MODULAR OPERADS AND BATALIN-VILKOVISKY GEOMETRY

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Abstract. We describe the noncommutative Batalin-Vilkovisky geometry associated naturally with arbitrary modular operad. The classical limit of this geometry is the noncommutative symplectic geometry of the related cyclic operad. We show, in particular, that the algebras over the Feynman transform of a twisted modular operad $\mathcal{P}$ are in one-to-one correspondence with solutions to quantum master equation of Batalin-Vilkovisky geometry on the affine $\mathcal{P}-$manifolds. As an application we give a construction of characteristic cohomology classes in the quotient of Deligne-Mumford moduli spaces. These classes are associated naturally with solutions to quantum master equation of Batalin-Vilkovisky geometry on affine $\mathbb{S}[t]-$manifolds, where $\mathbb{S}[t]$ is the twisted modular $Det-$operad generalizing the cyclic operad of assocciative algebras.

1. Introduction.

Modular operads, introduced in [GK], are the higher genus analogs of cyclic operads. The basic idea is to replace the trees, playing the central role in the theory of cyclic operads, by graphs. In particular, the graph complexes from [K2] arise naturally as the modular analog of cobar transformation. The calculation of homology of graph complexes is a very complicated combinatorial problem. The examples here are the homology of chord diagrams, which encode the Vassiliev invariants of knots, or the cohomology of moduli spaces of Riemann surfaces.

In this note we show that the modular operads are intimately related with a kind of noncommutative Batalin-Vilkovisky geometry. The classical limit of the latter is the noncommutative symplectic geometry described in [K1], [G] in connection with cyclic operads.

We show, in particular, that the algebras over the Feynman transform of a twisted modular operad $\mathcal{P}$ are in one-to-one correspondence with solutions to quantum master equation of Batalin-Vilkovisky geometry on the affine $\mathcal{P}-$manifolds.

As an application we give a construction of characteristic cohomology classes in the quotient of Deligne-Mumford moduli spaces. These classes are associated naturally with solutions to quantum master equation of Batalin-Vilkovisky geometry on affine $\mathbb{S}[t]-$manifolds, where $\mathbb{S}[t]$ is the twisted $Det-$operad generalizing the cyclic operad of assocciative algebras. This construction is based on the identification of the Feynman transform of $\mathbb{S}[t]$ with the stable ribbon graph complex.

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Notations: we denote by k a field of characteristic zero, if $V = \oplus_i V_i$ is a graded vector space over k then $V[i]$ denotes the vector space with graded components $V[i]_j = V_{i+j}$, if $x \in V_i$ then $\overline{x} = i \bmod 2$ denotes its degree modulo $\mathbb{Z}/\not{2}\mathbb{Z}$, the cardinality of a finite set I is denoted by $|I|$. Throughout the paper we work in the tensor symmetric category of $\mathbb{Z}-$graded vector spaces where the isomorphism $X \bigotimes Y \simeq Y \bigotimes X$ is

$$x \otimes y \to (-1)^{\overline{xy}} y \otimes x \tag{1.1}$$

For a module U over a group G we denote via U_G the $k-$vector space of coinvariants: $U/\{gu - u | u \in U, g \in G\}$ and via U^G the subspace of invariants: $\{\forall g \in G : gu = u | u \in U\}$.

2. Modular operads.

In this section we collect the definitions relative to the concepts of the modular operad and of the Feynman transformation of modular operad. These are the higher genus analogs of the cyclic operads and their (co)-Bar transformations. The material presented in this section is borrowed from [GK].

An $\mathbb{S}$-module $\mathcal{P}$ is a collection of chain complexes of $k-$vector spaces $\mathcal{P}((n))$, $n \in \mathbb{N}$, equipped with an action of $\mathbb{S}_n$, the group of automorphisms of the set $\{1, \ldots, n\}$. Given an $\mathbb{S}$-module $\mathcal{P}$ and a finite set I we extend $\mathcal{P}$ to functor on finite sets by putting $\mathcal{P}((I)) = \left(\bigoplus_{bijections: I \leftrightarrow \{1,\ldots,n\}} \mathcal{P}((n))\right)_{\mathbb{S}_n}$.

A graph G is a triple $(Flag(G), \lambda, \sigma)$, where $Flag(G)$ is a finite set, whose elements are called flags, λ is a partition of $Flag(G)$, and σ is an involution acting on $Flag(G)$. By partition here one understands a disjoint decomposition into unordered subsets, called blocks. The vertices of the graph are the blocks of the partition. The set of vertices is denoted by $Vert(G)$. The subset of $Flag(G)$ corresponding to vertex v is denoted by $Leg(v)$. The cardinality of $Leg(v)$ is called the valence of v and is denoted $n(v)$. The edges of the graph are the pairs of flags forming a non-trvial two-cycle of the involution σ. The set of edges is denoted $Edge(G)$. The legs of the graph are the fixed elements of the involution σ. The set of legs is denoted $Leg(G)$. The number of legs is denoted $n(G)$.

Stable graph G is a connected graph with a non-negative integer number $b(v)$ assigned to each vertex $v \in Vert(G)$, such that $2b(v) + n(v) - 2 > 0$ for any $v \in Vert(G)$. For a stable graph G we put

$$b(G) = \Sigma_{v \in Vert(G)} b(v) + b_1(G).$$

Stable $\mathbb{S}$-module $\mathcal{P}$ is an $\mathbb{S}$-module with extra grading by non-negative integers on each $\mathbb{S}_n$-module : $\mathcal{P}((n)) = \oplus_{b \geq 0} \mathcal{P}((n, b))$, such that $\mathcal{P}((n, b)) = 0$ if $2b + n - 2 \leq 0$. In the modular operad setting $\mathcal{P}((n))$ can be thought of space of all possibilities to get an $n-$tensor using contractions of indices and basic $\mathcal{P}$-operations.

Given a stable $\mathbb{S}$-module $\mathcal{P}$ and a stable graph G one defines

$$\mathcal{P}((G)) = \bigotimes_{v \in Vert(G)} \mathcal{P}((Leg(v), b(v)))$$

where the tensor product over $v \in Vert(G)$ is understood as

$$\left(\bigoplus_{bijections:Vert(G)\leftrightarrow\{1,\dots,l\}} V((Leg(v_1),b_1))\otimes\ldots\otimes\mathcal{P}((Leg(v_l),b_l))\right)_{\mathbb{S}_l}$$

