THOMAE TYPE FORMULAE FOR SINGULAR Z_N CURVES

V.Z. ENOLSKI AND T.GRAVA

Abstract. We give an elementary and rigorous proof of the Thomae type formula for the singular curves $\mu^N = \prod_{j=1}^m (\lambda - \lambda_{2j})^{N-1} \prod_{j=0}^m (\lambda - \lambda_{2j+1})$. To derive the Thomae formula we use the traditional variational method which goes back to Riemann, Thomae and Fuchs. An important step of the proof is the use of the Szegö kernel computed explicitly in algebraic form for non-singular $1/N$-periods. The proof inherits principal points of Nakayashiki's proof [31], obtained for non-singular Z_N curves.

1. Introduction

The original Thomae formula [36] expresses the zero values of the Riemann θ-functions with half integer characteristics as functions of the branch points of the hyperelliptic curve $\mu^2 = \prod_{i=1}^{2m}(\lambda - \lambda_i)$, that is

$$\theta[\boldsymbol{e}](\mathbf{0};\Pi)^8 = \left(\frac{\det \mathcal{A}}{(2\pi i)^{m-1}}\right)^4 \prod_{k<l} (\lambda_{i_k} - \lambda_{i_l})^2 (\lambda_{j_k} - \lambda_{j_l})^2, \tag{1.1}$$

where $\boldsymbol{e}$ is a non-singular even half-period corresponding to the partition of the branch points $\{1, \cdots, 2m\} = \{i_1 < \cdots < i_m\} \cup \{j_1 < \cdots < j_m\}$. The Riemann period matrix Π and the matrix of α-periods $\mathcal{A} = (\oint_{\alpha_i} \lambda^{j-1}/\mu)_{1\le i,j\le m-1}$ are computed in a canonical homology basis $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

The modern interest in the Thomae formulae was initially stimulated by the finite-gap integration of KdV and KP type equations where, according to the Dubrovin-Novikov program of "effectivization of the finite-gap solutions", it was necessary to express winding vectors in the Its-Matveev formula [20] in terms of θ-constants (see e.g. [10, 30]). In this context Thomae formulae were recently considered in [12].

Thomae formulae were used to describe the moduli space of hyperelliptic curves with level two structure in terms of theta constants [30]. They give a generalization of the λ function of an elliptic curve [30] (Umemura's appendix) and [13]. In [28] they were combined with the arithmetic-geometric mean to obtain criteria for the reducibility of ultraelliptic Jacobians. Thomae formulae also appear in a wider context of modern research on integrable systems, in particular in the Riemann-Hilbert problem and the associated Schlesinger equations (see, e. g.,[21]). In this set of problems the Thomae formulae were used to give explicit expression for the τ-function of the Schlesinger equation associated with hyperelliptic curve [21], general curve with simple branch points [27] and non simple branch points [11]. Thomae formulas are also used to compute action variables in conditions of complete integrability (e. g. [9]). In the conformal field theory Thomae formulae became relevant after the work of Knizhnik [25] who expressed correlation functions of the multi-loop string amplitude in terms of θ-constants of a Riemann surface with Z_N symmetry.

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A solution of Knizhnik-Zamolodchikov equations in terms of θ-constants was also obtained in [35] on the basis of Thomae formulae.

After the classical Thomae paper [36] the derivation of Thomae formulae in hyperelliptic case was given in many places: Fuchs [17], Bolza [4], Fay [14], Mumford [30]. But only recently Bershadsky and Radul [5, BR88] discovered a generalization of Thomae formula for Z_N curve $\mu^N = \prod_{i=1}^{Nm}(\lambda - \lambda_i)$ and gave a heuristic proof of them on the basis of path integral formulation of the conformal field theory. Afterwards Nakayashiki [31] developed a rigourous derivation of these formulae in the frame of the classical methods.

In this paper we derive the Thomae formula for the singular curve

$$\mu^N = \prod_{j=1}^{m}(\lambda - \lambda_{2j})^{N-1}\prod_{i=0}^{m}(\lambda - \lambda_{2i+1}), \quad N > 1 \quad N \in \mathbb{N}, \tag{1.2}$$

which has Z_N symmetry and it will be called singular Z_N curves.

The modular properties of certain family of curves (1.2) were investigated by Burhardt [6], Hutchinson [19], more recently by Shiga [34] and Koike [26], Diez [7] and others. This curve (1.2) appeared in Zverovich [40] as the curve related to a solvable Riemann-Hilbert problem with quasi-permutation monodromy matrices. Explicit solution of this problem was derived in our recent paper [11] which stimulated the present investigation.

Our proof of Thomae formulae goes up to the original Thomae paper [36] and inherits principal steps of the Nakayashiki's proof [31] for non singular Z_N curves. The non singular Z_N curves are invariant under the group generated by the permutations of branch points, while the singular Z_N curves are not. This is the main difference with the case treated by Nakayashiki.

The main steps of our proof are the explicit algebraic formula for the Szegö kernel associated with non-singular $1/N$-period and the explicit formula for the canonical bimeromorphic differential kernel. The above kernel functions can be realized in θ-functional form, and in algebraic forms. In the former case the theory was developed by Fay [14], in the latter case the theory has originated in the work of Klein [23, 24] and later it was developed by Baker [2, 1] and also [18],[3] and for nonsingular Z_N curves by Nakayashiki [31]. The comparison of these representations leads to a certain relation which is the important point of the proof. We also use the notion of variation of branch points by means of Rauch formulae [33],[27].

The paper is organized as follows. In Section 2 we are fixing the notations and remind basic notions from the theory of algebraic curves which we need for the foregoing development. In the Section 3 we describe characteristics of Abelian images of branch points. We introduce in this section families of non-special divisors supported on branch points and describe corresponding $1/N$-periods. We describe canonical bimeromorphic differential and Szegö kernels in the Section 4 and develop their expansions. In the Section 5 we briefly discuss Kleinian formulation of the kernel forms. All these results are summarized in the Section 6 where the formulation and proof of the Thomae formula is given. We briefly discuss further perspectives in the last Section.

2. The curve, its homologies, differentials, θ-functions and variations

Consider the Riemann surface $\mathcal{C}$ of the curve

$$\mu^N = p(\lambda)q(\lambda)^{N-1}, \tag{2.1}$$

$$p(\lambda) = \prod_{k=0}^{m}(\lambda - \lambda_{2k+1}), \quad q(\lambda) = \prod_{k=1}^{m}(\lambda - \lambda_{2k}). \tag{2.2}$$

The curve (2.1) has branch points $P_k = (\lambda_k, 0)$, $k = 1, \ldots, 2m+1$ and $P_{2m+2} = P_\infty = (\infty, \infty)$ and singularities at the points P_{2k}, $k = 1, \ldots, m+1$. These singularities can be easily resolved [29] to give rise to a compact Riemann surface which we still denote by $\mathcal{C}_{N,m}$. The genus g of $\mathcal{C}_{N,m}$ is equal to $(N-1)m$.

The projection $(\lambda, y) \to \lambda$, (which we still denote by λ) defines $\mathcal{C}_{N,m}$ as a $N-$sheeted covering of the complex plane $\mathbb{CP}^1$ branched over the points P_k, $k = 1, \dots 2m+2$. The pre-image of a non-branch point $\lambda \in \mathbb{CP}^1$ consists of N points. The N-cyclic automorphism J of $\mathcal{C}_{N,m}$ is given by the action $J : (\lambda, y) \to (\lambda, \rho y)$, where ρ is the N-primitive root of unity, namely $\rho = \mathrm{e}^{\frac{2\pi\imath}{N}}$. For P in a neighbourhood U_R of the point $R \in \mathcal{C}_{N,m}$, a local coordinate $x(P)$, with $x(R) = 0$, is the function defined by

$$x(P) = \begin{cases} \lambda(P) - \lambda(R), & \text{if} \quad R \quad \text{is an ordinary point,} \\ \sqrt[N]{\lambda(P) - \lambda(R)}, & \text{if} \quad R = P_k,\ k = 1, \dots, 2m+1, \\ \frac{1}{\sqrt[N]{\lambda(P)}}, & \text{if} \quad R = P_\infty. \end{cases} \tag{2.3}$$

The canonical homology basis, $(\alpha_1, \dots, \alpha_{(N-1)m}; \beta_1, \dots, \beta_{(N-1)m}) \in H(\mathcal{C}, \mathbb{Z})$ of the curve $\mathcal{C}$ is shown in the Figure 1. Namely the cycles α_{k+sm}, $k = 1, \dots, m$ lie on the $(s+1)-$th sheet, $s =$

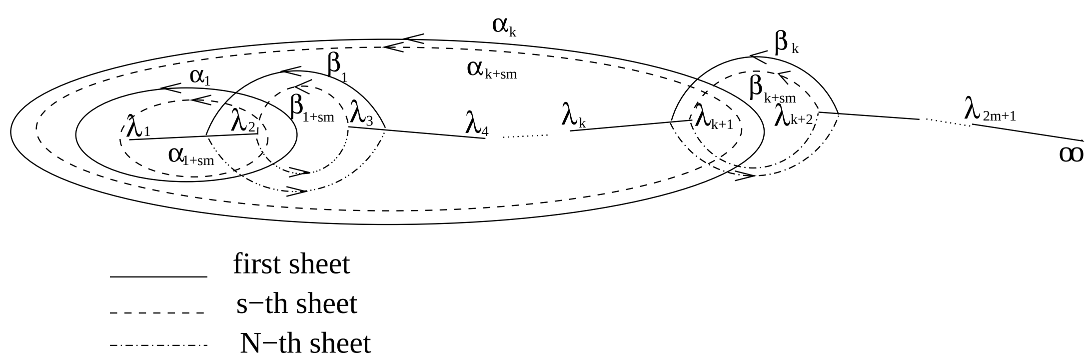

FIGURE 1. The homology basis.

$0, \dots, N-2$ and encircle anti-clockwise the cuts $\cup_{j=1}^{k}(\lambda_{2j-1}, \lambda_{2j})$. The cycles β_{k+sm}, $k = 1, \dots, m$, $s = 0, \dots, N-2$, emerges on the $(s+1)-$th sheet through the point $(\lambda_{2k+1}, 0)$ pass anti-clockwise to the N-th sheet through the point $(\lambda_{2k}, 0)$ and return to the initial point through the N-th sheet. We remark that on Figure 1, when $N > 3$, the β-cycles placed from the second to the $(N-2)$th sheet should intersect the cuts only in the branch points. It is possible to drop this requirement and plot appropriate number of loops around each branch point.

