zeta_a(\zeta_a-1)}\quad a=2,\dots,n,\qquad \rho_1=\frac{\sum_{j=1}^n h_j}{\zeta_1}$$

The transformation of coordinates connecting the ρ_i and the μ_i can be easily found noticing that they are connected to the coordinates h_i by a linear transformation with coefficients depending on ζ_i and λ_i respectively. Consequently, the transformation between the coordinates ρ_i and μ_i is a linear transformation with λ_i depending coefficients:

$$\rho_i = \sum_{j=1}^{n} A_{ij}(\lambda_1, \dots, \lambda_n)\mu_j \tag{5.8}$$

and it follows that it must be the lifting of the transformation defined by ((5.6) and (5.7)) among the ζ_i and the λ_i;

$$\rho_i = \sum_{j=1}^{n} ((J^t)^{-1})_{ij}\mu_j \tag{5.9}$$

where J is the Jacobian of the transformation (5.6).

Finally, we just mention some other problems which remain open. The first one is to compare our results with the picture of the generalized bending flows of [12]. The setting presented in that paper was aimed at providing a generalization of the previous paper [19], and, in our setting, should be obtained by reduction to the submanifold of matrices A_i having rank equal to one.

The second one is the application of the scheme herewith presented to the quantum $sl(r)$ case. Preliminary results for $r = 3$ indicate that this should be a viable procedure for giving explicit expressions to the quantum integrals whose existence has been proven in [6]. Work in both these directions is in progress.

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Appendix A : Proof of Proposition 3.4

The proof of Proposition 3.4 is divided into a couple of steps. We recall that we are considering a corank k affine bihamiltonian manifold $(M, P_1 - \lambda P_0)$, endowed with a transversal distribution $\mathcal{Z}$, satisfying the requirements of Proposition 3.2. $Z_a,\ a = 1, \dots, k$ is a set of normalized flat generators for $\mathcal{Z}$, and

$$H^{(a)}(\lambda) = \lambda^{n_a} H_0^a + \cdots + H_{n_a}^a,$$

are polynomial Casimirs of $P_1 - \lambda P_0$. Finally, the k^2 polynomials

$$D_b^a(\lambda) = \mathrm{Lie}_{Z_b} H^{(a)}(\lambda) = \lambda^{n_a}\delta_b^a - D_{b,1}^a \lambda^{n_a - 1} - \cdots - D_{b,n_a}^a. \tag{A.1}$$

are their deformations along the flat generators. Finally, we recall the definition

$$\widetilde{P}_1 = P_1 - \sum_{a=1}^{k} X_a \wedge Z_a, \quad X_a = P_1 dH_0^a.$$

Lemma 1 *The actions of $\widetilde{P}_1$ and P_0 on the deformation of the Casimirs of the pencil are related by the following formula:*

$$\widetilde{P}_1 dD^a_b(\lambda) = \lambda P_0 dD^a_b(\lambda) + \sum_{c=1}^{k} D^a_c(\lambda) P_0 dD^c_{b,1}. \tag{A.2}$$

Proof. We limit ourselves to sketch the proof of this Proposition, which is essentially contained in Section 7 of [11], although in a disguised form. We consider the characteristic property of a Casimir of the Poisson pencil,

$$P_\lambda dH^{(a)}(\lambda) = 0$$

and derive it w.r.t. Z_b. We get:

$$\mathrm{Lie}_{Z_b}(P_\lambda)\, dH^{(a)}(\lambda) + P_\lambda dD^a_b(\lambda) = 0. \tag{A.3}$$

Since $\mathrm{Lie}_{Z_b}(P_\lambda) = \sum_c [Z_b, X^1_c] \wedge Z_c$ with $X^1_c = P_1 dH^c_0 = P_0 dH^c_1$, we see that

$$[Z_b, X^1_c] = \mathrm{Lie}_{Z_b}(X^1_c) = \mathrm{Lie}_{Z_b}(P_0 dH^c_1) = -P_0 dD^c_{b,1}.$$

Thus eq.(A.3) takes the form

$$P_\lambda dD^a_b(\lambda) - \sum_c D^a_c(\lambda) P_0 dD^c_{b,1} - \sum_c \langle [Z_b, X^1_c], dH^{(a)}(\lambda)\rangle \cdot Z_c = 0. \tag{A.4}$$

