HIGHER GENUS ICOSAHEDRAL PAINLEVÉ CURVES

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Abstract. We will write down the higher genus algebraic curves supporting icosahedral solutions of the sixth Painlevé equation, including the largest (genus seven) curve.

1. Introduction

A Painlevé curve Π is an algebraic curve supporting a solution to Painlevé's sixth equation (henceforth $\mathrm{P_{VI}}$). That is, there should be rational functions y, t on Π such that

$$t : \Pi \longrightarrow \mathbb{P}^1 \tag{1}$$

is a Belyi map (so expresses Π as a branched cover, ramified only over $0, 1, \infty$) and y (viewed as a function of t) solves a $\mathrm{P_{VI}}$ equation.

This notion was introduced by Hitchin [6] who found an infinite family of examples related to the Poncelet problem. In essence he showed that all the modular curves $X_1(n)$ are Painlevé curves, at least for n prime. More precisely one should first pull back along the standard map $X(2) \to X(1)$ (with Galois group $\mathrm{Sym}_3 = \mathrm{PSL}_2(2)$), so there is a diagram:

$$\begin{array}{ccc} \Pi & \longrightarrow & X_1(n) \\ \big\downarrow{\scriptstyle t} & & \big\downarrow \\ \mathbb{P}^1 \cong X(2) & \longrightarrow & X(1). \end{array}$$

In particular, for $n = 5$, Hitchin wrote down the first explicit genus one Painlevé curve.

The aim of this article is to write down some other explicit Painlevé curves not in the above family of examples.

The (nonlinear) $\mathrm{P_{VI}}$ equation controls the "isomonodromic" (or monodromy preserving) deformations of (linear) rank two Fuchsian systems on $\mathbb{P}^1$ with four singularities, at $0, t, 1, \infty$. The monodromy of such a system is a representation

$$\rho : \mathcal{F}_3 = \pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

and one of the main properties of these Painlevé curves is that the monodromy of the cover (1), i.e. its permutation representation $\mathcal{F}_2 \to \{1, 2, \ldots, \deg(t)\}$, coincides with the standard action of the pure mapping class group of the four-punctured sphere ($\cong \mathcal{F}_2$) on the orbit it generates through the conjugacy class of the representation ρ.

Hitchin's examples arose by seeking such isomonodromic deformations when the image of ρ was equal to a binary dihedral group, and in a previous article [1] the author studied the case when the monodromy group is equal to the binary icosahedral group. All such solutions were classified and explicit formulae were written down for all but 8 of the 52 cases, including all those of genus zero and most of the genus one cases. (Five interesting cases had previously appeared in [4, 5, 7].)

Unfortunately the icosahedral Painlevé curves of genus ≥ 2 were not amenable to the method of construction used in [1], essentially due to the large degrees of the Belyi maps t. (The method used was to first obtain, from the icosahedral linear monodromy, the precise asymptotics of the P_{VI} solution, using (the author's correction of) Jimbo's asymptotic formula; this determined the Puiseux expansions to arbitrary order which in turn enabled the curve to be obtained algebraically.)

However it turns out that there is a trick to convert earlier icosahedral Painlevé curves (that were found in [1], or were previously known) into those of higher genus. Namely one may use the so-called "quadratic transformations" introduced by Kitaev [8] in 1991 and written in simpler form by Ramani et al. [10] (we learnt of them from the recent article [11]). Somewhat miraculously the solutions that can be obtained in this way are almost exactly the complement of those we were able to obtain by the previous method (there is a small overlap though).

Thus our aim is to explain how the quadratic transformations may be applied in this way and write down the resulting curves. (This is not entirely trivial since, if applied blindly, the quadratic transformations lead to badly parameterised solutions, for example with the wrong genus.) We also make some effort to obtain nice models (over $\mathbb{Q}$) of the resulting Painlevé curves.

For example the following result will be established:

Theorem. *There are precisely two non-hyperelliptic icosahedral Painlevé curves. The first supports two inequivalent Painlevé solutions and is of genus three and isomorphic to the smooth plane quartic with affine equation*

$$5(p^4+q^4)+6(p^2q^2+p^2+q^2)+1=0.$$

The second is of genus seven and is birationally isomorphic over $\mathbb{Q}$ to the affine curve cut out by the octic

$$9\,(p^6\,q^2+p^2\,q^6)+18\,p^4\,q^4+4\,(p^6+q^6)+26\,(p^4\,q^2+p^2\,q^4)+8\,(p^4+q^4)+57\,p^2\,q^2+20\,(p^2+q^2)+16$$

whose closure in $\mathbb{P}^2$ only has double point singularities. Moreover the obvious symmetries of these curves (negating and exchanging p and q, generating a dihedral group of order 8) correspond to the Okamoto symmetries of the Painlevé solutions.