The action of the automorphism J on the basis of cycles is given by

$$\begin{aligned} &J(\alpha_{i+sm}) = \alpha_{i+(s+1)m},\ i = 1, \dots, m,\ s = 0, \dots, N-3, \\ &J(\alpha_{i+(N-2)m}) = -\sum_{s=0}^{N-2} \alpha_{i+sm}, \quad i = 1, \dots, m, \\ &J(\beta_{i+sm}) = \beta_{i+(s+1)m} - \beta_i, \quad s = 0, \dots, N-3, \\ &J(\beta_{i+(N-2)m}) = -\beta_i, \quad i = 1, \dots, m. \end{aligned}$$

The basis of canonical holomorphic differentials reads

$$\mathrm{d}u_{j+sm}(P) = \frac{\lambda^{j-1} q(\lambda)^s}{\mu^{s+1}} \mathrm{d}\lambda, \quad j = 1, \dots, m, \quad s = 0, \dots, N-2. \tag{2.4}$$

The induced action of J on the holomorphic differential is given by

$$J(\mathrm{d}u_{j+sm}(P)) := \mathrm{d}u_{j+sm}(J(P)) = \frac{\lambda^{j-1}q(\lambda)^s}{\rho^{s+1}\mu^{s+1}}\mathrm{d}\lambda, \quad j=1,\ldots,m, \;\; s=0,\ldots,N-2,$$

where ρ is the N-th root of unity. The $(N-1)m \times (N-1)m$ matrices $\mathcal{A}$ of α-periods and $\mathcal{B}$ of β-periods are expressible in terms of $m \times m$-matrices

$$\mathcal{A}_{s+1} = \left(\oint_{\alpha_j} \mathrm{d}u_{k+ms}\right)_{k,j=1,\ldots,m}, \quad \mathcal{B}_{s+1} = \left(\oint_{\beta_j} \mathrm{d}u_{k+ms}\right)_{k,j=1,\ldots,m}, \quad s=0,\ldots,N-2, \tag{2.5}$$

in the following way. Let us introduce the $(N-1)m \times (N-1)m$ dimensional matrices

$$\mathcal{R}_{\mathcal{A}} = \left(\frac{\rho^{-i(k-1)} - \rho^{-ik}}{1-\rho^{-i}}\right)_{i,k=1,\ldots,N-1} \otimes 1_m, \tag{2.6}$$

$$\mathcal{R}_{\mathcal{B}} = \left(\frac{\rho^{-i(k-1)} - \rho^{-i(N-1)}}{1-\rho^{-(N-1)i}}\right)_{i,k=1,\ldots,N-1} \otimes 1_m. \tag{2.7}$$

Then

$$\mathcal{A} = \left(\oint_{\alpha_j} \mathrm{d}u_k\right)_{k,j=1,\ldots,(N-1)m} = \mathrm{Diag}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{N-1})\mathcal{R}_{\mathcal{A}}, \tag{2.8}$$

$$\mathcal{B} = \left(\oint_{\beta_j} \mathrm{d}u_k\right)_{k,j=1,\ldots,(N-1)m} = \mathrm{Diag}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{N-1})\mathcal{R}_{\mathcal{B}}, \tag{2.9}$$

where $\mathrm{Diag}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{N-1})$, $\mathrm{Diag}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{N-1})$ are the block diagonal matrices having as entries the matrices $\mathcal{A}_s$ and $\mathcal{B}_s$, $s=1,\ldots,N-1$, respectively.

The normalized holomorphic differentials $\mathrm{d}\boldsymbol{v} = (\mathrm{d}v_1, \ldots, \mathrm{d}v_{(N-1)m})$ are given as $\mathrm{d}\boldsymbol{v} = \mathrm{d}\boldsymbol{u}\mathcal{A}^{-1}$ and the Riemann period matrix Π,

$$\Pi = \left(\oint_{\beta_k} \mathrm{d}v_j\right)_{j,k=1,\ldots,(N-1)m}$$

is given by $\Pi = \mathcal{R}_{\mathcal{A}}^{-1}\mathrm{Diag}(\mathcal{A}_1^{-1}\mathcal{B}_1, \mathcal{A}_2^{-1}\mathcal{B}_2, \ldots, \mathcal{A}_{N-1}^{-1}\mathcal{B}_{N-1})\mathcal{R}_{\mathcal{B}}$ with $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{B}}$ defined in (2.6) and (2.7). The matrix Π is necessarily symmetric and has positively defined imaginary part. The space of such matrices is called Siegel half-space $\mathcal{H}_g$. Denote also Jacobi variety of the curve $\mathrm{Jac}(\mathcal{C}) = \mathbb{C}^g/1_g \oplus \Pi$.

To complete this Section we recall the important *Rauch variation formulae* [33] for the period matrix Π

$$\frac{\partial}{\partial \lambda_k}\Pi_{ij} = 2\pi\imath \operatorname*{Res}_{\lambda=\lambda_k}\left\{\frac{1}{(\mathrm{d}\lambda(P))^2}\sum_{s=1}^{N}\mathrm{d}v_i(P^{(s)})\mathrm{d}v_j(P^{(s)})\right\}, \tag{2.10}$$

where $i,j=1,\ldots,2m$, $k=1,\ldots,2m+1$. We remark that in the case of hyperelliptic curves the formula (2.10) appeared already in the Thomae article [36]. For general surfaces the infinitesimal variation of Abelian differentials and their periods with respect to Beltrami differentials is due to Fay [15]. Korotkin [27] reduced the variation formula to useful form (2.10). The proof of the variational formula for the holomorphic differentials can be found in [22].

2.1. **Theta functions.** Any point $\boldsymbol{e} \in \mathrm{Jac}(\mathcal{C})$ can be written uniquely as $\boldsymbol{e} = (\boldsymbol{\epsilon}, \boldsymbol{\delta})\begin{pmatrix} 1_g \\ \Pi \end{pmatrix}$, where $\boldsymbol{\epsilon}, \boldsymbol{\delta} \in \mathbb{R}^g$ are the characteristics of $\boldsymbol{e}$. We shall use the notation $[\boldsymbol{e}] = [{}^{\boldsymbol{\delta}}_{\boldsymbol{\epsilon}}]$. If $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ are half integer, then we say that the corresponding characteristics $[\boldsymbol{e}]$ are half-integer. The half-integer characteristics are odd or even, whenever $4\langle \boldsymbol{\delta}, \boldsymbol{\epsilon}\rangle$ is equal to 1 or 0 modulo 2. Here and below the brackets $\langle\ ,\ \rangle$ denote the standard Euclidean scalar product. The Riemann θ-function with characteristics $[\boldsymbol{e}]$ is defined in $\mathcal{H}_g \times \mathrm{Jac}(\mathcal{C})$ as the Fourier series

$$
\begin{aligned}
\theta\,[\boldsymbol{e}]\,(\boldsymbol{z};\Pi) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp(\pi\imath\,\langle \boldsymbol{n}\Pi + \boldsymbol{\delta}\Pi, \boldsymbol{n}+\boldsymbol{\delta}\rangle + 2\pi\imath\,\langle \boldsymbol{z}+\boldsymbol{\epsilon}, \boldsymbol{n}+\boldsymbol{\delta}\rangle) \\
&= \theta(\boldsymbol{z}+\boldsymbol{\epsilon}+\boldsymbol{\delta}\Pi;\Pi)\exp\{2\imath\pi\langle\boldsymbol{\delta}, (\boldsymbol{z}+\frac{1}{2}\boldsymbol{\delta}\Pi)\rangle + 2\imath\pi\langle\boldsymbol{\epsilon},\boldsymbol{\delta}\rangle\}.
\end{aligned}
\tag{2.11}
$$

For $[\boldsymbol{e}] = [\boldsymbol{0}]$ we write $\theta[\boldsymbol{0}](\boldsymbol{z};\Pi) = \theta(\boldsymbol{z};\Pi)$. The θ-function is an entire in $\boldsymbol{z} \in \mathbb{C}^g$ and has periodicity properties:

$$
\theta\,[\boldsymbol{e}]\,(\boldsymbol{z}+\boldsymbol{m}'+\boldsymbol{m}\Pi;\Pi) = \exp\{-2\imath\pi\langle\boldsymbol{m}', (\boldsymbol{z}+\frac{1}{2}\boldsymbol{m}'\Pi)\rangle - 2\imath\pi\langle\boldsymbol{\epsilon},\boldsymbol{m}'\rangle\}\theta\,[\boldsymbol{e}]\,(\boldsymbol{z};\Pi),
\tag{2.12}
$$

where $\boldsymbol{m}$ and $\boldsymbol{m}'$ are integer vectors.

The θ-function with arbitrary characteristics satisfies the heat equation

$$
\frac{\partial^2}{\partial z_k \partial z_l}\theta\,[\boldsymbol{e}]\,(\boldsymbol{z};\Pi) = 2\imath\pi(1+\delta_{k,l})\frac{\partial}{\partial \Pi_{kl}}\theta\,[\boldsymbol{e}]\,(\boldsymbol{z};\Pi), \quad k,l = 1,\ldots,g.
\tag{2.13}
$$

The zeros of the θ-function are described by the fundamental *Riemann singularity theorem.*

Theorem 2.1. *Let $\boldsymbol{e} \in \mathrm{Jac}(\mathcal{C})$ be an arbitrary vector and $Q_0 \in \mathcal{C}$- arbitrary point. Then the multi-valued function*

$$
P \to \theta\left(\int_{Q_0}^{P} \mathrm{d}\boldsymbol{v} - \boldsymbol{e};\Pi\right)
$$

has on $\mathcal{C}$ exactly g zeros $Q_1, Q_2, \ldots, Q_g$ provided it does not vanish identically and

$$
\boldsymbol{e} = \sum_{i=1}^{g}\int_{Q_0}^{Q_i} \mathrm{d}\boldsymbol{v} - \boldsymbol{K}_{Q_0},
\tag{2.14}
$$

where $\boldsymbol{K}_{Q_0}$ is the vector of Riemann constants

$$
(\boldsymbol{K}_{Q_0})_j = \frac{1+\Pi_{jj}}{2} - \sum_{i=1, i\neq j}^{g} \oint_{\alpha_i} \mathrm{d}v_i(P)\int_{Q_0}^{P}\mathrm{d}v_j.
\tag{2.15}
$$

Furthermore, the divisor $\sum_{i=1}^{g} Q_i$ is non special.

