Large Parameter Behavior of Equilibrium Measures

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Abstract: We study the equilibrium measure for a logarithmic potential in the presence of an external field $V_*(\xi) + t\, p(\xi)$, where t is a parameter, $V_*(\xi)$ is a smooth function and $p(\xi)$ a monic polynomial. When $p(\xi)$ is of an odd degree, the equilibrium measure is shown to be supported on a single interval as $|t|$ is sufficiently large. When $p(\xi)$ is of an even degree, the equilibrium measure is supported on two disjoint intervals as t is negatively large; it is supported on a single interval for convex $p(x)$ as t is positively large and is likely to be supported on multiple disjoint intervals for non-convex $p(x)$.

The support of the equilibrium measure shrinks to isolated points as $|t| \to +\infty$ in all the cases that we consider. For sufficiently large $|t|$, each topological component of the support contains a local minimizing point of the external field $V_*(\xi) + t\ p(\xi)$; a "potential well" phenomenon.

§ 1 Introduction

In this paper, we study the following minimization problem with constraints

$$\underset{\{\psi\geq 0,\, \int\psi d\xi=1\}}{\text{Minimize}} \left[-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \log|\xi-\eta|\psi(\xi)\psi(\eta)d\xi d\eta + \int_{-\infty}^{+\infty} V(\xi)\psi(\xi)d\xi \right] . \quad (1.1)$$

The external field $V(\xi)$ is a C^∞ function that satisfies

$$\lim_{\xi\to\pm\infty} \frac{V(\xi)}{log(1+\xi^2)} = +\infty . \quad (1.2)$$

Under this condition, the existence and uniqueness of the minimizer for (1.1) has been established [20]. The measure $\psi(\xi)d\xi$, where $\psi(\xi)$ is the minimizer of (1.1), is called the equilibrium measure under the external field $V(\xi)$.

Equilibrium measures find applications in many branches of mathematical sciences. It is used to describe the partition function of the Hermitian one-matrix model in random matrix theory of statistical physics [2, 18]. It is also intrinsically connected to the free energy of the Yang-Mills theory [10]. Finally, it plays an important role in orthogonal polynomials and approximation theory [20].

Although its importance to physics and approximation theory ([9, 20]) is well known, the minimization problem (1.1) is not well understood. The minimizer is explicitly known only for a few cases where the external fields $V(\xi)$ are the simplest polynomials [1, 7, 20]. It is therefore desirable to study the minimization problem for much more general C^∞ external fields.

Our method to solve the minimization problem (1.1) is the same as one [12] we used to solve a similar minimization problem for the zero dispersion limit of the KdV equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \quad \text{with } u(x,0;\epsilon) = u_0(x) ,$$

where the initial data $u_0(x)$ is a bounded decreasing function. The weak limit of the KdV solution $u(x,t;\epsilon)$ as $\epsilon\to 0$ is determined by a minimization problem [15, 16, 23]

$$\underset{\{\psi\geq 0,\ \psi\in L^1\}}{\text{Minimize}} \left[-\frac{1}{2\pi}\iint \log\left|\frac{\xi-\eta}{\xi+\eta}\right|\psi(\xi)\psi(\eta)d\xi d\eta + \int V(\xi,x,t)\psi(\xi)d\xi \right] , \quad (1.3)$$

where the space-time dependence is through the function $\tilde{V}$

$$\tilde{V}(\xi, x, t) = x\xi - 4\xi^3 t - \theta(\xi) \tag{1.4}$$

and $\theta(\xi)$ is encoded with the initial information $u_0(x)$. We found that the problem (1.3) is intrinsically connected to the Euler-Poisson-Darboux equations [12]. We utilized the solution of the equations to study the minimizer and hence the weak limit of the KdV solution.

The minimizer of (1.1) has been known to have a compact support if the external field $V(\xi)$ satisfies condition (1.2). The support is usually a union of a finite or infinite number of disjoint closed intervals. In this paper, we will study how the number of gaps in the support of the minimizer varies with respect to the external field. We would like to know whether there is any universality on the number of gaps.

We shall consider parameter-dependent external field $V(\xi) = V_*(\xi) + t\, p(\xi)$, where t is a parameter, $V_*(\xi)$ is a smooth function of ξ and $p(\xi)$ a monic polynomial. This is motivated by the space-time dependence of $\tilde{V}$ of (1.4) in the KdV minimization problem (1.3). We shall show that when $p(\xi)$ is of an odd degree, the minimizer of (1.1) has no gap in its compact support as $|t|$ is sufficiently large. When $p(\xi)$ is of an even degree, the minimizer has a single gap in its support as t is negatively large; it has no gap for convex $p(\xi)$ as t is positively large and is likely to have multi-gaps for non-convex $p(\xi)$. We note that the same minimizer may have an arbitrary number of gaps when $|t|$ is not large. A similar result has also been discovered in the case of the KdV weak limit [22]. Namely, the KdV weak limit is of zero or single phase for all x when t is sufficiently large, i.e., the minimizer of (1.1) has either zero or one gap in its support.

The support of the minimizer shrinks to isolated points as $|t| \to +\infty$ in all the cases that we consider. Each topological component of the support contains a local minimizing point of the external field $V(\xi)$; a "potential well" phenomenon.

The organization of the paper is as follows.

In Section 2, we will use function theoretical methods to solve the minimization problem. We will formulate the minimizer in terms of solutions of the Euler-Poisson-Darboux equations.

In Section 3, we will use the Euler-Poisson-Darboux solutions to study the behavior of the minimizer when the parameter in the external field is sufficiently large.

§ 2 Solution of the Minimization Problem

In this section, we will use the method that we have developed for the KdV zero dispersion limit [12] to solve the minimization problem (1.1).

Introducing a linear operator

$$L\psi(\xi) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \log|\xi - \mu| \psi(\mu) d\mu \ , \tag{2.1}$$

we rewrite the quadratic functional of (1.1) as $-\frac{1}{2} < L\psi, \psi > + < V, \psi >$. Here $< \ >$ is the standard L^2 inner product. The Euler-Lagrange equations take the form

$$L\psi(\xi) - V(\xi) \ = \ l \quad \text{where } \psi > 0 \ , \tag{2.2}$$

$$L\psi(\xi) - V(\xi) \ \le \ l \quad \text{where } \psi = 0 \ , \tag{2.3}$$

where l is the Lagrange multiplier. It can be shown [6] that ψ is the minimizer iff ψ is a nonnegative function that satisfies variational conditions (2.2-2.3) and the constraint

$$\int_{-\infty}^{+\infty} \psi(\xi) d\xi = 1 \ . \tag{2.4}$$

We make the ansatz that the support of ψ consists of a finite union of disjoint intervals. One denotes $I = \{\xi;\ \psi > 0\}$ and writes

$$I = \cup_{k=1}^{g+1} (u_{2k}, u_{2k-1}) \ , \tag{2.5}$$

where $u_{2g+2} < \cdots < u_2 < u_1$. Hence, the support is the closure of I.

We now consider a slightly stronger version of (2.2) and (2.3),

$$L\psi(\xi) - V(\xi) \ = \ l \quad \text{on } I \ , \tag{2.6}$$

$$L\psi(\xi) - V(\xi) \ < \ l \quad \text{on } \mathbb{R}\backslash \bar{I} \ , \tag{2.7}$$

where $\bar{I}$ denotes the closure of I. Since $\psi \geq 0$, we must also have

$$\psi > 0 \quad \text{on } I \;, \tag{2.8}$$

$$\psi = 0 \quad \text{off } I \;. \tag{2.9}$$

Our strategy to construct the minimizer is to first find the solution ψ of equations (2.4), (2.6) and (2.9) and then impose inequalities (2.7) and (2.8) on ψ. Since it is a non-negative function and since it satisfies (2.2), (2.3) and (2.4), ψ will then be the minimizer.

To solve (2.4), (2.6) and (2.9), we use the fact that the operator L of (2.1) is connected to the Hilbert transform H on the real line.

$$L\psi(\xi) = \int_0^{\xi} H\psi(\tau)d\tau \;, \tag{2.10}$$

where

$$H\psi(\xi) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\psi(\mu)d\mu}{\xi - \mu} \;.$$

This makes equations (2.6) and (2.9) amenable to the Riemann-Hilbert technique in function theory.

Differentiating (2.6) with respect to ξ and using (2.10), one obtains

$$H\psi(\xi) = V'(\xi) \qquad \text{on I} \;. \tag{2.11}$$

To recover equation (2.6) from (2.11) by integration, one must have for $k = 1, 2, \cdots, g$,

$$\int_{u_{2k+1}}^{u_{2k}} [H\psi(\xi) - V'(\xi)]d\xi = 0 \;. \tag{2.12}$$

Recalling the relation of the Hilbert transform to analytic function, one can write for real ξ

$$\mathcal{G}^+(\xi) = \psi(\xi) + \sqrt{-1}H\psi(\xi) \;,$$

where $\mathcal{G}^+(\xi)$ is the boundary value on the real axis of a function

$$\mathcal{G}(z) = \frac{1}{\pi\sqrt{-1}} \int_{-\infty}^{+\infty} \frac{\psi(\mu)d\mu}{\mu - z} \;,$$

which is analytic in the upper half complex plane. In view of the constraint (2.4), we expand the Cauchy integral to obtain $\mathcal{G}(z) = -\frac{1}{\sqrt{-1}\pi z} + O(1/z^2)$ for large $|z|$.

Conditions (2.12) now take a new form

$$\int_{u_{2k+1}}^{u_{2k}} [Im\mathcal{G}^+(\xi) - V'(\xi)]d\xi = 0 \tag{2.13}$$

for $k = 1, 2, \cdots, g$.

Equations (2.9) and (2.11) then become a Riemann-Hilbert problem in function theory

$$Im\mathcal{G}^+(\xi) = V'(\xi) \qquad \text{on } I \ , \tag{2.14}$$

$$Re\mathcal{G}^+(\xi) = 0 \qquad \text{off } I \ , \tag{2.15}$$

where $\mathcal{G}(z)$ is analytic in the upper half complex plane. It follows from the Plemelj formula that

$$\tilde{\mathcal{G}}(z) = \frac{R(z,\vec{u})}{\pi}\int_I \frac{V'(\mu)}{R(\mu,\vec{u})(\mu - z)}d\mu \ ,$$

where $\vec{u}$ denotes $(u_1, u_2, \cdots, u_{2g+2})$ and $R(\xi,\vec{u}) = \sqrt{(\xi-u_1)(\xi-u_2)\cdots(\xi-u_{2g+2})}$, is a solution to this Riemann-Hilbert problem. Here $R(\xi,\vec{u})$ is set to be positive for $\xi > u_1$. It defines a Riemann surface with branch cuts along the set I of (2.5).