Let us denote by $\Gamma((n,b))$ the set consisting of all pairs (G,ρ) where G is a stable graph with $n(G)=n$ and $b(G)=b$ and ρ is a bijection $Leg(G)\leftrightarrow\{1,\dots n\}$. A modular operad $\mathcal{P}$ is a stable $\mathbb{S}$-module $\mathcal{P}$ together with the composition maps

$$\mu_G^{\mathcal{P}}:\mathcal{P}((G))\to\mathcal{P}((n,b)) \tag{2.1}$$

defined for any stable graph $G\in\Gamma((n,b))$ and all possible n and b. These maps must be $\mathbb{S}_n-$equivariant with respect to relabeling of legs of G and satisfy the natural associativity condition with respect to the compositions in the category of stable graphs (see loc.cit. sections 2.13-21). Given a finite set I and a stable graph G with exterior legs marked by the elements of I, we extend the composition (2.1) to the map $\mu_G^{\mathcal{P}}:\mathcal{P}((G))\to\mathcal{P}((I,b(G)))$, using a bijection $I\leftrightarrow\{1,\dots,n\}$. Because of $\mathbb{S}_n-$equivariance this does not depend on the choice of the bijection. For any subset of edges I of G one has the stable graph G/I and naturally defined morphism of stable graphs $f:G\to G/I$. For such morphism one defines the natural map

$$\mu^{\mathcal{P}}_{G\to G/I}:\mathcal{P}((G))\to\mathcal{P}((G/I)),\quad \mu^{\mathcal{P}}_{G\to G/I}=\bigotimes_{v\in Vert(G/I)}\mu^{\mathcal{P}}_{f^{-1}(v)}$$

and the associativity condition tells that

$$\mu_G^{\mathcal{P}}=\mu^{\mathcal{P}}_{G/I}\circ\mu^{\mathcal{P}}_{G\to G/I}$$

It follows from the asssociativity condition, that it is sufficient to define μ_G just for the stable graphs with one edge. There are two types of such graphs. The first one, which we denote $G_{(I,J)}$, has two vertices so that the set of legs is decomposed into two substes $I\sqcup J=\{1,\dots,n\}$

$$\mu^{\mathcal{P}}_{G_{(I,J)}}:\mathcal{P}((I\sqcup\{f\},b))\otimes\mathcal{P}((J\sqcup\{f'\},b'))\to\mathcal{P}((n,b+b')) \tag{2.2}$$

where f , f' are the two flags corresponding to the edge joining the two vertices. The second type, which we denote by $G_{2,n}$, has one vertice and in this case the single edge is a loop

$$\mu^{\mathcal{P}}_{G_{2,n}}:\mathcal{P}((\{1,\dots n\}\sqcup\{f,f'\},b))\to\mathcal{P}((n,b+1)) \tag{2.3}$$

Moreover, since the symmetric group $\mathbb{S}_n$, acting on $\mathcal{P}((n,b+b'))$, acts transitevely on the set of all pairs I, J in (2.2) with $|I|=const$, it is sufficient to consider the composition (2.2) just for the subsets $I=\{1,\dots,m-1\}$, $J=\{m,\dots,n\}$ with any $m<n$. We denote this composition via

$$o^{\mathcal{P}}:\mathcal{P}((m,b))\otimes\mathcal{P}((l,b'))\to\mathcal{P}((l+m-2,b+b')) \tag{2.4}$$

where $l+m=n+2$ and where we use the natural bijections respecting the orders: $\{1,\dots,m-1\}\sqcup\{f\}\leftrightarrow\{1,\dots,m\}$ with the element f corresponding to m and $\{m,\dots,n\}\sqcup\{f'\}\longleftrightarrow\{1,\dots,l\}$ with the element f' corresponding to 1.

An example of modular operad is the endomorphism operad. Let $V=\oplus_i V_i[-i]$ be a chain complex over k equipped with symmetric pairing B of degree 0,

$$B(u,v)=(-1)^{\overline{u}\,\overline{v}}B(v,u),\quad B:V^{\otimes 2}\to k$$

so that $B(u,v) = 0$ unless $\deg u + \deg v = 0$. The $\mathbb{S}-$module underlying the endomorphism modular operad of V is defined as

$$\mathcal{E}[V]((n,b)) = V^{\otimes n} \tag{2.5}$$

with the standard $\mathbb{S}_n-$action. Then

$$\mathcal{E}[V]((G)) = V^{\otimes Flag(G)}$$

The composition (2.1) is the contraction with $B^{\otimes Edge(G)}$. This is compatible with the definition of the usual endomorphisms whose components are defined as $Hom_k(V^{\otimes n-1}, V)$: the isomorphism induced by B: $V \simeq V^{dual}$ gives the isomorphisms of the underlying operad

$$\mathcal{E}[V]((n,b)) \simeq Hom_k(V^{\otimes n-1}, V)$$

Any cyclic operad with $\mathcal{P}((m)) = 0$ for $m = 1, 2$ can be considered as a modular operad by putting $\mathcal{P}((m,b)) = 0$ for $b \geq 1$.

The image of a modular operad under Feynman transform is some modification of modular operad with extra signs involved. To take into account these signs one needs to introduce the twisting of modular operads. The twisting is also necessary when one wants to associate an endomorphism modular operad with chain complex with symmetric or antisymmetric inner products of arbitrary degree.

2.1. **Determinants.** To simplify the signs bookkeeping it is convenient to introduce for a $k-$vector space V the determinant

$$Det(V) = \Lambda^{\dim V}(V)[\dim V]$$

This is the top-dimensional exterior power of the $k-$vector space V concentrated in degree $-\dim V$. We shall mostly need the determinant of the vector space k^S associated with a finite set S. We denote it $Det(S)$:

$$Det(S) = Det(k^S)$$

Because of (1.1) one has the natural isomorphism for the disjoint union of sets $\sqcup_{i\in I} S_i$

$$Det(\coprod_{i\in I} S_i) \simeq \bigotimes_{i\in I} Det(S_i) \tag{2.6}$$

Another obvious property is $Det^{\otimes 2}(S) \simeq k[2|S|]$. We shall also put for a graded $k-$vector space V_*

$$Det(V_*) = \bigotimes_i \det(V_i)^{((-1)^{i \bmod 2})}$$

2.2. **Cocycles.** We will only consider cocycles with values in the Picard tensor symmetric category of invertible graded $k-$vector spaces. Such a cocycle $\mathcal{D}$ is a certain functor which assigns to a stable graph $G \in \Gamma((n,b))$ the graded one-dimensional vector space $\mathcal{D}(G)$ and to any morphism of stable graphs $f : G \to G/I$ the linear isomorphism

$$\nu_f : \mathcal{D}(G/I) \otimes \bigotimes_{v\in Vert(G/I)} \mathcal{D}(f^{-1}(v)) \to \mathcal{D}(G)$$

satisfying the natural associativity condition with respect to the composition of two morphisms, see loc.cit. section 4.1. It satisfies also $\mathcal{D}(*_{n,b}) = k$. Examples of such cocycles are

$$\mathcal{K}(G) = Det(Edge(G)) \tag{2.7}$$

$$\mathcal{L}(G) = Det(Flag(G))Det^{-1}(Leg(G))$$

By definition modular $\mathcal{D}-$operad $\mathcal{P}$ is a stable $\mathbb{S}$-module $\mathcal{P}$, $\mathcal{P}((n)) = \oplus_b \mathcal{P}((n,b))$, together with the composition maps

$$\mu_G^{\mathcal{P}} : \mathcal{D}(G) \bigotimes \mathcal{P}((G)) \to \mathcal{P}((n,b)) \tag{2.8}$$

for $G \in \Gamma((n,b))$ which should satisfy associativity and $\mathbb{S}_n-$equivariance conditions parallel to those of modular operad.