For a point $P \in \mathcal{C}$, we define the Abel map $\mathfrak{A} : \mathcal{C} \longrightarrow \mathrm{Jac}(\mathcal{C})$ by setting

$$
\mathfrak{A}(P) = \int_{Q_0}^{P} \mathrm{d}\boldsymbol{v},
\tag{2.16}
$$

for some base point $Q_0 \in \mathcal{C}$. For a positive divisor $\mathcal{D}$ of degree n the Abel map reads

$$
\mathfrak{A}(\mathcal{D}) = \int_{nQ_0}^{\mathcal{D}} \mathrm{d}\boldsymbol{v}.
$$

There exists a non-positive divisor Δ of degree $g-1$ such that

$$
\mathfrak{A}(\Delta - (g-1)Q_0) = \boldsymbol{K}_{Q_0},
\tag{2.17}
$$

where $\boldsymbol{K}_{Q_0}$ has been defined in (2.15). The divisor Δ is called the Riemann divisor and satisfies the condition $2\Delta = \mathcal{K}_{\mathcal{C}}$, where $\mathcal{K}_{\mathcal{C}}$ is the canonical class (that is the class of divisors of Abelian differentials). The vector $\boldsymbol{e}$ defined in (2.14) can be written in the form

$$\boldsymbol{e} = \mathfrak{A}\Big(\sum_{i=1}^{g} Q_i - Q_0 - \Delta\Big). \tag{2.18}$$

3. Characteristics supported on branch points

In this section we are going to compute the characteristics $[\boldsymbol{\mathcal{U}}_k]$ of the Abelian images of branch points $P_k = (\lambda_k, 0)$

$$\boldsymbol{\mathcal{U}}_k = \int_{P_\infty}^{P_k} \mathrm{d}\boldsymbol{v}, \quad k = 1, \dots, 2m+1, \quad P_\infty = (\infty, \infty),$$

in terms of the period matrix Π.

Lemma 3.1. *The relations*

$$\int_{P_{2k}}^{P_{2k-1}} \mathrm{d}v_{k+sm} = \frac{N-1-s}{N}, \qquad \int_{P_{2k+2}}^{P_{2k+1}} \mathrm{d}v_{k+sm} = -\frac{N-1-s}{N},$$
$$\int_{P_{2k+2}}^{P_{2k+1}} \mathrm{d}v_{j+sm} = 0, \;\; j \neq k, k+1, \;\; j = 1, \dots, m,$$
$$\int_{P_{2j}+1}^{P_{2j}} \mathrm{d}v_{k+sm} = \frac{N-1}{N}\Pi_{k+sm,j} - \frac{1}{N}\sum_{r=1}^{N-2} \Pi_{k+sm,j+rm}$$

are valid for $k, j = 1, \dots, m,\ s = 0, \dots, N-2.$

The proof represents a generalization to the case $N > 2$, of the derivation of half-integer characteristics given e.g. in [16]; for detail see the proof of this lemma in [11].

From the relations given in the Lemma 3.1 we are able to write the characteristics $[\boldsymbol{\mathcal{U}}_k]$ in the form

$$[\boldsymbol{\mathcal{U}}_{2m+1}] = \left[\underbrace{\begin{array}{cccc} 0 & \dots & 0 & \overset{m\downarrow}{0} \\ 0 & \dots & 0 & \frac{1}{N} \end{array}}_{m} \dots \underbrace{\begin{array}{cccc} 0 & \dots & 0 & \overset{sm\downarrow}{0} \\ 0 & \dots & 0 & \frac{s}{N} \end{array}}_{m} \dots \underbrace{\begin{array}{cccc} 0 & \dots & 0 & \overset{(N-1)m\downarrow}{0} \\ 0 & \dots & 0 & \frac{N-1}{N} \end{array}}_{m} \right],$$

$$[\boldsymbol{\mathcal{U}}_{2m}] = \left[\underbrace{\begin{array}{cccc} 0 & \dots & 0 & \overset{m\downarrow}{-\frac{1}{N}} \\ 0 & \dots & 0 & \frac{1}{N} \end{array}}_{m} \dots \underbrace{\begin{array}{cccc} 0 & \dots & 0 & \overset{sm\downarrow}{-\frac{1}{N}} \\ 0 & \dots & 0 & \frac{s}{N} \end{array}}_{m} \dots \underbrace{\begin{array}{cccc} 0 & \dots & 0 & \overset{(N-1)m\downarrow}{-\frac{1}{N}} \\ 0 & \dots & 0 & \frac{N-1}{N} \end{array}}_{m} \right],$$

$$\vdots$$

$$[\boldsymbol{\mathcal{U}}_{2k+1}] = \left[\underbrace{\begin{array}{ccccccc} 0 & \dots & \overset{k\downarrow}{0} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & \dots & \frac{1}{N} & 0 & \dots & 0 \end{array}}_{m} \dots \underbrace{\begin{array}{cccccc} 0 & \dots & \overset{k+(s-1)m\downarrow}{0} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & \dots & \frac{s}{N} & 0 & \dots & 0 \end{array}}_{m} \dots \right.$$

$$\cdots \underbrace{\begin{matrix} 0 & \cdots & \overset{k+(N-2)m\downarrow}{0} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & \cdots & \frac{N-1}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \Bigg],$$

$$[\boldsymbol{\mathcal{U}}_{2k}] = \Bigg[\underbrace{\begin{matrix} 0 & \cdots & \overset{k\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & \cdots & \frac{1}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \cdots \underbrace{\begin{matrix} 0 & \cdots & \overset{k+(s-1)m\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & \cdots & \frac{s}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \cdots$$

$$\cdots \underbrace{\begin{matrix} 0 & \cdots & \overset{k+(N-2)m\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & \cdots & \frac{N-1}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \Bigg],$$

$$\vdots$$

$$[\boldsymbol{\mathcal{U}}_{2}] = \Bigg[\underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ \frac{1}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \cdots \underbrace{\begin{matrix} \overset{1+(s-1)m\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ \frac{s}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \cdots \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ \frac{N-1}{N} & 0 & \cdots & 0 \end{matrix}}_{m} \Bigg],$$

$$[\boldsymbol{\mathcal{U}}_{1}] = \Bigg[\underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & 0 & \cdots & 0 \end{matrix}}_{m} \cdots \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & 0 & \cdots & 0 \end{matrix}}_{m} \cdots \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & 0 & \cdots & 0 \end{matrix}}_{m} \Bigg].$$

In the following we determine the vector of Riemann constants and Riemann divisor of the curve $\mathcal{C}$.

Lemma 3.2. *The vector of Riemann constants computed in the homology basis described in Figure 1 and with base point P_∞ equals*

$$\boldsymbol{K}_\infty = (N-1)\sum_{k=1}^{m} \int_{P_\infty}^{P_{2k}} \mathrm{d}\boldsymbol{v}, \quad P_{2k} = (\lambda_{2k}, 0). \tag{3.1}$$

The Riemann divisor Δ of the curve $\mathcal{C}$ in the homology basis described in Figure 1 is equivalent to

$$\Delta = (N-1)\sum_{k=1}^{m} P_{2k} - P_\infty. \tag{3.2}$$

Proof. The proof of (3.1) is obtained by direct calculations from the definition (2.15) and Lemma 3.1. The relation (3.2) follows immediately from (3.1). □

Following Diez [7], we describe a family of non-special divisors of degree g on $\mathcal{C}$, supported on the branch points. Denote by $\boldsymbol{s} = (s_1, \ldots, s_{2m+1})$ a $2m+1$-vector with non-negative entries s_k, satisfying the condition

$$\sum_{i=1}^{2m+1} s_i = (N-1)m, \quad 0 \le s_i \le N-1. \tag{3.3}$$

To each vector $\boldsymbol{s}$ we put into correspondence the divisors

$$\mathcal{D}_{\boldsymbol{s}} = \sum_{k=1}^{2m+1} s_k P_k \tag{3.4}$$

where $P_k=(\lambda_k,0)$, $k=1\dots,2m+1$, are branch points. In particular we shall consider the divisor class $\mathcal{D}_{\boldsymbol{m}}$ containing $\left(\begin{array}{c}2m+1\\ m\end{array}\right)$ divisors

$$\mathcal{D}_{\boldsymbol{m}}=(N-1)P_{i_1}+\ldots+(N-1)P_{i_{m-1}}+(N-1)P_{i_m}, \tag{3.5}$$

where the indices $\{i_1,i_2,\dots,i_m\}\subset\{1,\dots,2m+2\}$. Among the divisors with $m+1$ branch points we consider the divisor class $\mathcal{D}_{\boldsymbol{m+1}}$ which contains $\frac{1}{2}(m+2)(m+1)\left(\begin{array}{c}2m+1\\ m-1\end{array}\right)$ divisors

$$\mathcal{D}_{\boldsymbol{m+1}}=(N-1)P_{i_1}+\ldots+(N-1)P_{i_{m-1}}+(N-2)P_{i_m}+P_{i_{m+1}}, \tag{3.6}$$

where the indices $\{i_1,i_2,\dots,i_m,i_{m+1}\}\subset\{1,\dots,2m+2\}$. It is out of the scope of the present manuscript to classify all the non-special divisors of the form (3.4). However we can single out two families of non-special divisors.