To derive the equations governing the endpoints u_1, u_2, $\cdots$, u_{2g+2}, one usually imposes condition (2.13) and the asymptotics $\mathcal{G}(z) = -\frac{1}{\sqrt{-1}\pi z} + O(1/z^2)$ for large $|z|$ [7, 13]. The former gives g equations and the latter results in $g+2$ moment conditions. Consequently, one obtains exactly $2g+2$ algebraic equations on u_1, u_2, $\cdots$, u_{2g+2}.

In this paper, we take a slightly different approach. We observe that the Riemann-Hilbert problem (2.14-2.15) has many other solutions. Indeed, it is obvious that

$$\mathcal{G}(z) = \frac{\frac{R^2(z,\vec{u})}{\pi}\int_I \frac{V'(\mu)}{R(\mu,\vec{u})(\mu-z)}d\mu + \sqrt{-1}Q(z)}{R(z,\vec{u})} \ , \tag{2.16}$$

where $Q(z)$ is an arbitrary polynomial with real coefficients, is also a solution. We choose polynomial $Q(z)$ such that

1. it is a polynomial of degree $2g+1$.

2. $\mathcal{G}(z) = -\frac{1}{\sqrt{-1}\pi z} + O(1/z^2)$ for large $|z|$.

3. conditions (2.13) are satisfied.

Hence, constraint (2.4) and conditions (2.6) and (2.9) are built in the construction of $\mathcal{G}$.

It is quite obvious that such a polynomial Q is unique.

We first analyze the boundary value of $\mathcal{G}(z)$ on the real axis. Necessarily, the Cauchy integral in the numerator of (2.16) will become a singular integral. Our key observation is that the latter is, more or less, the solution of a boundary value problem for the Euler-Poisson-Darboux equations.

Proposition 2.1. *The boundary value of* $\mathcal{G}(z)$ *at* $\xi \neq u_i$, $i = 1, 2, \cdots, 2g+2$

$$\mathcal{G}^+(\xi) = \frac{-2\sqrt{-1}R^2(\xi,\vec{u})\Phi_g(\xi, u_1, u_2, \cdots, u_{2g+1}) + \sqrt{-1}Q(\xi)}{R(\xi,\vec{u})} + \sqrt{-1}V'(\xi) \ , \quad (2.17)$$

where $\Phi_g(\xi, u_1, \cdots, u_{2g+2})$ *satisfies the Euler-Poisson-Darboux equations*

$$2(u_i - u_j)\frac{\partial^2 \Phi_g}{\partial u_i \partial u_j} = \frac{\partial \Phi_g}{\partial u_i} - \frac{\partial \Phi_g}{\partial u_j} \ , \quad (2.18)$$

$$2(\xi - u_i)\frac{\partial^2 \Phi_g}{\partial \xi \partial u_i} = \frac{\partial \Phi_g}{\partial \xi} - 2\frac{\partial \Phi_g}{\partial u_i} \ , \quad (2.19)$$

$$\Phi_g(u, u, \cdots, u) = \frac{1}{2(g+1)!}\frac{d^{g+2}V(u)}{du^{g+2}} \ . \quad (2.20)$$

We will omit the proof here, since it is similar to the proof of an analogous result on the KdV zero dispersion limit [12].

A formula for Q can be derived from the loop conditions (2.13) and asymptotics $\mathcal{G}(z) = -\frac{1}{\sqrt{-1}\pi z} + O(1/z^2)$ for large $|z|$. To achieve this, we introduce a sequence of polynomials

$$P_{g,n}(\xi,\vec{u}) = \xi^{g+n} + a_{g,1}\xi^{g+n-1} + \cdots + a_{g,g+n} \quad (2.21)$$

whose coefficients are uniquely determined by

$$\frac{P_{g,n}(\xi,\vec{u})}{R(\xi,\vec{u})} = \xi^{n-1} + O(\frac{1}{\xi^2}) \qquad \text{for large } |\xi| \ , \quad (2.22)$$

and

$$\int_{u_{2k+1}}^{u_{2k}} \frac{P_{g,n}(\xi,\vec{u})}{R(\xi,\vec{u})} d\xi = 0 \qquad k = 1, 2, \cdots, g \ . \tag{2.23}$$

Proposition 2.2.

$$\begin{aligned} Q(\xi,\vec{u}) &= -\sum_{i=1}^{2g+2} [\prod_{l=1, l\neq i}^{2g+2} (\xi - u_l)] \Psi_g(u_i,\vec{u}) + \frac{1}{\pi} P_{g,0}(\xi,\vec{u}) + c_1 P_{g,1}(\xi,\vec{u}) \\ &+ \cdots + c_g P_{g,g}(\xi,\vec{u}) \ . \end{aligned}$$

Here $\Psi_g(\xi, u_1, u_2, \cdots, u_{2g+2})$ satisfies the Euler-Poisson-Darboux equations (2.18) and (2.19) with the diagonal boundary value

$$\Psi_g(u, u, \cdots, u) = \frac{1}{2(g+1)!} \frac{d^{g+1} V(u)}{du^{g+1}} \ . \tag{2.24}$$

The coefficients

$$c_k = 2k \sum_{l=0}^{g-k} \Gamma_l(\vec{u}) q_{g,k+l}(\vec{u}) \qquad k = 1, 2, \cdots, g,$$

where $\Gamma_l(\vec{u})$'s come from the expansion

$$R(\mu,\vec{u}) = \mu^{g+1}[\Gamma_0(\vec{u}) + \frac{\Gamma_1(\vec{u})}{\mu} + \frac{\Gamma_2(\vec{u})}{\mu^2} + \cdots] \ . \tag{2.25}$$

The function $q_{g,k}$ is

$$q_{g,k}(\vec{u}) = \frac{1}{2\pi\sqrt{-1}} \int_I \frac{V(\mu)\mu^{g-k}}{R(\mu,\vec{u})} \, d\mu \ . \tag{2.26}$$

The proof is also similar to that of the KdV case [12].

The function $q_{g,k}(\vec{u})$ also satisfies equations of Euler-Poisson-Darboux type [12].

Finally, we postulate the boundedness of the minimizer, which is equal to the real part of $\mathcal{G}^+(\xi)$. The numerator in (2.17) must then vanish at the end points of set I

$$P(u_i,\vec{u}) = 0 \qquad i = 1, 2, \cdots, 2g+2 \ , \tag{2.27}$$

where

$$P(\xi,\vec{u}) = 2R^2(\xi,\vec{u})\Phi_g(\xi,\vec{u}) - Q(\xi,\vec{u}) \ . \tag{2.28}$$

Since it is of degree $2g+1$ in ξ and has $2g+2$ zeros because of (2.27), the polynomial $Q(\xi, \vec{u})$ must be identically zero. The real part of $\mathcal{G}^+$ of (2.17), on the solution $\vec{u}$ of equations (2.27), becomes

$$\psi(\xi) = -2Re\{\sqrt{-1}R(\xi, \vec{u})\}\Phi_g(\xi, \vec{u}) \ . \tag{2.29}$$

Hence, $\psi(\xi)$ of (2.29) satisfies conditions (2.4), (2.6) and (2.9).

Inequality (2.8) is equivalent to

$$Re\{\sqrt{-1}R(\xi, \vec{u})\}\Phi_g(\xi, \vec{u}) < 0 \quad \text{for } \xi \text{ on I} \ , \tag{2.30}$$

and (2.7) is equivalent to

$$\begin{aligned} \int_{u_{2k+1}}^{\xi} R(\mu, \vec{u})\Phi_g(\mu, \vec{u})d\mu &> 0 \quad \text{for } u_{2k+1} < \xi < u_{2k} \text{ and } 1 \le k \le 2g+2 \ , \\ \int_{u_1}^{\xi} R(\mu, \vec{u})\Phi_g(\mu, \vec{u})d\mu &> 0 \quad \text{for } \xi > u_1 \ , \\ \int_{\xi}^{u_{2g+2}} R(\mu, \vec{u})\Phi_g(\mu, \vec{u})d\mu &< 0 \quad \text{for } \xi < u_{2g+2} \ . \end{aligned} \tag{2.31}$$

We summarize the above results in the following theorem

Theorem 2.3. *If $(u_1, u_2, \cdots, u_{2g+2})$ satisfies equations (2.27) and if inequalities (2.30-2.31) are satisfied, then $\psi(\xi)$ of (2.29) is a non-negative function that satisfies variational conditions (2.2) and (2.3) and constraint (2.4); so $\psi(\xi)$ must be the minimizer of the minimization problem (1.1).*

§ 3 Large Parameter Results

In this section, we shall consider a one-parameter family of external field

$$V(\xi) = V_*(\xi) + t\ p(\xi) \ . \tag{3.1}$$

Here, t is a parameter, $p(\xi)$ is a monic polynomial of degree n and $V_*(\xi)$ is a $C^\infty(-\infty, +\infty)$ function. We are interested in the behavior of the equilibrium measure when $|t|$ is large.

Our strategy for studying the large parameter behavior of the minimizer of (1.1) is as follows. For simplicity, we will assume $V_*(\xi)$ to have a power function growth at $\xi = -\infty$ or $\xi = +\infty$. This allows us to use the scaling technique to study the equilibrium measure when the parameter t is sufficiently large.

§ 3.1 The degree of polynomial $p(\xi)$ is odd

We shall show that the equilibrium measure is supported on a single interval $[u_2, u_1]$ when $|t|$ is large. This corresponds to (2.5) for $g = 0$.

We will use Theorem 2.3 to construct the minimizer of (1.1). Hence, we need to solve equations (2.27) for $g = 0$ and verify inequalities (2.30-2.31).