Let us consider the coefficient $\langle [Z_b, X^1_c], dH^{(a)}(\lambda)\rangle$ in the last sum. This is, by definition,

$$\mathrm{Lie}_{[Z_b, X^1_c]}(H^{(a)}(\lambda)) = \mathrm{Lie}_{Z_b}\mathrm{Lie}_{X^1_c}(H^{(a)}(\lambda)) - \mathrm{Lie}_{X^1_c}\mathrm{Lie}_{Z_b}(H^{(a)}(\lambda)).$$

Since $\{H^c_1, H^{(a)}(\lambda)\}_0 = 0$ only the second term is non identically vanishing, and equals $-\mathrm{Lie}_{X^1_c}(D^a_b(\lambda))$.

Furthermore, thanks to the affinity of the GZ manifold, we see that all terms of the form $\mathrm{Lie}_{Z_c}(D^a_b(\lambda))$ identically vanish.

So, we see that A.4 can be written as

$$P_\lambda dD^a_b(\lambda) - \sum_c D^a_c(\lambda) P_0 dD^c_{b,1} - \sum_c (X^1_c \wedge Z_c) \cdot (dD^a_b(\lambda)) = 0, \tag{A.5}$$

which, in view of (3.5), yields the statement.

□

Proposition 5.1 *Let D be a $k \times k$ polynomial matrix of the form*

$$D^a_b(\lambda) = \lambda^{n_a}\delta^a_b - \lambda^{n_a-1} D^a_{b,1} - \cdots - D^a_{b,n_a},\ a, b = 1, \ldots, k, \tag{A.6}$$

where the $D^a_{b,p}$ are smooth independent functions on a bihamiltonian manifold M, satisfying equation (A.2). Then:

1. *Its determinant* $\Delta(\lambda)$ *has the form*

$$\Delta(\lambda) = \lambda^{\nu} - \Delta_1\lambda^{\nu-1} + \cdots + \Delta_{\nu}, \tag{A.7}$$

with $\nu = \sum_a n_a$ *and satisfies*

$$\widetilde{P}_1 d\Delta(\lambda) = \lambda P_0 d\Delta(\lambda) + \Delta(\lambda) P_0 d\Delta_1. \tag{A.8}$$

2. *The roots* λ_i *of* $\Delta(\lambda)$ *satisfy*

$$\widetilde{P}_1 d\lambda_i = \lambda_i d\lambda_i. \tag{A.9}$$

3. *Let* $\widetilde{D}(\lambda)$ *denote the classical adjoint matrix of* $D(\lambda)$*, and let* $[\widetilde{D}(\lambda)]_{a,c}$ *be non identically vanishing. Then any ratio* $\rho(\lambda) := [\widetilde{D}(\lambda)]_{a,b}/[\widetilde{D}(\lambda)]_{a,c}$ *of elements belonging to the* a*-th row of* $\widetilde{D}(\lambda)$*, evaluated at the roots* λ_i *of* $\Delta(\lambda)$ *satisfy the equation*

$$\widetilde{P}_1 d\rho(\lambda_i) = \lambda_i P_0 d\rho(\lambda_i). \tag{A.10}$$

Proof. The power expansion (A.6) simply states that the (a, a) entry of D^a_b is a monic degree n_a polynomial, while all other entries in the a-th row are of degree not exceeding $n_a - 1$. We preliminarily notice that

$$\Delta(\lambda) = \prod_{a=1}^{k} D^a_a(\lambda) + O(\lambda^{\nu-2}), \text{ whence } \Delta_1 = \sum_{a=1}^{k} D^a_{a,1}. \tag{A.11}$$

We multiply the matrix equation (A.2) say, on the left, by the classical adjoint $\widetilde{D}$, to get

$$\begin{gathered}\sum_c \widetilde{P}_1(\widetilde{D}^a_c d(\lambda) D^c_b(\lambda)) - \sum_c \lambda P_0 \widetilde{D}(\lambda)^a_c dD^c_b(\lambda) = \\ \sum_{c,d} \widetilde{D}^a_c(\lambda) D^c_d(\lambda) P_0 dD^d_{b,1}.\end{gathered} \tag{A.12}$$

Recalling that $\sum_c \widetilde{D}^a_c(\lambda) D^c_d(\lambda) = \delta_{ad}\Delta(\lambda)$ and

$$\sum_{a,c} \widetilde{D}^a_c(\lambda) dD^c_a(\lambda) = \mathrm{Tr}\widetilde{D}(\lambda) dD = d\Delta(\lambda),$$

taking the trace of the matrix equation A.12 and taking into account (A.11) we get the proof of the first item.