2.3. **Coboundaries.** Let s be an $\mathbb{S}$-module such that $\dim_k s((g,n)) = 1$ for all g, n. Then s defines a cocycle

$$\mathcal{D}_s(G) = s((n,b)) \otimes \bigotimes_{v\in Vert(G)} s^{-1}((n(v),b(v)))$$

This is called the coboundary of s. Tensoring underlying $\mathbb{S}$-modules by s defines equivalence of the category of modular $\mathcal{D}-$operad with the category of modular $\mathcal{D}\otimes\mathcal{D}_s-$operad. Examples of such coboundaries are

$$\Sigma((n,b)) = k[1]$$

$$\alpha((n,b)) = k[n]$$

$$\beta((n,b)) = k[b-1]$$

$$\widetilde{\mathfrak{s}} = sgn_n[n]$$

in the first three examples the $\mathbb{S}_n-$action is trivial, and in the last example it is the alternating representation.

2.4. **Free modular operads.** The forgettful functor

$$modular\ operads \to stable\ S-modules$$

has the left adjoint functor which associates to a stable $\mathbb{S}-$module $\mathcal{A}$ the free modular operad $\mathbb{M}\mathcal{A}$ generated by $\mathcal{A}$:

$$\mathbb{M}\mathcal{A}((n,b)) = \bigoplus_{G\in[\Gamma((n,b))]} \mathcal{A}((G))_{Aut(G)}$$

here $[\Gamma((n,b))]$ denotes the set of isomorphisms classes of pairs (G,ρ) where G is a stable graph with $n(G)=n$, $b(G)=b$ and ρ is a bijection $Leg(G)\leftrightarrow\{1,\dots n\}$.

Similarly one defines the free modular $\mathcal{D}-$operad $\mathbb{M}_{\mathcal{D}}\mathcal{A}$ generated by stable $\mathbb{S}-$module $\mathcal{A}$:

$$\mathbb{M}_{\mathcal{D}}\mathcal{A}((n,b)) = \bigoplus_{G\in[\Gamma((n,b))]} (\mathcal{D}(G)\otimes\mathcal{A}((G)))_{Aut(G)}$$

On the subspace of generators the composition map μ_G is simply the projection $\mathcal{D}(G)\otimes\mathcal{A}((G)) \to (\mathcal{D}(G)\otimes\mathcal{A}((G)))_{Aut(G)}$

2.5. **Feynman transform.** Let $\mathcal{P}$ be a modular $\mathcal{D}-$operad. Let us put $\mathcal{D}^\vee = \mathcal{K}\mathcal{D}^{-1}$, where $\mathcal{K}$ is the cocycle (The Feynman transform of a modular $\mathcal{D}-$operad $\mathcal{P}$ is a modular $\mathcal{D}^\vee-$operad $\mathcal{F}_\mathcal{D}\mathcal{P}$, defined in the following way. As a stable $\mathbb{S}-$module, forgetting the differential, $\mathcal{F}_\mathcal{D}\mathcal{P}$ is the free modular $\mathcal{D}^\vee-$operad generated by stable $\mathbb{S}-$module $\{\mathcal{P}((n,b))^{dual}\}$. The differential on $\mathcal{F}_\mathcal{D}\mathcal{P}$ is the sum $d_\mathcal{F} = \partial_{\mathcal{P}^{dual}} + \partial_c$ of the differential $\partial_{\mathcal{P}^{dual}}$ induced on $\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}$ by the differential on $\mathcal{P}$ and of the differential ∂_c, whose value on the term $(\mathcal{D}^\vee(G)\otimes\mathcal{P}^{dual}((G)))_{Aut(G)}$ is a sum over all graphs $\widetilde{G}$ such that $\widetilde{G}/\{e\}\simeq G$ of the map dual to the composition $\mu^{\mathcal{P}}_{\widetilde{G}\to G}$ multiplied by the element $e[1]\in Det(\{e\})$:

$$\partial_c|_{(\mathcal{D}^\vee(G)\otimes\mathcal{P}^{dual}((G)))_{Aut(G)}} = \sum_{\widetilde{G}/\{e\}\simeq G} e[1]\otimes(\mu^{\mathcal{P}}_{\widetilde{G}\to G})^{dual}$$

see section 5 of loc.cit. .

The Feynman transform is a generalisation of graph complexes from [K1]. The Lie, commutative and associative graph complexes correspond to the $|Leg(G)|=0$ part of the Feynman transforms of the corresponding cyclic operads, considered as modular operads with $\mathcal{P}((n,b))=0$ for $b\geq 1$.

3. Endomphorism operad for inner products of arbitrary degree.

Here we discuss the natural twisted modular operads of endomorphisms associated with symmetric or antisymmetric inner products of arbitrary degree.

3.1. **Symmetric inner product of degree $l\in\mathbb{Z}$.** Let V be a chain complex with

symmetric inner product B of arbitrary degree, $\deg B = l$, $l\in\mathbb{Z}$:

$$B(u,v) = (-1)^{\overline{u}\,\overline{v}}B(v,u),\quad B: V^{\otimes 2}\to k[-l],\ l\in\mathbb{Z}$$

so that $B(u,v)=0$ unless $\deg u+\deg v = l$. If we put for underlying $\mathbb{S}$-modules

$$\mathcal{E}[V]((n,b)) = V^{\otimes n}$$

then the contraction with $B^{Edge(G)}$defines naturally the composition map

$$\mu_G^{\mathcal{E}[V]}:\mathcal{E}[V]((G))\otimes\mathcal{K}^{\otimes l}(G)\to\mathcal{E}[V]((n,b))$$

of the modular $\mathcal{K}^{\otimes l}-$operad where

$$\mathcal{K}^{\otimes l}(G) = Det^{\otimes l}(Edge(G))$$

Indeed, for even l, $l=2l'$, the cocycle acts simply as the degree shift

$$\mathcal{K}^{\otimes 2l'}(G) = k[2l'|Edge(G)|]$$

and the contraction with $B^{Edge(G)}$ acting on $V^{Flag(G)}$ decreases the total degree exactly by $2l'|Edge(G)|$. For odd l, $l=2l'+1$, the cocycle is the degree shift tensored by the top exterior power of $k^{Edge(G)}$

$$\mathcal{K}^{\otimes 2l'+1}(G) = \Lambda^{|Edge(G)|}(k^{Edge(G)})[(2l'+1)|Edge(G)|]$$

Notice that the permutation of any two edges inverses the sign of the value of $B^{\otimes Edge(G)}$ on $V^{\otimes Flag(G)}$ since B is of odd degree. This explains the necessity for the term $\Lambda^{|Edge(G)|}(k^{|Edge(G)|})$ in the case of odd degree.

3.2. **Antisymmetric inner product of degree $l \in \mathbb{Z}$.** Let V be a chain complex with *antisymmetric* inner product

$$B(u,v) = -(-1)^{\overline{u}\,\overline{v}} B(v,u), \ \deg B = l, l \in \mathbb{Z}$$

The cocycle corresponding to such inner product is

$$\mathcal{K}^{\otimes l-2}\mathcal{L}(G) = Det^{\otimes l-2}(Edge(G))Det(Flag(G))Det^{-1}(Leg(G))$$

Using (2.6) we get

$$\mathcal{K}^{\otimes l-2}\mathcal{L}(G) = \mathcal{K}^{\otimes l} \bigotimes_{e\in Edge(G)} \Lambda^2(k^{\{s_e,t_e\}})$$

where $\{s_e, t_e\}$ is the set of two flags corresponding to the edge e. We put now $\mathcal{E}[V]((n,b)) = V^{\otimes n}$ and define the composition map (2.8) as in the previous cases: identify $\mathcal{E}[V](G)$ with $V^{Flag(G)}$ and contract with $B^{Edge(G)}$. The composition map is well defined since permutation of two flags s_e, t_e reverses the sign of $\Lambda^2(k^{\{s_e,t_e\}})$.