We now consider the algebraic equations (2.27) for $g = 0$. The function $P(\xi, u_1, u_2)$ of (2.28) is

$$2(\xi - u_1)(\xi - u_2)\Phi_0(\xi, u_1, u_2) + (\xi - u_2)\Psi_0(u_1, u_1, u_2) + (\xi - u_1)\Psi_0(u_2, u_1, u_2) - \frac{1}{\pi} .$$

Equations (2.27) exactly become

$$(u_1 - u_2)\Psi_0(u_1, u_1, u_2) - \frac{1}{\pi} = 0 , \tag{3.2}$$

$$(u_2 - u_1)\Psi_0(u_2, u_1, u_2) - \frac{1}{\pi} = 0 , \tag{3.3}$$

which are equivalent to

$$(u_1 - u_2)\sqrt{\frac{\partial}{\partial u_2}\Psi_0(u_1, u_1, u_2)} - \sqrt{\frac{1}{\pi}} = 0 , \tag{3.4}$$

$$\Psi_0(u_1, u_1, u_2) + \Psi_0(u_2, u_1, u_2) = 0 . \tag{3.5}$$

In the derivation of (3.4), we have used an identity

$$\Psi_0(u_1, u_1, u_2) - \Psi_0(u_2, u_1, u_2) = 2(u_1 - u_2)\frac{\partial}{\partial u_2}\Psi_0(u_1, u_1, u_2) . \tag{3.6}$$

This is easily verified by calculating

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left[2(\xi - u_2)\frac{\partial}{u_2}\Psi_0(\xi, u_1, u_2) - \Psi_0(\xi, u_1, u_2)\right] \\ &= \; 2(\xi - u_2)\frac{\partial^2}{\partial u_2 \partial \xi}\Psi_0 + 2\frac{\partial}{\partial u_2}\Psi_0 - \frac{\partial}{\partial \xi}\Psi_0 . \end{aligned}$$

The right hand side vanishes because Ψ_g satisfies the Euler-Poisson-Darboux equation (2.19). The function in the parenthesis is then independent of ξ; so we obtain

$$2(\xi - u_2)\frac{\partial}{\partial u_2}\Psi_0(\xi, u_1, u_2) - \Psi_0(\xi, u_1, u_2) = -\Psi_0(u_2, u_1, u_2) \ .$$

Letting $\xi = u_1$ yields the identity (3.6).

We will rewrite equation (3.5) in another useful form. Its left hand side is a function of u_1 and u_2. Denote this function by $H(u_1, u_2)$. Since $\Psi_0(\xi, u_1, u_2)$ satisfies equations (2.18-2.19), we derive an equation for $H(u_1, u_2)$

$$2(u_1 - u_2)\frac{\partial^2 H}{\partial u_1 \partial u_2} = \frac{\partial H}{\partial u_1} - \frac{\partial H}{\partial u_2} \ .$$

The boundary condition (2.24) for $\Psi_0(\xi, u_1, u_2)$ implies $H(u, u) = V'(u)$. We then use formula (B.3) to obtain

$$H(u_1, u_2) = \frac{1}{\pi}\int_{u_2}^{u_1} \frac{V'(\mu) d\mu}{\sqrt{(u_1 - \mu)(\mu - u_2)}} \ .$$

Hence, equation (3.5) is equivalent to

$$\int_{u_2}^{u_1} \frac{V'(\mu) d\mu}{\sqrt{(u_1 - \mu)(\mu - u_2)}} = 0 \ . \tag{3.7}$$

We shall study the case when t is negatively large in detail. The other case when t is positively large can be handled in the same way.

We now make assumptions on $V_*(\xi)$. Since $V(\xi) = V_*(\xi) + t\ p(\xi)$ satisfies the growth condition (1.2), $V_*(\xi)$ must grow faster than $p(\xi)$ as $\xi \to +\infty$ because of $t < 0$. In contrast, we will make a very mild assumption on $V_*(\xi)$ as $\xi \ll -1$

$$V_*(\xi) = C_+ \xi^{M_+} + h_+(\xi) \qquad \text{for } \xi \gg 1 \ , \tag{3.8}$$

$$V_*'(\xi) \leq 0 \qquad \text{for } \xi \ll -1 \ . \tag{3.9}$$

Here C_+ is a positive constant and M_+ is a positive constant that is bigger than n and 2. The number n is the degree of the polynomial $p(\xi)$. Function $h_+(\xi)$ has an order less than n. More generally, we assume

$$\lim_{\xi \to +\infty} \frac{h_+'''(\xi)}{\xi^{M_+ - 3}} = 0 \ . \tag{3.10}$$

This immediately implies

$$\lim_{\xi\to+\infty}\frac{h_+(\xi)}{\xi^{M_+}}=0\ ,\quad \lim_{\xi\to+\infty}\frac{h'_+(\xi)}{\xi^{M_+-1}}=0\ ,\quad \lim_{\xi\to+\infty}\frac{h''_+(\xi)}{\xi^{M_+-2}}=0\ . \tag{3.11}$$

We will solve equations (3.4) and (3.5) for large u_1 and u_2. This is motivated by the fact that $V(\xi)$ has a critical point between u_2 and u_1 in view of equation (3.7) and that the minimizing point of $V(\xi)=V_*(\xi)+t\ p(\xi)$ moves to $+\infty$ as $t\to-\infty$.

We will first consider the case $p(\xi)=\xi^n$.

We will split Ψ_0 of (3.4-3.5) into simpler terms. In view of its boundary data (2.24) for $g=0$, Ψ_0 depends linearly on V. The decomposition of $V=t\xi^n+C_+\xi^{M_+-1}+h_+(\xi)$ allows us to write

$$\Psi_0=t\Psi_{\xi^n}+C_+\Psi_{\xi^{M_+}}+\Psi_{h_+(\xi)}\ . \tag{3.12}$$

In view of the integral formula (B.4), Ψ_{ξ^n} and $\Psi_{\xi^{M_+}}$ are homogeneous functions of (ξ,u_1,u_2) of orders $n-1$ and M_+-1, respectively.

Substituting (3.12) into equation (3.4) and then dividing it by $|t|^{\frac{M_+}{2(M_+-n)}}$, we use the homogeneity of Ψ_{ξ^n} and $\Psi_{\xi^{M_+}}$ to obtain

$$\begin{aligned}(U_1-U_2)\{-&\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U_1,U_1,U_2)+C_+\frac{\partial}{\partial U_2}\Psi_{\xi^{M_+}}(U_1,U_1,U_2)\\&+\frac{1}{|t|^{\frac{M_+-2}{M_+-n}}}\frac{\partial}{\partial u_2}\Psi_{h_+(\xi)}(u_1,u_1,u_2)\}^{\frac{1}{2}}-\frac{1}{\sqrt{\pi}|t|^{\frac{M_+}{2(M_+-n)}}}=0\ .\end{aligned} \tag{3.13}$$

Here U_1 and U_2 are

$$U_1=\frac{u_1}{|t|^{\frac{1}{M_+-n}}}\ ,\quad U_2=\frac{u_2}{|t|^{\frac{1}{M_+-n}}}\ . \tag{3.14}$$

Similarly, substituting (3.12) into equation (3.5) and then dividing it by $|t|^{\frac{M_+-1}{M_+-n}}$, we obtain

$$\begin{aligned}&-\Psi_{\xi^n}(U_1,U_1,U_2)+C_+\Psi_{\xi^{M_+}}(U_1,U_1,U_2)+\frac{1}{|t|^{\frac{M_+-1}{M_+-n}}}\Psi_{h_+(\xi)}(u_1,u_1,u_2)\\&\quad-\Psi_{\xi^n}(U_2,U_1,U_2)+C_+\Psi_{\xi^{M_+}}(U_2,U_1,U_2)+\frac{1}{|t|^{\frac{M_+-1}{M_+-n}}}\Psi_{h_+(\xi)}(u_2,u_1,u_2)=0\ .\end{aligned} \tag{3.15}$$

It follows from the behavior (3.10-3.11) of $h_+(\xi)$ that the three terms involving $\Psi_{h_+(\xi)}$ in (3.13-3.15), together with their first derivatives with respect to U_1 and U_2, decay to zero as $t \to -\infty$ if U_1 and U_2 of (3.14) are kept bounded.

We now denote $1/t$ by T and solve equations (3.13) and (3.15) for U_1 and U_2 as functions of T in the neighborhood of $T = 0$.

First, equations (3.13) and (3.15) at $T = 0$ become

$$\begin{aligned}
(U_1 - U_2)\sqrt{-\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U_1,U_1,U_2) + C_+\frac{\partial}{\partial U_2}\Psi_{\xi^{M_+}}(U_1,U_1,U_2)} &= 0 \ , \qquad (3.16)\\
-\Psi_{\xi^n}(U_1,U_1,U_2) + C_+\Psi_{\xi^{M_+}}(U_1,U_1,U_2) & \\
-\Psi_{\xi^n}(U_2,U_1,U_2) + C_+\Psi_{\xi^{M_+}}(U_2,U_1,U_2) &= 0 \ . \qquad (3.17)
\end{aligned}$$

These two equations have a solution $U_1 = U_2 = U^*$, where U^* is determined by

$$-\Psi_{\xi^n}(U^*,U^*,U^*) + C_+\Psi_{\xi^{M_+}}(U^*,U^*,U^*) = \frac{1}{2}\frac{d}{d\xi}[-\xi^n + C_+\xi^{M_+}]|_{\xi=U^*} = 0 \ ,$$

where we have used the boundary condition (2.24) for Ψ_{ξ^n} and $\Psi_{\xi^{M_+}}$. We hence obtain

$$U_1 = U_2 = U^* = [\frac{n}{C_+M_+}]^{\frac{1}{M_+-n}}$$

as a solution of (3.16-3.17).

Second, we calculate the Jacobian of equations (3.13) and (3.15) at $U_1 = U_2 = U^*$ and $T = 0$. Denote the left hand side of (3.13) by $F_1(U_1,U_2,T)$ and that of (3.15) by $F_2(U_1,U_2,T)$. Since Ψ_g satisfies the Euler-Poisson-Darboux equations (2.18), (2.19) and (2.24), we obtain

$$\begin{aligned}
\frac{\partial F_1}{\partial U_1} = -\frac{\partial F_1}{\partial U_2} &= \sqrt{-\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U^*,U^*,U^*) + C_+\frac{\partial}{\partial U_2}\Psi_{\xi^{M_+}}(U^*,U^*,U^*)}\\
&= \sqrt{\frac{1}{8}\frac{d^2}{d\xi^2}[-\xi^n + C_+\xi^{M_+}]|_{\xi=U^*}} > 0 \ ,\\
\frac{\partial F_2}{\partial U_1} = \frac{\partial F_2}{\partial U_2} &= 4[-\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U^*,U^*,U^*) + C_+\frac{\partial}{\partial U_2}\Psi_{\xi^{M_+}}(U^*,U^*,U^*)]\\
&= \frac{1}{2}\frac{d^2}{d\xi^2}[-\xi^n + C_+\xi^{M_+}]|_{\xi=U^*} > 0 \ .
\end{aligned}$$

Hence, the Jacobian of (3.13) and (3.15) is nonzero. Equations (3.13) and (3.15) then give U_1 and U_2 as functions of T near $T = 0$.

