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Double Scaling and Finite Size Corrections in $\mathfrak{sl}(2)$ Spin Chain

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Abstract

We find explicit expressions for two first finite size corrections to the distribution of Bethe roots, the asymptotics of energy and high conserved charges in the $\mathfrak{sl}(2)$ quantum Heisenberg spin chain of length J in the thermodynamical limit $J \to \infty$ for low lying states with energies $E \sim 1/J$. This limit was recently studied in the context of integrability in perturbative $\mathcal{N} = 4$ super-Yang-Mills theory. We applied the double scaling technique to Baxter equation, similarly to the one used for large random matrices near the edge of the eigenvalue distribution. The positions of Bethe roots are described near the edge by zeros of Airy function. Our method can be generalized to any order in $1/J$. It should also work for other quantum integrable models.

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1 Introduction

We study in this paper the integrable periodic Heisenberg XXX_s chain of noncompact quantum spins transforming under the representation $s = -1/2$ of $\mathfrak{sl}(2)$, in the so called thermodynamical limit of large J, where J is the length of the chain, in the ferromagnetic regime of low energies $E \sim 1/J$. [1]

The problem is known to be solvable by the Bethe ansatz approach (see for example [1]) and the energy of a state of S magnons in dimensionless units is given by a simple formula

$$E = \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4}, \tag{1}$$

where the Bethe roots $u_j, \ j = 1, 2, \cdots, S$, parametrizing the momenta of magnons, are solutions of a system of polynomial Bethe ansatz equations (BAE)

$$-\left(\frac{u_j - i/2}{u_j + i/2}\right)^J = \prod_{k=1}^{S} \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad j = 1, \ldots, S. \tag{2}$$

It can be proven that for this model the roots are always real.

In the thermodynamical limit we will also consider a large number of magnons $S \sim J$. It is clear then from eq.(1) that in order to focus on the low lying states with energies $E \sim 1/J$ we should take the characteristic Bethe roots of the order $u_j \sim J$. It means that the chain is very long and the spins are rarely changing along it. The typical length of spin-waves (magnons) is of the order of the length J. Our goal is to study the limiting $J \to \infty$ distributions of Bethe roots and the finite volume $1/J$ corrections to these distributions, to the energy and higher conserved charges. In the main order this thermodynamical limit for the compact Heisenberg $XXX_{1/2}$ chain of $\mathfrak{su}(2)$ spins was already considered in [2], and later in [3] in relation to the integrable dilatation hamiltonian in planar perturbative superconformal $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. Its description and the general solution in terms of algebraic curves was proposed in [4] for the $\mathfrak{su}(2)$ case[2] and in [6, 7] for the $\mathfrak{sl}(2)$ chain. We will concentrate in the current paper on this last case. Generalization to the $\mathfrak{su}(2)$ case is straightforward.

The study of $1/J$ corrections in these systems was started recently in the papers [8, 9] for the simplest single support, or one cut distribution, whereas a similar quantum $\hbar$ correction to the classical KdV solitons was already found earlier in the general multi-cut case in [10].

The main results of our paper are:

1. The explicit formulas for the $1/J$ and $1/J^2$ corrections to the general multi-cut distribution of Bethe roots and to the corresponding energy of a Bethe state in terms of the underlying algebraic curve.

2. The universal description of the distribution of Bethe roots in the vicinity of an edge of a support in terms of zeroes of the Airy function, similar to the double scaling limit in the matrix models.

3. Asymptotics of conserved local charges $Q_n(S, J)$ in the large n limit.

4. Asymptotics of conserved global (non-local) charges of high order.

Unlike the papers [8, 9] using the method of singular integral equation corrected by so called anomaly term[3], we will use here the exact Baxter equation written directly for the analytical function - the resolvent of the root distribution (similar approach was used in [6]). This equation is valid before any limit.

These results might be interesting for different kinds of specialists.

First, for those who are studying large N random matrix or random partition ensembles. In particular, the distributions of eigenvalues in matrix models are described by similar (although not the same) algebraic

[1] It is different from a more traditional regime $E \sim J$ widely studied since many years, especially in the condensed matter literature.

[2] Following a similar approach of [5] to a somewhat different limit of large spin

[3] This phenomenon of anomaly, or the contribution of close eigenvalues in the thermodynamical limit of BAE was first observed in [11]

curves. The $1/J$ corrections remind the $1/N$ corrections, and the Airy edge distribution observed in our work is known to describe the edge behaviour in the double scaled matrix integrals as well [12]. No doubt that by choosing particular solutions of eq.(2) or modifying its l.h.s. (by considering inhomogeneous chains) we can also find here various multi-critical phenomena similar to those found in the matrix models in the context of quantum 2D gravity applications [13].

Second, it might be interesting to those who work on various aspects of AdS/CFT correspondence in supersymmetric string and gauge theories and in particular on the integrability in $\mathcal{N}=4$ SYM theory (see [14] and references therein). They might shed some light on quantum corrections to the classical limit of the AdS dual of SYM, the superstring on the $AdS_5 \times S_5$ background, now known only for particular classical solitonic solutions [15] of rotating strings. AdS/CFT correspondence should manifest itself as the coincidence of such corrections in these so different integrable systems. Their similarity, and even the coincidence in a certain regime, was already observed on particular string and chain solutions, having only one support for the Bethe roots distribution [8, 9, 16, 17, 18, 19, 43, 42]. 1/J corrections were first studied for BMN states in [40], where the integrable spin chain for $\mathcal{N}=4$ SYM was first proposed, and then in [41]. The Airy edge behavior also seems to be universal enough to manifest itself on both sides of AdS/CFT duality. This behaviour observed in the present paper for spin chains describing the spectrum of anomalous dimensions in the perturbative SYM in thermodynamical limit, is natural to expect also in the quantized string theory where the classical limit, the analogue of the thermodynamical limit, is described by similar algebraic curves obtained by the finite gap method [4]. The universal edge behaviour might be completely driven by the similarity of integrable structures of these two seemingly different systems.

The paper is organized as follows. In section 2 we summarize the explicit formulas for the Hamiltonian and the local and non-local conserved charges of the model. In section 3 we review the general solution of the model in thermodynamical limit when $J \to \infty$ and $E \sim 1/J$. In section 4 we calculate the first $1/J$ correction to the general multi-cut solution from Baxter equation. In section 5 we apply the method of the double scaling limit to the Baxter equation and find the near branch point behavior. In section 6 we combine the results of section 4 and 5 to find the explicit formula for the second, $1/J^2$ correction. In section 7 we give the formulas for $1/J$ corrections to the energy and compute the asymptotics of high conserved local and non-local charges in all orders in $1/J$. Conclusions are devoted to unsolved problems and perspectives, as well as to the discussion of parallels with the matrix models. In appendix A we write explicit formulas for $1/J$ expansion of BAE and in Appendix B we express $1/J^2$ corrections to the energy for one-cut solution through certain double sums in mode numbers, generalizing the results of [8].

2 Hamiltonian, Transfer-matrix and Higher Charges of $\mathfrak{sl}(2)$ chain

The hamiltonian of interaction of the neighboring spins s_l, s_{l+1} can be written in an explicit way [14]

$$H_{-1/2} = \sum_{l=1}^{J} \hat{H}_{-1/2}^{l,l+1} \tag{3}$$

with the Hamiltonian density

$$\hat{H}_{-1/2}^{l,l+1}|k, m-k\rangle = \sum_{k'=0}^{m} \left(\delta_{k=k'} \left(h(k) + h(m-k) \right) - \frac{\delta_{k\neq k'}}{|k-k'|} \right) |k', m-k'\rangle, \tag{4}$$

where $|k_1, \ldots, k_l, k_{l+1}, \ldots, k_J\rangle$ is a state vector labeled by J integers k_j ($s=-1/2$ spin components) and $h(k) = \sum_{j=1}^{k} \frac{1}{j}$ are harmonic numbers.

The total momentum $P(u)$

$$e^{iP(u_j)} = \frac{u_j - i/2}{u_j + i/2} \tag{5}$$

[4] See [11, 20] for this approach on both sides of duality

satisfies the (quasi-)periodicity condition following directly from eq.(2)

$$P_{tot} = \sum_{j=1}^{S} P(u_j) = 2\pi k/J, \quad k \in \mathbf{Z}. \tag{6}$$

In some applications, such as the spectrum of anomalous dimensions of operators [5] in $\mathcal{N} = 4$ SYM theory, we select only purely periodic Bethe states

$$P_{tot} = 2\pi m, \quad m \in \mathbf{Z}. \tag{7}$$

We can also study other physically interesting quantities of this model, such as the local conserved charges $\hat{Q}_r$. They are defined as follows

$$\hat{T}(v) = \exp\left(i\sum_{r=1}^{\infty} \hat{Q}_r v^{r-1}\right), \tag{8}$$

where the quantum transfer matrix $\hat{T}(v) \equiv \hat{T}(v; 0, 0, \cdots, 0)$ is a particular case of the inhomogeneous transfer matrix

$$\hat{T}(v; v_1, \cdots, v_J) = \mathrm{Tr}_0\left[\hat{R}_{0,1}(v - v_1)\cdots\hat{R}_{0,J}(v - v_J)\right] \tag{9}$$

and $\hat{R}_{0,j}$ is the universal $\mathfrak{sl}(2)$ R-matrix defined as [21]

$$\hat{R}_{0,1}(v) = \sum_{j=0}^{\infty} R_j(v)\mathcal{P}_{0,1}^{(j)}, \qquad R_j(v) = \prod_{k=1}^{j} \frac{v - ik}{v + ik} \tag{10}$$

with $\mathcal{P}_{01}^{(j)}$ being the operator projecting the direct product of two neighboring spins $s_0 = s_1 = -1/2$ to the representation j. Recall that

$$\left[\hat{T}(v; v_1, \cdots, v_J), \hat{T}(v'; v_1, \cdots, v_J)\right] = 0 \tag{11}$$

for any pair v, v', due to Yang-Baxter equations on the $\hat{R}$-matrix.