Lemma 3.3. *The divisors $\mathcal{D}_{\boldsymbol{m}}$ defined in (3.5) are non-special and the divisors $\mathcal{D}_{\boldsymbol{m+1}}$ defined in (3.6) are non-special for $N>3$. At $N=3$ the divisors*

$$\mathcal{D}_{\boldsymbol{m+1}}=2P_{i_1}+\ldots+2P_{i_{m-1}}+P_{i_m}+P_{i_{m+1}}, \tag{3.7}$$

are non-special if one of the following conditions are satisfied

- *i_m and i_{m+1} have different parity,*
- *i_m and i_k have different parity for $k<m$, $m>1$,*
- *i_k and i_j have different parity for $k,j<m$ and $m>1$.*

Proof. Assume the opposite: suppose that the divisor $\mathcal{D}_{\boldsymbol{m}}$ or $\mathcal{D}_{\boldsymbol{m+1}}$ is special, this means that there exists a non-constant meromorphic function $f(\lambda,\mu)$ whose divisor of poles is $\mathcal{D}_{\boldsymbol{m}}$ or $\mathcal{D}_{\boldsymbol{m+1}}$. For simplify the proof, we assume that $P_{i_k}\neq P_\infty$, $k=1,\dots,m+1$. Let $\mathbb{I}_l$ be a subset of $\{1,3,5,\dots,2m+1\}$ with l distinct elements. The function

$$\phi(\lambda,y)=f(\lambda,y)\prod_{i_j\in\mathbb{I}_l}(\lambda-\lambda_{i_j}),\quad l=m,m+1,$$

has poles only at infinity. It follows from the *Weierstrass gap theorem*, that the ring of meromorphic functions with poles at infinity is generated in the case of the curve $y^N=p(\lambda)q^{N-1}(\lambda)$ by powers of λ and functions $y^k/q(\lambda)^{k-1}$, $k=1,\dots,N-1$ with $q(\lambda)=\prod_{j=1}^m(\lambda-\lambda_{2j})$. Therefore the function $\phi(\lambda,y)$ can be written in the form

$$\phi(\lambda,y)=R_0(\lambda)+\sum_{k=1}^{N-1}R_k(\lambda)\frac{y^k}{q^{k-1}(\lambda)}, \tag{3.8}$$

where $R_k(\lambda)$ are polynomials in λ.[1]

We remark that $\mathrm{ord}_\infty\left(R_k(\lambda)\frac{y^k}{q^{k-1}}\right)\neq\mathrm{ord}_\infty\left(R_j(\lambda)\frac{y^j}{q^{j-1}}\right)$ for $k\neq j$ because otherwise

$$N\mathrm{ord}_\lambda R_k(\lambda)+k=N\mathrm{ord}_\lambda R_j(\lambda)+j, \tag{3.9}$$

which implies $k=j$. From this observation it follows that

$$\mathrm{ord}_\infty(f(\lambda,y)\prod_{i_n\in\mathbb{I}_l}(\lambda-\lambda_{i_n}))=\mathrm{ord}_\infty\left(R_j(\lambda)\frac{y^j}{q^{j-1}(\lambda)}\right) \tag{3.10}$$

[1] In this point our proof differs from that given in [7] which is working for Galois covers of the form $y^N=\prod_{k=1}^{mN}(\lambda-\lambda_k)$ where the ansatz for the function (3.8) can be written as $\sum R_iy^i$.

for some $0 \le j \le N-1$. Moreover

$$\mathrm{ord}_\infty(f(\lambda,y)\prod_{i_n\in\mathbb{I}_l}(\lambda-\lambda_{i_n})) = -N|\mathbb{I}_l| + k_l, \quad l = m, m+1,$$

where k_l, $l = m, m+1$, is the order at infinity of $f(\lambda, y)$ and $|\mathbb{I}_l| = l$. From the equation of the curve we get $\deg y = mN + 1$. Therefore the equality (3.10) can be written as

$$N|\mathbb{I}_l| - k_l = N(r_j + m) + j, \quad l = m,\, m+1,$$

where r_j is the degree of $R_j(\lambda)$, so that

$$j = N(|\mathbb{I}_l| - r_j - m) - k_l \ge 0.$$

When $l = m$ that is $|\mathbb{I}_l| = m$, it follow that $r_j = 0$, $j = 0$, $k_m = 0$ and

$$f(\lambda,y) = \frac{1}{\prod_{i_n\in\mathbb{I}_m}(\lambda-\lambda_{i_n})}$$

and contradicts the assumption that $f(\lambda, y)$ has divisor $\mathcal{D}_{\boldsymbol{m}}$.

When $l = m + 1$, that is $|\mathbb{I}_l| = m + 1$, two possibilities occurs: (i) $r_j = 0,\ j = N - k_{m+1},\ 0 \le k_{m+1} < N$ and (ii) $r_j = 1$, $k_l = 0$, $j = 0$. This latter case can be easily excluded while for the former one we have

$$f(\lambda,y) = \frac{1}{\prod_{i_j\in\mathbb{I}_{m+1}}(\lambda-\lambda_{i_j})}\frac{y^{N-k_{m+1}}}{q^{N-k_{m+1}-1}(\lambda)}, \quad \mathrm{ord}_\infty(f(\lambda,y)) = k_{m+1},$$

which has divisor

$$\mathrm{Div} f(\lambda,y) = -N\sum_{i_n\in\mathbb{I}_{m+1}} P_{i_n} + (N-k_{m+1})\sum_{j=1}^{m+1} P_{2j+1} + k_{m+1}\sum_{j=1}^{m} P_{2j}.$$

Namely the divisors of poles of $f(\lambda, y)$ is

$$\mathrm{Div}_{\mathrm{poles}} f(\lambda,y) = (N-k_{m+1})\sum_{i_n\in\mathbb{I}_{m+1}, i_n \text{ even}} P_{i_n} + k_{m+1}\sum_{i_n\in\mathbb{I}_{m+1}, i_n \text{ odd}} P_{i_n}$$

and for $N > 3$, differs from $\mathcal{D}_{\boldsymbol{m+1}}$. This contradicts the assumption unless f is constant. For $N = 3$ the divisor of poles of $f(\lambda, y)$ coincides with $\mathcal{D}_{\boldsymbol{m+1}}$ in the following two cases:

$$\mathcal{D}_{\boldsymbol{m+1}} = 2\sum_{k=1}^{m-1} P_{i_k} + P_{i_m} + P_{i_{m+1}},$$

with

$$i_m,\, i_{m+1} \in \{2, 4, 6\ldots, 2m\}, \quad i_k \in \{1, 3, 5, \ldots, 2m+1\},\ k = 1, \ldots, m-1$$

or

$$i_m,\, i_{m+1} \in \{1, 3, 5\ldots, 2m+1\}, \quad i_k \in \{2, 4, 6, \ldots, 2m\},\ k = 1, \ldots, m-1.$$

We conclude that the divisors (3.7) are non special when one of the following conditions are satisfied: 1) i_m and i_{m+1} have different parity; 2) i_m and i_k, $k < m$, $m > 1$ have different parity; 3) i_j and i_k have different parity for $j, k < m$, $m > 1$. □

By the above lemma and by the Riemann singularity theorem 2.1, the vector

$$\boldsymbol{e_m} = \mathfrak{A}(\mathcal{D}_{\boldsymbol{m}} - Q_0 - \Delta), \quad Q_0 \notin \mathcal{D}_{\boldsymbol{m}}.$$

is a non singular $1/N$ period, namely $\theta(\boldsymbol{e_m}; \Pi) \ne 0$. In the above expression we can get rid of the base point Q_0. For the purpose, we introduce the divisor

$$\mathcal{D} = Q + J(Q) + \cdots + J^{N-1}Q,$$

which is independent from the point Q and satisfies the relation

$$\mathcal{D} \equiv NP_j, \quad j = 1, \ldots, 2m+2, \;\; P_{2m+2} = P_\infty.$$

Let $J_0 = \{2, 4, \ldots, 2m+2\}$ and $I_0 = \{1, 3, \ldots, 2m+1\}$ be a partition of the branch points in odd and even and $P_\infty = P_{2m+2}$. Let $J_1 \subset J_0$ and $I_1 \subset I_0$ be a partition such that

$$|J_1| + |I_1| = m + 1.$$

Consider the vector

$$\boldsymbol{e_m} = \mathfrak{A}((N-1)\sum_{i \in I_1} P_i + (N-1)\sum_{j \in J_1} P_j - \mathcal{D} - \Delta). \tag{3.11}$$

Then from Lemma 3.3, the vector $\boldsymbol{e_m}$ is non singular, namely $\theta(\boldsymbol{e_m}; \Pi) \neq 0$. In the same way we define the vector obtained from the divisors $\mathcal{D}_{\boldsymbol{m+1}}$ in the following way

$$\boldsymbol{e_{m+1}} = \mathfrak{A}((N-1)\sum_{i \in I_1} P_i + (N-1)\sum_{j \in J_1} P_j + (N-2)P_{i_m} + P_{i_{m+1}} - \mathcal{D} - \Delta), \tag{3.12}$$

where now

$$|I_1| + |J_1| = m - 1, \quad I_1 \subset I_0, \;\; J_1 \subset J_0,$$

and

$$i_m \in (I_0 - I_1), \quad i_{m+1} \in (J_0 - J_1).$$

From Lemma 3.3, the vector $\boldsymbol{e_{m+1}}$ is non singular.

In the next sections, we are going to study the Szegö kernel associated to the characteristics (3.11) and (3.12).

4. Kernel-forms

The Schottky-Klein prime form $E(P,Q)$, $P, Q \in \mathcal{C}$ is a skew-symmetric $(-\frac{1}{2}, -\frac{1}{2})$-form on $\mathcal{C} \times \mathcal{C}$ [14]

$$E(P,Q) = \frac{\theta[\boldsymbol{\gamma}]\left(\int\limits_P^Q \mathrm{d}\boldsymbol{v}; \Pi\right)}{\sqrt{\sum_{j=1}^g \frac{\partial}{\partial z_j}\theta[\boldsymbol{\gamma}](\boldsymbol{0};\Pi)\mathrm{d}v_j(P)}\sqrt{\sum_{j=1}^g \frac{\partial}{\partial z_j}\theta[\boldsymbol{\gamma}](\boldsymbol{0};\Pi)\mathrm{d}v_j(Q)}}, \tag{4.1}$$

where $[\boldsymbol{\gamma}]$ are non-singular odd half-integer characteristics.

The prime form does not depend on the characteristics $[\boldsymbol{\gamma}]$. The automorphic factors of the prime form along all the cycles α_k are trivial; the automorphic factor along each β_k cycle in the Q variable equals $\exp\{-\pi\imath\Pi_{kk} - 2\pi\imath\int_P^Q \mathrm{d}v_k\}$. If the points P and Q are placed in the vicinity of the point R with local coordinate x, $x(R) = 0$, then the prime form has the following local behavior as $Q \to P$

$$E(P,Q) = \frac{x(Q) - x(P)}{\sqrt{\mathrm{d}x(P)}\sqrt{\mathrm{d}x(Q)}}\left(1 + O(1)\right). \tag{4.2}$$

The canonical bimeromorphic differential $\omega(P,Q)$ is defined as a symmetric 2-differential,

$$\omega(P,Q) = \mathrm{d}_{x(P)}\mathrm{d}_{x(Q)} \log E(P,Q). \tag{4.3}$$

All α-periods of $\omega(P,Q)$ with respect to any of its two variables vanish. The period of the 2-differential $\omega(P,Q)$ with respect to the variable P or Q, along the β_k cycle, is equal to $2\pi\imath\mathrm{d}v_k(Q)$ or $2\pi\imath\mathrm{d}v_k(P)$ respectively.