2. Background

We will constrain ourselves to giving the notation and terminology that we will use, referring the reader to [1] or the review article [2] and references therein for more details and geometrical background.

The sixth Painlevé equation (P_{VI}) is:

$$\begin{aligned}\frac{d^2y}{dt^2} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\ &+\frac{y(y-1)(y-t)}{2\,t^2(t-1)^2}\left((\theta_4-1)^2-\frac{\theta_1^2\,t}{y^2}+\frac{\theta_3^2(t-1)}{(y-1)^2}+\frac{(1-\theta_2^2)t(t-1)}{(y-t)^2}\right)\end{aligned}$$

where $\theta=(\theta_1,\theta_2,\theta_3,\theta_4)$ are (complex) constants. This arises naturally when one tries to isomonodromically deform Fuchsian systems of the form

$$\frac{d}{dz}-\left(\frac{A_1}{z}+\frac{A_2}{z-t}+\frac{A_3}{z-1}\right), \qquad A_i\in\mathfrak{g}:=\mathfrak{sl}_2(\mathbb{C}) \tag{2}$$

as the second pole position t varies in $\mathbb{P}^1\setminus\{0,1,\infty\}$. (The parameters θ specify the eigenvalues of the residues: namely A_i has eigenvalues $\pm\theta_i/2$ for $i=1,2,3,4$, where $A_4=-\sum_1^3 A_i$.) Geometrically P_{VI} can (thus) be thought of as the explicit form of the simplest nonabelian Gauss–Manin connection.

Definition 1. An algebraic solution of P_{VI} consists of a triple (Π,y,t) where Π is a compact (possibly singular) algebraic curve and y,t are rational functions on Π such that:
• $t:\Pi\to\mathbb{P}^1$ is a Belyi map (i.e. t expresses Π as a branched cover of $\mathbb{P}^1$ which only ramifies over $0,1,\infty$), and
• Using t as a local coordinate on Π away from ramification points, $y(t)$ should solve P_{VI}, for some value of the parameters θ.

Indeed given an algebraic solution in the form of a polynomial relation $F(y,t)=0$ one may take Π to be the closure in $\mathbb{P}^2$ of the affine plane curve defined by F. That t is a Belyi map on Π follows from the Painlevé property of P_{VI}: solutions will only branch at $t=0,1,\infty$ and all other singularities are just poles. The reason we prefer this reformulation is that often the polynomial F is quite complicated and usually there are much simpler models of the plane curve defined by F. (The polynomial F can of course be recovered as the minimal polynomial of y over $\mathbb{C}(t)$.)

We will say a Painlevé curve Π is 'minimal' or an 'efficient parameterisation' if y generates the field of rational functions on Π, over $\mathbb{C}(t)$, so that y and t are not pulled back from another curve covered by Π (i.e. that Π is birational to the curve defined by F).

The main invariants of an algebraic solution are the genus of a (minimal) Painlevé curve Π and the degree of the corresponding Belyi map t (the number of branches the solution has over the t-line).

We will say that two solutions of P_{VI} are *equivalent* if they are related by Okamoto's affine F_4 Weyl group symmetries [9] of P_{VI} (which act on the set of parameters $\{\theta\}\cong\mathbb{C}^4$ in the standard way). (See e.g. [11, 2] for formulae for this action.) For an algebraic solution, this acts within the set of rational functions on the curve Π, and preserves the degree and genus of the solution (at least if the linear monodromy representation is irreducible and not rigid).

We are interested here in the case where the monodromy group of the linear system (2) is equal to the binary icosahedral group[1] $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. To understand the different cases that may occur essentially amounts to studying the different conjugacy classes of the local *projective* monodromies. Recall that the icosahedral rotation group $\Gamma/\pm \cong A_5 \subset \mathrm{SO}_3(\mathbb{R})$ has four non-trivial conjugacy classes, which we will label a, b, c, d corresponding to rotations by $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}$-of a turn, respectively. Thus we define, as in [1], the A_5-*type* of a representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}) \to \Gamma$$

to be the corresponding unordered set of four conjugacy classes of projective local monodromies (i.e. take the conjugacy classes of the images in A_5 of the elements $\rho(\gamma_i)$ for simple loops γ_i encircling one of $0, t, 1$ or ∞ once). The different cases that occur are tabulated in [1].