The direct calculation shows that $\hat{P}_{tot} = -\hat{Q}_1$ is the operator of the momentum, such that and $\hat{H}_{-1/2} = \hat{Q}_2$ is the hamiltonian eq.(3), etc. Those charges are local, in the sence that the charge density of Q_k contains $\leq k$ consecutive spins.

There is a more efficient way to generate higher charges than using eq.(8). We can use the recurrence relation (see [14]):

$$r\hat{Q}_{r+1} = [B, \hat{Q}_r], \tag{12}$$

where $B = i\sum_{l=1}^{J} l\ \hat{H}_{-1/2}^{l,l+1}$ is the boost operator. The last formula is true up to some boundary terms destroying the periodicity, which should be dropped.

Due to the integrability manifestly expressed by eq.(11) all these charges commute and their eigenvalues on a Bethe state characterized by a set of Bethe roots satisfying eq.(2) (enforcing the periodicity of the chain or the quasi-periodicity of the Bethe state) are given by [22]

$$Q_r = \sum_{j=1}^{S} \frac{i}{r-1}\left(\frac{1}{(u_j + i/2)^{r-1}} - \frac{1}{(u_j - i/2)^{r-1}}\right). \tag{13}$$

We can also study the non-local charges. They can be defined in many different ways. The definition could be similar to eq.(8), but the expansion of $T(u)$ goes around $u = \infty$. However, the most natural charges are defined through the resolvent of Bethe roots:

$$G(x) = \sum_{k=1}^{S} \frac{1}{xJ - u_k} = \sum_{n=1}^{\infty} d_n x^{-n}. \tag{14}$$

[5] The operators of the type $\mathrm{Tr}\left(\nabla^{k_1} Z \cdots \nabla^{k_J} Z\right)$ in SYM, where $\nabla = \partial + A$ is a covariant derivative in a null direction and Z is a complex scalar, represent the state vectors $|k_1, \ldots, k_l, k_{l+1}, \ldots, k_J\rangle$ and the dilatation hamiltonian is given at one loop by the $XXX_{-1/2}$ hamiltonian.

The j-th charge is the j-th symmetric polynomial of Bethe roots

$$d_n = \frac{1}{J}\sum_{k=1}^{S}\left(\frac{u_k}{J}\right)^{n-1}, \qquad d_n \equiv \sum_{k=0}^{\infty} d_{k,n} J^{-k}. \tag{15}$$

We will later estimate the behavior of $d_{k,n}$ at $n \to \infty$ and high orders of $1/J$ expansion. This asymptotics will be universal and a few leading terms of it will be the same for various definitions of non-local charges.

3 $1/J$ expansion of BAE

Let us start from reviewing the "old" method of solving eq.(2) in the thermodynamical limit $J \to \infty$, $u_k \sim J \sim S$, before sticking with the most efficient one using the Baxter equation.

As we mentioned in the introduction the eq.(2) has only real solutions, i.e. all the roots lie on the real axis. We label the roots so that $u_{j+1} > u_j$. Suppose there exists a smooth function $X(x)$ parametrizing the Bethe roots

$$u_k = JX(k/J), \qquad \varrho(X(x)) \equiv \frac{1}{X'(x)} \simeq \frac{1}{u_{k+1}-u_k}. \tag{16}$$

For large S the function $\varrho(x)$ has a meaning of density of Bethe roots. As follows from definition (16) its normalization is

$$\int dx \varrho(x) = \alpha \tag{17}$$

with $\alpha = S/J$. Taking log of both parts of eq.(2) we have[6]

$$2\pi i m_j + J \log\frac{u_j - i/2}{u_j + i/2} = -\sum_{k=1}^{S}{}' 2i\arctan(u_j - u_k), \tag{18}$$

where m_j are strictly ordered integers $m_{j+1} > m_j$. In terms of the logarithm we can write

$$2\pi i n_j + J \log\frac{u_j - i/2}{u_j + i/2} = \sum_{k=1}^{S}{}' \log\frac{u_j - u_k + i}{u_j - u_k - i}, \tag{19}$$

where $n_j = m_j - j + \frac{S+1}{2}$ are non-decreasing integers [7].

Now we have instead of "fermionic" m_j's the "bosonic" n_j's which are simply ordered $n_{j+1} \geq n_j$ and different Bethe roots can have the same magnon numbers $n_j = n_{j+1} = \cdots$. In the thermodynamical limit we can rewrite eq.(19) assuming k to be far from the edges, as follows (see also Appendix A)

$$\sum_j{}' i\log\left(\frac{u_j-u_k+i}{u_j-u_k-i}\right) = -2\sum_j{}'\frac{1}{u_j-u_k} + \frac{2}{3}\sum_j{}'\frac{1}{(u_j-u_k)^3} - \frac{2}{5}\sum_j{}'\frac{1}{(u_j-u_k)^5} + \frac{2}{7}\sum_j{}'\frac{1}{(u_j-u_k)^7} \tag{20}$$
$$+\frac{\pi\varrho'[\coth(\pi\varrho)]_6}{J} - \frac{1}{12J^3}\left((\pi\varrho')^3\left[\frac{\coth(\pi\varrho)}{\sinh^2(\pi\varrho)}\right]_2 - 2\pi^2\varrho'\varrho''\left[\frac{1}{\sinh(\pi\varrho)}\right]_3 + \pi\varrho^{(3)}[\coth(\pi\varrho)]_4\right) + \left(\frac{1}{J^5}\right),$$

were we introduce the notation defined by $[f(\varrho)]_n \equiv f(\varrho) - \sum_{i=0}^{n-1} f^{(i)}(0)\frac{\varrho^i}{i!}$ for the functions regular at zero. For singular functions the Taylor series should be substituted by the Laurent series so that $[f(\varrho)]_n$ is zero for $\varrho = 0$ and has first $n-1$ zero derivatives at this point. The terms in the first line represent the naive expansion of the l.h.s. in $1/(u_j - u_k)$. It works well for the terms in the sum with $u_j \gg u_k$. The terms in

[6] Note that $\frac{i}{2}\log\frac{x+i}{x-i} = \arctan(x) - \frac{\pi}{2}\mathrm{sign}(x)$ for standard definition of the log i.e. $\log(x) = -\log(x - i0)$ for $x < 0$.

[7] This fact is obvious when $S \ll J$, for $J \to \infty$, when we can neglect the r.h.s. of eq.(19). We believe that this is true also in general, although this fact is irrelevant for the rest of the paper

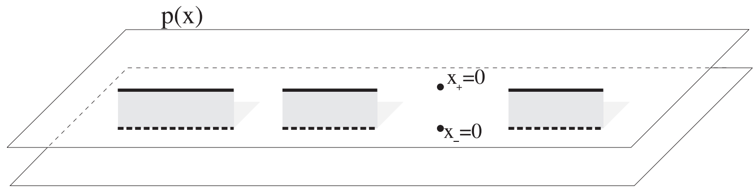

Figure 1: *Hyperelliptic Riemann surface*

the second line describe the anomalous contribution at $u_j \sim u_k$, for close roots with $i \sim j$. In this case we can expand

$$u_j - u_k = \frac{j-k}{\varrho(u_j/J)} + (1/J) \tag{21}$$

and calculate the corresponding converging sum giving the terms in the second line. This anomaly was noticed in the Bethe ansatz context in [11] although this phenomenon was known since long in the large N matrix integrals or similar character expansions [23, 24]. It was proven in [11] to happily cancel in the main order of $1/J$, even for a more complicated nested Bethe ansatz.

In our case when $J \to \infty$ it is obvious from eqs.(20,19) that the anomaly does not contribute to the main order and the Bethe ansatz equation becomes a singular integral equation [2, 3]

$$2\pi n_k - \frac{1}{x} = 2 \int_{C_{tot}} \frac{dy \ \varrho_0(y)}{x-y}, \qquad x \in C_k, \ k = 1, \cdots, K. \tag{22}$$

Now we introduce the resolvent and the quasi-momentum:

$$G(x) \equiv \sum_j \frac{1}{xJ - u_j}, \qquad p(x) \equiv G(x) + \frac{1}{2x} \tag{23}$$

as well as the standard definition of density [8]

$$\rho(x) = \frac{1}{2\pi i} \left(p(x - i\epsilon) - p(x + i\epsilon) \right) \tag{24}$$

and rewrite it in terms of the following Riemann-Hilbert problem

$$2\not{p}(x) = 2\pi n_k, \qquad x \in C_k, \quad n_k \in \mathbf{Z}, \tag{25}$$

where $\not{p}(x) = \frac{1}{2}\left[p(x+i0) + p(x-i0)\right]$ denotes the symmetric (real) part of the quasi-momentum on a cut C_k, $k = 1, \cdots, K$.

To solve it [7] we notice that $p(x)$ is a function on a hyperelliptic Riemann surface with two sheets related by K cuts along the real axis, as shown on the fig.1. It has a known behavior $p(x) \simeq P_0 + (x)$ at $x = 0$, and $p(x) \sim \frac{S/J+1/2}{x} + (1/x^2)$ at $x \to \infty$ on the first sheet. There are no other singularities. However, as we see from eq.(25) $p(x)$ is not a single valued function but its derivative is. This information is enough to fix $p'(x)$ as a single-valued function on hyperelliptic Riemann surface $f^2 = \prod_{j=1}^{2K}(x - \mathrm{x}_j)$

$$p'(x) = \frac{1}{f(x)} \sum_{k=-1}^{K-1} a_k x^{k-1}. \tag{26}$$

The single-valuedness imposes $\oint_{A_l} dp = 0$, $l = 1, \ldots, K-1$, where A_l, $l = 1, \cdots, K$ are the A-cycles each surrounding a cut C_l.

[8] These two densities $\rho(x)$ and $\varrho(x)$ are clearly diferent. By definition $\varrho(x)$ is a smooth function, whereas $\rho(x)$ is a sum of δ-functions. However their $1/J$ expansion is the same at least for the first two orders. It is also important to note that to obtain the $1/J$ expansion of $\rho(x)$ one should first expand the rhs of eq.(24) in powers of $1/J$ and then take the limit $\epsilon \to 0$.