The 2-differential $\omega(P,Q)$ has a double pole along the diagonal with the following local behavior [39, 14]

$$\omega(P,Q)=\left(\frac{1}{(x(P)-x(Q))^2}+H(x(P),x(Q))+\text{higher order terms}\right)\mathrm{d}x(P)\mathrm{d}x(Q), \tag{4.4}$$

where $H(x(P),x(Q))\mathrm{d}x(P)\mathrm{d}x(Q)$ is the non-singular part of $\omega(P,Q)$ in each coordinate chart. The restriction of H on the diagonal is the Bergman projective connection (see for example [37])

$$R(x(P))=6H(x(P),x(P)), \tag{4.5}$$

which depends non-trivially on the chosen system of local coordinate $x(P)$. Namely the projective connection transforms as follows with respect to a change of local coordinates $x\to f(x)$

$$R(x)\to R(f(x))[f'(x)]^2+\{f(x),x\},$$

where $\{\,,\,\}$ is the Schwarzian derivative.

The Szegö kernel $S[\boldsymbol{e}](P,Q)$ is defined for all non-singular characteristics $[\boldsymbol{e}]$ as the $(\frac12,\frac12)$-form on $\mathcal{C}\times\mathcal{C}$ [14]

$$S[\boldsymbol{e}](P,Q)=\frac{\theta[\boldsymbol{e}]\left(\int\limits_P^Q\mathrm{d}\boldsymbol{v};\Pi\right)}{\theta[\boldsymbol{e}](\mathbf{0};\Pi)E(P,Q)}. \tag{4.6}$$

The Szegö kernel transforms when the variable Q goes around α_k and β_k-cycles as follows

$$\begin{aligned}S[\boldsymbol{e}](P,\,Q+\alpha_k)&=\mathrm{e}^{2\pi\imath\delta_k}S[\boldsymbol{e}](P,Q),\\ S[\boldsymbol{e}](P,\,Q+\beta_k)&=\mathrm{e}^{-2\pi\imath\epsilon_k}S[\boldsymbol{e}](P,Q),\end{aligned}\qquad k=1,\ldots,g \tag{4.7}$$

The local behaviour of the Szegö kernel when $Q\to P$ is

$$S[\boldsymbol{e}](P,Q)=\frac{\sqrt{\mathrm{d}x(P)}\sqrt{\mathrm{d}x(Q)}}{x(Q)-x(P)}\left[1+T(x(Q))(x(Q)-x(P))+O((x(Q)-x(P)^2)\right], \tag{4.8}$$

where

$$T(x(Q))dx(Q)=\sum_{k=1}^{g}\frac{\partial}{\partial z_k}\log\theta[\boldsymbol{e}](\mathbf{0};\Pi)\mathrm{d}v_k(x(Q)). \tag{4.9}$$

The important relation [14], Cor. 2.12, connects the Szegö kernel with canonical bimeromorphic differential

$$S[\boldsymbol{e}](P,Q)S[-\boldsymbol{e}](P,Q)=\omega(P,Q)+\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\theta[\boldsymbol{e}](\mathbf{0};\Pi)\mathrm{d}v_k(P)\mathrm{d}v_l(Q). \tag{4.10}$$

In the following we are going to give an algebraic expression for the Szegö kernel associated with the characteristics

$$\boldsymbol{e_m}=\mathfrak{A}((N-1)\sum_{i\in I_1}P_i+(N-1)\sum_{j\in J_1}P_j-\mathcal{D}-\Delta),$$

defined in (3.11). For simplicity, we assume $2m+2\notin J_1$. We define the functions

$$\psi_k(P,Q)=\frac{x(Q)-\lambda_k}{x(P)-\lambda_k},\quad k=1,\ldots,2m+1. \tag{4.11}$$

Theorem 4.1. *The Szegö kernel with characteristics $[\boldsymbol{e_m}]$ defined in (3.11) is given by the formula*

$$
\begin{aligned}
S[\boldsymbol{e_m}](P,Q) = & \frac{\sqrt{dx(P)dx(Q)}}{N(x(Q)-x(P))} \\
& \times \sum_{s=0}^{N-1}\left[\left(\frac{\prod_{i\in I_1}\psi_i}{\prod_{j\in J_2}\psi_j}\right)^{\frac{N-1}{2N}-\frac{s}{N}}\left(\frac{\prod_{i\in I_2}\psi_i}{\prod_{j\in J_1}\psi_j}\right)^{\frac{N-1}{2N}-\left\{\frac{s+1}{N}\right\}}\right],
\end{aligned}
\tag{4.12}
$$

where $\{\ \}$ *is the fractional part and*

$$J_2 = J_0 - J_1 - 2m + 2, \quad I_2 = I_0 - I_1.$$

In particular, the Szegö kernel with zero characteristics is obtained by fixing $I_1 = I_0$ *and* $J_1 = \emptyset$ *and takes the form*

$$S[0](P,Q) = \frac{1}{N}\frac{\sqrt{\mathrm{d}x(P)\mathrm{d}x(Q)}}{x(Q)-x(P)}\sum_{s=0}^{N-1}\left(\frac{q(x(P))}{p(x(P))}\frac{p(x(Q))}{q(x(Q))}\right)^{-\frac{s}{N}+\frac{N-1}{2N}}, \tag{4.13}$$

where the polynomials $p(\lambda)$ *and* $q(\lambda)$ *have been defined in (2.2).*

The proof is based on the uniqueness of the Szegö kernel (see e.g. Narasimhan [32]) and the results of Nakayashiki [31]. Indeed it is sufficient to check that the expression (4.12) is regular everywhere but on the diagonal where $P = Q$ and that its divisor in the variables P and Q coincides with the divisor of the Szegö kernel given by the formula (4.6). Using the coordinate chart given in (2.3), the regularity of the expression (4.12) can be checked in a straighforward way. In the same way, by first fixing $P = P_{i_1}$ and then $Q = P_{i_1}$ with $i_1 \in I_0 \cup J_0$ it is straighforward to obtain the divisor class of (4.12).

Corollary 4.2. *The expansion of the Szegö kernel (4.12) along the diagonal takes the form*

$$
\begin{aligned}
S[\boldsymbol{e_m}](P,Q) \simeq & \frac{\sqrt{dx(P)dx(Q)}}{x(Q)-x(P)} \\
& \times\left\{1+\left(\frac{1}{12}\{x(Q),Q\}+\phi[\boldsymbol{e_m}](Q)\right)(x(P)-x(Q))^2\right\},
\end{aligned}
$$

where the function $\phi[\boldsymbol{e_m}](\lambda)$ *takes the form*

$$
\begin{aligned}
\phi[\boldsymbol{e_m}](\lambda) = & \frac{N^2-1}{24N^2}\left[\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_1}(\lambda-\lambda_i)}{\prod_{j\in J_2}(\lambda-\lambda_j)}\right)^2+\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_2}(\lambda-\lambda_i)}{\prod_{j\in J_1}(\lambda-\lambda_j)}\right)^2\right] \\
& + \frac{2(N-1)(N-5)}{24N^2}\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_1}(\lambda-\lambda_i)}{\prod_{j\in J_2}(\lambda-\lambda_j)}\right)\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_2}(\lambda-\lambda_i)}{\prod_{j\in J_1}(\lambda-\lambda_j)}\right).
\end{aligned}
\tag{4.14}
$$

The proof of the above corollary is obtained by direct calculation. We remark that for $I_1 = I_0$ and $J_1 = \emptyset$ the second term in (4.14) disappears and we obtain

$$\phi[\mathbf{0}](\lambda) = \frac{N^2-1}{24N^2}\left(\frac{d}{d\lambda}\log\frac{p(\lambda)}{q(\lambda)}\right)^2,$$

where $p(\lambda)$ and $q(\lambda)$ have been defined in (2.2).
Since

$$-\boldsymbol{e_m} = \mathfrak{A}((N-1)\sum_{i\in I_0-I_1}P_i + (N-1)\sum_{j\in J_0-J_1}P_j - \mathcal{D} - \Delta)$$

because

$$(N-1)\sum_{k\in I_0\cup J_0} P_k - 2\mathcal{D} - 2\Delta \equiv 0$$

we conclude that

$$\phi[\boldsymbol{e_m}](Q) = \phi[-\boldsymbol{e_m}](Q).$$

Therefore, the following identity holds

$$S[\boldsymbol{e_m}](P,Q)S[-\boldsymbol{e_m}](P,Q) = \frac{\mathrm{d}x(P)\mathrm{d}x(Q)}{(x(P)-x(Q))^2}$$
$$\times\left\{1+\left[\frac{1}{6}\{x(P),P\}+2\phi[\boldsymbol{e_m}](P)\right](x(P)-x(Q))^2+\dots\right\}.$$

The relations (4.8), (4.9) and the corollary 4.2 imply

$$\frac{\partial}{\partial z_k}\theta[\boldsymbol{e_m}](\boldsymbol{z};\Pi)\,|_{\boldsymbol{z}=0} = 0\,, \quad k=1,\dots,g, \tag{4.15}$$

for the vector $\boldsymbol{e_m}$ defined in (3.11).

In the following we are going to give an algebraic expression for the Szegö kernel associated to the characteristics

$$\boldsymbol{e_{m+1}} = \mathfrak{A}((N-1)\sum_{i\in I_1} P_i + (N-1)\sum_{j\in J_1} P_j + (N-2)P_{j_m} + P_{i_m} - \mathcal{D} - \Delta), \tag{4.16}$$

where

$$|I_1| + |J_1| = m-1, \quad i_m \in I_0 - I_1, \;\; j_m \in J_0 - J_1$$

The construction is very similar to the previous case. Also in this case, for simplifying the notation, we assume that $2m+2 \notin J_1$ and $j_m \neq 2m+2$. We define the following sets

$$I_2 = I_0 - I_1 - i_m, \;\; J_2 = J_0 - J_1 - j_m - 2m+2.$$

Theorem 4.3. *The Szegö kernel with characteristics* $[\boldsymbol{e_{m+1}}]$ *defined in (4.16) is given by the formula*

$$S[\boldsymbol{e_{m+1}}](P,Q) = \frac{\sqrt{dx(P)dx(Q)}}{N(x(Q)-x(P))}\times$$
$$\sum_{s=0}^{N-1}\left[\left(\frac{\prod_{i\in I_1}\psi_i}{\prod_{j\in J_2}\psi_j}\right)^{\frac{N-1}{2N}-\frac{s}{N}}\left(\frac{\prod_{i\in I_2}\psi_i}{\prod_{j\in J_1}\psi_j}\right)^{\frac{N-1}{2N}-\left\{\frac{s+1}{N}\right\}}\left(\frac{\psi_{i_m}}{\psi_{j_m}}\right)^{\frac{N-1}{2N}-\left\{\frac{s+2}{N}\right\}}\right], \tag{4.17}$$

where $\{\ \}$ *is the fractional part and the function* ψ_i *has been defined in (4.11).*

The proof is based on the uniqueness of the Szegö kernel (see e.g. Narasimhan [32]) and the results of Nakayashiki [31].