Therefore, equations (3.2-3.3) have a solution $u_1(t)$, $u_2(t)$ for negatively large t. The solution has the following asymptotics

$$u_1(t) \approx U^*|t|^{\frac{1}{M_+-n}} \ , \quad u_2(t) \approx U^*|t|^{\frac{1}{M_+-n}} \qquad \text{for } t \ll -1 \ .$$

Moreover, it follows from (3.13) and (3.14) that the length of the interval $[u_1(t), u_1(t)]$, the support of the minimizer, shrinks to zero as $t \to -\infty$.

To make sure that $\psi(\xi)$ of (2.29) is the minimizer, we also need to verify that inequalities (2.30-2.31) are satisfied.

We split Φ_0 in the fashion of (3.12)

$$\Phi_0 = t\Phi_{\xi^n} + C_+\Phi_{\xi^{M_+}} + \Phi_{h_+(\xi)} \ .$$

Using the scaling (3.14), we write $\Phi_0(\xi, u_1(t), u_2(t))$ as

$$|t|^{\frac{M_+-1}{M_+-n}}[-\Phi_{\xi^n}(\Xi, U_1, U_2) + C_+\Phi_{\xi^{M_+}}(\Xi, U_1, U_2) + \frac{1}{|t|^{\frac{M_+-1}{M_+-n}}}\Phi_{h_+(\xi)}(\xi, u_1, u_2)] \ , \qquad (3.18)$$

where $\Xi = \xi/|t|^{\frac{1}{M_+-n}}$. We use (3.11) to deduct that if Ξ is kept bounded, say, $U^*/3 \le \Xi \le 3U^*$, the last term in the parenthesis goes to zero uniformly as $t \to -\infty$. The sum in the parenthesis then has the limit

$$\begin{aligned}
&-\Phi_{\xi^n}(\Xi, U^*, U^*) + C_+\Phi_{\xi^{M_+}}(\Xi, U^*, U*) \\
&= \frac{\frac{d}{d\Xi}[-\Xi^n + C_+\Xi^{M_+}] - \frac{d}{d\xi}[-\xi^n + C_+\xi^{M_+}]|_{\xi=U^*}}{2(\Xi - U^*)} \\
&= \frac{\frac{d}{d\Xi}[-\Xi^n + C_+\Xi^{M_+}]}{2(\Xi - U^*)} \\
&= \frac{C_+M_+\Xi^{n-1}[\Xi^{M_+-n} - {U^*}^{M_+-n}]}{2(\Xi - U^*)} > 0 \quad \text{for } \tfrac{U^*}{3} \le \Xi \le 3U^* \ .
\end{aligned}$$

Here we have used formula (B.5) for Φ_{ξ^n} and $\Phi_{\xi^{M_+}}$ in the first equality and $U^* = [n/C_+M_+]^{1/(M_+-n)}$ in the last two equalities. In view of (3.18), we have the inequality

$\Phi_0(\xi, u_1(t), u_2(t)) > 0$ for $\frac{1}{2}u_2(t) \le \xi \le 2u_1(t)$ when t is negatively large. This proves the inequality (2.30).

To prove inequality (2.31), we first show that $\Phi_0(\xi, u_1(t), u_2(t)) > 0$ uniformly for $\xi < -1$ and $\xi > 2u_1(t)$ when t is negatively large.

We shall prove the uniform positivity for $\xi < -1$ first. We use formula (B.6) to write $\Phi_0(\xi, u_1(t), u_2(t))$ as

$$\begin{aligned}
&\frac{1}{2\sqrt{(\xi-u_1)(\xi-u_2)}}[V'(\xi) \\
&\quad - \frac{1}{\pi}\int_{u_2}^{u_1} \frac{\sqrt{(\xi-u_1)(\xi-u_2)}}{\xi-\mu}\frac{V'(\mu)d\mu}{\sqrt{(u_1-\mu)(\mu-u_2)}}] \\
&= \frac{1}{2\sqrt{(\xi-u_1)(\xi-u_2)}}[V'(\xi) \\
&\quad - \frac{1}{\pi}\int_{u_2}^{u_1} (\frac{\sqrt{(\xi-u_1)(\xi-u_2)}}{\xi-\mu} - 1)\frac{V'(\mu)d\mu}{\sqrt{(u_1-\mu)(\mu-u_2)}}] \ , \qquad (3.19)
\end{aligned}$$

where in the last equality we have used equation (3.7).

We now show that the integral of (3.19) tends to zero uniformly for $\xi < 0$ as $t \to -\infty$. Because of the inequality

$$|\frac{\sqrt{(\xi-u_1)(\xi-u_2)}}{\xi-\mu} - 1| \le \frac{u_1-u_2}{u_2-\xi} \le \frac{u_1-u_2}{u_2} \quad \text{for } \xi < 0 < u_2 < u_1 \ ,$$

the integral is bounded by

$$\frac{u_1-u_2}{u_2}\int_{u_2}^{u_1} \frac{|V'(\mu)|}{\sqrt{(u_1-\mu)(\mu-u_2)}}d\mu$$

for $\xi < 0$. This, in view of the scaling (3.14), is further bounded by a constant times

$$\frac{[U_1(t)-U_2(t)]^2}{u_2(t)}|t|^{\frac{M_+}{M_+-n}} \ ,$$

where we have used $V'(\mu) = O((u_1(t)-u_2(t))t^{\frac{M_+-2}{M_+-n}})$ for $u_2(t) \le \mu \le u_1(t)$. This asymptotics can be verified by observing that $V'(\mu)$ has a zero between $u_2(t)$ and $u_1(t)$ because of (3.7) and by expanding $V'(\mu)$ around the zero. Since $U_1(t) - U_2(t) = O(|t|^{-\frac{M_+}{2(M_+-n)}})$ because of (3.13), the integral of (3.19) tends to zero uniformly for $\xi < 0$ as $t \to -\infty$.

The first term $V'(\xi) = tn\xi^{n-1} + V_*'(\xi)$ in the parenthesis of (3.19) is bounded from above by a negative constant for $\xi \le -1$ when t is negatively large. This immediately follows from condition (3.9).

Therefore, $\Phi_0(\xi, u_1(t), u_2(t)) > 0$ uniformly for $\xi < -1$ when t is negatively large.

In the same way, we can prove that $\Phi_0(\xi, u_1(t), u_2(t)) > 0$ uniformly for $\xi > 2u_1(t)$ when t is negatively large. This together with $\Phi_0(\xi, u_1(t), u_2(t)) > 0$ for $\frac{1}{2}u_2(t) \le \xi \le 2u_1(t)$ proves the first half of (2.31).

It remains to prove the rest of (2.31), i.e.,

$$\int_{\xi}^{u_2(t)} \sqrt{(\mu - u_1(t))(\mu - u_2(t))}\ \Phi_0(\mu, u_1(t), u_2(t))d\mu < 0 \tag{3.20}$$

for $\xi < u_2(t)$. Since $\Phi_0(\mu, u_1(t), u_2(t)) > 0$ uniformly for $\xi < -1$ and $\xi \ge u_2(t)/2$ when t is negatively large, it suffices to prove the inequality for $-1 \le \xi \le u_2(t)/2$.

Using formula (B.6) for $\Phi_0(\xi, u_1(t), u_2(t))$, we write the left hand side of the inequality of (3.20) as half of

$$\begin{aligned} &-V(\xi) + V(u_2(t)) \\ &\quad - \frac{1}{\pi}\int_{\xi}^{u_2(t)} \sqrt{(\xi - u_1)(\xi - u_2)}\ [\int_{u_2}^{u_1} \frac{V'(\mu)d\mu}{(\xi - \mu)\sqrt{(u_1 - \mu)(\mu - u_2)}}]d\xi\ . \end{aligned} \tag{3.21}$$

The first two terms $-V(\xi) + V(u_2(t)) \le -V(u_2(t)/2) + V(u_2(t) < 0$ for $-1 \le \xi \le u_2(t)/2$. The whole third term, in view of the scaling (3.14), has an order in t lower than that of $-V(u_2(t)/2) + V(u_2(t))$. These prove that (3.21) is negative for $-1 \le \xi \le u_2(t)/2$ when t is negatively large. We have therefore proved (3.20).

Inequalities (2.30-2.31) have thus been verified. By Theorem 2.3, the equilibrium measure for the external field $V(\xi) = t\xi^n + V_*(\xi)$ has no gap in its support when t is negatively large enough.

The scaling nature of the above approach allows us to extend the result from $p(\xi) = \xi^n$ to an arbitrary monic polynomial of degree n.

Theorem 3.1. *Under conditions (3.8-3.10) on $V_*(\xi)$, the equilibrium measure for the external field $V(\xi, t) = V_*(\xi) + t\ p(\xi)$, where $p(\xi)$ is a monic polynomial of an odd degree n, has no gap in its support when t is negatively large enough.*

The case when t is positively large can be treated in the same way. Here, we replace conditions (3.8-3.11) on $V_*(\xi)$ by analogous ones

$$
\begin{aligned}
V_*(\xi) &= C_-|\xi|^{M_-} + h_-(\xi) \qquad &&\text{for } \xi \ll -1 \ , &(3.22)\\
V_*'(\xi) &\geq 0 &&\text{for } \xi \gg 1 \ , &(3.23)
\end{aligned}
$$

where $C_- > 0$ and $M_- > max\{2, n\}$. Function $h_+(\xi)$ has the following behavior

$$
\lim_{\xi \to -\infty} \frac{h_-'''(\xi)}{|\xi|^{M-3}} = 0 \ . \tag{3.24}
$$

Theorem 3.2. *Under conditions (3.22-3.24) on $V_*(\xi)$, the equilibrium measure for the external field $V(\xi,t) = V_*(\xi) + t\ p(\xi)$, where $p(\xi)$ is a monic polynomial of an odd degree n, has no gap in its support when t is positively large enough.*

§ 3.2 The degree of polynomial $p(\xi)$ is even

We shall show that the equilibrium measure for the $V_*(\xi) + t\ p(\xi)$ has a single gap in its support when t is negatively large. When t is positively large, there are many possibilities. The equilibrium measure has no gap when $p(\xi)$ is convex and may have multi-gaps when $p(\xi)$ is non-convex.