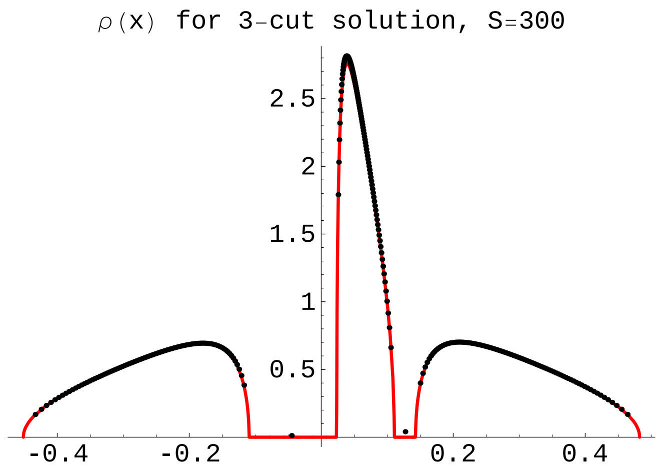

Figure 2: *Density of roots. The dots correspond to numerical 3-cut solution with total number of Bethe roots $S = 300$ and equal fractions $\alpha_i = 1/6$, and $n_i = \{-1, 3, 1\}$. They are fixed from the numerical values of the roots by the eq.(16). Solid line is the density at $J = \infty$ computed analytically from the corresponding hyper-elliptic curve. x coordinates of the dots are $\frac{u_j+u_{j+1}}{2J}$ so that the solitary points in the middle of empty cuts are artifacts of this definition.*

The BAE eq.(25) become the integer B-period conditions:

$$\oint_{B_j} dp = 2\pi n_j \qquad n_j = 1, \ldots, K, \tag{27}$$

where the cycle B_j starts at $x = \infty_+$ on the upper sheet, goes through the cut C_J to the lower sheet and ends up at its infinity $x = \infty_-$.

From eq.(2) we have the quasi-periodicity condition for the total momentum expressed by eq.(6). In some applications, such as the above mentioned spectrum of the integrable matrix of anomalous dimensions in $\mathcal{N} = 4$ SYM theory, we select only purely periodic Bethe states (7). This information, together with the filling fractions $\alpha_k = S_k/J$, $k = 1, \cdots, K$, such that $\sum_{k=1}^{K} \alpha_k = S/J$, or

$$\alpha_k = \oint_{A_k} dx \; p(x) \tag{28}$$

fixes completely a solution. For the energy we obtain in this limit

$$E_0 = -G_0'(0). \tag{29}$$

A particular 3-cut solution is demonstrated on the fig.2 and is compared with the numerical solution of exact Bethe ansatz equation. We see that already for 300 roots and $J = 600$ the description in terms of the algebraic curve becomes excellent.

4 Large J limit and $1/J$-corrections from Baxter equation

Eq. (2) can be also obtained as the condition that the transfer matrix eigenvalue defining a Bethe state (see for example [25])

$$T(u) = W(u + i/2)\frac{Q(u+i)}{Q(u)} + W(u - i/2)\frac{Q(u-i)}{Q(u)}, \tag{30}$$

where $Q(u) = \prod_{k=1}^{S}(u - u_k)$, $W(u) = u^J$. $T(u)$ is a polynomial of degree J, which is clear from the very construction of a Bethe state in the algebraic Bethe ansatz approach [1]. The Bethe equations (2) follow immediately from eq.(30) assuming analyticity (polynomiality) of $T(u)$.

Introduce the notations: $x = u/J$, $\Phi(x) = \frac{1}{J}\sum_{k=1}^{S}\log(x - x_k)$, $V(x) = \log x$, $2t(x) = T(Jx)/(Jx)^J$ and rewrite (30) as

$$2t(x) = \exp J\left[\Phi\left(x + \frac{i}{J}\right) - \Phi(x) + V\left(x + \frac{i}{2J}\right) - V(x)\right] + \text{c.c.}. \qquad (31)$$

Defining the quasi-momentum (now at all orders in $1/J$)

$$p(x) \equiv \Phi' + V'/2 \qquad (32)$$

and expanding the Baxter equation in $1/J$ we get

$$\begin{aligned} t(x) \;&=\; \cos p(x)\left[1 - \frac{1}{J}\left(\frac{p'(x)}{2} - \frac{V''(x)}{8}\right) + \frac{1}{2J^2}\left(\frac{p'(x)}{2} - \frac{V''(x)}{8}\right)^2\right] \\ &+\; \frac{1}{J^2}\sin p(x)\left(\frac{p''(x)}{6} - \frac{V^{(3)}(x)}{16}\right) + O\left(\frac{1}{J^3}\right). \end{aligned} \qquad (33)$$

To find the $1/J$ corrections to the quasi-momentum, we will expand $p(x) = p_0(x) + \frac{1}{J}p_1(x) + \frac{1}{J^2}p_2(x) + O(1/J^3)$, $t(x) = t_0(x) + \frac{1}{J}t_1(x) + \frac{1}{J^2}t_2(x) + O(1/J^3)$ and plug it into the last equation. We assume that the coefficients of expansion $t_0(x), t_1(x), t_2(x), \ldots$ are the entire functions on the plane x with no cuts, having only an essential singularity at $x = 0$.

The quasi-periodicity property of the total momentum reads up to 3 first orders as follows

$$P_{tot} = -\sum_j \frac{1}{u_j} + \sum_j \frac{1}{12u_j^3} + (1/J^4) = 2\pi k/J \qquad (34)$$

and in the purely periodic case we select only the states with $k = mJ$, with integer m.

4.1 Zero order from Baxter equation

Let us restore from the Baxter equation the zero order result of the previous section. In the zero order approximation we get from eq.(33)

$$\cos p_0(x) = t_0(x) \qquad (35)$$

or

$$p_0'(x) = \frac{2t_0'(x)}{\sqrt{1 - t_0^2}}. \qquad (36)$$

As is usual in the finite gap method [26], we expect that since $t_0(x)$ is an entire functions all the branch cuts of eq.(26) come from the square root in denominator, after the Bethe roots condense to a set $C_1, \cdots, C_K$ of dense supports in the $J \to \infty$ limit. It is easy to see from the definition eq.(32) that $p_0(x) = \frac{\alpha + 1/2}{x} + (1/x^2)$ when $x \to \infty$ and $p_0(x) \sim 2\pi\frac{k}{J} + (x)$ when $x \to 0$. Consequently, we reproduced the general solution (26-29) of [7].

4.2 $1/J$ correction from Baxter equation

To find the next, $1/J$ approximation to the density of roots and to the energy we deduce from eq.(33)

$$p_1 = (-p_0'/2 + V''/8)\cot p_0 - \frac{t_1}{\sin p_0}, \qquad (37)$$

where $t_1(x)$ is an entire function on the plane as was mentioned before. We know about $p_0(x)$ that

$$p_0^+ = \pi n_j - \pi i\rho_0, \quad p_0^- = \pi n_j + \pi i\rho_0, \quad \sin p_0^+ = -\sin p_0^- \qquad (38)$$

and thus we have for the real and imaginary parts of $p_0(x)$ on the cuts

$$\pi i \rho_1 = \left(\frac{V''}{8} t_0 - t_1\right) \frac{1}{\sin p_0^-}, \qquad \not{p}_1 = -p_0' \cot p_0/2. \tag{39}$$

We will solve these equations below and restore the explicit p_1.

Since $t_1(x)$ is a regular function on the cuts ([44])

$$\not{p}_1 = -\frac{1}{2}\pi\rho' \coth \pi\rho = -p_0^{\pm\prime} \cot p_0^{\pm}/2. \tag{40}$$

Moreover we see from eq.(34) that

$$p_1(0) = 0 \tag{41}$$

and for large x $p_1(x)$ should decreases as $(1/x^2)$.

We can write from (40) the general solution of this Riemann-Hilbert problem

$$p_1(x) = \frac{x}{4\pi i f(x)} \oint_{\mathcal{C}} \frac{f(y) p_0'(y) \cot p_0(y)}{y(y-x)} dy + \sum_{j=1}^{K-2} \frac{a_j x^j}{f(x)} , \tag{42}$$

where $f^2(x) = \prod_{j=1}^{2K}(x - x_j)$ and the contour encircles all cuts C_k (but no other singularities). The first term in the r.h.s. represents the Cauchy integral restoring the function from its real part on the cuts and having a zero at the origin (the value of the quasi-momentum $p(x)$ at $x = 0, \infty$ was already fixed for p_0) whereas the second one is purely imaginary on the cuts, with the polynomial in the numerator chosen in such a way that it does not spoil the behavior of $p(x)$ at $x = 0, \infty$.[9]

Thus for $K < 3$ the solution is unique. In particular, for $K = 1$ we restore from here the 1-cut solution of [8]. For $K \geq 3$ we have to fix $K - 2$ parameters a_j. To do this we can use K additional conditions ensuring the right fractions α_j of the roots already chosen for p_0:

$$\oint_{C_l} p_1(x) dx = 0, \qquad l = 1, \ldots, K, \tag{43}$$

in fact only $K - 2$ of them are linear independent (since we have already fixed the total filling fraction by the asymptotic properties of eq.(42) at $x = \infty$: $p_1(x) = O(1/x^2)$. Eq.(41) also restricts some linear combination of the conditions (43)). Hence we completely fixed all parameters of our K-cut solution for the $1/J$ correction p_1 knowing the zero order solution (algebraic curve) for p_0.