Two inequivalent icosahedral solutions will be said to be *siblings* if their monodromy representations ρ are related by the nontrivial outer automorphism of A_5 (swapping the conjugacy classes c, d). They will have the same Belyi map t, just a different solution function y. (In general it is useful to generalize this notion by considering Galois conjugate representations, e.g. for representations into the 237 triangle group there are sometimes three siblings, cf. [3].)

3. Quadratic transformations

The basic idea [8] behind the quadratic transformations is as follows. Given an icosahedral Fuchsian system A with A_5 type $a^2\xi\eta$ for some $\xi, \eta \in \{a, b, c, d\}$ (i.e. with two local monodromies, say at 0 and ∞, of order two in $\mathrm{PSL}_2(\mathbb{C})$) we can pull back along the map $w \mapsto z = w^2$ to get a Fuchsian system with two apparent singularities at $0, \infty$ and four non-apparent singularities at $\pm 1, \pm\sqrt{t}$. Removing the apparent singularities (using Schlesinger transformations) yields a system B with A_5 type $\xi^2\eta^2$, which may be put in the form (2) by a coordinate transformation. Isomonodromic deformations of A correspond to isomonodromic deformations of B, and one can obtain formulae relating the corresponding $\mathrm{P_{VI}}$ solutions. In practice the formulae are much simpler at different (Okamoto equivalent) values of the parameters (see Ramani et al. [10] (2.7)). We should emphasise that these transformations are not really symmetries of the family of Painlevé VI equations since the conditions on the parameters restrict us to a co-dimension two subset of the four-dimensional parameter space. Nonetheless they are precisely what is needed to obtain the eight outstanding icosahedral solutions, since they all have the desired factor of a^2 in their A_5 types. Indeed for these cases, this procedure gives an algebraic relation with a solution having half the number of branches; Examining table 1 of [1] we see solution 31 $\Rightarrow$ solution 44 and in turn solution 44 $\Rightarrow$ solution 50. Similarly

$$32 \Rightarrow 45 \Rightarrow 51, \qquad 39 \Rightarrow 47, \qquad 40 \Rightarrow 48, \qquad 41 \Rightarrow 49 \Rightarrow 52.$$

[1] more precisely we are interested in the solutions equivalent to such; one should bear in mind that the Okamoto transformations can change the monodromy group, and it will in fact be simpler to work at different equivalent values of the parameters θ. cf. Remark 4

The formula of Ramani et al. that we will use to construct these outstanding solutions from known solutions is as follows. (In fact this is the inverse of the formula [10] (2.7), having converted their parameters to our conventions.)

Proposition 2 ([10])**.** *Given a solution* (y_0, t_0) *of* P_{VI} *with parameters of the form* $\theta = (0, \theta_2, \theta_3, 1)$ *then, by taking two square roots, one obtains a new solution* (y, t) *with parameters* $\theta = (\theta_3, \theta_2, \theta_2, 2 - \theta_3)/2$ *where*

$$y = \frac{(\tau - 1)(\eta + 1)}{(\tau + 1)(\eta - 1)}, \qquad t = \left(\frac{\tau - 1}{\tau + 1}\right)^2$$

with $\eta^2 = y_0, \tau^2 = t_0$*.*

Note that negating τ corresponds to the Okamoto symmetry $(y, t) \mapsto (y/t, 1/t)$ and negating both η and τ corresponds to $(y, t) \mapsto (1/y, 1/t)$.

In practice this will usually lead to an inefficiently parameterised Painlevé curve. In the cases at hand this may be remedied as follows. (In the process we will convert the formula to that most directly useful to us.) The relation between the Painlevé curve Π' we end up with and the original curve Π may be summarised by the diagram:

$$\begin{array}{ccc} & \widetilde{\Pi} & \\ {}^{4}\swarrow & & \searrow^{2} \\ \Pi & & \Pi' \end{array}$$

where the numbers indicate the degrees of the maps, and $\widetilde{\Pi}$ is the intermediate curve obtained by adjoining the two square roots to the function field of Π.