We will first study the case when t is negatively large. The minimizer will be supported on $[u_4, u_3] \cup [u_2, u_1]$. This corresponds to $g = 1$ in (2.5).

For simplicity, we assume that $V_*(\xi)$ of (3.1) is an even function. The evenness of $V(\xi) = t\xi^n + V_*(\xi)$ implies the evenness of the equilibrium measure $\psi(\xi)$. Consequently, its support must be symmetric about the origin. This means that $u_3 = -u_2$ and $u_4 = -u_1$. We therefore obtain from (2.28) that

$$
\begin{aligned}
P(\xi, \vec{u}) &= 2(\xi^2 - u_1^2)(\xi^2 - u_2^2)\Phi_1(\xi, \vec{u}) + (\xi + u_1)(\xi^2 - u_2^2)\Psi_1(u_1, \vec{u}) \\
&+ (\xi^2 - u_1^2)(\xi + u_2)\Psi_1(u_2, \vec{u}) + (\xi^2 - u_1^2)(\xi - u_2)\Psi_1(-u_2, \vec{u}) \\
&\quad +(\xi - u_1)(\xi^2 - u_2^2)\Psi_1(-u_1, \vec{u}) - \frac{1}{\pi}\xi \ .
\end{aligned}
$$

Equations (2.27) exactly become

$$\begin{aligned} 2u_1(u_1^2-u_2^2)\Psi_1(u_1,\vec{u})-\frac{1}{\pi}u_1 &= 0 \ , \\ 2u_2(u_2^2-u_1^2)\Psi_1(u_2,\vec{u})-\frac{1}{\pi}u_2 &= 0 \ , \end{aligned}$$

which are equivalent to

$$(u_1-u_2)\sqrt{2(u_1+u_2)}\sqrt{\frac{\partial\Psi_1(u_1,\vec{u})}{\partial u_2}}-\sqrt{\frac{1}{\pi}} = 0 \ , \tag{3.25}$$

$$\Psi_1(u_1,\vec{u})+\Psi_1(u_2,\vec{u}) = 0 \ . \tag{3.26}$$

We have used the following identity when deriving (3.25)

$$\Psi_1(u_1,\vec{u})-\Psi_1(u_2,\vec{u})=2(u_1-u_2)\frac{\partial}{\partial u_2}\Psi_1(u_1,\vec{u}) \ .$$

Its proof is the same as the one for a similar identity (3.6).

In view of the integral formula (B.10) for its left hand side, equation (3.26) is equivalent to

$$\int_{u_2}^{u_1}\frac{V'(\mu)d\mu}{\sqrt{(u_1^2-\mu^2)(\mu^2-u_2^2)}}=0 \ . \tag{3.27}$$

We further assume V_* satisfies the condition

$$V_*(\xi)=C|\xi|^M+h(\xi) \qquad \text{for } |\xi|\gg 1 \ , \tag{3.28}$$

where $C>0$ and $M>max\{3,n\}$. Here $h(\xi)$ satisfies

$$\lim_{\xi\to\pm\infty}\frac{h''''(\xi)}{\xi^{M-4}}=0 \ . \tag{3.29}$$

This implies

$$\lim_{\xi\to\pm\infty}\frac{h(\xi)}{\xi^M}=0 \ , \quad \lim_{\xi\to\pm\infty}\frac{h'(\xi)}{\xi^{M-1}}=0 \ , \quad \lim_{\xi\to\pm\infty}\frac{h''(\xi)}{\xi^{M-2}}=0 \ , \quad \lim_{\xi\to\pm\infty}\frac{h'''(\xi)}{\xi^{M-3}}=0 \ . \tag{3.30}$$

We will solve equations (3.25-3.26) for large u_1 and u_2.

Again, we will first consider $p(\xi)=\xi^n$.

We split Ψ_1 of (3.25-3.26) into simpler terms

$$\Psi_1=t\Psi_{\xi^n}+C\Psi_{\xi^M}+h(\xi) \ ,$$

where Ψ_{ξ^n} and Ψ_{ξ^M} are homogeneous functions of (ξ, u_1, u_2) of orders $n-2$ and $M-2$, respectively.

Introducing

$$U_1 = \frac{u_1}{|t|^{\frac{1}{M-n}}} \ , \quad U_2 = \frac{u_2}{|t|^{\frac{1}{M-n}}} \ , \tag{3.31}$$

we scale equations (3.25-3.26) as

$$\begin{aligned}
&(U_1 - U_2)\sqrt{2(U_1+U_2)}\{-\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U_1, U_1, U_2, -U_2, -U_1) \\
&\quad + C\frac{\partial}{\partial U_2}\Psi_{\xi^M}(U_1, U_1, U_2, -U_2, -U_1) \qquad (3.32)\\
&\quad + \frac{1}{|t|^{\frac{M-3}{M-n}}}\frac{\partial}{\partial u_2}\Psi_{h(\xi)}(u_1, u_1, u_2, -u_2, -u_1)\}^{\frac{1}{2}} - \frac{1}{\sqrt{\pi}|t|^{\frac{M}{2(M-n)}}} = 0 \ , \\
&-\Psi_{\xi^n}(U_1, U_1, U_2, -U_2, -U_1) + C\Psi_{\xi^M}(U_1, U_1, U_2, -U_2, -U_1) \\
&\quad + \frac{1}{|t|^{\frac{M-2}{M-n}}}\Psi_{h(\xi)}(u_1, u_1, u_2, -u_2, -u_1) - \Psi_{\xi^n}(U_2, U_1, U_2, -U_2, -U_1) \qquad (3.33)\\
&\quad + C\Psi_{\xi^M}(U_2, U_1, U_2, -U_2, -U_1) + \frac{1}{|t|^{\frac{M-2}{M-n}}}\Psi_{h(\xi)}(u_2, u_1, u_2, -u_2, -u_1) = 0 \ .
\end{aligned}$$

At $t = -\infty$, equations (3.32-3.33) become

$$\begin{aligned}
&(U_1 - U_2)\sqrt{2(U_1+U_2)}\{-\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U_1, U_1, U_2, -U_2, -U_1) \\
&\quad + C\frac{\partial}{\partial U_2}\Psi_{\xi^M}(U_1, U_1, U_2, -U_2, -U_1)\}^{\frac{1}{2}} = 0 \ , \qquad (3.34)\\
&-\Psi_{\xi^n}(U_1, U_1, U_2, -U_2, -U_1) + C\Psi_{\xi^M}(U_1, U_1, U_2, -U_2, -U_1) \\
&\quad - \Psi_{\xi^n}(U_2, U_1, U_2, -U_2, -U_1) + C\Psi_{\xi^M}(U_2, U_1, U_2, -U_2, -U_1) = 0 \ . \qquad (3.35)
\end{aligned}$$

They have a solution $U_1 = U_2 = \hat{U}$, where $\hat{U}$ is defined by

$$-\Psi_{\xi^n}(\hat{U}, \hat{U}, \hat{U}, -\hat{U}, -\hat{U}) + C\Psi_{\xi^M}(\hat{U}, \hat{U}, \hat{U}, -\hat{U}, -\hat{U}) = \frac{1}{4\hat{U}}\frac{d}{d\xi}[-\xi^n + C\xi^M]|_{\xi=\hat{U}} = 0 \ ,$$

where we have used formula (B.9) for Ψ_{ξ^n} and Ψ_{ξ^M} in the first equality. We hence obtain

$$U_1 = U_2 = \hat{U} = [\frac{n}{CM}]^{\frac{1}{M-n}}$$

as a solution of (3.34-3.35).

It is straight forward to use the Implicit Function Theory to determine the solution of equations (3.32-3.33) in the neighborhood of $U_1 = U_2 = \hat{U}$, $t = -\infty$. This in turn gives solution $u_1(t)$, $u_2(t)$ of (3.25-3.26) for $t \ll -1$. The solution has the asymptotics

$$u_1(t) \approx \hat{U}|t|^{\frac{1}{M-n}} \ , \quad u_1(t) \approx \hat{U}|t|^{\frac{1}{M-n}} \quad \text{for } t \ll -1 \ .$$

We now verify inequalities (2.30-2.31). We will first prove (2.30) and the last two inequalities of (2.31). It suffices to show that $\Phi_1(\xi, u_1(t), u_2(t), -u_2(t), -u_1(t))$ is negative for $\xi \leq -u_2(t)/2$ and positive for $\xi \geq u_2(t)/2$ when $t \ll -1$. Since $\Phi_1(\xi, u_1(t), u_2(t), -u_2(t), -u_1(t))$ is odd in ξ in view of formula (B.8), we only need to prove its positivity for $\xi \geq u_2(t)/2$.

We split Φ_1 into simpler terms and scale these terms

$$\begin{aligned}
&\Phi_1(\xi, u_1, u_2, -u_2, -u_1) \\
&= |t|^{\frac{M-3}{M-n}}[-\Phi_{\xi^n}(\Xi, U_1, U_2, -U_2, -U_1) + C\Phi_{\xi^M}(\Xi, U_1, U_2, -U_2, -U_1) \\
&\quad + \frac{1}{t^{\frac{M-3}{M-n}}}\Phi_{h(\xi)}(\xi, u_1, u_2, -u_2, -u_1)] \ ,
\end{aligned} \tag{3.36}$$

where $\Xi = \xi/|t|^{\frac{1}{M-n}}$. In view of condition (3.30) and the scaling (3.31), the last term in the parenthesis goes to zero uniformly for $\hat{U}/3 \leq \xi/|t|^{\frac{1}{M-n}} \leq 3\hat{U}$ as $t \to -\infty$. The sum in the parenthesis then has the limit

$$\begin{aligned}
&-\Phi_{\xi^n}(\Xi, \hat{U}, \hat{U}, -\hat{U}, -\hat{U}) + C\Phi_{\xi^M}(\Xi, \hat{U}, \hat{U}, -\hat{U}, -\hat{U}) \\
&= \frac{\hat{U}\frac{d}{d\Xi}[-\Xi^n + C\Xi^M] - \Xi\frac{d}{d\xi}[-\xi^n + C\xi^M]|_{\xi=\hat{U}}}{2\hat{U}(\Xi^2 - \hat{U}^2)} \\
&= \frac{CM\Xi^{n-1}[\Xi^{M-n} - \hat{U}^{M-n}]}{\Xi^2 - \hat{U}^2} \ ,
\end{aligned}$$

where we have used (B.7) in the first equality and $\hat{U} = [n/CM]^{1/M-n}$ in the last one. The limit is positive for $\hat{U}/3 \leq \Xi \leq 3\hat{U}$. It then follows from equation (3.36) that $\Phi_1(\xi, u_1(t), u_2(t), -u_2(t), -u_1(t))$ is positive for $u_2(t)/2 \leq \xi \leq 2u_1(t)$ when $t \ll -1$.