It is also useful to rewrite eq.(42) in the following way

$$p_1(x) = \frac{Q(x)}{4\pi i f(x)} \oint_{\mathcal{C}} \frac{f(y) p_0'(y) \cot p_0(y)}{Q(y)(y-x)} dy , \tag{44}$$

where $Q(x) = \sum_{k=1}^{K-2} \tilde{a}_j x^j$ and the contour of integration encircles all the cuts. Again, $\tilde{a}_j$ are fixed by eq.(43). Equivalence of this formula to eq.(42) can be seen from the coincidence of their analytical properties: blowing up the contour in the contour integrals of any of these representations we obtain [10]

$$p_1(x) = -p_0'(x) \cot p_0(x)/2 + \frac{1}{f(x)} \sum_n \frac{c_n}{x_n - x} + \sum_{j=1}^{K-2} \frac{a_j x^j}{f(x)}, \tag{45}$$

where the first term comes from the residue at $y = x$. Note that only this term contributes to the r.h.s of eq.(40) thus showing that we satisfied this equation. The second term exactly cancels all the poles of the first term, so that p_1 is regular everywhere except the cuts. It is a meromorphic function in the x plane, having no cuts. The last term reflects the freedom of adding $K - 2$ coefficients before fixing them by conditions eq.(43).

---

[9] We could also add terms $\frac{1}{f^3}, \frac{1}{f^5}, \ldots$ but they are too singular at the branch points as we shell see in the next section.

[10] In fact one should take into account an infinite number of residues at $y = 0$, see [8]. A more regular procedure is to express cot as a sum and then do the integration.

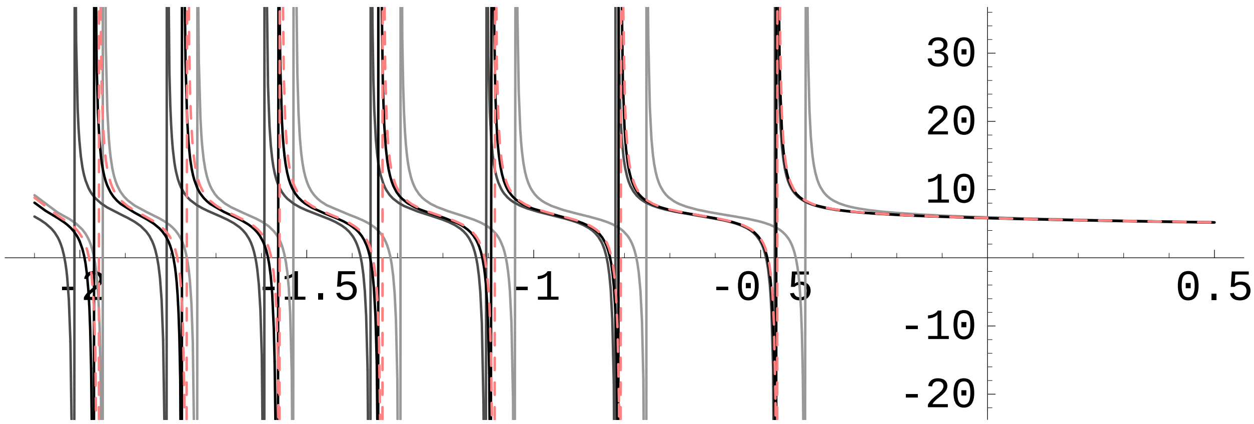

Figure 3: *Quasi-momentum near branch point as a function of the scaling variable v for $S = 200$. The poles corresponds to the positions of Bethe roots u_i. Red dashed line - "exact" numerical value, light grey - zero order approximation given by Airy function* $\mathrm{Ai}(a^{1/3}x)$, *grey - first order and black - second order approximation.*

4.3 Equations for $1/J^2$ corrections from Baxter relation

Expanding eq.(33) up to $1/J^2$ we obtain

$$p_2 = -\frac{1}{2}\partial_x[\cot(p_0)I] - \frac{1}{8x^3} - \frac{\tilde{t}_2}{2\sin(p_0)}, \tag{46}$$

where

$$I = -\frac{\tilde{t}_1}{\sin(p_0)} = p_1 + \frac{p_0'}{2}\cot p_0. \tag{47}$$

We introduced here the notations

$$\tilde{t}_1 = t_1 + \frac{\cos p_0}{8x^2}, \qquad \tilde{t}_2 = t_2 - \frac{\cos p_0}{128x^4} + \frac{\tilde{t}_1}{8x^2} - \frac{\cos(2p_0)+5}{24\sin p_0}p_0'' + \frac{\cos p_0}{8\sin^2 p_0}\left(3(p_0')^2 + 4\tilde{t}_1^2\right) \tag{48}$$

so that $\tilde{t}_1$ and $\tilde{t}_2$ are single valued functions.

Note that above the cut $I^+ = \pi i \rho_1$. We will find the explicit solution of these equations later, but we will need for that some results of the next section where we study the behavior of $p(x)$ near the branch points.

5 Double scaling solution near the branch point

As we stated above the branch point singularities come only from the square roots of the denominator of eq.(36). We define an exact branch point as a point x_* where $t(x_*) = \pm 1$. If we approach one of the branch points $x \to x_*$ we can expand

$$t(x) \simeq \pm[1 - a(x-x_*)/2 - b(x-x_*)^2/2]. \tag{49}$$

Note that x_*, a, b themselves depend on J. We assume that they have a regular expansion in $1/J$ (justified by self-consistency of further calculations and by numerics) and define $x_* = x_0 + x_1/J + \dots$. We call x_0 a classical branch point and x_1/J a branch point displacement.

Denoting $v = (x - x_*)J^{2/3}$ which will be our double scaling variable $v \sim 1$, we get from eq.(30) up to $1/J^2$ terms

$$\pm 2\left(1 - \frac{av}{2J^{2/3}} - \frac{bv^2}{2J^{4/3}}\right)Q(u) = Q(u+i)\frac{W(u+i/2)}{W(u)} + Q(u-i)\frac{W(u-i/2)}{W(u)}. \tag{50}$$

In terms of a new function $q(v)=e^{-n\pi v J^{1/3}}e^{\frac{vJ^{1/3}}{2x_*}}Q(x_*J+vJ^{1/3})$, where n is such that $t(x^*)=e^{i\pi n}$, and after expansion in $1/J$ the last eq. takes the form

$$q''-avq=\frac{1}{J^{1/3}}\frac{4vq'+q}{4x_*^2}+\frac{1}{J^{2/3}}\left[\frac{1}{12}q^{(4)}(v)-\frac{v^2q(v)}{4}\left(\frac{1}{x_*^4}-4b\right)\right]+\left(\frac{1}{J}\right). \tag{51}$$

In fact, this equation can be easily solved in terms of q_0

$$q\propto\left[1+\frac{v^2}{4x_*^2J^{1/3}}+\frac{1}{J^{2/3}}\left(\frac{v^4}{32x_*^4}-\frac{3b-a^2}{15a}v\right)\right]q_0\left(v-\frac{1}{4ax_*^2J^{1/3}}+\frac{a^2+12b}{60aJ^{2/3}}v^2\right), \tag{52}$$

where $q_0(v)=\mathrm{Ai}(a^{1/3}v)$ (the Airy function). The second solution of the eq.(51), $\mathrm{Bi}(a^{1/3}v)$ has a wrong asymptotic as we will see. The sign $\propto$ means that the solution is defined up to a constant multiplier but this unknown multiplier doesn't affect the quasi-momentum. Now we can express the quasi-momentum only through our scaling function $q(v)$

$$p\left(x_*+\frac{v}{J^{2/3}}\right)=\frac{\partial_v q(v,J)}{q(v,J)J^{1/3}}+\pi n+\frac{1}{2x_*}\left(\frac{1}{1+\frac{v}{x_*J^{2/3}}}-1\right). \tag{53}$$

The first two terms in the r.h.s., if we substitute $q(v)\to q_0(v)$, represent the principal contribution to the double scaling limit near the edge, valid up to the corrections of the order $1/J^{2/3}$. We see from the definition (23) that the zeros of $q(v)$ are nothing but the positions u_i of Bethe roots. Thus we know these positions with a precision $1/J^{2/3}$ (see fig.3).

The large v asymptotic will be very helpful in fixing some unknown constant in the $1/J^2$ corrections given in the next section

$$p(x_*+vJ^{-2/3})=\pi n+\frac{1}{J^{1/3}}\left(-\underbrace{\sqrt{av}}_{1}-\underbrace{\frac{1}{4v}}_{1/J}+\underbrace{\frac{5}{32v^2\sqrt{av}}}_{1/J^2}+\dots\right)+\frac{1}{J^{2/3}}\left(\underbrace{\frac{1}{8x_*^2\sqrt{av}}}_{1/J}-\underbrace{\frac{1}{16ax_*^2v^2}}_{1/J^2}+\dots\right)+\dots, \tag{54}$$

where the cut corresponds to negative v for $a>0$. Introducing the notation $y=vJ^{-2/3}$ and rearranging the terms by the powers $1/J$ we have

$$\begin{aligned}p(x_*+y) \quad &= \quad \pi n+\left[-\sqrt{ay}-\frac{(a^2+12b)y^{3/2}}{24\sqrt{a}}+\dots\right]+\frac{1}{J}\left[-\frac{1}{4y}+\frac{1}{8x_*^2\sqrt{ay}}+\frac{a^2-4b}{16a}+\dots\right] \\ &+\frac{1}{J^2}\left[\frac{5}{32y^2\sqrt{ay}}-\frac{1}{16ay^2x_*^2}+\frac{6-x_*^4(a^2+12b)}{768x_*^4(ay)^{3/2}}+\dots\right]+\dots.\end{aligned} \tag{55}$$

Doing this re-expansion we assume that $J^{-1}\ll y\ll 1$, trying to sew together the double scaling region with the $1/J$ corrections to the thermodynamical limit. This procedure is similar to the one used in higher orders of the WKB approximation in the usual one dimensional quantum mechanics (see for example [27]).