3.3. **Suspension.** The suspension coboundary $\mathcal{D}_{\mathfrak{s}}$ from loc.cit. is associated with the $\mathbb{S}-$module

$$\mathfrak{s}((n,b)) = sgn_n[2(b-1)+n]$$

where sgn_n is the standard alternating representation of $\mathbb{S}_n$. Identifying $sgn_{n(v)}[n(v)]$ with $Det(Leg(v))$ we get

$$\mathcal{D}_{\mathfrak{s}}(G) = Det(Leg(G))[2(b(G)-1)] \bigotimes_{v\in Vert(G)} Det^{-1}(Leg(v))[2(1-b(v))]$$

This is equal to $\mathcal{L}^{-1}\mathcal{K}^{\otimes 2}(G)$ because of the formula

$$\Sigma_{v\in Vert(G)}(b(v)-1) = b(G) - 1 - |Edge(G)|$$

We see that the multiplication by $\mathfrak{s}$ transforms modular $\mathcal{K}^{\otimes l-2}\mathcal{L}$-operads to modular $\mathcal{K}^{\otimes l}$-operads and vice versa. In particular, the degrees of the corresponding inner products must be the same. Perhaps more useful in the situation of inner products of arbitrary degree is the coboundary associated with

$$\widetilde{\mathfrak{s}} = sgn_n[n]$$

so that

$$\mathcal{D}_{\widetilde{\mathfrak{s}}} = \mathcal{L}^{-1}$$

The multiplication by $\widetilde{\mathfrak{s}}$ of the underlying $\mathbb{S}-$module transforms modular $\mathcal{K}^{\otimes l-2}\mathcal{L}$-operad to modular $\mathcal{K}^{\otimes l-2}$-operad and, since $\mathcal{L}^2 \simeq \mathcal{K}^{\otimes 4}$, it transforms modular $\mathcal{K}^{\otimes l+2}$-operad to modular $\mathcal{K}^{\otimes l-2}\mathcal{L}$-operad. If V is a chain complex with symmetric (respectively antisymmetric) inner product B of degree $\deg B = l$, then the suspension of V is a chain complex $V[1]$ with the antisymmetric (respectively symmetric) inner product $\widetilde{B}$ of degree $l-2$ defined via

$$\widetilde{B}(x[1], y[1]) = (-1)^{\overline{x}} B(x,y)$$

The multiplication by $\widetilde{\mathfrak{s}}$ of the modular $\mathcal{K}^{\otimes l}$-operad $\mathcal{E}[V]$ gives the modular $\mathcal{K}^{\otimes l-4}\mathcal{L}$-operad $\mathcal{E}[V[1]]$. We see that if $\mathcal{P}$ is a $\mathcal{K}^{\otimes l}-$operad and V is a chain complex with symmetric inner product B of degree l, then the modular $\mathcal{P}-$algebra structure on V corresponds under the suspension to the modular $\widetilde{\mathfrak{s}}\mathcal{P}$-algebra structure on the chain complex $V[1]$ equipped with the antisymmetric inner product $\widetilde{B}$ of degree $l-2$.

3.4. **Twisted modular $\mathcal{P}$-algebras.** Twisted modular $\mathcal{P}$-algebra structure on chain complex V with symmetric inner product of degree l is a morphism of modular $\mathcal{K}^{\otimes l}-$operads : $\mathcal{P} \to \mathcal{E}[V]$. Twisted modular $\mathcal{P}$-algebra structure on chain complex V with *antisymmetric* inner product of degree l is a morphism of modular $\mathcal{K}^{\otimes l-2}\mathcal{L}-$operads : $\mathcal{P} \to \mathcal{E}[V]$.

The twistings corresponding to the inner products can be reduced using coboundaries to just two cocycles: the trivial cocycle, if l is even, and the cocycle Det whose value on a graph G is

$$Det(G) = Det(H_1(G)),$$

if l is odd. The last cocycle is isomorphic to

$$Det(G) \simeq \mathcal{K}^{-1}\mathcal{D}_{\widetilde{\mathfrak{s}}}^{-1}\mathcal{D}_{\Sigma}^{-1}$$

see loc.cit. proposition 4.14. Notice that both cocycles are trivial on trees. A possible choices of such coboundaries given by the following identities :

$$(\mathcal{K}^{\otimes 2l})\mathcal{D}_{\beta}^{\otimes -2l} \simeq k, \ (\mathcal{K}^{\otimes 2l}\mathcal{L})\mathcal{D}_{\widetilde{\mathfrak{s}}}\mathcal{D}_{\beta}^{\otimes -2l} \simeq k,$$
$$(\mathcal{K}^{\otimes 2l-1})\mathcal{D}_{\widetilde{\mathfrak{s}}}^{-1}\mathcal{D}_{\beta}^{\otimes -2l}\mathcal{D}_{\Sigma}^{-1} \simeq Det(G) \ \ (\mathcal{K}^{\otimes 2l-1}\mathcal{L})\mathcal{D}_{\beta}^{\otimes -2l}\mathcal{D}_{\Sigma}^{-1} \simeq Det(G).$$

One can use these twistings by coboundaries in order to extend the notion of algebra over a cyclic operad from complexes with degree zero inner product to complexes with inner products of arbitrary degree.

4. Algebras over Feynman transform.

In this section we consider the equation defining the structure of algebra over Feynman transform of a modular $\mathcal{D}-$operad.

Let us consider first the case of $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structure on the chain complex V with symmetric inner product B of degree l, $B : V^{\otimes 2} \to k[-l]$, this implies that $\mathcal{D} \simeq \mathcal{K}^{\otimes 1-l}$. The $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structure on V is a morphism of modular operads $\widehat{m} : \mathcal{F}_{\mathcal{D}}\mathcal{P} \to \mathcal{E}[V]$. Since $\mathcal{F}_{\mathcal{D}}\mathcal{P}$ is generated by $\mathcal{P}((n,b))^{dual}$, the $\mathcal{F}_{\mathcal{D}}\mathcal{P}$-algebra structure on V is determined by a set of $\mathbb{S}_n-$equivariant linear maps

$$\widehat{m}_{n,b} : \mathcal{P}((n,b))^{dual} \to V^{\otimes n}$$

or, equivalently, of degree zero elements

$$m_{n,b} \in (V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n} \tag{4.1}$$