We now show that $\Phi_1(\xi, u_1(t), u_2(t), -u_2(t), -u_1(t)) > 0$ uniformly for $\xi \geq 2u_1(t)$.

We use formula (B.8) to write $\Phi_1(\xi, u_1, u_2, -u_2, -u_1)$ as

$$\begin{aligned}
& \frac{\xi}{\sqrt{(\xi^2-u_1^2)(\xi^2-u_2^2)}}[\frac{V'(\xi)}{2\xi} \\
& \qquad +\frac{1}{\pi}\int_{u_2}^{u_1} \frac{\sqrt{(\mu^2-u_1^2)(\mu^2-u_2^2)}}{(\mu^2-\xi^2)} \frac{V'(\mu)d\mu}{\sqrt{(u_1^2-\mu^2)(\mu^2-u_2^2)}}] \\
= \quad & \frac{\xi}{\sqrt{(\xi^2-u_1^2)(\xi^2-u_2^2)}}[\frac{V'(\xi)}{2\xi} \\
& \qquad +\frac{1}{\pi}\int_{u_2}^{u_1} (\frac{\sqrt{(\mu^2-u_1^2)(\mu^2-u_2^2)}}{\mu^2-\xi^2} - 1) \frac{V'(\mu)d\mu}{\sqrt{(u_1^2-\mu^2)(\mu^2-u_2^2)}}] \ ,
\end{aligned} \tag{3.37}$$

where we have used (3.27) in the equality.

It is then straight forward to use the argument below (3.19) to show that the integral of (3.37) tends to zero uniformly for $\xi \geq 2u_1(t)$ as $t \to -\infty$.

The first term $V'(\xi) = tn\xi^{n-1} + V_*'(\xi)$ in the parenthesis is bounded from below by a positive constant uniformly for $\xi \geq 2u_1(t)$ when $t \ll -1$. This follows from the conditions (3.28-3.30) on $V_*(\xi)$.

Function $\Phi_1(\xi, u_1(t), u_2(t), -u_2(t), -u_1(t))$ is therefore positive in view of (3.37). This together with the similar result for $u_2(t)/2 \leq \xi \leq 2u_1(t)$ proves that it is positive uniformly for $\xi \geq u_2(t)/2$ when $t \ll -1$.

We have therefore proved (2.30) and the last two inequalities of (2.31).

It remains to prove the rest of (2.31), i.e.,

$$\int_{-u_2(t)}^{\xi} \sqrt{(\mu^2-u_1(t)^2)(\mu^2-u_2(t)^2)}\ \Phi_1(\mu, u_1(t), u_2(t), -u_2(t), -u_1(t))d\mu > 0 \tag{3.38}$$

for $|\xi| < u_2(t)$ when $t \ll -1$.

First, the inequality (3.38) is valid for $-u_2(t) < \xi \leq -u_2(t)/2$ since Φ_1 is negative for $-2u_1(t) \leq \xi \leq -u_2(t)/2$. It suffices to prove the inequality for $-u_2(t)/2 \leq \xi \leq u_2(t)$.

We use formula (B.8) to write the integral of (3.38) as

$$\begin{aligned}
& \frac{V(\xi)-V(-u_2)}{2} \\
& \qquad +\frac{1}{\pi}\int_{-u_2}^{\xi} \sqrt{(\nu^2-u_1^2)(\nu^2-u_2^2)}\ [\int_{u_2}^{u_1} \frac{V'(\mu)d\mu}{(\mu^2-\nu^2)\sqrt{(u_1^2-\mu^2)(\mu^2-u_2^2)}}]\ \nu d\nu \ .
\end{aligned}$$

It is easy to use the scaling (3.31) to show that the above is positive for $-u_2(t)/2 < \xi \leq 0$; hence, inequality (3.38) is verified for $-u_2(t) < \xi \leq 0$. Since $\Phi_1(\xi, u_1, u_2, -u_2, -u_1)$ is odd in ξ in view of (B.8), the inequality (3.38) can be extended from $-u_2(t) < \xi \leq 0$ to $-u_2(t) < \xi \leq u_2(t)$. This completes the verification of inequalities (2.30-2.31).

By Theorem 2.3, the equilibrium measure for the external field $V(\xi) = t\xi^n + V_*(\xi)$ has a single gap in its support when t is negatively large.

It is also easy to extend the result from $p(\xi) = \xi^n$ to any even monic polynomial of degree n.

Theorem 3.3. *Suppose $V_*(\xi)$ is an even function that satisfies condition (3.28-3.29). The equilibrium measure for the external field $V(\xi, t) = V_*(\xi) + t\ p(\xi)$, where $p(\xi)$ is an even monic polynomial of degree n, has a single gap in its support when t is negatively large.*

For the case that t is positively large, we assume

$$V_*''(\xi) \geq 0 \qquad \text{for } |\xi| \gg 1\ . \tag{3.39}$$

Again, we first consider the case $p(\xi) = \xi^n$

Since $V_*''(\xi)$ is uniformly bounded in $|\xi| \leq A$ for some large constant A, we may choose t positively large enough so that $V(\xi) = t\xi^n + V_*(\xi)$ is a convex function for $|\xi| \geq \epsilon_0$, where ϵ_0 is a tiny positive number. It is well known that an everywhere convex external field has an equilibrium measure whose support is a connected finite interval [20]. Hence, if $V(\xi) = t\xi^n + V_*(\xi)$ is also convex in $|\xi| < \epsilon_0$ for large $t > 0$, the corresponding equilibrium measure has no gap in its support.

It is therefore interesting to note that, even if $V(\xi) = t\xi^n + V_*(\xi)$ is never convex in a small neighborhood of $\xi = 0$, the equilibrium measure can still be shown to have no gap in its support for large $t > 0$.

The approach is similar to one we use in § 3.1. We will present the proof briefly.

We still need to solve equations (3.2-3.3). Using

$$U_1 = u_1 t^{\frac{1}{n}}\ , \quad U_2 = u_2 t^{\frac{1}{n}}\ ,$$

we scale the equations as

$$(U_1 - U_2)\sqrt{\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U_1, U_1, U_2) + \frac{1}{t^{\frac{1}{n}}}\frac{\partial}{\partial u_2}\Psi_{V_*}(u_1, u_1, u_2)} - \sqrt{\frac{1}{\pi}} = 0 \ , \tag{3.40}$$

$$\begin{aligned}&\Psi_{\xi^n}(U_1, U_1, U_2)\\ &\quad + \frac{1}{t^{\frac{1}{n}}}\Psi_{V_*}(u_1, u_1, u_2) + \Psi_{\xi^n}(U_2, U_1, U_2) + \frac{1}{t^{\frac{1}{n}}}\Psi_{V_*}(u_2, u_1, u_2) = 0 \ .\end{aligned} \tag{3.41}$$

At $t = +\infty$, these equations become

$$\begin{aligned}(U_1 - U_2)\sqrt{\frac{\partial}{\partial U_2}\Psi_{\xi^n}(U_1, U_1, U_2)} - \sqrt{\frac{1}{\pi}} &= 0 \ ,\\ \Psi_{\xi^n}(U_1, U_1, U_2) + \Psi_{\xi^n}(U_2, U_1, U_2) &= 0 \ .\end{aligned}$$

They have a solution $U_1 = \tilde{U}, U_2 = -\tilde{U}$, where $\tilde{U}$ is defined by

$$2\tilde{U}\sqrt{\frac{\partial}{\partial U_2}\Psi_{\xi^n}(\tilde{U}, \tilde{U}, -\tilde{U})} - \sqrt{\frac{1}{\pi}} = 0 \ .$$

It is not hard to get from a formula of type (B.11) for Ψ_{ξ^n} that

$$\frac{\partial}{\partial U_2}\Psi_{\xi^n}(\tilde{U}, \tilde{U}, -\tilde{U}) = \frac{1}{4}\frac{(n-1)!!}{(n-2)!!}U^{n-2} \ .$$

We hence obtain

$$U_1 = \tilde{U} \ , \quad U_2 = -\tilde{U} \ , \quad \tilde{U} = [\frac{(n-2)!!}{\pi(n-1)!!}]^{\frac{1}{n}} \ .$$

as a solution of equations (3.40-3.41) at $t = +\infty$.

One can then use the Implicit Function Theory to show that (3.40-3.41) give U_1 and U_2 as functions of t near $t = +\infty$ and

$$lim_{t\to+\infty}U_1(t) = \tilde{U} \ , \quad lim_{t\to+\infty}U_2(t) = -\tilde{U} \ .$$

Therefore, equations (3.2-3.3) can be inverted to give $u_1(t)$ and $u_2(t)$, which have the asymptotics

$$u_1(t) \approx \frac{\tilde{U}}{t^{\frac{1}{n}}} \ , \quad u_2(t) \approx -\frac{\tilde{U}}{t^{\frac{1}{n}}} \quad \text{as } t \gg 1 \ .$$

To verify the inequalities (2.30-2.31), it suffices to prove that $\Phi_0(\xi, u_1(t), u_2(t)) > 0$ uniformly for all ξ when t is sufficiently large.

We first decompose Φ_0

$$\begin{aligned}\Phi_0(\xi, u_1(t), u_2(t)) &= t\Phi_{\xi^n}(\xi, u_1(t), u_2(t)) + \Phi_{V_*(\xi)}(\xi, u_1(t), u_2(t)) \\ &= t^{\frac{2}{n}}[\Phi_{\xi^n}(\frac{\xi}{t^{\frac{1}{n}}}, U_1(t), U_2(t)) + \frac{1}{t^{\frac{2}{n}}}\Phi_{V_*(\xi)}(\xi, u_1(t), u_2(t))] \ .\end{aligned}$$

The first term in the parenthesis is bigger than a positive constant for all ξ and large t. To see this, function Φ_{ξ^n}, in view of formula (B.4), can be written as a multiple integral of the second derivative of ξ^n. Hence, when $\tilde{U}/2 \leq U_1 \leq 2\tilde{U}$ and $-2\tilde{U} \leq U_2 \leq -\tilde{U}/2$, the first term is bounded from below by a positive constant for all $\xi/t^{1/n}$.