Suppose our initial solution is a pair of functions of the form

$$Y = \frac{1}{2} + a_Y(s)u, \qquad T = \frac{1}{2} + a_T(s)u \tag{3}$$

for parameters of the form $\theta = (0, \theta_2, 0, \theta_4)$ on a curve of the form

$$\Pi := \{u^2 = u_2(s)\}$$

where u_2 is a polynomial, and a_Y, a_T are rational functions of s. In other words Π is a double cover of the s-line $\mathbb{P}^1_s$, and the symmetry of Π (negating u) corresponds to the symmetry $(y, t) \mapsto (1 - y, 1 - t)$. Our basic observation is that the parameter u will drop out in the solution obtained, as follows.

Applying the Okamoto transformation $(Y, T) \mapsto (Y/(Y - 1), T/(T - 1))$ yields a solution to which we may apply Proposition 2. Thus we need to take square roots of $Y/(Y-1), T/(T-1)$, i.e. of expressions of the form $(A + u)/(A - u)$ where $A = 2au_2$ is still a rational function of s. A useful trick is to look for square roots of similar form: i.e. to find B such that

$$\left(\frac{B + u}{B - u}\right)^2 = \frac{A + u}{A - u}.$$

Taking the square root of both sides and solving for B we find

$$B = A \pm \sqrt{A^2 - u_2}$$

which does not involve u. Carrying this out for both Y and T we obtain

$$\eta = \frac{B_Y + u}{B_Y - u}, \qquad \tau = \frac{B_T + u}{B_T - u},$$

where $B_i = A_i \pm \sqrt{A_i^2 - u_2}$ for $i = Y, T$. Then the formulae of Proposition 2 yield

$$y = \frac{B_Y}{B_T}, \qquad t = \frac{u_2}{B_T^2}$$

neither of which involves u. Thus Π' can be viewed as either the quotient of $\widetilde{\Pi}$ by the involution negating u or as the four-fold cover of the s-line obtained by adjoining functions v, w with

$$v^2 = A_Y^2 - u_2, \qquad w^2 = A_T^2 - u_2$$

where $A_i = 2u_2 a_i$ for $i = Y, T$. (The reader may verify that the involution of Π' negating both v and w together yields the transformation $(y, t) \mapsto (1/y, 1/t)$.)

In turn if we apply the transformation $(y, t) \mapsto (y/(y-1), t/(t-1))$ we will obtain a solution of form similar to (3). In summary (after some relabelling) the version of the quadratic transformations we will actually use is as follows:

Corollary 3. *If the functions y_0, t_0 of the form*

$$y_0 = \frac{1}{2} + a_y(s)u, \qquad t_0 = \frac{1}{2} + a_t(s)u$$

are a P_{VI} solution with parameters $\theta = (0, \theta_2, 0, \theta_4)$ on a Painlevé curve of the form

$$\Pi := \{u^2 = u_2(s)\}$$

for a polynomial $u_2(s)$, then the functions

$$y = \frac{1}{2} + \frac{w + v}{2(A_y - A_t)}, \qquad t = \frac{1}{2} - \frac{A_t}{2w}$$

are a P_{VI} solution for parameters $\theta = (1 - \theta_4, \theta_2, 1 - \theta_4, 2 - \theta_2)/2$ on the curve obtained by adjoining to $\mathbb{C}(s)$ the functions v, w where

$$v^2 = A_y^2 - u_2, \qquad w^2 = A_t^2 - u_2$$

and $A_i = 2a_i u_2$ for $i = y, t$.

Of course, a similar result is true upon replacing $\mathbb{P}^1_s$ by an arbitrary genus curve, but this will be sufficient for us here. Note that negating both v and w now corresponds to the Okamoto transformation $(y, t) \mapsto (1 - y, 1 - t)$.

4. Solutions

We will now carry out the following steps to find the formulae for the outstanding icosahedral solutions:

1) Choose an icosahedral solution from the table in [1] and if possible convert it, via Okamoto transformations, into a solution with parameters of the form $(0, \theta_2, 0, \theta_4)$,

2) Apply Corollary 3 to obtain a new solution, which will (in the examples here) have twice the number of branches (and larger genus) than the original solution,

3) Look for a simple model of the resulting Painlevé curve (either as a double cover of some $\mathbb{P}^1$, if it is hyperelliptic, or as a low degree plane curve otherwise).

A priori suitable solutions for step 1) are easily detected by looking for two zero coordinates in the solution's alcove point listed in table 1 of [1].