As above, for any finite set I one can extend this to the collection of elements $\{m_{I,b}\}$: $m_{I,b} \in (V^{\otimes I} \otimes \mathcal{P}((I,b)))^{Aut(I)}$ using an arbitrary bijection $I \leftrightarrow \{1, \ldots, |I|\}$. On the subspace $\mathbb{M}_{\mathcal{K}^{\otimes l}}\mathcal{P}^{dual}((G))_{Aut(G)}$ of $\mathcal{F}_{\mathcal{D}}\mathcal{P}((n,b))$ corresponding to a stable graph $G \in [\Gamma((n,b))]$ the map

$$\widehat{m} : \mathbb{M}_{\mathcal{K}^{\otimes l}}\mathcal{P}^{dual}((G))_{Aut(G)} \to V^{\otimes n}$$

is given by

$$\mu_G^{\mathcal{E}[V]} \circ \left(\bigotimes_{v \in Vert(G)} \widehat{m}_{n(v),b(v)} \right) \tag{4.2}$$

since an element from $\mathbb{M}_{\mathcal{K}^{\otimes l}}\mathcal{P}^{dual}((G))_{Aut(G)}$ is represented by the composition along G of elements from $\mathcal{P}((n(v),b(v)))^{dual}$, $v \in Vert(G)$. The map $\widehat{m} : \mathcal{F}_{\mathcal{D}}\mathcal{P} \to \mathcal{E}[V]$ corresponding to the set $\{m_{n,b}\}$ is a morphism of modular $\mathcal{K}^{\otimes l}$-operads iff it respect the differentials on $\mathcal{F}_{\mathcal{D}}\mathcal{P}$ and $\mathcal{E}[V]$. It is sufficient to check this condition on the generators of $\mathcal{F}_{\mathcal{D}}\mathcal{P}$. On the subspace $\mathcal{P}((n,b))^{dual}$ the differential $d_{\mathcal{F}}$ is a sum of $d_{\mathcal{P}^{dual}}$ plus sum of the adjoints to the structure maps $\mu^{\mathcal{P}}_{\mathbb{G}}$ (2.8) corresponding to the graphs with single edge, multiplied by $e[1]$, the canonical element of degree -1 from $Det(\{e\})$ where e is the unique edge of the graph G:

$$d_{\mathcal{F}} = d_{\mathcal{P}^{dual}} + \sum_{\{1,\dots,n\}=I\sqcup J} e[1]\otimes (\mu^{\mathcal{P}}_{G_{(I,J)}})^{dual} + e[1]\otimes(\mu^{\mathcal{P}}_{\mathbb{G}_{2,n}})^{dual}$$

We see that the condition

$$\widehat{m}\circ d_{\mathcal{F}} = d_V\circ\widehat{m}$$

is equavalent to

$$\begin{aligned}(4.3)\quad &\widehat{m}_{n,b}d_{\mathcal{P}^{dual}} - d_V\widehat{m}_{n,b} + \mu^{\mathcal{E}[V]}_{G_{2,n}}(e[1]\otimes(\widehat{m}_{\{1,\dots,n\}\sqcup\{f,f'\},b-1}\mu^{\mathcal{P}}_{\mathbb{G}_{2,n}}{}^{dual}))+\\ &+\frac{1}{2}\sum_{\substack{\{1,\dots,n\}=I_1\sqcup I_2\\ b_1+b_2=b}} \mu^{\mathcal{E}[V]}_{G_1(I_1,I_2)}(e[1]\otimes(\widehat{m}_{I_1\sqcup\{f\},b_1}\otimes\widehat{m}_{I_2\sqcup\{f'\},b_2}\mu^{\mathcal{P}}_{G_{(I_1,I_2)}}{}^{dual}))=0\end{aligned}$$

where $\mu^{\mathcal{E}[V]}_{\mathbb{G}_{2,n}}$ is the contraction $(k[1])^{\otimes l}\otimes V^{\otimes\{f,f'\}}\otimes V^{\otimes n}\to V^{\otimes n}$ with the bilinear form B applied to the factors corresponding to f,f' and $\mu^{\mathcal{E}[V]}_{G_1}$ is the similar contraction $(k[1])^{\otimes l}\otimes V^{\otimes I_1\sqcup\{f\}}\otimes V^{\otimes I_2\sqcup\{f'\}}\to V^{\otimes n}$. In terms of m this can be written as the equation

$$\begin{aligned}(4.4)\quad &(d_{\mathcal{P}}+d_V)m_{n,b} - B_{ff'}\xi^{\mathcal{P}}_{ff'}m_{\{1,\dots,n\}\sqcup\{f,f'\},b-1})-\\ &-\frac{1}{2}\sum_{\substack{\{1,\dots,n\}=I_1\sqcup I_2\\ b_1+b_2=b}} B_{ff'}\phi^{\mathcal{P}}_{ff'}(m_{I_1\sqcup\{f\},b_1}\otimes m_{I_2\sqcup\{f'\},b_2}))=0\end{aligned}$$

where the degree $(l-1)$ maps

$$(4.5)\qquad \phi^{\mathcal{P}}_{ff'} = \mu^{\mathcal{P}}_{G_1(I_1,I_2)}(e[1]^{\otimes 1-l})$$

$$\phi^{\mathcal{P}}_{ff'}:\mathcal{P}((I_1\sqcup\{f\},b_1))\otimes\mathcal{P}((I_2\sqcup\{f'\},b_2))\to\mathcal{P}((n,b_1+b_2))[l-1]$$

$$(4.6)\qquad \xi^{\mathcal{P}}_{ff'} = \mu^{\mathcal{P}}_{\mathbb{G}_{2,n}}(e[1]^{\otimes 1-l})$$

$$\xi^{\mathcal{P}}_{ff'}:\mathcal{P}((\{1,\dots n\}\sqcup\{f,f'\},b))\to\mathcal{P}((n,b+1))[l-1]$$

are the evaluation on $(e[1])^{\otimes 1-l}$ of the $\mathcal{P}-$compositions, and $B_{ff'}:V^{\otimes\{f,f'\}}\to k[-l]$ is the degree $(-l)$ contraction with B. Let us put $m_n=\sum_b z^b m_{n,b}$ then

$$\begin{aligned}(4.7)\quad &(d_{\mathcal{P}}+d_V)m_n - zB_{ff'}\xi^{\mathcal{P}}_{ff'}m_{\{1,\dots,n\}\sqcup\{f,f'\}}-\\ &-\frac{1}{2}\sum_{\{1,\dots,n\}=I_1\sqcup I_2} B_{ff'}\phi^{\mathcal{P}}_{ff'}(m_{I_1\sqcup\{f\}}\otimes m_{I_2\sqcup\{f'\}})=0\end{aligned}$$

We can rewrite this equation directly in terms of $\{m_n\}$ using the canonical projection to the $\mathbb{S}_n-$invariant subspace $\frac{1}{n!}(\Sigma_{\sigma\in\mathbb{S}_n}\sigma)$. Notice that the terms of the last summand in (4.4) are invariant with respect to the action of the subgroup $\mathbb{S}_{card(I_1)}\times\mathbb{S}_{card(I_2)}$. The $\mathbb{S}_n$-equivariance of (2.1) implies that the result of the action of arbitrary element σ of $\mathbb{S}_n$ on such a term is $(B_{ff'}\phi_{\widetilde{I}_1\widetilde{I}_2})(m_{\widetilde{I}_1\sqcup\{f\},b_1}\otimes m_{\widetilde{I}_2\sqcup\{f'\},b_2})$ with $\widetilde{I}_1=\sigma(I_1)$, $\widetilde{I}_2=\sigma(I_2)$. Let us single out the term $B_{ff'}\phi^{\mathcal{P}}_{ff'}(m_{I_1\sqcup\{f\}}\otimes m_{I_2\sqcup\{f'\}})$

with $I_1 = \{1, \ldots, n_1\}$ and denote via $o^{\mathcal{P}}$ the composition $\phi^{\mathcal{P}}_{ff'}$after the identifications: $I_1 \sqcup \{f\}$ with $\{1, \ldots, n_1+1\}$ such that $i \leftrightarrow i$ for $1 \le i \le n_1$ and $f \leftrightarrow n_1+1$, and $I_2 \sqcup \{f'\}$ with $\{1, \ldots, n_2+1\}$ such that $f' \to 1$, $i \to i - n_1 + 1$ for $n_1+1 \le i \le n$:

$$o^{\mathcal{P}} : \mathcal{P}((n_1+1, b)) \otimes \mathcal{P}((n-n_1+1, b')) \to \mathcal{P}((n, b+b'))[l-1]$$