To show $\Phi_0(\xi, u_1(t), u_2(t)) > 0$ for all ξ, it is enough to show that the second term $\Phi_{V_*(\xi)}$ is bounded from below for all ξ and large t. To accomplish this, we deduct from condition (3.39) that $V_*''(\xi)$ is bounded from below for all ξ. Function $\Phi_{V_*(\xi)}$, which can be written as a multiple integral of V_*'' in view of formula (B.4), is therefore bounded from below for all ξ, u_1 and u_2.

We have therefore verified inequalities (2.30-2.31). By Theorem 2.3, the equilibrium measure for the external field $V(\xi, t) = V_*(\xi) + t\xi^n$ is supported on a single interval when $t \gg 1$.

To generalize $p(\xi)$ from ξ^n to any convex polynomial $p(\xi)$, we notice that the lowest order term ξ^m in $p(\xi)$ must be of an even order and that its coefficient must also be positive. The previous analysis centered around ξ^n can be applied to ξ^m.

Theorem 3.4. *Under condition (3.39), the equilibrium measure for the external field $V(\xi, t) = V_*(\xi) + t\ p(\xi)$, where $p(\xi)$ is a convex polynomial, has no gap in its support when $t \gg 1$.*

We conclude this section by making an observation on the case of non-convex $p(\xi)$. Such a polynomial $p(\xi)$ has multiple "wells". They will be amplified and become the "wells" of the external field $V(\xi, t) = V_*(\xi) + t\ p(\xi)$ as t is positively large if $V_*(\xi)$

satisfies condition (3.39). Its equilibrium measure is therefore likely to be supported on multiple disjoint intervals.

Appendix A. Algebro-Geometric Solution of the Riemann-Hilbert Problem

In Section 2, we use function theoretical methods to solve the Riemann-Hilbert problem (2.14) and (2.15). Our solution is given by formula (2.16); it is the cornerstone of Propositions 2.1 and 2.2. In this appendix, we will present yet another approach to the Riemann-Hilbert problem. We will solve it using an algebro-geometric method. More precisely, we will give another expression for the solution (2.16) and hence re-derive the formulae of Propositions 2.1 and 2.2.

The Riemann-Hilbert problem (2.14) and (2.15) has an intrinsic algebro-geometric structure. All its solutions are connected to the Riemann surface defined by the equation $w^2 = (\mu - u_1)(\mu - u_2) \cdots (\mu - u_{2g+2})$. We choose the branch cuts along the set of I of (2.5). It is remarkable that our algebro-geometric approach does not require the external field $V(\xi)$ to be an analytic function.

Our starting point is to generalize the Cauchy kernel. On a Riemann surface, there are many analogues of the Cauchy kernel. The most convenient one for the Riemann-Hilbert problem (2.14) and (2.15) is an Abelian differential of the third kind, denoted by $K(\mu, z)d\mu$, with two simple poles at the points $(z, \pm R(z, \vec{u}))$ with residues ± 1, respectively. Here, $\pm R(z, \vec{u})$ are the upper and lower sheets of the Riemann surface. This means that $K(\mu, z)d\mu$ takes the form

$$K(\mu, z)d\mu = \frac{d\mu}{R(\mu, \vec{u})} \frac{R(z, \vec{u})}{\mu - z} + \text{holomorphic terms}$$

and that its behavior, as μ is near the poles, is

$$K(\mu, z)d\mu = \pm \frac{d\mu}{\mu - z} + \text{regular terms.}$$

We may further require that

$$\int_{u_{2k+1}}^{u_{2k}} K(\mu, z)d\mu = 0 \qquad k = 1, 2, \cdots, g \ , \tag{A.1}$$

which is analogous to condition (2.13).

The differential $K(\mu, z)d\mu$ then takes the form

$$K(\mu, z)d\mu = \frac{d\mu}{R(\mu, \vec{u})} \frac{R(z, \vec{u})}{\mu - z} + \sum_{k=1}^{g} \omega_k(\mu)d\mu \int_{u_{2k+1}}^{u_{2k}} \frac{d\xi}{R(\xi, \vec{u})} \frac{R(z, \vec{u})}{\xi - z} ,$$

where $\omega_k(\mu)d\mu$ is the basis of holomorphic differentials normalized along the intervals $[u_{2k+1}, u_{2k}]$, $k = 1, \ldots, g$.

Using the Riemann bilinear relations between $K(\mu, z)d\mu$ and the differentials $\frac{P_{g,n}(\eta, \vec{u})}{R(\eta, \vec{u})}d\eta$ defined in (2.22), it is possible to reduce the above formula to the form

$$K(\mu, z)d\mu = \frac{d\mu}{R(\mu, \vec{u})} \frac{R(z, \vec{u})}{\mu - z} + \frac{1}{2} \sum_{m=1}^{g} \frac{\mu^{g-m}d\mu}{R(\mu, \vec{u})} \sum_{k=1}^{m} k\Gamma_{m-k}(\vec{u}) \int_{p_-}^{p_+} \frac{P_{g,k}(\eta, \vec{u})}{R(\eta, \vec{u})}d\eta , \quad \text{(A.2)}$$

where $p_\pm = (z, \pm R(z, \vec{u}))$ and $\Gamma_k(\vec{u})$ are the coefficients of the expansion (2.25). Formula (A.2) can also be derived from the explicit form of the Bergmann kernel on the Riemann surface $w^2 = R^2(\xi, \vec{u})$ [4].

We next point out a remarkable symmetry property

$$\frac{\partial}{\partial z}K(\mu, z) = \frac{\partial}{\partial \mu}K(z, \mu) .$$

It also follows from the Riemann bilinear relations [24].

We now use $K(\mu, z)d\mu$ to construct a solution of the Riemann-Hilbert problem (2.14) and (2.15)

$$\mathcal{G}(z) = -\frac{1}{\pi} \int_I V'(\mu)K(z, \mu)d\mu + \frac{\sqrt{-1}}{\pi} \frac{P_{g,0}(z, \vec{u})}{R(z, \vec{u})} \quad \text{(A.3)}$$

$$= \frac{\partial}{\partial z} \left[\frac{1}{\pi} \int_I V(\mu)K(\mu, z)d\mu \right] + \frac{\sqrt{-1}}{\pi} \frac{P_{g,0}(z, \vec{u})}{R(z, \vec{u})} . \quad \text{(A.4)}$$

To see this, we derive from (A.3) the boundary value of $\mathcal{G}$ at real ξ

$$\mathcal{G}^+(\xi) = \sqrt{-1}V'(\xi) - \frac{1}{\pi}P.V. \int_I V'(\mu)K(\xi, \mu)d\mu + \frac{\sqrt{-1}}{\pi} \frac{P_{g,0}(\xi)}{R(\xi, \vec{u})} .$$

When $\xi \in I$ and $\mu \in I$, the kernel $K(\xi, \mu)$ is real and $R(\xi, \vec{u})$ is pure imaginary. We then derive $Im\, \mathcal{G}^+(\xi) = V'(\xi)$, which is (2.14). When $\xi \in \mathbb{R}\backslash\bar{I}$ and $\mu \in I$, the kernel

$K(\xi,\mu)$ is pure imaginary and $R(\xi,\vec{u})$ is real. We instead have $Re\,\mathcal{G}^+(\xi)=0$, which is exactly (2.15).

We further claim that $\mathcal{G}$ of (A.3) equals the function defined in (2.16). To see this, it suffices to prove that $\mathcal{G}$ of (A.3) satisfies both the loop conditions (2.13) and the asymptotics $\mathcal{G}(z)=-\frac{1}{\sqrt{-1}\pi z}+O(1/z^2)$ for large $|z|$. Conditions (2.13) are easily verified, in view of similar conditions (A.1) on $K(\mu,z)d\mu$ and (2.23) on $P_{g,0}$. For large $|z|$, the kernel $K(z,\mu)$ behaves like $O(\frac{1}{z^2})$. Indeed the first term of $K(z,\mu)$ is proportional to $\frac{1}{R(z,\vec{u})(z-\mu)}$, which clearly decays at least as fast as $\frac{1}{z^2}$ as $|z|\to\infty$. The other terms of $K(z,\mu)$ are of the form $\frac{z^{g-m}d\mu}{R(z,\vec{u})}=O(\frac{1}{z^{m+1}})$ for $|z|$ large and $m=1,\ldots,g$. This, together with the behavior (2.22) of $P_{g,0}$, justifies the asymptotics $\mathcal{G}(z)=-\frac{1}{\sqrt{-1}\pi z}+O(1/z^2)$ for large $|z|$.

Inserting the explicit form (A.2) into the expression (A.4), we obtain

$$\begin{aligned}\mathcal{G}(z) &= \frac{\partial}{\partial z}\left[\frac{R(z,\vec{u})}{\pi}\int_I \frac{V(\mu)d\mu}{R(\mu,\vec{u})(\mu-z)}\right] \\ &\quad + 2\sqrt{-1}\sum_{m=1}^{g} q_{g,m}(\vec{u})\sum_{k=1}^{m} k\Gamma_{m-k}(\vec{u})\frac{P_{g,k}(z,\vec{u})}{R(z,\vec{u})}+\frac{\sqrt{-1}}{\pi}\frac{P_{g,0}(z,\vec{u})}{R(z,\vec{u})}\ ,\end{aligned} \tag{A.5}$$

where $q_{g,m}(\vec{u})$ is given in (2.26).