Remark 4. To aid the interested reader, and avoid typos, a Maple text file of the solutions of this article has been included with the source file (obtained by clicking on "Other formats") for the preprint version on the math arxiv. This file also contains solutions equivalent to those written here for which the corresponding isomonodromic family of Fuchsian systems has finite (icosahedral) monodromy group.

10 branch genus zero ⇒ 20 branch genus one.

Applying some Okamoto transformations to the H_3 solution from [4] E.33, [5], which is equivalent to icosahedral solution 32, one obtains the solution

$$y_0 = \frac{1}{2} - \frac{(3\,s^2 + 6\,s - 1)\,u}{16s^2}, \quad t_0 = \frac{1}{2} + \frac{u\,P}{256\;(5\,s - 1)\,s^3}$$

for parameters $\theta = (0, 1/5, 0, 1)$ where $u^2 = s$ and $P = 27\,s^5 - 315\,s^4 - 370\,s^3 + 170\,s^2 - 25\,s + 1$. Applying Corollary 3 to this (and adjusting v, w slightly to remove square factors) yields the solution

$$y = \frac{1}{2} - \frac{16\,s\,(5\,s - 1) + vw}{2\,(s-1)\,(3\,s+1)\,v}, \qquad t = \frac{1}{2} - \frac{P}{2\,(s-1)\,v^2 w}$$

for parameters $\theta = (0, 1, 0, 9)/10$ with P as above and where $w = vw_1$ and

$$v^2 = (9s-1)(s-1), \qquad w_1^2 = s^2 - 18s + 1. \tag{4}$$

One may check directly that this is a genus one solution with twenty branches, and is equivalent to icosahedral solution 45. (It is reassuring to compute the monodromy of the cover $t : \Pi' \to \mathbb{P}^1$ and find it has the properties listed in table 1 of [1].) Our next aim is to find a good model of the elliptic curve defined by (4), preserving the symmetry negating v. We will do this by parameterising the conic $w_1^2 = s^2 - 18s + 1$ as follows:

$$s = \frac{j^2 - 1}{2j - 18}, \qquad w_1 = \frac{j^2 - 18 + 1}{2j - 18}.$$

Then if we define $v = \frac{z}{2j-18}$ the condition that $v^2 = (9s-1)(s-1)$ says that (z, j) is a point of the elliptic curve

$$z^2 = (9j^2 - 2j + 9)(j^2 - 2j + 17), \tag{5}$$

and the above formulae give y, t explicitly as functions on this curve. (One may show, using Magma for example, that this elliptic curve corresponds to entry 200B1 of Cremona's tables of elliptic curves and for example is isomorphic over $\mathbb{Q}$ to the plane cubic $u^2 = s(s^2-5s+5)$, but this model hides the symmetry of the Painlevé solution.)

Similarly we can proceed with the sibling solution to that above, to obtain the solution:

$$y = \frac{1}{2} - \frac{64\,(5\,s-1)\,s^2 + (s-1)\,vw}{2\,(3\,s^3+75\,s^2-15\,s+1)\,v}$$

with t, s, v, w as above but $\theta = (0,3,0,7)/10$. This is equivalent to icosahedral solution 44.

20 branch genus one $\Rightarrow$ 40 branch genus three.

We can apply Corollary 3 again to the resulting solutions above, since their parameters are again of the desired form. Solution 45 then yields the solution

$$y = \frac{1}{2} + \frac{(j^2-18\,j+1)\,z^2 + 16\,(j+3)\,(j+1)\,vw}{8\,(3\,j-7)\,(j-9)\,(j-1)^2\,v}, \qquad t = \frac{1}{2} + \frac{u\,P}{256\,(5\,s-1)\,s^3}$$

with $\theta = (1,1,1,19)/20$, where $P(s), z^2$ are as in the previous subsection,

$$s = \frac{j^2-1}{2j-18}, \qquad u = \frac{w}{2j-18}$$

and now

$$v^2 = -(j-1)(j-9)(5j^2-2j+13), \qquad w^2 = 2(j-9)(j^2-1). \tag{6}$$

One may check directly that this is a genus 3 solution with forty branches and is equivalent to icosahedral solution 51. (Note that t is simply the pullback of the original degree 10 function t_0.) The curve defined by (6) is not hyperelliptic, so we can find a plane model by taking the canonical embedding. (Eliminating s from the equations (6) yields a singular plane sextic, and we compute three independent differentials directly on this.) This gives the following model of the Painlevé curve as a smooth plane quartic, with affine equation:

$$5(p^4+q^4) + 6(p^2q^2+p^2+q^2) + 1 = 0. \tag{7}$$

The solution functions (y, t) become functions on this quartic by setting

$$v = \frac{200p\,(6p^2+5q^2+1)}{84p^2q^2-55q^4-166q^2-156p^2-31}, \qquad w = qv, \qquad s = \frac{28v^2-4w^2+800}{3v^2+15w^2-800}.$$