To prove item # 2, we first notice that, for any function f on M, the evaluation of a polynomial (or rational) function $F(\lambda)$ in the parameter λ, whose coefficients are themselves functions on M gives rise to a new function $F(f)$ on M. Its differential can be written as follows:

$$d(F(f)) = dF(\lambda)\Big|_{\lambda=f} + \frac{\partial F(\lambda)}{\partial \lambda}\Big|_{\lambda=f} df$$

To clarify the notations, the first term in the RHS of the above equations means the differential of $F(\lambda)$, with λ taken as a parameter, then evaluated for $\lambda = f$, and the second the partial derivative of $F(\lambda)$ w.r.t. λ, subsequently evaluated for $\lambda = f$. Keeping this proviso in mind, we consider now $F(\lambda) = \Delta(\lambda)$, and $f = \lambda_i$, $i = 1, \ldots, n$. We have:

$$d(\Delta(\lambda_i)) = d\Delta(\lambda)\Big|_{\lambda=\lambda_i} + \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} d\lambda_i, \tag{A.13}$$

where $d\Delta(\lambda) = -\sum_j \lambda^{p-j} d\Delta_j$. Taking into account the relation (A.13), we get

$$0 = (\widetilde{P}_1 - \lambda_i P_0) d\Delta(\lambda_i) = \Delta(\lambda_i) P_0 d\Delta_1 + \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} (\widetilde{P}_1 - \lambda_i P_0) d\lambda_i, \tag{A.14}$$

which implies the assertion, since

$$\Delta(\lambda_i) = 0, \text{ while } \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} \neq 0,$$

thanks to the fact that being the coefficients $\Delta_{a,i}$ functionally independent, the roots are generically simple.

The proof of the third assertion is basically contained in the proof of Proposition 8.4 of [11]. We limit ourselves to sketch it.

By using the relations (A.2) and (A.8), together with the defining relation $\widetilde{D}^a_c(\lambda) D^c_d(\lambda) = \delta_{ad} \Delta(\lambda)$, one arrives at the matrix equation

$$\sum_c \big(\widetilde{P}_1 d\widetilde{D}^a_c - \lambda P_0 d\widetilde{D}^a_c\big) D^c_b = \Delta(\lambda) P_0 \big(d\Delta_1 \delta^a_b - dD^a_{b,1}\big). \tag{A.15}$$

If $\boldsymbol{\sigma}$ denotes one row of the adjoint matrix $\widetilde{D}(\lambda)$, we can rewrite the above equation as

$$\big(\widetilde{P}_1 \boldsymbol{\sigma} - \lambda P_0 \boldsymbol{\sigma}\big) D(\lambda) = \Delta(\lambda) \cdot X$$

where X is the corresponding row of the RHS of (A.15). If we consider the normalized row $\boldsymbol{\rho} = \boldsymbol{\sigma}/\sigma_j$, we see that, since

$$\boldsymbol{\rho} D(\lambda) = \Delta(\lambda) \boldsymbol{\sigma}/\sigma_j,$$

it holds

$$\big(\widetilde{P}_1 \boldsymbol{\rho} - \lambda P_0 \boldsymbol{\rho}\big) D = \Delta(\lambda) \cdot Y, \tag{A.16}$$

for some suitable Y whose form is irrelevant here. Evaluating this equation for $\lambda = \lambda_i$, we see that

$$\big(\widetilde{P}_1 \boldsymbol{\rho} - \lambda P_0 \boldsymbol{\rho}\big) D(\lambda)\Big|_{\lambda=\lambda_i} = 0 \tag{A.17}$$

Taking into account that Δ has simple eigenvalues, we see that each row $(\widetilde{P}_1 d\boldsymbol{\rho} - \lambda P_0 d\boldsymbol{\rho})$, evaluated at $\lambda = \lambda_i$ must be proportional to the corresponding row of $\widetilde{D}(\lambda)$, that is, there must exist vector fields X' such that

$$(\widetilde{P}_1 d\boldsymbol{\rho} - \lambda P_0 d\boldsymbol{\rho})|_{\lambda=\lambda_i} = X' \cdot \boldsymbol{\rho}\Big|_{\lambda=\lambda_i}.$$

Since one element of $\boldsymbol{\rho}$ is normalized to 1, we thus see that X' must vanish, whence the thesis.