To compare with p_0, p_1 and p_2 we have to re-expand around x_0

$$p(x_0+y)=p(x_*+y)+\frac{x_1}{J}\frac{\sqrt{a}}{2\sqrt{y}}+\frac{1}{J^2}\left[-\frac{x_1}{4y^2}+\frac{x_1}{16x_0^2y\sqrt{ay}}+\frac{\sqrt{a}x_1^2}{8y\sqrt{y}}\right] \tag{56}$$

or, introducing notation

$$x_1=\frac{2A}{\sqrt{a}}-\frac{1}{4x_0^2a} \tag{57}$$

we get

$$\begin{aligned}p(x_0+y) \quad &= \quad \pi n+\left[-\sqrt{ay}-\frac{(a^2+12b)y^2}{24\sqrt{ay}}+\dots\right]+\frac{1}{J}\left[-\frac{1}{4y}+\frac{A}{\sqrt{y}}+\frac{a^2-4b}{16a}+\dots\right] \\ &+\frac{1}{J^2}\left[\frac{5}{32y^2\sqrt{ay}}-\frac{A}{2\sqrt{a}y^2}+\left(\frac{A^2}{2y\sqrt{ay}}-\frac{b}{64(ay)^{3/2}}-\frac{\sqrt{ay}}{768y^2}\right)+\dots\right]+\dots.\end{aligned} \tag{58}$$

Near the left branch point (i.e for $a < 0$ and $y < 0$) we have

$$p(x_0 + y) \quad = \quad \pi n + \left[\sqrt{ay} + \frac{(a^2+12b)y^2}{24\sqrt{ay}} + \dots\right] + \frac{1}{J}\left[-\frac{1}{4y} - \frac{A}{\sqrt{-y}} + \frac{a^2-4b}{16a} + \dots\right] \tag{59}$$
$$+\frac{1}{J^2}\left[-\frac{5}{32y^2\sqrt{ay}} + \frac{A}{2\sqrt{-a}y^2} + \left(\frac{A^2}{2y\sqrt{ay}} + \frac{b}{64(ay)^{3/2}} + \frac{\sqrt{ay}}{768y^2}\right) + \dots\right] + \dots .$$

Now we can compare it with our results of the previous sections and fix a, b and x_1.

Let us note that similar Airy type oscillations were observed in the papers on random matrices where this behavior occurs near an endpoint of a distribution of eigenvalues [12]. This shows the intrinsic similarity of these two seemingly different problems. The large J limit for the spin chain is rather similar to the large size N limit in random matrices. We will see this analogy even clearer in the section 7 where we will calculate the asymptotics of conserved charges using the results of this section.

5.1 Comparison with p_0 and p_1

It is instructive to establish the relations between a, b, A and the parameters of the algebraic curve which completely defines, as we know, $p = p_0 + \frac{1}{J}p_1 + \left(\frac{1}{J^2}\right)$ up to the first two orders.

For that we use the expansion (49) defining a, b and find from eq.(35) for $y > 0$

$$p_0(x_0 + y) = \pi n + \arccos t_0 \simeq \pi n - \sqrt{ay} - \frac{a^2+12b}{24\sqrt{a}}y^{3/2} + (y^{5/2}), \tag{60}$$

in agreement with eqs.(58,59). We can fix a and b up to $(1/J)$ corrections from here through the parameters of the solution for p_0 given by eq.(26).

To calculate a and b up to $(1/J)$ and to fix A, we use the expansion eq.(49) with eq.(37). Note that we have the minus sign in front of $\sqrt{ay}$ which ensures the positivity of the density (24) on the cut (i.e. for $y < 0$ and $a > 0$) $\rho(y) \simeq \sqrt{a(-y)}/\pi$. If we had Bi instead of Ai the sign would be plus and the density would be negative.

Now we compare this near-cut behaviour to p_1. Consider the regular part first

$$\not{p}_1 = -\frac{1}{2}p_0' \cot p_0 \simeq -\frac{1}{4y} + \frac{a^2-4b}{16a} + (y), \tag{61}$$

which is in full agreement with eq.(55). From eq.(42) we see that

$$p_1(x_0 + y) - \not{p}_1(x_0 + y) \simeq \frac{A}{\sqrt{y}} + \left(\frac{1}{y^{3/2}}\right), \tag{62}$$

where A can be written explicitly, again using the parameters of p_0 given by eq.(26).

For the example of one-cut solution see eq.(101).

6 General solution for p_2 and E_2

Now we have enough of information to construct p_2 in the most general situation of an arbitrary number of cuts.

We start from a formula which immediately follows from eq.(46)

$$\not{p}_2 = -\frac{1}{2}\partial_x\left[\cot(p_0)\left(p_1 + \frac{p_0'}{2}\cot p_0\right)\right] - \frac{1}{8x^3}, \tag{63}$$

where p_1 is given by eq.(42). The behaviors near zero and at infinity are the following. Since from eq.(6) and eq.(34) it follows that $G(0) - \frac{1}{24J^2}G''(0) = 2\pi k/J + (\frac{1}{J^4})$ we can conclude that

$$p_2(0) = \frac{1}{24}G_0''(0). \tag{64}$$

For large x we have again

$$p_2(x) = O\left(1/x^2\right). \tag{65}$$

Repeating the arguments of the previous subsection we have

$$p_2(x) = \frac{x}{4\pi i f(x)} \oint_{\mathcal{C}} \frac{f(y)}{y(y-x)} \left(\frac{1}{4y^3} + \partial_y \left[\cot(p_0) p_1 \right] \right) + \sum_{j=0}^{5K-1} \frac{c_j x^j}{f^5(x)}, \tag{66}$$

where the path $\mathcal{C}$ is defined as in eq.(42). Again the first term guarantees that p_2 satisfies the eq.(63). We drop out the $p_0' \coth p_0$ for simplicity. We can do this since together with $f(y)$ it forms a single-valued function without cuts and the integral is given by the poles inside of the path of integration. In fact there are only poles at each branch point so that the result can be absorbed into the second term in eq.(66).

So far the second term in eq.(66) was restricted only by the conditions (64) and (65). Of cause this does not explain why we should restrict ourselves by the fifth power of $f(x)$ in denominator. A natural explanation comes from the known behaviour near the branch points (58,59) from where we can see that

$$p_2(x_0^i + y) = \left\{ \begin{array}{ll} \frac{5}{32y^2\sqrt{a_i y}} - \frac{A_i}{2\sqrt{a_i}y^2} + \left(\frac{A_i^2}{2y\sqrt{a_i y}} - \frac{b_i}{64(a_i y)^{3/2}} - \frac{\sqrt{a_i y}}{768 y^2} \right) + \left(\frac{1}{y}\right), & a_i > 0,\ y > 0 \\ -\frac{5}{32y^2\sqrt{a_i y}} + \frac{A_i}{2\sqrt{-a_i}y^2} + \left(\frac{A_i^2}{2y\sqrt{a_i y}} + \frac{b_i}{64(a_i y)^{3/2}} + \frac{\sqrt{a_i y}}{768 y^2} \right) + \left(\frac{1}{y}\right), & a_i < 0,\ y < 0 \end{array} \right. , \tag{67}$$

where all $6K$ constants $a_i,\ b_,\ A_i$ for $i = 1, \ldots, 2K$ are known since they can be determined from the near branch point behaviour of p_0 and p_1 (58,59). a_i and b_i follow from p_0

$$p_0(x_0^i + y) = \left\{ \begin{array}{ll} -\sqrt{a_i y} - \frac{(a_i^2 + 12 b_i) y^2}{24\sqrt{a_i y}} + \left(y^{5/2}\right), & a_i > 0,\ y > 0 \\ \sqrt{a_i y} + \frac{(a_i^2 + 12 b_i) y^2}{24\sqrt{a_i y}} + \left(y^{5/2}\right), & a_i < 0,\ y < 0 \end{array} \right. \tag{68}$$

and A_i comes from p_1

$$p_1(x_0^i + y) = \left\{ \begin{array}{ll} -\frac{1}{4y} + \frac{A_i}{\sqrt{y}} + \left(y^0\right), & a_i > 0,\ y > 0 \\ -\frac{1}{4y} - \frac{A_i}{\sqrt{-y}} + \left(y^0\right), & a_i < 0,\ y < 0 \end{array} \right. . \tag{69}$$

In fact eq.(67) gives only two nontrivial conditions for each branch point which are the coefficient before the half-integer power of y so that we have $4K$ conditions. The extra K conditions come from zero A-period constraints signifying the absence of corrections to the filling fractions α_i.

$$\oint_{\mathcal{C}_l} p_2(x) dx = 0, \qquad l = 1, \ldots, K. \tag{70}$$

To reduce the number of unknown constants consider a branch point x_0. We can see that for small $y = x - x_0$ (for simplicity we assume that the cut is on the left i.e. $a_i > 0$)

$$I_1 \equiv \frac{x}{4\pi i f(x)} \oint_{\mathcal{C}} \frac{f(z)}{z(z-x)} \left(\frac{1}{4z^3} + \partial_z (p_1 \cot p_0) \right) = \frac{3}{16y^2\sqrt{ay}} - \frac{A}{2\sqrt{a}y^2} + \frac{1}{y^{3/2}} \left(\frac{b}{32a^{3/2}} - \frac{5\sqrt{a}}{128} \right) + \left(\frac{1}{y}\right). \tag{71}$$

Introduce the following integral

$$I_2 \equiv \frac{x}{4\pi i f(x)} \oint_{\mathcal{C}} \frac{f(z)}{z(z-x)} \left((p_1 + p_0' \cot p_0) p_1 \cot p_0 - \frac{p_0''}{12} \right) = -\frac{1}{32y^2\sqrt{ay}} + \frac{1}{y^{3/2}} \left(\frac{A^2}{2\sqrt{a}} - \frac{3b}{64a^{3/2}} + \frac{29\sqrt{a}}{768} \right) + \left(\frac{1}{y}\right) \tag{72}$$