Notice that this curve has three involutions (generating a group isomorphic to the dihedral group of order eight). These correspond to the Okamoto symmetries coming from the three hyperplanes on which the solution's parameters lie (as listed in table 1 of [1]). In more detail the symmetries mapping (p, q) to $(-p, q), (p, -q), (q, p)$ correspond to the Okamoto symmetries mapping (y, t) to

$$(1-y, 1-t), \qquad \left(\frac{y(t-1)}{t-y}, 1-t\right), \qquad \left(\frac{y-t}{y-1}, t\right)$$

respectively.

Similarly, from the sibling solution 44 one obtains the following, which is equivalent to icosahedral solution 50:

$$y = \frac{1}{2} + \frac{(j^2 - 18\,j + 1)\,(j^2 - 2\,j + 17)\,z^2 + 8\,\,(j-1)\,(j^3 + 57\,j^2 - 69\,j + 75)\,vw}{8\,(3\,j^3 - 21\,j^2 - 15\,j - 31)\,w^2 v}$$

with t, v, w, z^2 as above and $\theta = (3, 3, 3, 17)/20$.

15 branch genus one $\Rightarrow$ 30 branch genus two.

If we apply some Okamoto transformations to icosahedral solution 39 (from [1]) then we can obtain the solution

$$y_0 = \frac{1}{2} - \frac{u\,(2\,s^2 + 3\,s - 3)}{6\,(s+1)\,(4\,s^2 + 15\,s + 15)}, \qquad t_0 = \frac{1}{2} - \frac{u\,P}{18\,(4\,s^2 + 15\,s + 15)^2\,(s^2 - 5)}$$

with $\theta = (0, 7/15, 0, 13/15)$, where $u^2 = 3(s+5)(4s^2 + 15s + 15)$ and

$$P = 2\,s^7 + 10\,s^6 - 90\,s^4 - 135\,s^3 + 297\,s^2 + 945\,s + 675.$$

Applying Corollary 3 to this, and again adjusting v, w to remove square factors, yields the solution:

$$y = \frac{1}{2} + \frac{(s^2 - 5)\,u^2 v + s\,(s-3)\,(s+1)\,w^3}{2\,(s-3)\,(s+5)\,(s^3 + s^2 - 9\,s - 15)\,w^2}, \quad t = \frac{1}{2} + \frac{(s+5)^2\,P}{4s\,(s^2 - 9)\,w^3}$$

with $\theta = (2, 7, 2, 23)/30$ where P and u^2 are as above and

$$v^2 = s\,(s+5)\,(s+2)\,(s-3)\,, \qquad w^2 = s\,(s+5)\,(s+2)\,(s+3)\,. \tag{8}$$

This has thirty branches, genus two and is equivalent to icosahedral solution 47. Being of genus two, the curve (8) is hyperelliptic. We will express it as a double cover of a $\mathbb{P}^1$ branched at six points. Indeed by choosing a parameter j on the conic $x^2 = s^2 - 9$, the Painlevé curve (8) becomes isomorphic to the hyperelliptic curve

$$z^2 = (j^2 + 9)(j + 9)(j + 1)(j^2 + 4j + 9)$$

via the map

$$v = \frac{j-3}{4j^2} z, \qquad w = \frac{j+3}{4j^2} z, \qquad s = \frac{j^2 + 9}{2j}.$$

Similarly we can repeat starting with solution 40 (the sibling of 39) and obtain solution 48 (the sibling of solution 47). The result is

$$y = \frac{1}{2} + \frac{(s^2 - 5)\,u^2 + (s^2 - 6\,s - 15)\,vw}{2\,s\,(s+5)\,(s+3)^2\,v}$$

with $\theta = (4, 1, 4, 29)/30$ and with t, v, w, u^2, s as for solution 47 above.

18 branch genus one $\Rightarrow$ 36 branch genus three.