□

Appendix B : Proof of Proposition 3.5

The key point is the following observation on the (ordinary) Lie-Poisson brackets on a single copy of $M = sl(r)$. The Poisson bracket of two functions F, G on M, is given by $\{F,G\} = \mathrm{Tr}(\frac{\partial F}{\partial A}\cdot[A, \frac{\partial G}{\partial A}]) = -\mathrm{Tr}(A\cdot[\frac{\partial F}{\partial A}, \frac{\partial G}{\partial A}])$. Let A_{ij} denote the ij-th entry of A, and consider the family of $r-1$ vector fields on M defined by $W^\alpha = \frac{\partial}{\partial A_{r,\alpha}}$, $\alpha = 1, \ldots, r-1$, as well as the distribution $\mathcal{Z} \subset TM$ defined by the W^α.

We notice that differentials of functions vanishing along $\mathcal{Z}$ admit a very simple matrix representation. Indeed W^α is represented via its action on the matrix A as the elementary matrix $e_{\alpha,r}$, having 1 in the α-th place of the last column. So $\mathrm{Lie}_{W^\alpha}(F) = 0$ iff $\left(\frac{\partial F}{\partial A}\right)_{r,a} = 0, a = 1, \ldots, r-1$, i.e., iff $\frac{\partial F}{\partial A}$ lies in the lower maximal parabolic subalgebra $\mathfrak{p}_-$ of $sl(r)$.

Let now W denote any element in $\mathcal{Z}$, and let F, G be functions such that $\mathrm{Lie}_Z F = \mathrm{Lie}_Z G = 0$, and let us compute $\mathrm{Lie}_Z(\{F,G\})$. Thanks to the Leibniz property of the Lie derivative and the fact that Z is a constant vector field we have that

$$\mathrm{Lie}_W(\{F,G\}) = -\mathrm{Tr}\left(\mathrm{Lie}_W(A)\cdot\left[\frac{\partial F}{\partial A}, \frac{\partial G}{\partial A}\right]\right) \tag{B.1}$$

which vanishes as well since $\mathfrak{p}_-$ is indeed a Lie subalgebra of $sl(r)$.

In the case of the n-particle $sl(r)$ Gaudin model, whose phase space is parametrized by n matrices A_i, we consider the family of $n\cdot(r-1)$ vector fields defined by

$$\mathrm{Lie}_{W_i^\alpha}(A_1, \ldots, A_n) = (0, 0, \cdots, \underbrace{e_{\alpha,r}}_{\text{i-th place}}, \cdots, 0). \tag{B.2}$$

The distribution $\mathcal{Z}$ generated by these vector fields is generically transversal to the symplectic leaves of the Lie–Poisson product structure on $sl(r)^N$. We now prove that the space of functions vanishing along $\mathcal{Z}$ is a Poisson subalgebra for any *affine* Poisson tensor Q. The brackets $\{F,G\}_Q = \langle dF, QdG\rangle$ are given by the multiple sum

$$\{F,G\}_Q = \sum_{i,j,k=1}^{N} \mathrm{Tr}\left(\frac{\partial F}{\partial A_i}\cdot\left(\sum_{k=1}^{N} c_{i,j}^k\left[A_k, \frac{\partial G}{\partial A_j}\right]\right) + d_{i,j}^k\left[\sigma_k, \frac{\partial G}{\partial A_j}\right]\right)$$

where σ_k denote constant matrices. Noticing that the differentials of functions F vanishing along $\mathcal{Z}$ are represented by n-tuples of matrices $dF = (\frac{\partial F}{\partial A_1}, \frac{\partial F}{\partial A_2}, \ldots, \frac{\partial F}{\partial A_n})$ with $\frac{\partial F}{\partial A_i} \in \mathfrak{p}_-$, $i = 1, \ldots, N$, we see that the Lie derivatives $\mathrm{Lie}_{W_i^\alpha}\{F,G\}_Q$ are given by multiple sums of terms like those of Eq. (B.1), and so vanish whenever $\mathrm{Lie}_{W_i^\alpha} F = \mathrm{Lie}_{W_i^\alpha} G = 0$.