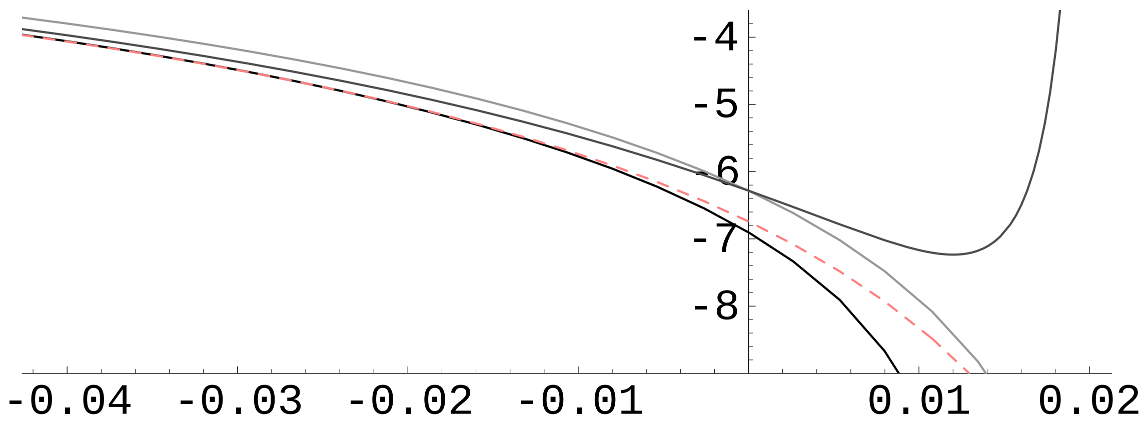

Figure 4: *Resolvent far from branch point as a function of x. Red dashed line - "exact" numerical value for one cut solution with $S = 10$, $n = 2$, $m = 1$, light grey - zero order approximation, grey - first order given by eq.(42) and black - second order approximation given by eq.(73). Note that near branch point ($x_0 = 0.02$) the approximation does not work and instead of it we should use the Airy function of eq.(53), like in the usual WKB near a turning point.*

so that $I_1 + I_2$ reproduces the right series expansion near the branch points given by eq.(54) and eq.(55). Moreover, on the cuts $I_2(x + i0) + I_2(x - i0) = 0$ since the function under integral is single valued. We can simply take

$$p_2(x) = I_1(x) + I_2(x) + \sum_{j=0}^{K-1} \frac{\tilde{c}_j x^j}{f(x)}, \tag{73}$$

where the remaining K constants are fixed from eq.(70). Using that $p_2(0) = G''(0)/24$ we can fix one constant $\tilde{c}_0 = \frac{G''(0)f(0)}{24}$ before imposing the condition (70).

This is our final result for the second quantum correction to the quasi-momentum. In the Appendix B we will specify this result for the example of the one-cut solution where it can be made much more explicit.

7 Energy and higher charges

7.1 Energy

To find $1/J$ corrections to the energy we represent the exact formula eq.(1) as follows

$$E = -\frac{1}{J}G'(0) + \frac{1}{24J^3}G^{(3)}(0) + \left(\frac{1}{J^5}\right), \tag{74}$$

where $G(x)$ is defined in eq.(23). We still have to expand $G(x) = -\frac{1}{2x} + p_0(x) + \frac{1}{J}p_1(x) + \frac{1}{J^2}p_2(x) + O(1/J^3)$.

Finally, we obtain for the energy:

$$E = \frac{1}{J}E_0(x) + \frac{1}{J^2}E_1 + \frac{1}{J^3}E_2 + \left(\frac{1}{J^4}\right), \tag{75}$$

where

$$E_0 = -G_0'(0), \tag{76}$$

$$E_1 = -p_1'(0) = -\frac{Q'(0)}{4\pi i f(0)} \oint_{\mathcal{C}} \frac{f(y)p'(y)\cot p(y)}{Q(y)y} dy, \tag{77}$$

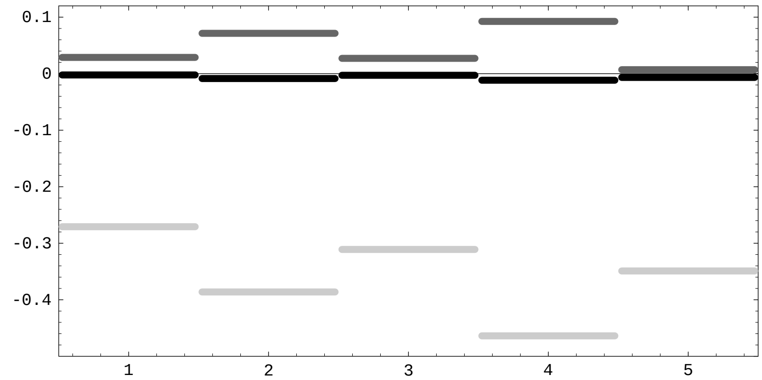

Figure 5: *Relative deviation $\delta E(S)/E(S)$ of analytical computations of the energy $E(S)$ from its "exact" value $E_{exact}(S)$ for the one cut distribution found numerically by Mathematica (solid line corresponds to $\delta E(S)=0$), for a finite number of roots S and a finite length J for zero order (light gray), first order (gray) and second order (black) approximation. Details are summarized in the table*

#	1	2	3	4	5
m,n	$1,2$	$2,1$	$1,3$	$2,2$	$1,5$
E_0	$12\pi^2$	$24\pi^2$	$16\pi^2$	$32\pi^2$	$24\pi^2$
E_1	-558.4	-1563	-855.3	-2401	-1563
E_2	1160	5464	1592	8982	1504
S	10	40	7	20	5
J	20	20	21	20	25
$E(S)$	4.66004	8.54515	5.7359	10.7876	7.0232
$E^{(2)}(S)$	4.670	8.619	5.752	10.912	7.070

and $Q(x) = \sum_{k=1}^{K-2} b_k x^k$ is related to the last term in (42). For E_2 we have from eq.(73) the following representation

$$E_2 = \frac{G_0^{(3)}(0)}{24} - p_2'(0) = \frac{-1}{4\pi i f(0)} \oint \frac{f(y)}{y^2} \left(\frac{1}{4z^3} + \partial_z (p_1 \cot p_0) - \frac{p_0''}{12} + (p_1 + p_0' \cot p_0) p_1 \cot p_0 \right)$$
$$- \frac{c_1}{f(0)} + \frac{G_0''(0) f'(0)}{24 f(0)} + \frac{G_0^{(3)}(0)}{24}. \tag{78}$$

Note that for 1-cut we should take $c_1 = 0$. We can compare our results with numerical calculations, as it is done for a few 1-cut solutions in the fig.5

7.2 Local charges

In this and the next subsection we will calculate local and non-local, or global charges Q_r in *all* powers of $1/J$ for the large r from the behavior near the relevant branch point. The idea of this calculation is taken from the double scaling approach in matrix models. Namely, we can compare it to the calculation of the resolvent of eigenvalues in a gaussian unitary matrix ensemble

$$H_N(x) = \int \frac{d^{N^2} M}{(2\pi)^{N^2}} \exp\left(-\frac{N}{2} \mathrm{Tr}\ M^2 \right) \mathrm{Tr}(x - M)^{-1} = \sum_{g=1}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} x^{-2n-1} H_{(g,n)} \tag{79}$$

where M is a hermitian matrix of large size N. The coefficients $H_{(g,n)}$ actually give the number of specific planar graphs: it is given by the number of surfaces of genus g which can be done from a polygon with $2n$

edges, by the pairwise gluing of these edges. To extract the large n asymptotics of $H_{(g,n)}$ for any g one can use that in the large N limit the density (which is the imaginary part of the resolvent on the support of eigenvalues) is given by the Wigner's semi-circle law, and the near-edge behavior is described by the Airy functional asymptotics [12, 28] showing the traces of individual eigenvalues in the continuous semi-circle distribution. We will try to extract the similar asymptotics for the distribution of Bethe roots. The role of $1/N$ expansion will be played by the $1/J$ expansion, whether as the order of the $1/x$ expansion in the matrix model will be now played by the label r of the charge.

We start from expanding eq.(13)

$$Q_r = \sum_{m=0}^{\infty} \frac{1}{J^{r+2m-1}} \frac{(-1)^{m+1} G^{(r+2m-1)}(0)}{(2m+1)!(r-1)!2^{2m}} \,. \tag{80}$$

As we shell see, for large r only the $m = 0$ term contributes. We express the derivative as a contour integral around cuts

$$G^{(n)}(0) = -\frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{G(x)}{x^{n+1}} dx. \tag{81}$$

For large n only a small neighborhood of the closest to zero branch point x_0 contributes due to the exponential suppression by the $1/x^{n+1}$ factor. Near the branch point x_0 we have from eq.(58) (see also eqs.(59,53))

$$G_k(x) = \delta_{k0}\left(\pi n_i - \frac{1}{2x_0}\right) + \begin{cases} c_k (x-x_0)^{\frac{1}{2}-\frac{3k}{2}} |a|^{\frac{1}{2}-\frac{k}{2}} + \left((x-x_0)^{1-\frac{3k}{2}}\right), & a>0,\ x_0<0 \\ (-1)^{k+1} c_k (x_0-x)^{\frac{1}{2}-\frac{3k}{2}} |a|^{\frac{1}{2}-\frac{k}{2}} + \left((x_0-x)^{1-\frac{3k}{2}}\right), & a<0,\ x_0>0 \end{cases}, \tag{82}$$

where the universal constants c_k can be computed from the known asymptotic of Airy function

$$c_k = \left.\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right|_{z^{-\frac{3k-1}{2}}}, \qquad \mathrm{Ai}(z) = \frac{e^{-\frac{2z^{3/2}}{3}}}{2\sqrt{\pi} z^{1/4}} \left[\sum_{k=0}^{n} \frac{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}{k!} \left(-\frac{3}{4z^{3/2}}\right)^k + \left(\frac{1}{z^{3(n+1)/2}}\right)\right] \tag{83}$$

in particular $c_0 = -1,\ c_1 = -\frac{1}{4},\ c_2 = \frac{5}{32},\ c_3 = -\frac{15}{64},\ c_4 = \frac{1105}{2048},\ c_5 = -\frac{1695}{1024},\ c_6 = \frac{414125}{65536},\ c_7 = -\frac{59025}{2048}$.

These coefficients behave asymptotically as $c_k \sim (-1)^k k!$ at $k \to \infty$.