Let us identife also in the first summand $\{1, \ldots, n\} \sqcup \{f, f'\}$ with $\{1, \ldots, n+2\}$ in such a way that $i \leftrightarrow i$ for $i \in \{1, \ldots, n\}$ and $f \leftrightarrow n+1$, $f' \leftrightarrow n+2$. Then we can write the equation (4.4) as

$$(4.8)\quad (d_{\mathcal{P}} + d_V)m_n - zB_{n+1,n+2}\xi_{n+1,n+2}m_{n+2} + \\ -\frac{1}{2}\sum_{n_1+n_2=n}\frac{1}{n_1!n_2!}\sum_{\sigma\in\mathbb{S}_n}\sigma(B_{n_1+1,1}o^{\mathcal{P}}(m_{n_1+1}\otimes m_{n_2+1})) = 0$$

In the case when V is a chain complexes with antisymmetric inner product of degree l the $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structures on V is described again by the equation (4.3). In this case $\mathcal{P}$ must be a $\mathcal{D}-$operad with $\mathcal{D} = \mathcal{K}^{\otimes -1-l}\mathcal{L}$. If one rewrites this equation in terms of $\{m_n\}$ then one gets the equations (4.7,4.8) with degree $(l-1)$ compositions

$$(4.9)\qquad \phi^{\mathcal{P}}_{ff'} = \mu^{\mathcal{P}}_{G_1(I_1,I_2)}(e[1]^{\otimes -1-l} \otimes (f[1] \wedge f'[1]))$$

$$(4.10)\qquad \xi^{\mathcal{P}}_{ff'} = \mu^{\mathcal{P}}_{\mathbb{G}_{2,n}}(e[1]^{\otimes -1-l} \otimes (f[1] \wedge f'[1]))$$

and $B_{ff'} : V^{\otimes\{f\}} \otimes V^{\otimes\{f'\}} \to k[-l]$ the degree $(-l)$ contraction with B where the order of flags f and f' is taken the same as in (4.9,4.10). We have proven the following result.

Proposition 1. *The element m from $\bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}$ defines the modular $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structure on the chain complex V with symmetric (respectively antisymmetric) inner product B of degree l, $B : V^{\otimes 2} \to k[-l]$, where $\mathcal{P}$ is a twisted $\mathcal{K}^{\otimes 1-l}-$modular operad (respectively $\mathcal{K}^{\otimes -1-l}\mathcal{L}-$moduar operad) iff m satisfies the equation (4.7).*

5. Differential graded Lie algebra $\bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}[1]$.

Recall that the equation defining the algebra structure over the *cobar*$-$transformation of some cyclic operad $\mathcal{A}$ can be written as $[h, h] = 0$, where h is a function on the symplectic affine $\mathcal{A}-$manifold. The *cobar*$-$transformation is the tree-level part of the Feynman transform. As we shall see below for the twisted modular operad $\mathcal{P}$ the equation (4.7) describing the algebra over the Feynman transform is the principal equation of the Batalin-Vilkovisky $\mathcal{P}-$geometry on the affine $\mathcal{P}$-manifold.

5.1. **Odd vector field on the space of morphisms $\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual} \to \mathcal{E}[V]$.** Let V be chain complex with symmetric or antisymmetric inner product B of degree l. Let $\mathcal{P}$ be a modular $\mathcal{D}-$operad, so that $\mathcal{D}^\vee$ is the cocycle corresponding to the twisting of $\mathcal{E}[V]$, i.e. $\mathcal{D} = \mathcal{K}^{\otimes 1-l}$ for symmetric B and $\mathcal{D} = \mathcal{K}^{\otimes -1-l}\mathcal{L}$ for antisymmetric B. We explain in this subsection that the linear and quadratic terms in the equation (4.3) define the structure of differential graded Lie algebra on the graded k-vector space $\bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}[1]$.

Let $Mor(\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}, \mathcal{E}[V])$ denotes the space of operad morphisms from $\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}$ to $\mathcal{E}[V]$. Since $\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}$ is a free modular operad we have

$$Mor(\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}, \mathcal{E}[V]) = (\bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n})_0$$

One can consider the corresponding graded version of the space of morphism $\underline{Mor}(\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}, \mathcal{E}[V])$. It is the affine $\mathbb{Z}-$ graded scheme representing the functor $R \to Mor(\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual} \otimes R, \mathcal{E}[V])$ where R is a graded commutatitve $k-$algebra. For the graded version we have

$$\underline{Mor}(\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}, \mathcal{E}[V]) = \bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}$$

The differential $d_{\mathcal{F}}$ acting on $\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}$ induces canonical odd vector field on $\underline{Mor}(\mathbb{M}_{\mathcal{D}^\vee}\mathcal{P}^{dual}, \mathcal{E}[V])$:

$$Q(\varphi) = d_V\varphi - \varphi d_{\mathcal{F}} \tag{5.1}$$

Since $d_{\mathcal{F}}^2 = 0$ it follows that

$$[Q, Q] = 0. \tag{5.2}$$

The equation (4.3) describing the $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structures on V is precisely the equation

$$Q(\varphi) = 0$$

on the subspace $\deg\varphi = 0$. The same calculation as in the previous subsection shows that the vector field Q has only linear and quadratic components and they are given by the linear in $\{m_n\}$ and the quadratic in $\{m_n\}$ terms in (4.3). The vector field Q induces the odd derivation of cocommutative coalgebra generated by the $\mathbb{Z}-$ graded vector space $F = \bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}$. It follows from (5.2) that the linear and the quadratic components in (4.3) define the structure of differential graded Lie algebra on $F[1]$ (in the category of chain complexes). In particular,

$$\{m_1, m_2\} = \frac{(-1)^{\overline{m}_1\overline{m}_2}}{n_1!n_2!}\sum_{\sigma\in\mathbb{S}_n}\sigma(o^{\mathcal{E}[V]}o^{\mathcal{P}}(m_1 \otimes m_2)), \tag{5.3}$$

$$m_i \in (V^{\otimes n_i+1} \otimes \mathcal{P}((n_i+1, b_i)))^{\mathbb{S}_{n_i+1}}, i = 1, 2$$

is the odd Lie bracket and

$$\Delta m = (-1)^{\overline{m}}\xi^{\mathcal{E}[V]}_{n-1,n}\xi^{\mathcal{P}}_{n-1,n}m, \;\; m \in (V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n} \tag{5.4}$$