Corollary 4.4. *The expansion of the Szegö kernel (4.12) along the diagonal takes the form*

$$S[\boldsymbol{e_{m+1}}](P,Q) \simeq \frac{\sqrt{dx(P)dx(Q)}}{x(Q)-x(P)}$$
$$\times\left\{1+\left(\frac{1}{12}\{x(Q),Q\}+\phi[\boldsymbol{e_{m+1}}](Q)\right)(x(P)-x(Q))^2\right\},$$

where the function $\phi[\boldsymbol{e_{m+1}}](\lambda)$ *is*

$$
\begin{aligned}
\phi[\boldsymbol{e_{m+1}}](\lambda) = \frac{N^2-1}{24N^2}\Bigg[&\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_1}(\lambda-\lambda_i)}{\prod_{j\in J_2}(\lambda-\lambda_j)}\right)^2 + \left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_2}(\lambda-\lambda_i)}{\prod_{j\in J_1}(\lambda-\lambda_j)}\right)^2 + \\
&+\left(\frac{d}{d\lambda}\log\frac{(\lambda-\lambda_{i_m})}{(\lambda-\lambda_{j_m})}\right)^2\Bigg] \\
&+\frac{2(N-1)(N-5)}{24N^2}\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_1'}(\lambda-\lambda_i)}{\prod_{j\in J_2'}(\lambda-\lambda_j)}\right)\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_2}(\lambda-\lambda_i)}{\prod_{j\in J_1}(\lambda-\lambda_j)}\right) \\
&+\frac{2(N^2-12N+23)}{24N^2}\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_1}(\lambda-\lambda_i)}{\prod_{j\in J_2}(\lambda-\lambda_j)}\right)\left(\frac{d}{d\lambda}\log\frac{(\lambda-\lambda_{i_m})}{(\lambda-\lambda_{j_m})}\right),
\end{aligned}
\tag{4.18}
$$

where

$$I_1' = I_1 + i_m, \quad J_2' = J_2 + j_m.$$

The proof of the above corollary is obtained by direct calculation.
Since

$$-\boldsymbol{e_{m+1}} = \mathfrak{A}(N-1)\sum_{i\in I_2}P_i + (N-1)\sum_{j\in J_0-J_1-j_m}P_j + (N-2)P_{i_m} - P_{j_m} - \mathcal{D} - \Delta)$$

because

$$(N-1)\sum_{k\in I_0\cup J_0}P_k - 2\mathcal{D} - 2\Delta \equiv 0$$

we conclude that

$$\phi[\boldsymbol{e_{m+1}}](Q) = \phi[-\boldsymbol{e_{m+1}}](Q).$$

Therefore, the following identity holds

$$
\begin{aligned}
S[\boldsymbol{e_{m+1}}](P,Q)S[-\boldsymbol{e_{m+1}}](P,Q) &= \frac{\mathrm{d}x(P)\mathrm{d}x(Q)}{(x(P)-x(Q))^2} \\
&\times\left\{1+\left[\frac{1}{6}\{x(P),P\}+2\phi[\boldsymbol{e_{m+1}}](P)\right](x(P)-x(Q))^2+\dots\right\},
\end{aligned}
$$

We remark that from (4.8), (4.9) and the corollary 4.2

$$\frac{\partial}{\partial z_k}\theta[\boldsymbol{e_{m+1}}](\boldsymbol{z};\Pi)\,|_{\boldsymbol{z}=0} = 0\,, \quad k=1,\dots,g, \tag{4.19}$$

for the characteristics $\boldsymbol{e_{m+1}}$ defined in (3.12).

5. Algebraic realization of the canoncal bimeromorphic differential

The canoncal bimeromorphic differential can be given in an algebraic form due to Klein [23],[24] also [2],[18] and [14]. To develop this approach we first write the third kind differential with poles in two arbitrary points. For the purpose we consider an arbitrary curve $\mathcal{C}$ given by the polynomial equation $f(\lambda,\mu)=0$ of degree N in the variable μ. We suppose that the curve $\mathcal{C}$ has a branch point at infinity. Let

$$\boldsymbol{\Psi}(\lambda,\mu) = (1,\psi_1(\lambda,\mu),\dots,\psi_{N-1}(\lambda,\mu)) \tag{5.1}$$

be the basis in the ring $\mathcal{O}(\mathcal{C})$ of meromorphic functions on $\mathcal{C}$ with the only pole at infinity. There exists a vector function

$$\boldsymbol{\Phi}(\lambda,\mu) = (1,\Phi_1(\lambda,\mu),\dots,\Phi_{N-1}(\lambda,\mu)) \tag{5.2}$$

for which

$$\langle \mathbf{\Psi}(\lambda,\mu), \mathbf{\Phi}(\lambda,\mu')\rangle = \frac{f(\lambda,\mu')-f(\lambda,\mu)}{\mu'-\mu}. \tag{5.3}$$

Let $Q=(\lambda',\mu')$ and $R=(\lambda'',\mu'')$ be two arbitrary points of the curve $\mathcal{C}$. The third kind differential $\Omega_{Q,R}(P)$ with simple poles with residues ±1 in the points $P=Q$ and $P=R$ is given by the formula

$$\Omega_{Q,R}(P) = \left(\frac{\langle \mathbf{\Psi}(Q),\mathbf{\Phi}(P)\rangle}{\lambda-\lambda'} - \frac{\langle \mathbf{\Psi}(R),\mathbf{\Phi}(P)\rangle}{\lambda-\lambda''}\right)\frac{\mathrm{d}\lambda}{f_\mu(\lambda,\mu)}, \tag{5.4}$$

with $f_\mu(\lambda,\mu)=\partial f(\lambda,\mu)/\partial\mu$. In the case of the curve (2.1) the vectors $\mathbf{\Psi}(\lambda,\mu)$ (see the proof of Lemma 3.3) and $\mathbf{\Phi}(\lambda,\mu)$ take the form

$$\mathbf{\Psi}(\lambda,\mu) = \left(1,\mu,\frac{\mu^2}{q(\lambda)},\ldots,\frac{\mu^{N-1}}{q(\lambda)^{N-2}}\right), \tag{5.5}$$

$$\mathbf{\Phi}(\lambda,\mu) = \left(\mu^{N-1},\mu^{N-2},q(\lambda)\mu^{N-3},\ldots,q(\lambda)^{N-2}\right). \tag{5.6}$$

The canoncal bimeromorphic differential can be obtained by differentiation of (5.4) as follows

$$\omega(P,Q) = \mathrm{d}\lambda'\frac{\partial}{\partial\lambda'}\frac{\langle \mathbf{\Psi}(Q),\mathbf{\Phi}(P)\rangle}{\lambda-\lambda'}\frac{\mathrm{d}\lambda}{f_\mu(\lambda,\mu)} + \mathrm{d}\chi(P,Q), \tag{5.7}$$

where $\mathrm{d}\chi(P,Q)$ is the 2-form uniquely defined by the requirement that (5.7) is a symmetric bi-differential normalized with respect to the α cycles and with the only pole of second order along the diagonal. The above arguments lead to the following algebraic expression for the canoncal bimeromorphic differential:

$$\omega(P,Q) = \frac{\partial}{\partial\lambda'}\frac{1}{\lambda-\lambda'}\left[1+\sum_{s=1}^{N-1}\frac{{\mu'}^s q(\lambda)^{s-1}}{\mu^s q(\lambda')^{s-1}}\right]\frac{\mathrm{d}\lambda\mathrm{d}\lambda'}{N} + \mathrm{d}\chi(P,Q), \tag{5.8}$$

$$\mathrm{d}\chi(P,Q) = -\frac{1}{N}\sum_{s=1}^{N}\sum_{j=1}^{m}\lambda^{j-1}\frac{q(\lambda)^{s-1}}{\mu^s}\frac{q(\lambda')^{N-s-1}}{{\mu'}^{N-s}}\mathcal{R}_{s,j}(\lambda')\mathrm{d}\lambda\mathrm{d}\lambda', \tag{5.9}$$

with certain polynomials $\mathcal{R}_{s,j}(\lambda,\mu)$. Here we do not need the exact form of these polynomials.

Expanding the above expression along the diagonal, we obtain the Bergman projective connection

$$\begin{aligned}\mathcal{H}(P) &= \frac{1}{6}\{x(P),P\} + \frac{\mathrm{d}\chi(P,P)}{(\mathrm{d}x(P))^2}\\ &- \frac{1}{4}\frac{N-1}{N}\left(\frac{\frac{\partial^2}{\partial\lambda^2}p(\lambda)}{p(\lambda)} + \frac{\frac{\partial^2}{\partial\lambda^2}q(\lambda)}{q(\lambda)}\right) + \frac{N^2-1}{12N^2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\frac{p(\lambda)}{q(\lambda)}\right)^2.\end{aligned} \tag{5.10}$$

Combining the above relation with the expansion (4.15) of the Szegö kernel along the diagonal and the Fay relation (4.10) which connects the canoncal bimeromorphic differential and Szegö kernel, we obtain an algebraic expression for the second derivatives of the theta function, namely

$$\begin{aligned}&\frac{1}{(\mathrm{d}x(P))^2}\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\theta[\boldsymbol{e_m}](\mathbf{0};\Pi)\mathrm{d}v_k(P)\mathrm{d}v_l(P)\\ &= \frac{1}{4}\frac{N-1}{N}\left(\frac{\frac{\partial^2}{\partial\lambda^2}p(\lambda)}{p(\lambda)} + \frac{\frac{\partial^2}{\partial\lambda^2}q(\lambda)}{q(\lambda)}\right) - \frac{N^2-1}{12N^2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\frac{p(\lambda)}{q(\lambda)}\right)^2 + 2\phi[\boldsymbol{e_m}](P),\end{aligned} \tag{5.11}$$

where $\phi[\boldsymbol{e_m}](P)$ has been defined in (4.14).

The following relation will be useful for the proof of the Thomae type formula. It connects the function $\mathrm{d}\chi(P,Q)$ defined in (5.9) with the derivative with respect to the branch points of the determinant of the matrix $\mathcal{A}$ of α-periods.