Next we will start with icosahedral solution 41 (from [1]; the 10 page implicit form of this solution in the preprint version of [5] is not useful here). Applying some Okamoto transformations yields the solution:

$$y_0 = \frac{1}{2} - \frac{8\,s^3 - 12\,s^2 + 3\,s - 4}{6u}, \qquad t_0 = \frac{1}{2} + \frac{P}{54\,s\,(s-1)\,u^3}.$$

for $\theta = (0, 1/3, 0, 1)$ where

$$u^2 = s(8s^2 - 11s + 8), \tag{9}$$

and

$$P = (s+1)\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right).$$

Applying Corollary 3 to this yields the solution:

$$y = \frac{1}{2} - \frac{9\,s\,(s-1)\,u^2 + (s-2)\,wv}{2\,(s^3+12\,s^2-12\,s+4)\,(2\,s-1)\,v}, \quad t = \frac{1}{2} - \frac{P}{4\,(2\,s-1)\,(s-2)\;v^2\,w}$$

with P, u^2 as above, $\theta = (0, 1, 0, 5)/6$ and $w = vw_1$ where

$$v^2 = (s-2)(2s-1)(2s^2+s+2), \qquad w_1^2 = s^2 - 7s + 1. \tag{10}$$

One may check this is a 36 branch genus three solution and is equivalent to icosahedral solution 49. However in this case the curve defined by (10) is hyperelliptic. Indeed let j be a parameter on the conic $w_1^2 = s^2 - 7s + 1$, so for example

$$w_1 = \frac{j^2 - 7j + 1}{2j - 7}, \qquad s = \frac{j^2 - 1}{2j - 7}. \tag{11}$$

Then the Painlevé curve (10) becomes isomorphic to

$$z^2 = \left(j^2 - 4\,j + 13\right)\left(2\,j^2 - 2\,j + 5\right)\left(2\,j^4 + 2\,j^3 - 3\,j^2 - 58\,j + 107\right)$$

via (11) and the assignment $v = z/(2j-7)^2$.

36 branch genus three $\Rightarrow$ 72 branch genus seven.

Finally we can apply Corollary 3 again to the solution above (since the parameters are of the desired form) to obtain the largest icosahedral solution. The solution is given by

$$y = \frac{1}{2} + \frac{9\,(j-1)\,(j^3+27j^2-57j+79)\,wv + 2\,(2j^2-2j+5)\,(j^2-7j+1)\,(2j^4+2j^3-3j^2-58j+107)\,(j^2-4j+13)^2}{6\,(j^2-1)\,(2j^2+j+17)\,(j^3-3j^2+3j-11)\,(2j-7)^2\,v},$$

$$t = \frac{1}{2} + \frac{P}{54\,s\,(s-1)\,u^3},$$

where $P(s)$ is the polynomial in the previous subsection,

$$s = \frac{j^2-1}{2j-7}, \qquad u = \frac{w}{(2j-7)^2},$$

and

$$v^2 = -\,(j+1)\left(6+j^2-2\,j\right)\left(4\,j^2-13\,j+19\right), \tag{12}$$

$$w^2 = (j-1)\,(2\,j-7)\,(j+1)\left(2\,j^2+j+17\right)\left(4\,j^2-13\,j+19\right). \tag{13}$$

Note that equation (13) is equivalent to equation (9) so t is the pullback of the original degree 18 function t_0.

One may check directly that this does indeed define a genus seven, 72 branch Painlevé solution and is equivalent to icosahedral solution 52. Of course being of genus 7 the degree–genus formula implies we cannot hope to find a non-singular plane model of the Painlevé curve. Instead we will look for a low degree plane model with mild singularities. (The curve obtained upon eliminating j from (12), (13) is a highly singular degree 14 plane curve, with large coefficients.) We do this by selecting a three-dimensional subspace of the space of holomorphic one-forms on the curve, and taking the corresponding plane curve. After some trial and error choosing a good subspace we found the following plane octic with only double points (ten nodes and two tacnodes):

$$9\,(p^6\,q^2+p^2\,q^6)+18\,p^4\,q^4+4\,(p^6+q^6)+26\,(p^4\,q^2+p^2\,q^4)+8\,(p^4+q^4)+57\,p^2\,q^2+20\,(p^2+q^2)+16.$$

The map between the curves is given by

$$p = \frac{w}{3\,(j-1)\,v}, \qquad q = \frac{v}{3(j^2-2\,j+6)}$$

and, if needed, the (rather long) inverse appears in the accompanying computer file (see Remark 4).

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