is a degree (-1) differential. Notice that the bracket (5.3) is defined in terms of the compositions of the underlying cyclic operads, while the operator (5.4) is defined in terms of the extra modular compositions along the graphs $G_{2,n}$. Remark that this bracket is closely related with the bracket defined in [KM] in the case of usual operads. If we put $dm = ((-1)^{\overline{m}}d_{\mathcal{P}} - d_V)m$ then our basic equation 4.7 becomes the familiar quantum master equation

$$dm + z\Delta m + \frac{1}{2}[m, m] = 0 \tag{5.5}$$

Using this and the proposition 1 we get the following result

Theorem 1. *The modular $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structure on the chain complex V with symmetric (respectively antisymmetric) inner product B of degree l, $B : V^{\otimes 2} \to k[-l]$, where $\mathcal{P}$ is an arbitrary twisted $\mathcal{K}^{\otimes 1-l}-$modular operad (respectively $\mathcal{K}^{\otimes -1-l}\mathcal{L}$-moduar operad), are in one-to-one correspondence with solutions of the quantum master equation (5.5) in the space $(\bigoplus_{n,b}(V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n})_0$.*

If one considers the modular $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structures over some commutative graded algebra C then they are in one-to-one correspondence with solutions to (5.5) in the space $(F \otimes C)_0$. One can define, using the algebra $C = k[\varepsilon]/\varepsilon^2$, the modular homotopy equivalence of the modular $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structures and the gauge equivalence on the solutions to (5.5). Then equivalence classes of the modular $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structure are in-to one correspondence with the gauge equivalence classes of solutions to the quantum master equation.

We must add that a characterisation of algebras over the Feynman transform of the modular completion of the commutative operad was given in [M] via higher order coderivations of a free cocommutative coalgebra. We are not aware of extension of this approach to other operads.

One may notice that the definitions of the odd bracket (5.3) and the odd differential (5.4) work in fact for an arbitrary pair $(\mathcal{P}, \widetilde{\mathcal{P}})$ where $\mathcal{P}$ is a modular $\mathcal{D}-$operad with arbitrary $\mathcal{D}$ and $\widetilde{\mathcal{P}}$ is a modular $\mathcal{D}^{\vee}-$operad. Then there is a natural differential graded Lie algebra structure on

$$\bigoplus_{n,b}(\widetilde{\mathcal{P}}((n,b)) \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}[1]$$

defined by the components of the vector field (5.1). The solutions to the corresponding Maurer-Cartan equation in $\bigoplus_{n,b}(\widetilde{\mathcal{P}}((n,b)) \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}[1]$ are in one-to one correspondence with morphisms of operads $\mathcal{F}_{\mathcal{D}}\mathcal{P} \to \widetilde{\mathcal{P}}$. Notice that this equation coincides with the equation describing the morphisms $\mathcal{F}_{\mathcal{D}^{\vee}}\widetilde{\mathcal{P}} \to \mathcal{P}$.

6. Free $\mathcal{P}-$algebra.

If $\mathcal{P}$ is a cyclic operad then the free $\mathcal{P}-$algebra generated by the graded $k-$vector space V is

$$C = \bigoplus_{n}(V^{\otimes n} \otimes \mathcal{P}((n+1)))^{\mathbb{S}_n}$$

and it was argued in ([K1],[G]) that the vector space

$$F = \bigoplus_{n}(V^{\otimes n} \otimes \mathcal{P}((n)))^{\mathbb{S}_n}$$

can be considered naturally as the analog of the space of functons on $Spec(C)$. If $\mathcal{P}$ is a twisted modular operad then the compositions along trees form twisted version of cyclic operad. There is the corresponding version of the free $\mathcal{P}$ algebra and the arguing can be repeated that F can be seen as the space of functions on $Spec(C)$ in the twisted case.

Let $\mathcal{P}$ be a modular $Det-$operad in the category of graded vector spaces and let $Cyc\mathcal{P} = \oplus_n \mathcal{P}((n,0))$ is the cyclic operad, which is $b = 0$ part of $\mathcal{P}$. Then $Cyc(\Sigma^{-1}\mathcal{F}_{Det}\mathcal{P})$, the $b = 0$ part of $\Sigma^{-1}\mathcal{F}_{Det}\mathcal{P} = \mathcal{F}_{\mathcal{D}_{\Sigma}Det}\Sigma\mathcal{P}$, is equal to $\mathfrak{s}BCyc\mathcal{P}$, that is the suspension of the cobar transformation of $Cyc\mathcal{P}$. In the framework of non-commutative symplectic geometry of [K1], see also [G], the $BCyc\mathcal{P}-$algebra structures on a vector space V with symmetric inner product B of degree zero are described by functions on affine $Cyc\mathcal{P}-$manifold $\mathfrak{s}V$, that is the elements $h \in \bigoplus_n ((\mathfrak{s}V)^{\otimes n} \otimes \mathcal{P}((n,0)))^{\mathbb{S}_n}$, of degree $\deg h = 1$, such that

$$[h,h] = 0, \tag{6.1}$$

where the bracket is the Poisson bracket which is associated with the antisymmetric inner product $\mathfrak{s}B$. Let us put $\widetilde{\mathcal{P}} = \Sigma\mathcal{P}$ and consider the $\mathcal{F}_{\mathcal{D}}\widetilde{\mathcal{P}}-$algebra stractures on $\mathfrak{s}V$, where $\mathcal{D} = \mathcal{D}_{\Sigma}Det$. Then $b = 0$ part of such structure correspond to $BCyc\mathcal{P}-$algebra structure on V. The $\mathcal{F}_{\mathcal{D}}\widetilde{\mathcal{P}}-$algebra stractures on $\mathfrak{s}V$ is described, as we proved in section 4, by an element $\widehat{h}(z) = \sum_{b\geq 0} h_b z^b$, $h_b \in \bigoplus_n ((\mathfrak{s}V)^{\otimes n} \otimes \Sigma\mathcal{P}((n,b)))^{\mathbb{S}_n}$, $\deg h_b = 0$, such that

$$z\Delta\widehat{h}(z) + \frac{1}{2}[\widehat{h}(z), \widehat{h}(z)] = 0$$

We see that in the "classical" limit $z \to 0$, $\widehat{h}(z)$ becomes a solution to (6.1), the "classical" master equation. The operator Δ can be seen as the odd second order operator and the bracket is the odd Poisson bracket extending the previous bracket, which was defined on the subspace $b = 0$. The whole picture is a noncommutative $\mathcal{P}-$analog of the usual commutative Batalin-Vilkovisky geometry, described for example in [S]. We plan to return elsewhere to the corresponding analog of the theory of integration of $\exp(\widehat{h}(z)/z)$ and to the combinatorial consequences of the invariance of such integrals under deformations.