Lemma 5.1. *For $s=1,\ldots,N-1$ the following identities are valid*

$$\frac{\partial}{\partial\lambda_i}\log\det\mathcal{A}_s=\frac{1}{\prod\limits_{l=1,l\neq i}^{2m+1}(\lambda_i-\lambda_l)}\sum_{j=1}^{m}\lambda_i^{j-1}\mathcal{R}_{s,j}(\lambda_i),\quad i=1,\ldots,2m+1. \tag{5.12}$$

Proof. We integrate $\omega(P,Q)$ in the variable P along the α_k-cycle and expand $\oint_{\alpha_k}\omega(P,Q)=0$ in the variable Q in the vicinity of the branch point $(\lambda_i,0)$ where the local coordinate is introduced as $x(Q)=\lambda_i+\xi^N$, $i\in\{1,\ldots,2m+1\}$. In this way we obtain for every fixed s the rule of variation with respect to the branch points of the first line of $\mathcal{A}_s$-block of the period matrix $\mathcal{A}$:

$$\frac{\partial}{\partial\lambda_i}(\mathcal{A}_s)_{1,k}=\frac{1}{\prod\limits_{l=1,l\neq i}^{2m+1}(\lambda_i-\lambda_l)}\sum_{j=1}^{m}\mathcal{R}_{s,j}(\mathcal{A}_s)_{j,k},\quad k=1,\ldots,m.$$

The derivative of all other lines can be obtained from the equivalence

$$\frac{\partial}{\partial\lambda_i}(\mathcal{A}_s)_{n,k}=\lambda_i^{n-1}\frac{\partial}{\partial\lambda_i}(\mathcal{A}_s)_{1,k}+\frac{1}{N}\sum_{l=1}^{n-1}(\mathcal{A}_s)_{l,k}\lambda_i^{n-l-1},\quad n=2,\ldots,m,\ k=1,\ldots,m.$$

Therefore one can write

$$\frac{\partial}{\partial\lambda_i}(\mathcal{A}_s)=\mathcal{P}_s(\lambda_i)\mathcal{A}_s,\quad \mathrm{Tr}\,\mathcal{P}=\sum_{j=1}^{m}\lambda_i^{j-1}\mathcal{R}_{s,j}(\lambda_i).$$

The equality (5.12) follows. □

Proposition 5.2. *The following relation is valid*

$$\frac{\partial}{\partial\lambda_i}\log\det\mathcal{A}=-\operatorname*{Res}_{P=P_i}\frac{\mathrm{d}\chi(P,P)}{(\mathrm{d}x(P))^2}. \tag{5.13}$$

Proof. The residue of (5.9) is

$$\frac{1}{\prod\limits_{l=1,l\neq i}^{2m+1}(\lambda_i-\lambda_l)}\sum_{s=1}^{N-1}\sum_{j=1}^{m}\lambda_i^{j-1}\mathcal{R}_{s,j}(\lambda_i).$$

Then use the decomposition (2.8) and the Lemma 5.1. □

Example 5.3. In the case of elliptic functions the formula (5.13) represents the known relation

$$\frac{\partial\omega}{\partial\lambda_i}=-\frac{1}{2}\frac{\eta+\lambda_i\omega}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)},\quad i\neq j\neq k=1,2,3,$$

where the Weierstrass notations are used.

6. Derivation of the Thomae type formula

Now we are in a position to derive Thomae type formula

Theorem 6.1. *Let $[\boldsymbol{e_m}]$ be nonsingular $1/N$ characteristics, given by the formula*

$$\boldsymbol{e_m} = \mathfrak{A}((N-1)\sum_{i\in I_1} P_i + (N-1)\sum_{j\in J_1} P_j - \mathcal{D} - \Delta),$$

where $J_1 \subset J_0 = \{2,4,\dots,2m+2\}$ and $I_1 \subset I_0 = \{1,3,\dots,2m+1\}$, with

$$|J_1| + |I_1| = m+1.$$

The Thomae type formula takes the form

(6.1)

$$\theta[\boldsymbol{e_m}](0;\Pi)^{4N} = \frac{\prod_{i=1}^{N-1}\det\mathcal{A}_i^{2N}}{(2\pi\imath)^{2mN(N-1)}} \prod_{1\le i<k\le m}(\lambda_{2i}-\lambda_{2k})^{N(N-1)} \prod_{0\le i<k\le m}(\lambda_{2i+1}-\lambda_{2k+1})^{N(N-1)}$$
$$\times\left(\frac{\prod_{i\in I_1,j\in J_1}(\lambda_i-\lambda_j)\prod_{i\in I_2,j\in J_2}(\lambda_i-\lambda_j)}{\prod_{i\in I_1,k\in I_2}(\lambda_i-\lambda_k)\prod_{j\in J_1,k\in J_2}(\lambda_j-\lambda_k)}\right)^{2(N-1)},$$

where

$$I_2 = I_0 - I_1, \quad J_2 = J_0 - J_1 - 2m + 2.$$

When $I_1 = I_0$ and $J_1 = \emptyset$, the above formula reduces to

$$\theta[\mathbf{0}](0;\Pi)^{4N} = \frac{\prod_{i=1}^{N-1}\det\mathcal{A}_i^{2N}}{(2\pi\imath)^{2mN(N-1)}} \prod_{1\le i<k\le m}(\lambda_{2i}-\lambda_{2k})^{N(N-1)} \prod_{0\le i<k\le m}(\lambda_{2i+1}-\lambda_{2k+1})^{N(N-1)}. \tag{6.2}$$

Proof. Using the heat equation (2.13), the Rauch variational formula (2.10) and (4.15) we have

$$\frac{\partial}{\partial\lambda_i}\log\theta[\boldsymbol{e_m}](\mathbf{0};\Pi) = \sum_{k,r=1}^{(N-1)m}\frac{\partial}{\partial\Pi_{k,r}}\log\theta[\boldsymbol{e_m}](\mathbf{0};\Pi)\frac{\partial\Pi_{k,r}}{\partial\lambda_i} \tag{6.3}$$
$$= \frac{1}{2}\mathop{\mathrm{Res}}_{P=(\lambda_i,0)}\left\{\frac{1}{(\mathrm{d}x(P))^2}\sum_{k,r=1}^{(N-1)m}\frac{\partial^2}{\partial z_k\partial z_r}\log\theta[\boldsymbol{e_m}](\mathbf{0};\Pi)\sum_{s=1}^{N}\mathrm{d}v_k(P^{(s)})\mathrm{d}v_r(P^{(s)})\right\}.$$

To proceed we shall represent the residue as a logarithmic derivative. The equality (5.11) and the proposition 5.2 enable us to compute the residue in the r.h.s of (6.3) obtaining

$$\frac{\partial}{\partial\lambda_i}\log\theta[\boldsymbol{e_m}](\mathbf{0};\Pi)$$
$$= \frac{N-1}{8}\mathop{\mathrm{Res}}_{\lambda=\lambda_i}\left(\frac{\frac{\partial^2}{\partial\lambda^2}p(\lambda)}{p(\lambda)} + \frac{\frac{\partial^2}{\partial\lambda^2}q(\lambda)}{q(\lambda)}\right) + \frac{1}{2}\frac{\partial}{\partial\lambda_i}\log\det\mathcal{A}$$
$$+ \mathop{\mathrm{Res}}_{P=(\lambda_i,0)}\phi[\boldsymbol{e_m}](P) - \frac{N^2-1}{24N}\mathop{\mathrm{Res}}_{\lambda=\lambda_i}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\frac{p(\lambda)}{q(\lambda)}\right)^2,$$

where $\phi[\boldsymbol{e_m}](P)$ is defined in (4.14).

Let us compute each residue in the above formula:

$$\mathop{\mathrm{Res}}_{\lambda=\lambda_{2i}}\frac{\frac{\partial^2}{\partial\lambda^2}p(\lambda)}{p(\lambda)} = 2\frac{\partial}{\partial\lambda_{2i}}\log\prod_{1\le i<k\le m}(\lambda_{2i}-\lambda_{2k}) \text{ and } \mathop{\mathrm{Res}}_{\lambda=\lambda_{2i+1}}\frac{\frac{\partial^2}{\partial\lambda^2}p(\lambda)}{p(\lambda)} = 0,$$

$$\operatorname*{Res}_{\lambda=\lambda_{2i+1}} \frac{\frac{\partial^2}{\partial\lambda^2}q(\lambda)}{q(\lambda)} = 2\frac{\partial}{\partial\lambda_{2i+1}}\log \prod_{1\le i<k\le m}(\lambda_{2i+1}-\lambda_{2k+1}) \text{ and } \operatorname*{Res}_{\lambda=\lambda_{2i}} \frac{\frac{\partial^2}{\partial\lambda^2}q(\lambda)}{q(\lambda)} = 0.$$

Also we have for any partition $I\cup J$ and any λ_n

$$\operatorname*{Res}_{\lambda=\lambda_n}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\frac{\prod\limits_{i\in I}(\lambda-\lambda_i)}{\prod\limits_{j\in J}(\lambda-\lambda_j)}\right)^2 = 2\frac{\partial}{\partial\lambda_n}\log\frac{\prod\limits_{i<k\in I}(\lambda_i-\lambda_k)\prod\limits_{i<k\in J}(\lambda_i-\lambda_k)}{\prod\limits_{i\in I,j\in J}(\lambda_i-\lambda_j)}.$$

Furthermore

$$\operatorname*{Res}_{\lambda=\lambda_n}\left[\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_1}(\lambda-\lambda_i)}{\prod_{j\in J_2}(\lambda-\lambda_j)}\right)\left(\frac{d}{d\lambda}\log\frac{\prod_{i\in I_2}(\lambda-\lambda_i)}{\prod_{j\in J_1}(\lambda-\lambda_j)}\right)\right]$$
$$= \frac{\partial}{\partial\lambda_n}\log\left(\frac{\prod_{i\in I_1,k\in I_2}(\lambda_i-\lambda_k)\prod_{j\in J_1 k\in J_2}(\lambda_j-\lambda_k)}{\prod_{i\in I_1,j\in J_1}(\lambda_i-\lambda_j)\prod_{i\in I_2,j\in J_2}(\lambda_i-\lambda_j)}\right).$$

The integration in λ_n then gives

(6.4)
$$\theta[\boldsymbol{e_m}](0;\Pi)^{4N} = C^{2N}\prod_{i=1}^{N-1}\det\mathcal{A}_i^{2N}\prod_{1\le i<k\le m}(\lambda_{2i}-\lambda_{2k})^{N(N-1)}\prod_{0\le i<k\le m}(\lambda_{2i+1}-\lambda_{2k+1})^{N(N-1)}$$
$$\times\left(\frac{\prod_{i\in I_1,j\in J_1}(\lambda_i-\lambda_j)\prod_{i\in I_2,j\in J_2}(\lambda_i-\lambda_j)}{\prod_{i\in I_1,k\in I_2}(\lambda_i-\lambda_k)\prod_{j\in J_1,k\in J_2}(\lambda_j-\lambda_k)}\right)^{2(N-1)},$$

where C is a constant independent of Π and the branch points. To derive the above relation we also used the decomposition of the period matrix $\mathcal{A}$ into blocks $\mathcal{A}_i$.