$\square$

Appendix C : Proof of Proposition 3.6

The proof of Proposition 3.6 follows from a few elementary but important facts following from the definitions of the Casimirs of the Poisson pencil $K^{(i)}(\lambda)$ and of the normalized transversal vector fields $Z_i^\alpha, \alpha = 1, \ldots, r-1,\ i = 1, \ldots, n$. Recall that we defined n^2 matrices D_{ij} by

$$\big(D_{ij}\big)^\alpha_\beta = \mathrm{Lie}_{Z_j^\beta}(K_\alpha^{(i)}(\lambda)), \tag{C.1}$$

where the polynomial Casimirs $K_\alpha^{(i)}$ are defined by

$$\mathrm{Det}(M_i(\lambda, \xi)) = \xi^r + \sum_{\alpha=1}^{r-1} K_\alpha^{(i)}(\lambda)\xi^{r-\alpha-1}. \tag{C.2}$$

One has:

1. The matrix $\big(D_{11}\big)$ is the identity. This trivially follows form our choice of $L_1 = A_1$.

2. The $n(r-1) \times n(r-1)$ matrix of the deformations of the Casimirs w.r.t. the transversal vector fields has the following block form:

$$\mathbb{D} = \begin{bmatrix} D_{11} & 0 & \cdots & \cdots & 0 \\ D_{21} & D_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & 0 \\ D_{n1} & D_{k3} & \cdots & & D_{nn} \end{bmatrix} \tag{C.3}$$

   This follows form the fact that, for $j > i$ and every α, $\mathrm{Lie}_{Z_j^\alpha} L_j(\lambda) = 0$. Thanks to point 1 above, we now consider the non trivial matrices $M_a(\lambda, \xi),\ a = 2, \ldots, n$, and the corresponding fields Z_b^α.

   Taking into account that:

   i) $\mathrm{Lie}_{Z_b^\alpha}(\mathrm{Det}(M_a(\lambda, \xi))) = -\mathrm{Tr}(\widetilde{M}_a(\lambda, \xi)\mathrm{Lie}_{Z_b^\alpha}(L_a))$;

   ii) $\mathrm{Lie}_{Z_a^\alpha}(L_a) = (\lambda - a + 2)\mathrm{Lie}_{Z_a^\alpha}(A_a)$;

   iii) The determinants of the diagonal blocks $\mathrm{Det}D_{aa} = \mathcal{D}^a(\lambda)$ are monic polynomials of degree $r(r+1)/2 - 1$ in λ;

   We can factorize $\mathcal{D}^a(\lambda)$ as

$$\mathcal{D}^a(\lambda) = (\lambda - a + 2)^{r-1}\Delta^a(\lambda) \tag{C.4}$$

   where $\Delta^a(\lambda)$ is a monic polynomial of degree $r(r-1)/2$.

3. Thanks to the lower diagonal block form of the matrix $\mathbb{D}$ of equation (C.3), *every* diagonal block D_{aa} satisfies Proposition (1). So its determinant satisfy, according to Proposition 5.1 $(\widetilde{R}-\lambda P)d\mathcal{D}^a(\lambda)=\mathcal{D}^a(\lambda)Pd\mathcal{D}^a_1$, and thanks to the factorization property (C.4), we have

$$(\widetilde{R}-\lambda P)d\Delta^a(\lambda)=\Delta^a(\lambda)Pd\Delta^a_1. \tag{C.5}$$

We now recall that the Casimir functions of P are given by the highest order terms $K^{(i)}_{\alpha,0}$ of the expansion of the Casimirs of the Poisson pencil $K^{(i)}_\alpha(\lambda)$ in powers of λ. If we call

$$G_i^{\alpha,\beta}=\mathrm{Lie}_{W_i^\alpha}K^{(i)}_{\beta,0} \tag{C.6}$$

the matrix of the deformations of the Casimirs of P with respect to the corresponding vector fields W_i^α introduced in eq. (3.10), noticing that $\mathrm{Lie}_{W_i^\alpha}K^{(j)}_{\beta,0}$ vanishes for $j\neq i$, we see that the normalized generators Z_i^α and the "constant" ones W_i^α are related by