We assume that $k \ll n, r$ and expand (for $x_0 < 0$)

$$\oint_{-y_0}^{0} (y+x_0)^{-n} y^{\beta} dy = |x_0|^{\beta+1-n} (-1)^n \oint_{-y_0}^{0} y^{\beta} e^{-n\log(1-y)} dy \simeq |x_0|^{\beta+1-n} (-1)^n \oint_{-\infty}^{0} y^{\beta} e^{ny} dy. \tag{84}$$

For the last integral the path of integration starts at $-\infty - i0$, encircles the origin in the counterclockwise direction, and returns to the point $-\infty + i0$. For the first integral the path is finite: it starts at some point $-y_0 - i0$ where $0 < y_0 < |x_0|$ and ends at $-y_0 + i0$. The dependence on the y_0 is exponentially suppressed. The last integral is nothing but the Hankel's contour integral

$$\oint_{-y_0}^{0} (y+x_0)^{-n} y^{\beta} dy = (-1)^n |x_0|^{\beta+1-n} n^{-\beta-1} \frac{2\pi i}{\Gamma(-\beta)} \left(1 + \left(\frac{1}{n}\right)\right) \tag{85}$$

similary

$$\oint_{0}^{y_0} (y+x_0)^{-n} (-y)^{\beta} dy = -|x_0|^{\beta+1-n} n^{-\beta-1} \frac{2\pi i}{\Gamma(-\beta)} \left(1 + \left(\frac{1}{n}\right)\right) \tag{86}$$

so that

$$\frac{G_k^{(n)}(0)}{n!} = \begin{cases} (-1)^n \frac{c_k |a|^{\frac{1}{2}-\frac{k}{2}} n^{\frac{3k}{2}-\frac{3}{2}} |x_0|^{\frac{1}{2}-\frac{3k}{2}-n}}{\Gamma\left(\frac{3k}{2}-\frac{1}{2}\right)} \left(1 + \left(\frac{1}{n}\right)\right), & a>0,\ x_0<0 \\ (-1)^{k+1} \frac{c_k |a|^{\frac{1}{2}-\frac{k}{2}} n^{\frac{3k}{2}-\frac{3}{2}} |x_0|^{\frac{1}{2}-\frac{3k}{2}-n}}{\Gamma\left(\frac{3k}{2}-\frac{1}{2}\right)} \left(1 + \left(\frac{1}{n}\right)\right), & a<0,\ x_0>0 \end{cases}. \tag{87}$$

As we can see from here, only the term with $m = 0$ in eq.(80) contributes at large n. The others are suppressed as $1/n$ and the final result is

$$Q_{k,r} = \begin{cases} (-1)^r \frac{c_k |a|^{\frac{1}{2}-\frac{k}{2}} r^{\frac{3k}{2}-\frac{3}{2}} |x_0|^{\frac{3}{2}-\frac{3k}{2}-r}}{\Gamma\left(\frac{3k}{2}-\frac{1}{2}\right)} \left(1 + \left(r^{-1/2}\right)\right), & a > 0,\ x_0 < 0 \\ (-1)^k \frac{c_k |a|^{\frac{1}{2}-\frac{k}{2}} r^{\frac{3k}{2}-\frac{3}{2}} |x_0|^{\frac{3}{2}-\frac{3k}{2}-r}}{\Gamma\left(\frac{3k}{2}-\frac{1}{2}\right)} \left(1 + \left(r^{-1/2}\right)\right), & a < 0,\ x_0 > 0 \end{cases}, \tag{88}$$

where we introduced the notation

$$Q_r = \frac{1}{J^{r-1}} \sum_{k=0}^{\infty} Q_{k,r} \frac{1}{J^k}. \tag{89}$$

Note that $Q_{k,r}$ is similar to $H_{g,n}$ of the matrix model.

7.3 Non-local charges

Now we will find the coefficients of the $1/x^n$ expansion for large n and arbitrary k

$$G(x) = \sum_{k=0}^{\infty} \frac{1}{J^k} \sum_{n=1}^{\infty} \frac{d_{k,n}}{x^n}, \tag{90}$$

in other words

$$d_{k,n} = \frac{1}{2\pi i} \oint_{\mathcal{C}} x^{n-1} G_k(x) dx. \tag{91}$$

In fact only the cut with minimal n_i contributes for large n, or rather its branch point closest to $x = \infty$. The contributions of other branch points are exponentially suppressed. It is enough to consider only a small neighborhood of the branch point with maximal $|x_{0,i}|$. Near the branch point $p_k(x)$ will behave as

$$G_k(x) = \delta_{k0}\left(\pi n_i - \frac{1}{2x_0}\right) + \begin{cases} c_k (x - x_0)^{\frac{1}{2}-\frac{3k}{2}} |a|^{\frac{1}{2}-\frac{k}{2}} + (x - x_0)^{1-\frac{3k}{2}}, & a > 0,\ x_0 > 0 \\ (-1)^{k+1} c_k (x_0 - x)^{\frac{1}{2}-\frac{3k}{2}} |a|^{\frac{1}{2}-\frac{k}{2}} + (x_0 - x)^{1-\frac{3k}{2}}, & a < 0,\ x_0 < 0 \end{cases} \tag{92}$$

similarly to the previous section we obtain

$$d_{k,n} = \begin{cases} \frac{c_k\ a^{\frac{1}{2}-\frac{k}{2}} n^{\frac{3k}{2}-\frac{3}{2}} x_0^{n+\frac{1}{2}-\frac{3k}{2}}}{\Gamma\left(\frac{3k}{2}-\frac{1}{2}\right)} \left(1 + \left(n^{-1/2}\right)\right), & a > 0,\ x_0 > 0 \\ (-1)^{k+n-1} \frac{c_k\ a^{\frac{1}{2}-\frac{k}{2}} n^{\frac{3k}{2}-\frac{3}{2}} x_0^{n+\frac{1}{2}-\frac{3k}{2}}}{\Gamma\left(\frac{3k}{2}-\frac{1}{2}\right)} \left(1 + \left(n^{-1/2}\right)\right), & a < 0,\ x_0 < 0 \end{cases}. \tag{93}$$

8 Conclusions

We showed in this paper on the example of $\mathfrak{sl}(2)$ Heisenberg spin chain, how to find general solutions to quantum integrable problems in a specific thermodynamical limit proposed by Sutherland, and how to find various finite size corrections to it. We also propose a double scaling analysis of the near edge distribution of Bethe roots giving some interesting results for the asymptotics of high conserved charges for the finite size corrections of any order.

Our methods borrow some ideas from the matrix models: the size of the chain J is somewhat similar to the size of a random matrix N, and the first two $1/J$ finite size corrections to the main limit which we calculated are of a similar mathematical nature as the $1/N$ corrections in matrix models. The asymptotics of high conserved charges found here in all orders of $1/J$ remind very much the double scaling approach in matrix models [12, 29, 30]. To calculate them we used for that the Airy asymptotics on the edge of the Bethe root distribution, similar to the generic edge behavior in the matrix models. However, it is obvious that by finetuning the parameters of the chain and of its large J solutions we can reach various multicritical points with a different from Airy classes of universality, probably also similar to those of the multicritical matrix models [13] or a two cut model [31, 32]. Probably, some modern methods of analysis of the doubly scaled matrix models, like the Riemann-Hilbert method for the asymptotics of orthogonal polynomials [33] should work as well for the quantum integrable chains in this thermodynamical limit. Indeed, it is known that the polynomials $Q_S(u) = \prod_{k=1}^{S}(u - u_k)$ are orthogonal for different S with a specific S-independent measure [34].

Our methods might be useful for finding the asymptotics of numbers of various combinatorial objects on regular lattices related to the integrable models, such the tilting patterns, spanning trees, dimers etc., as its analogue was useful for finding the asymptotics of various large planar graphs via the matrix models.

The methods presented here can be easily carried over to the $\mathfrak{su}(2)$ quantum chain as well, though some peculiarities of this model, like complex distributions of roots and the presence of "string" condensates with equally distributed roots [3], should be taken into account. Only slight modifications of our results will allow to find the $1/J$ corrections in the nonlocal integrable deformations of the $\mathfrak{su}(2)$ spin chain described in [35, 36]. As for more complicated models solved by nested Bethe ansatz and with the thermodynamical limit described by non-hyperelliptic algebraic curves, the $1/J$ and $1/J^2$ correactions are left to be established.

In the context of integrability in the 4D Yang-Mills theory with $\mathcal{N} = 4$ supersymmetries (SYM), where the hamiltonian of the $\mathfrak{psu}(2, 2|4)$ integrable spin chains describes the matrix of anomalous dimensions, our methods could be especially useful. The appropriate thermodynamical limit in terms of non-hyperelliptic curve, corresponding to very long operators in SYM was constructed in particular sectors in [4, 7, 20, 37] and for the full $\mathfrak{psu}(2, 2|4)$ SYM chain in [11]. Finding the finite size corrections to these solutions might be extremely useful for getting some clues for the quantization of the string sigma model on the other side of AdS/CFT correspondence. The finite size correction for the simplest $\mathfrak{su}(3)$ solution of SYM theory found in [38] are not available yet, let alone more complicated solutions and sectors. The finite size corrections to this solution, if found, could be compared to the string quantum correction computed in [39]. The corrections found here for the most general multi-cut solution in $\mathfrak{sl}(2)$ sector could shed some light on the structure of the first quantum corrections to string solitons describing spinning strings, where for the moment only the direct, very cumbersome quasi-classical methods of evaluation of the functional integral near the simplest classical solutions are known [15].

As for the double scaling limit used here for the refined analysis near the edge of a distribution and giving the asymptotics of high conserved charges to any order of $1/J$, it is very universal and is not based on any particular form of solutions. Near the edge we can forget about the rest of the sheets and cuts in any algebraic curve, and the asymptotics will be governed in all cases by the Airy functions. In fact, we think that this near edge behavior will persist also in the near-classical regime of the string theory on $AdS_5 \times S^5$, and even for some of its recently considered deformations. We expect then that the asymptotics of high conserved charges at all orders of $1/J$ will be also dominated by the same Airy type solution as found here. It is already a deeply quantum regime (reminding that of the non-critical string theories in ≤ 2 dimensions) and it could be extremely useful as a step to the full quantization and solution of the Metsaev-Tseytlin supersgtring and hence of the $\mathcal{N} = 4$ SYM theory.