To compute C we shall use the original Thomae arguments [36]. We consider the Thomae type formula (6.4) at zero characteristics, that is, the partition $I_1 = I_0$ and $J_1 = \emptyset$ and pinch the branch points in the following way

$$\lambda_{2k} = v_k - \epsilon,\quad \lambda_{2k-1} = v_k + \epsilon \quad k = 1,\dots,m,\quad 0<\epsilon\ll 1.$$

In this case the l.h.s of (6.4) becomes $\theta[\mathbf{0}](\mathbf{0};\Pi) = 1 + O(\epsilon)$. Regarding the r.h.s the following relations are needed:

$$\lim_{\epsilon\to 0}(\mathcal{A}_s)_{i,j} = 2\pi\imath\frac{v_i^{j-1}}{\prod\limits_{\substack{k\neq i\\k=1}}^{m}(v_i-v_k)(v_i-\lambda_{2m+1})^{\frac{s}{N}}}$$

so that

$$\lim_{\epsilon\to 0}(\det\mathcal{A}_s) = (2\pi\imath)^m\frac{1}{\prod\limits_{\substack{k<j\\k,j=1}}^{m}(v_k-v_j)}\frac{1}{\prod\limits_{k=1}^{m}(v_k-v_{2m+1})^{\frac{s}{N}}}. \tag{6.5}$$

Combining the above relations the expression for the constant $C = (2\pi\imath)^{-m(N-1)}$. □

It is straightforward to check that for $N = 2$ the formula (6.1) coincides with the original Thomae formula (1.1).

To complete the section we give the Thomae formula for the characteristics $[\boldsymbol{e_{m+1}}]$ defined in (3.12).

Theorem 6.2. *Let $[\boldsymbol{e_{m+1}}]$ are nonsingular $1/N$ characteristics, given by the formula*

$$\boldsymbol{e_{m+1}} = \mathfrak{A}((N-1)\sum_{i\in I_1} P_i + (N-1)\sum_{j\in J_1} P_j + (N-2)P_{j_m} + P_{i_m} - \mathcal{D} - \Delta),$$

where $J_1 \subset J_0 = \{2,4,\dots,2m+2\} - \{2m+2\}$ and $I_1 \subset I_0 = \{1,3,\dots,2m+1\}$, with

$$|J_1| + |I_1| = m-1$$

and $j_m \in J_0 - J_1 - 2m+2$, $i_m \in I_0 - I_1$. The Thomae type formula takes the form

(6.6)

$$\theta[\boldsymbol{e_{m+1}}](0;\Pi)^{4N} = \frac{\prod_{i=1}^{N-1}\det \mathcal{A}_i^{2N}}{(2\pi)^{2mN(N-1)}} \prod_{1\le i<k\le m}(\lambda_{2i}-\lambda_{2k})^{N(N-1)} \prod_{0\le i<k\le m}(\lambda_{2i+1}-\lambda_{2k+1})^{N(N-1)}$$
$$\times\left(\frac{\prod_{i\in I_1',j\in J_1}(\lambda_i-\lambda_j)\prod_{i\in I_2,j\in J_2'}(\lambda_i-\lambda_j)}{\prod_{i\in I_1',k\in I_2}(\lambda_i-\lambda_k)\prod_{j\in J_1,k\in J_2'}(\lambda_j-\lambda_k)}\right)^{2(N-1)}$$
$$\times\left(\frac{\prod_{j\in J_2}(\lambda_{i_m}-\lambda_j)\prod_{i\in I_1}(\lambda_i-\lambda_{j_m})}{\prod_{i\in I_1}(\lambda_{i_m}-\lambda_i)\prod_{j\in J_2}(\lambda_{j_m}-\lambda_j)}\right)^{4(N-2)},$$

where

$$I_1' = I_1 + i_m, \quad I_2 = I_0 - I_1 - i_m, \;\; J_2 = J_0 - J_1 - j_m - 2m + 2, \;\; J_2' = J_2 + j_m.$$

The proof of the above formula follows the steps of the proof of (6.1).

Example 6.3. We consider as an example the trigonal curve of genus two (studied by Hutchinson [19], see also [11] for more details)

$$\mu^3 = (\lambda-\lambda_1)(\lambda-\lambda_3)(\lambda-\lambda_2)^2. \tag{6.7}$$

In the homology basis given in Figure 1, the Riemann period matrix is of the form

$$\Pi = \begin{pmatrix} 2T & T \\ T & 2T \end{pmatrix}, \quad T = \frac{\imath\sqrt{3}}{3}\frac{F\left(\frac13,\frac23;1;1-t\right)}{F\left(\frac13,\frac23;1;t\right)}, \tag{6.8}$$

where $F(a,b,c;t)$ is standard hypergeometric function and $t=(\lambda_2-\lambda_1)/(\lambda_3-\lambda_1)$. In the homology basis given in figure 1 the characteristics of the branch points are

$$[\mathfrak{A}_1] = \begin{bmatrix} \frac23 & \frac23 \\ 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_2] = \begin{bmatrix} \frac23 & \frac23 \\ \frac13 & \frac23 \end{bmatrix}, \quad [\mathfrak{A}_3] = \begin{bmatrix} 0 & 0 \\ \frac13 & \frac23 \end{bmatrix}, \quad [\mathfrak{A}_4] = [0],$$

whilst characteristics of the vector of Riemann constants are $[\boldsymbol{K}_{P_\infty}] = 2[\mathfrak{A}_2]$. Given

$$I_0 = \{1,3\}, \quad J_0 = \{2,4\}, \quad P_4 = P_\infty,$$

we introduce the following partitions of the branch points.

(1) The partition

$$I_1 = \{1,3\},\; J_1 = \emptyset \quad \text{or} \quad I_1 = \emptyset,\; J_1 = \{2,4\},$$

which corresponds to the characteristics $[\boldsymbol{e_m}] = [\boldsymbol{0}]$.

(2) The partition

$$I_1 = \{1\}, \;\; J_1 = \{2\},$$

or

$$I_1 = \emptyset, \quad J_1 = \emptyset, \;\; i_m = 3, \; j_m = 2,$$

which corresponds to the characteristics $[\boldsymbol{e_m}] = \begin{bmatrix} \frac13 & \frac13 \\ 0 & 0 \end{bmatrix}$.

(3) The partition

$$I_1 = \{3\}, \quad J_1 = 2,$$

or

$$I_1 = \emptyset, \quad J_1 = \emptyset, \; i_m = 1, \; j_m = 2.$$

which corresponds to the characteristics $[\boldsymbol{e_m}] = \left[\begin{array}{cc} 0 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{array}\right]$.

In the first case, the Thomae formula takes the form

$$\theta[\mathbf{0}](\mathbf{0};\Pi)^4 = \frac{\mathcal{A}_1^2\mathcal{A}_2^2}{16\pi^4}(\lambda_1-\lambda_3)^2,$$

where

$$\mathcal{A}_1 = \frac{2\sqrt{3}\pi}{3}\frac{1-\rho^2}{(\lambda_3-\lambda_1)^{1/3}}F\left(\frac{1}{3},\frac{2}{3};1;t,\right), \quad \mathcal{A}_2 = -\rho\mathcal{A}_1, \tag{6.9}$$

where $\rho^3 = 1$. The second partition gives

$$\theta\left[\begin{array}{cc} \frac{1}{3} & \frac{1}{3} \\ 0 & 0 \end{array}\right](\mathbf{0};\Pi)^{12} = \left(\frac{\mathcal{A}_1\mathcal{A}_2}{4\pi^2}\right)^6(\lambda_1-\lambda_2)^4(\lambda_1-\lambda_3)^2$$

and the third partition

$$\theta\left[\begin{array}{cc} 0 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{array}\right](\mathbf{0};\Pi)^{12} = \left(\frac{\mathcal{A}_1\mathcal{A}_2}{4\pi^2}\right)^6(\lambda_3-\lambda_2)^4(\lambda_3-\lambda_1)^2.$$

The quotient of the first and third relations leads to

$$\left(\frac{\theta\left[\begin{array}{cc} 0 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{array}\right](\mathbf{0};\Pi)}{\theta(\mathbf{0};\Pi)}\right)^6 = \frac{(\lambda_3-\lambda_2)^2}{(\lambda_3-\lambda_1)^2},$$

which we mentioned in [11].

7. Conclusion

In this paper we presented the derivation of the Thomae type formula for singular curves with Z_N symmetry. We computed explicitly the Riemann period matrix of the curve in the fixed homology basis given in figure 1 taking into account the action of automorphism. We also described $1/N$ periods of the curve in terms of its characteristics given in the form of $g\times 2$-matrices with rational entries.

We considered in the papers a family of non-special divisors supported on the branch points and derived the Thomae formula for this family only. But our derivation is general and is working for any other family of non-singular $1/N$-periods. The proof given above goes up to the original Thomae proof and involved a number of steps such as Rauch variational formulae, calculation of the holomorphic projective connection, certain results about the Bergman kernel. But the key-point of the proof is the derivation of an algebraic expression for Szegö kernel associated to the aforementioned 1/N-period. We show that the values of the exponents in the Thomae formula follows directly from this expression.

The results of the paper can be generalized to the family of curves

$$\mu^N = (\lambda-\lambda_1)^{m_1}\ldots(\lambda-\lambda_k)^{m_k}, \quad m_1+\ldots+m_k \quad \text{divisible by} \quad N.$$

However even for the curve considered a number of open problems remain, among which the complete classification of non-special divisors supported on the branch points and the derivation of an algebraic expressions for the associated Szegö kernels. We remark that a family which is different from the one discussed in the present manuscript has already been studied in [11].

Another set of interesting problems consist of the derivation of Thomae formulae for θ-derivatives, which generalize the formulae given by Thomae in the case of hyperelliptic curve [36]. Such formulae are important for obtaining the expressions of the "winding vectors" in terms of θ-constants and which are generalization of the Rosenhain formulae known for genus two.

We can also point out the problem of the derivation for the given curve of the Jacobi-Riemann derivative formula, one of the most mysterious θ-formula, which Riemann called as a "pearl of mathematics".

The answers to these questions are of interest for the applied problems pointed out in the Introduction. The investigation of the particular curve undertaken above was stimulated by its links with the solvable Riemann-Hilbert problem of rank N considered in [11].

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