$$W_i^\alpha=\sum_\beta G_i^{\alpha,\beta}Z_i^\beta \ \ i=1,\ldots,n.$$

Thus, considering only the nontrivial indexes $a=2,\ldots,n$,

$$\mathrm{Lie}_{W_a^\alpha}\mathrm{Det}(M_a(\xi,\lambda))=\sum_\beta G_a^{\alpha,\beta}\mathrm{Lie}_{Z_a^\alpha}\mathrm{Det}(M_a(\xi,\lambda)), \tag{C.7}$$

we can argue as follows. Since $G_a^{\alpha,\beta}$ are independent of λ, we see that the common solutions (ξ_a,λ_a) of the two sets of $r-1$-tuple of equations

$$\begin{cases}\mathrm{Lie}_{W_a^2}\mathrm{Det}(M_a(\xi,\lambda))\\ \qquad\qquad\vdots\\ \mathrm{Lie}_{W_a^r}\mathrm{Det}(M_a(\xi,\lambda))\end{cases} \tag{C.8}$$

and

$$\begin{cases}\mathrm{Lie}_{Z_a^2}\mathrm{Det}(M_a(\xi,\lambda))\\ \vdots\\ \mathrm{Lie}_{Z_a^r}\mathrm{Det}(M_a(\xi,\lambda))\end{cases} \tag{C.9}$$

coincide for every (fixed) $a=2,\ldots,n$.

Expanding the RHS of the equations of the system (C.9) as

$$\mathrm{Lie}_{Z_a^\alpha}\mathrm{Det}(M_a(\xi,\lambda))=\sum_{\beta=1}^{r-1}\xi^{r-1-\beta}\mathrm{Lie}_{Z_a^\alpha}K^{(a)}_\beta=\sum_{\beta=1}^{r-1}\xi^{r-\beta}(D_a)^\alpha_\beta, \tag{C.10}$$

we see that:

a) The roots of $\mathcal{D}^a(\lambda)$, introduced in eq. (C.4) are those values of λ for which the $r-1$ equations, defined for $a = 2, \ldots, n$,

$$\sum_{\beta=1}^{r-1} \xi^{r-1-\alpha} (D_a)^\alpha_\beta = 0 \qquad \text{(C.11)}$$

admit solutions. In particular, the roots of $\Delta^a(\lambda)$ define non trivial elements λ^s_a, $s = 1, \ldots, r(r-1)/2$.

b) The values ξ^s_a corresponding to the roots λ^s_a of $\Delta^a(\lambda)$ are given by suitably normalized elements of the adjoint matrix $\widetilde{D}_a$, evaluated at $\lambda = \lambda^s_a$.

Thus, from the bihamiltonian theory, we can conclude that the only non vanishing Poisson brackets between such functions admit the separate form:

$$\{\lambda^s_b, \xi^t_a\}_P = \delta^{st}\delta_{ab}\varphi^s_a(\xi^s_a, \lambda^s_a), \qquad \{\lambda^s_b, \xi^t_a\}_{\widetilde{R}} = \delta^{st}\delta_{ab}\lambda^s_a\varphi^s_a(\xi^s_a, \lambda^s_a) \qquad \text{(C.12)}$$

We now consider eq. (C.8), taking into account the observation that the pairs (λ^s_b, ξ^t_a) are solutions of this system as well. We notice that from the definition of W^α_a, $\mathrm{Lie}_{W^\alpha_a}\mathrm{Det}(M_a(\xi, \lambda))$ is nothing but the determinant of the minor of $M_a(\xi, \lambda)$ relative to the α, r entry. Since the $r(r-1)/2$ pairs (ξ^a_i, λ^a_i), for every fixed a, annihilate the $r-1$ minors of the matrix $M_a(\xi, \lambda)$ relative to the entries $(1, r), \ldots (r-1, r)$, they annihilate the minor relative to the (r, r) entry as well. Hence they annihilate the last row of the adjoint matrix of $M_a(\xi, \lambda)$, and so satisfy the characteristic equation

$$\mathrm{Det}\big(M_a(\xi^s_a, \lambda^s_a)\big) = 0, \quad a = 2, \ldots, n, \ s = 1, \ldots, r(r-1)/2.$$

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