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Appendix A, BAE $1/J$ expansion

In this Appendix we give the formulas for the expansion of the rhs of BAE (2) in powers of $1/J$. The density $\varrho(x)$ is defined in eq.(16). We assume $k, S-k \sim J$ (i.e. far from the ends of the cut)

$$\sum_j{}' i\log\left(\frac{u_j-u_k+i}{u_j-u_k-i}\right) \simeq -2\sum_j{}'\frac{1}{u_j-u_k} + \frac{\pi\varrho'[\coth(\pi\varrho)]_0}{J} + \left(\frac{1}{J^2}\right) \tag{94}$$

$$\sum_j{}' i\log\left(\frac{u_j-u_k+i}{u_j-u_k-i}\right) \simeq -2\sum_j{}'\frac{1}{u_j-u_k} + \frac{2}{3}\sum_j{}'\frac{1}{(u_j-u_k)^3} + \frac{\pi\varrho'[\coth(\pi\varrho)]_2}{J} + \left(\frac{1}{J^3}\right)$$

$$\sum_j{}' i\log\left(\frac{u_j-u_k+i}{u_j-u_k-i}\right) \simeq -2\sum_j{}'\frac{1}{u_j-u_k} + \frac{2}{3}\sum_j{}'\frac{1}{(u_j-u_k)^3} - \frac{2}{5}\sum_j{}'\frac{1}{(u_j-u_k)^5} + \frac{\pi\varrho'[\coth(\pi\varrho)]_4}{J}$$

$$-\frac{1}{12J^3}\left((\pi\varrho')^3\left[\frac{\coth(\pi\varrho)}{\sinh^2(\pi\varrho)}\right]_0 - 2\pi^2\varrho'\varrho''\left[\frac{1}{\sinh(\pi\varrho)}\right]_1 + \pi\varrho'''[\coth(\pi\varrho)]_2\right) + \left(\frac{1}{J^4}\right)$$

$$\sum_j{}' i\log\left(\frac{u_j-u_k+i}{u_j-u_k-i}\right) \simeq -2\sum_j{}'\frac{1}{u_j-u_k} + \frac{2}{3}\sum_j{}'\frac{1}{(u_j-u_k)^3} - \frac{2}{5}\sum_j{}'\frac{1}{(u_j-u_k)^5} + \frac{2}{7}\sum_j{}'\frac{1}{(u_j-u_k)^7}$$

$$+\frac{\pi\varrho'[\coth(\pi\varrho)]_6}{J} - \frac{1}{12J^3}\left((\pi\varrho')^3\left[\frac{\coth(\pi\varrho)}{\sinh^2(\pi\varrho)}\right]_2 - 2\pi^2\varrho'\varrho''\left[\frac{1}{\sinh(\pi\varrho)}\right]_3 + \pi\varrho'''[\coth(\pi\varrho)]_4\right) + \left(\frac{1}{J^5}\right),$$

where we introduced the notation $[f(\varrho)]_n \equiv f(\varrho) - \sum_{i=0}^{n-1} f^{(i)}(0)\frac{\varrho^i}{i!}$ for the functions regular at zero. For singular functions the Taylor series should be substituted by the Laurent series, so that $[f(\varrho)]_n$ is zero for $\varrho = 0$ and has first $n-1$ zero derivatives at this point. For example $[\coth(\pi\varrho)]_2 \equiv \coth(\pi\varrho) - \frac{1}{\pi\varrho} - \frac{\pi\varrho}{3}$.

Appendix B, Example: 1-cut

In this Appendix we express corrections to the energy in terms of infinite sums for the simplest case of one-cut solution. For this solution the hyperelliptic curve is a sphere. It is two complex planes connected by a single cut. The density of the Bethe roots is given by a simple formula [7]

$$\rho(x) = \frac{\sqrt{8\pi m x - (2\pi n x - 1)^2}}{2\pi x}. \tag{95}$$

We can easily find explicit expressions for a_i and b_i of eq.(67). With the notation $M = \sqrt{m(m+n)}$ a_i and b_i become

$$a_1 = -\frac{8Mn^4\pi^3}{(\sqrt{4M^2+n^2}-2M)^2}, \qquad b_1 = \frac{4\pi^4 n^6}{3(\sqrt{4M^2+n^2}-2M)^4}\left(12M\sqrt{4M^2+n^2} + 3n^2 - 4n^2\pi^2M^2 - 24M^2\right) \tag{96}$$

and

$$a_2 = \frac{8Mn^4\pi^3}{(\sqrt{4M^2+n^2}+2M)^2}, \qquad b_2 = -\frac{4\pi^4 n^6}{3(\sqrt{4M^2+n^2}+2M)^4}\left(12M\sqrt{4M^2+n^2} - 3n^2 + 4n^2\pi^2M^2 + 24M^2\right). \tag{97}$$

It may be more convenient for comparison with string theory results [15] to express A defined by eq.(58) as an infinite sum. We have to evaluate the integral in eq.(42) and find A from the behavior near a branch point. We compute the integral by poles. To that end we use that the solutions to the equation $\sin(p_0(x_l^\pm)) = 0$ are

$$x_l^\pm = \frac{1}{2\pi}\frac{1}{\sqrt{4M^2+n^2} \mp \sqrt{4M^2+l^2}}, \qquad l \geq 0. \tag{98}$$

The points $x^{\pm}_{l=0}$ are the branch points. They are inside the contour of integration and thus do not contribute.

Using that $f(x^{\pm}_l)/x^{\pm}_l = \pm\frac{l}{n}$ and

$$\begin{aligned}\frac{1}{x_l^+ - x_{0,1}} - \frac{1}{x_l^- - x_{0,1}} &= -\frac{\sqrt{l^2+4M^2}}{l^2}\frac{1}{\pi x_{0,1}^2} \\ \frac{1}{x_l^+ - x_{0,2}} - \frac{1}{x_l^- - x_{0,2}} &= -\frac{\sqrt{l^2+4M^2}}{l^2}\frac{1}{\pi x_{0,2}^2}.\end{aligned} \tag{99}$$

We can evaluate the integral eq.(42) for $x \to x_0$ (we also take x inside the contour to drop irrelevant symmetric part of p_1)

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(y)p'(y)\cot p(y)}{y(y-x)}dy \to -\frac{1}{i\pi n x_0^2}\sum_{l=1}^{\infty}\frac{\sqrt{l^2+4M^2}}{l} \tag{100}$$

we can conclude that

$$\begin{aligned}A_2 &= -\frac{1}{2x_2^2\sqrt{a_2}}\sum_{l=1}^{\infty}\frac{\sqrt{l^2+4M^2}}{l} \\ A_1 &= -\frac{1}{2x_1^2\sqrt{-a_1}}\sum_{l=1}^{\infty}\frac{\sqrt{l^2+4M^2}}{l}\ ,\end{aligned} \tag{101}$$

where the sum should be understood in the zeta-function regularization (a natural explanation why this regularization gives the right result is given in [8]. A more regular way to express the integral as a sum is to expand cot into the sum before integration)

$$\sum_{l=1}^{\infty}\frac{\sqrt{l^2+4M^2}}{l} \equiv \sum_{l=1}^{\infty}\left(\frac{\sqrt{l^2+4M^2}}{l}-1\right)-\frac{1}{2}. \tag{102}$$

We can easily reproduce the result of [8] for E_1 in terms of a sum from eq.(77)

$$E_1 = -p_1'(0) = 4\pi^2\sum_{l=1}^{\infty} l\sqrt{l^2+4M^2} \tag{103}$$

with ζ-function regularization assumed.

We can also express our result for the next correction to the energy E_2 given by eq.(78) as a double sum. We will need the following quantity

$$p_1(x_k^{\pm}) = \frac{\pm 1}{2\pi (x_k^{\pm})^2 k}\left[\sum_{l=1}^{\infty}\left(\frac{l\sqrt{l^2+4M^2}-k\sqrt{k^2+4M^2}}{l^2-k^2}-1\right)+\frac{\sqrt{k^2+4M^2}}{2k}-\frac{1}{2}\right]. \tag{104}$$

Evaluating the integrals in eq.(78) we express E_2 as a double sum

$$E_2 = -(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4), \tag{105}$$

where

$$\begin{aligned}\mathcal{I}_1 &\equiv \frac{1}{4\pi i f(0)}\oint\frac{f(z)}{z^2}\partial_z(p_1\cot p_0) = -2p_1'(0)+\sum_{k=1}^{\infty}\left[2\pi\sum_{\pm}\left(\sqrt{4M^2+n^2}\pm 2\frac{k^2+2M^2}{\sqrt{k^2+4M^2}}\right)p_1(x_k^{\pm})-4p_1'(0)\right] \\ \mathcal{I}_2 &\equiv \frac{1}{4\pi i f(0)}\oint\frac{f(z)}{4z^5} = 4\pi^4M^2(n^2+5M^2) \\ \mathcal{I}_3 &\equiv I_2'(0) = \frac{1}{16}\left(\frac{1}{x_{0,1}^2}+\frac{1}{x_{0,2}^2}\right)+\frac{1}{x_{0,1}}\left(\frac{7a_1}{96}-\frac{b_1}{8a_1}-A_1^2\right)+\frac{1}{x_{0,2}}\left(\frac{7a_2}{96}-\frac{b_2}{8a_2}+A_2^2\right) \\ \mathcal{I}_4 &\equiv -\frac{G_0''(0)f'(0)}{24f(0)}-\frac{G_0^{(3)}(0)}{24} = \frac{4}{3}M^2(2n^2+11M^2)\pi^4.\end{aligned} \tag{106}$$

Note that in our new notations $1/x_{0,i} = 4\pi M \pm 2\pi\sqrt{4M^2 + n^2}$. Expressions for a_i, b_i and A_i are given in eqs.(96,97) and eq.(101).

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