The boundary value of $\mathcal{G}(z)$ of (A.5) on the real axis can be obtained by observing that the first term has the boundary value at real ξ

$$\frac{\partial}{\partial \xi}\left[-2\sqrt{-1}R(\xi,\vec{u})\Psi_g(\xi,\vec{u})+\sqrt{-1}V(\xi)\right]\ ,$$

where Ψ_g is as given in Proposition 2.2. The function $\Psi_g(\xi,\vec{u})$ is related to $\Phi_g(z,\vec{u})$ of Proposition 2.1 by an identity [12]

$$\Phi_g(\xi,\vec{u})=\frac{\partial}{\partial\xi}\Psi_g(\xi,\vec{u})+\frac{1}{2}\sum_{i=1}^{2g+2}\frac{\Psi_g(\xi,\vec{u})-\Psi_g(u_i,\vec{u})}{\xi-u_i}. \tag{A.6}$$

We hence arrive at the boundary value of $\mathcal{G}$ at the real ξ

$$\mathcal{G}^+(\xi)=\frac{-2\sqrt{-1}R^2(\xi,\vec{u})\Phi_g(\xi,\vec{u})+\sqrt{-1}Q(\xi)}{R(\xi,\vec{u})}+\sqrt{-1}V'(\xi)\ ,$$

where the polynomial $Q(\xi, \vec{u})$ is

$$-R^2(\xi, \vec{u}) \sum_{i=1}^{2g+2} \frac{\Psi_g(u_i, \vec{u})}{\xi - u_i} + 2 \sum_{m=1}^{g} q_{k,m}(\vec{u}) \sum_{k=1}^{m} k\Gamma_{m-k}(\vec{u}) P_{g,k}(\xi, \vec{u}) + \frac{1}{\pi} P_{g,0}(\xi, \vec{u}).$$

These are equivalent to the formulae of Propositions 2.1 and 2.2.

Appendix B. Euler-Poisson-Darboux Equations

The boundary value problem for the Euler-Poisson-Darboux equations (2.18-2.20) has one and only one solution. Its solution can be constructed using those of the following simpler problem as building blocks [21, 12]

$$\begin{aligned} 2(x_1 - x_2)\frac{\partial^2 q}{\partial x_1 \partial x_2} &= \frac{\partial q}{\partial x_1} - \rho \frac{\partial q}{\partial x_2} \ , \quad \rho > 0 \ \textit{is a constant} \ , && \text{(B.1)} \\ q(x_1, x_1) &= g(x_1) \ . && \text{(B.2)} \end{aligned}$$

A simple calculation shows that the solution of (B.1-B.2) is given by the formula [21]

$$q(x_1, x_2) = C_0 \int_{-1}^{1} \frac{g(\frac{1+\mu}{2}x_1 + \frac{1-\mu}{2}x_2)}{\sqrt{1-\mu^2}} \ (1+\mu)^{\frac{\rho-1}{2}} d\mu \ ,$$

where

$$C_0 = \frac{1}{\int_{-1}^{1} \frac{(1+\mu)^{\frac{\rho-1}{2}}}{\sqrt{1-\mu^2}} d\mu} \ .$$

A change of integration variable gives another formula for the solution

$$q(x_1, x_2) = C_0 \left[\frac{2}{x_1 - x_2} \right]^{\frac{\rho-1}{2}} \int_{x_2}^{x_1} g(x) \frac{(x - x_2)^{\frac{\rho-2}{2}}}{(x_1 - x)^{\frac{1}{2}}} dx \ , \tag{B.3}$$

where the square root is set to be positive for x between x_1 and x_2.

Using a solution method of [21], one is able to construct the solution of (2.18-2.20) using a multiple integral[12]

$$\Phi_g(\xi, \vec{u}) = M_0 \int_{-1}^{1} \cdots \int_{-1}^{1} \tilde{f} \ \frac{\prod_{k=2}^{2g+2} (1+\mu_k)^{\frac{k-1}{2}}}{\sqrt{\prod_{k=1}^{2g+2} (1-\mu_k)}} \ d\mu_1 \cdots d\mu_{2g+2} \ , \tag{B.4}$$

where $\tilde{f}$ denotes

$$f^{(g+2)}(\frac{1+\mu_{2g+2}}{2}(\cdots(\frac{1+\mu_2}{2}(\frac{1+\mu_1}{2}\xi + \frac{1-\mu_1}{2}u_1) + \cdots) + \frac{1-\mu_{2g+2}}{2}u_{2g+2})$$

and the constant M_0 is chosen so that the boundary condition (2.20) is satisfied. Here $f^{(g+2)}$ denotes the $(g+2)th$ derivative of f.

The function $\Psi_g(\xi, \vec{u})$ satisfies the same equations (2.18-2.19) as $\Phi_g(\xi, \vec{u})$; but, instead of the boundary condition (2.20), it satisfies (2.24) with one derivative lower.

We now list some of the properties concerning Φ_0, Φ_1 and Ψ_1. They are useful in the calculations in Section 3.

Property B.1.

$$\Phi_0(\xi, u, u) = \frac{V'(\xi) - V'(u)}{2(\xi - u)} , \tag{B.5}$$

$$\begin{aligned}\Phi_0(\xi, u_1, u_2) &= \frac{V'(\xi)}{2\sqrt{(\xi - u_1)(\xi - u_2)}} \\ &\quad - \frac{1}{2\pi} \int_{u_2}^{u_1} \frac{V'(\mu) d\mu}{(\xi - u)\sqrt{(u_1 - u)(u - u_2)}} ,\end{aligned} \tag{B.6}$$

$$\Phi_1(\xi, u, u, -u, -u) = \frac{uV'(\xi) - \xi V'(u)}{2(\xi^2 - u^2)u} , \tag{B.7}$$

$$\begin{aligned}\Phi_1(\xi, u_1, u_2, -u_2, -u_1) &= \frac{V'(\xi)}{2\sqrt{(\xi^2 - u_1^2)(\xi^2 - u_2^2)}} \\ &\quad + \frac{\xi}{\pi} \int_{u_2}^{u_1} \frac{V'(\mu) d\mu}{(\mu^2 - \xi^2)\sqrt{(u_1^2 - \mu^2)(\mu^2 - u_2^2)}} ,\end{aligned} \tag{B.8}$$

$$\Psi_1(u, u, u, -u, -u) = \frac{V'(u)}{4u} , \tag{B.9}$$

$$\begin{aligned}\Psi_1(u_1, u_1, u_2, -u_2, -u_1) &+ \Psi_1(u_2, u_1, u_2, -u_2, -u_1) \\ &= \frac{1}{2} \int_{u_2}^{u_1} \frac{V'(\mu) d\mu}{\sqrt{(u_1^2 - \mu^2)(\mu^2 - u_2^2)}} .\end{aligned} \tag{B.10}$$

Function $V(\xi)$ is assumed to be an even function in (B.7), (B.8), (B.9) and (B.10).

Proof. In formula (B.4), Φ_0, Φ_1 and Ψ_1 are written as multiple integrals of $V''(\xi)$, $V'''(\xi)$ and $V''(\xi)$, respectively. Since the smooth function $V(\xi)$ can be approximated by polynomials in $C^3(S)$ on every compact set S as close as possible, it suffices to prove Property B.1 when $V(\xi)$ is simply a polynomial.

We will rely on the contour integral formulation of Φ_g

$$\Phi_g(\xi, \vec{u}) = \frac{1}{4\pi\sqrt{-1}} \oint_\gamma \frac{V'(\mu) d\mu}{(\mu - \xi) R(\mu, \vec{u})} , \tag{B.11}$$

where γ is a contour enclosing the point ξ and all the cuts along $[u_{2k}, u_{2k-1}]$, $k = 1, 2, \cdots, g+1$. To see this , it is easy to check that the right hand side, as a function of ξ and $\vec{u}$, satisfies (2.18) and (2.19). Boundary condition (2.20) is also easily verified using the Cauchy Integral Formula.

Identity (B.5) is an easy consequence of (B.11) when $u_1 = u_2 = u$.

To prove (B.6), we replace the contour γ by γ', which still encloses the cuts $[u_2, u_1]$, $k = 1, 2, \cdots, g+1$, but excludes the point ξ, and write $\Phi_0(\xi, u_1, u_2)$ as

$$\frac{1}{2} Res_{\mu=\xi} \left[\frac{V'(\mu)}{(\mu - \xi)\sqrt{(\mu - u_1)(\mu - u_2)}} \right] + \frac{1}{4\pi\sqrt{-1}} \oint_{\gamma'} \frac{V'(\mu) d\mu}{(\mu - \xi) R(\mu, \vec{u})} \ .$$

The first term gives the first term on the right of (B.6). Rewriting the second term as an integral along the cut $[u_2, u_1]$ gives the second term of (B.6).

Identities (B.7) and (B.8) can be proved in the same way.

To prove (B.9), we use an analogous formula of (B.11) for $\Psi_1(\xi, u_1, u_2, -u_2, -u_1)$

$$\frac{1}{4\pi\sqrt{-1}} \oint_{\gamma} \frac{V(\mu) d\mu}{(\mu - \xi)\sqrt{(\mu^2 - u_1^2)(\mu^2 - u_2^2)}} \ . \tag{B.12}$$

Identity (B.9) is an immediate consequence of this formula and the Cauchy Integral Formula.

To prove (B.10), we notice from the formula of type (B.4) that Ψ_1 can be written as a multiple integral of $V''(\xi)$. Since $V(\xi)$ is an even function of ξ, so is $\Psi_1(\xi, u_1, u_2, -u_2, -u_1)$. This evenness allows us to modify formula (B.12) to get a new contour integral formulation for $\Psi(\xi, u_1, u_2, -u_2, -u_1)$

$$\frac{1}{4\pi\sqrt{-1}} \oint_{\gamma} \frac{V(\mu) \mu d\mu}{(\mu^2 - \xi^2)\sqrt{(\mu^2 - u_1^2)(\mu^2 - u_2^2)}} \ .$$

The left hand side of (B.10) then equals

$$\begin{aligned} & \frac{1}{4\pi\sqrt{-1}} \oint_{\gamma} \left(\frac{1}{\mu^2 - u_1^2} + \frac{1}{\mu^2 - u_2^2} \right) \frac{V(\mu) \mu d\mu}{\sqrt{(\mu^2 - u_1^2)(\mu^2 - u_2^2)}} \\ = & \ -\frac{1}{4\pi\sqrt{-1}} \oint_{\gamma} \frac{\partial}{\partial \mu} \left(\frac{1}{\sqrt{(\mu^2 - u_1^2)(\mu^2 - u_2^2)}} \right) V(\mu) d\mu \\ = & \ \frac{1}{4\pi\sqrt{-1}} \oint_{\gamma} \frac{V'(\mu) d\mu}{\sqrt{(\mu^2 - u_1^2)(\mu^2 - u_2^2)}} \ . \end{aligned}$$

We obtain the right hand side of (B.10) by writing the last integral along the cuts $[-u_1, -u_2]$ and $[u_2, u_1]$ and using the oddness of $V'(\mu)$.

The proof of Property B.1 is completed.

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