7. Characteristic classes of $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebras.

Let $\widehat{m} : \mathcal{F}_{\mathcal{D}}\mathcal{P} \to \mathcal{E}[V]$ be an $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$algebra structure on the chain complex V with symmetric or antisymmetric inner product B of degree l. Here $\mathcal{P}$ is a modular $\mathcal{D}-$operad, such that $\mathcal{D}^{\vee}$ is the cocycle corresponding to the twisting of $\mathcal{E}[V]$, i.e. $\mathcal{D} = \mathcal{K}^{\otimes 1-l}$ for symmetric B and $\mathcal{D} = \mathcal{K}^{\otimes -1-l}\mathcal{L}$ for antisymmetric B. It is one of the main application of the formalism developped in [GK] that taking the $n = 0$ component of the morphism $\widehat{m}$

$$\widehat{m}((0,*)) : \bigoplus_{G\in[\Gamma((0,b))], b>1} (\mathcal{D}^{\vee}(G) \otimes \mathcal{P}^{dual}((G)))_{Aut(G)} \to k$$

we get a cocycle on the subcomplex of $\mathcal{F}_{\mathcal{D}}\mathcal{P}$ corresponding to graphs with no external legs:

$$\widehat{m}((0,*))(d_{\mathcal{F}}\gamma) = 0,\ \gamma \in (\oplus_b \mathcal{F}_{\mathcal{D}}\mathcal{P}((0,b))). \tag{7.1}$$

More generally if $\widehat{m}_t \in ...$ is the $\mathcal{F}_{\mathcal{D}}\mathcal{P}-$operad structure depending on some graded parameters t then all Taylor cofficient of expansion of the component $\widehat{m}_t((0,*))$ at $t = 0$ are cocycles on the subcomplex of graphs with no external legs

$$\frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \dots \partial t_n^{\alpha_n}} \widehat{m}_t((0,*))|_{t=0}(d_{\mathcal{F}}\gamma) = 0,\ \gamma \in (\oplus_b \mathcal{F}_{\mathcal{D}}\mathcal{P}((0,b))). \tag{7.2}$$

It follows from (4.2) that the value $\widehat{m}((0,*))$ on an element from $(\mathcal{D}(G)\otimes\mathcal{P}((G)))_{Aut(G)}$ corresponding to the stable graph G is given by a partition function by the tensors $m_{n,b} \in (V^{\otimes n} \otimes \mathcal{P}((n,b)))^{\mathbb{S}_n}$ and the inner product B. Similarly the cocycle (7.2) is a partition function involving insertions of the derivatives $\frac{\partial^{|\beta|}}{\partial t_1^{\beta_1} \dots \partial t_n^{\beta_n}} m_{n,b}((0,*))|_{t=0}$ with $\beta_i \leq \alpha_i$ so that for all $1 \leq i \leq n$ the total sum of β_i for all such insertions in the given graph is equal to α_i.

8. Stable ribbon graphs.

In this section we give an application of the above construction and explain how it gives the cohomology classes in Deligne -Mumford moduli spaces associated with solutions to the master equation 5.5 on the space of functions on affine $\mathcal{P}-$manifold where $\mathcal{P}$ is the modular $Det-$operad $\mathbb{S}[t]$ which is introduced below. We use the complex of stable ribbon graphs and its relation with a compactification of moduli spaces of algebraic curves described in [K3], see also [L]. This is a generalisation of the equivalence of "decorated" moduli spaces of algebraic curves and moduli spaces of ribbon graphs due to J.Harer, D.Mumford, R.C.Penner, W.Thurston and others.

Stable ribbon graph is a connected graph G together with :

- partitions of the set of flags adjacent to every vertex into $i(v)$ subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \ldots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- fixed cyclic order on every subset $Leg(v)^{(k)}$,
- a map $g : Vert(G) \to \mathbb{Z}_{\geq 0}$ such that for any vertex $2(2g(v) + i(v) - 2) + n(v) > 0$, so that putting $b(v) = 2g(v) + i(v) - 1$ defines a stable graph.

Let us denote via $SR_{(n,b)}$ the set of all stable ribbon graphs with n exterrior legs and $b(G) = b$. The usual ribbon graphs correspond to the case $b(v) = 0$ for all $v \in Vert(G)$,.

It is easy to see that for graphs from $SR_{(0,b)}$ our definition is equivalent to the definition given in [K3] as the limit of certain functor on ribbon graphs. A metric on the stable ribbon graph is a function $l : Edge(G) \to \mathbb{R}_{>0}$. Given a stable ribbon graph $G \in SR_{(0,b)}$ a metric on G one can construct by the standard procedure a punctured Riemann surface $\Sigma(G)$, which will be singular in general. Namely one should replace every edge by oriented open strip $[0, l]\times] - i\infty, +i\infty[$ and glue them for each cyclically ordered subset according to the cyclic order. In this way one gets several punctured Riemann surfaces and for every vertex of the graph G one should identify the points on these surfaces corresponding to different cyclically ordered subsets associated with the given vertex of G. We also have the nonnegative integer $g(v)$ associated to every singular point of the Riemann surface $\Sigma(G)$. The graph G is naturally realised as a one-dimensional subset of $\Sigma(G)$, which we denote by $|G|$. One can also construct the Riemman surface associated with the general stable ribbon graph G having the legs. In such case one gets the singular Riemann surface associated with the graph $G/Legs(G)$, i.e. the stable ribbon graph G with legs removed, plus the extra structure, which consists of the lines on $\Sigma(G/Legs(G))$,one for each leg, which connect the vertice with the corresponding adjacent puncture, so that G is again naturally realised as a subset of $\Sigma(G)$. We shall denote the set of punctures of the surface $\Sigma(G)$ via $P_{\Sigma(G)}$.

One can consider the moduli space $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}$ parametrizing the equivalence classes of data (G, l), where G is a graph from $SR_{(0,\gamma)}$ whose associated surface $\Sigma(G)$ has exactly ν punctures numbered from 1 to ν, and l is a metric on G. It can be shown, see loc.cit. and [L], that there is a natural factor space $\overline{\mathcal{M}}'_{\gamma,\nu}$ of the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{\gamma,\nu}$ so that $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}$ is homeomorphic to $\overline{\mathcal{M}}'_{\gamma,\nu} \times \mathbb{R}^{\nu}_{>o}$ and the projection to $\mathbb{R}^{\nu}_{>o}$ corresponds to the map which sends stable graph with metric and numbered punctures to the set of perimeters of edges surrounding the punctures. In particular the preimage of $p = (p_1, \ldots, p_\nu)$, $p \in \mathbb{R}^{\nu}_{>o}$

in $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}$ can be considered as the moduli space space $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}(p)$ of data (G,l) such that the perimeters around punctures are equal to $p_1,\ldots,p_\nu$. This moduli space $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}(p)$ is then homeomorphic to $\overline{\mathcal{M}}'_{\gamma,\nu}$. The space $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}(p)$ has natural structure of a cell complex, or better say, orbi-cell complex, with (orbi-)cells indexed by equivalence classes of stable graphs G with numbered punctures as above.

8.1. **Modular $Det-$operad $\mathbb{S}[t]$.** Let us introduce the following modular $Det-$operad

$\mathbb{S}[t]$. Let $k[\mathbb{S}_n]\otimes Det(Cycle)$ denotes the graded $k-$vector space with the basis indexed by elements $\sigma\otimes a_\sigma$, where $\sigma\in\mathbb{S}_n$ is a permutation with $i(\sigma)$ cycles σ_α and $a_\sigma=\sigma_1\wedge\ldots\wedge\sigma_{i(\sigma)}$, $a_\sigma\in Det(Cycle(\sigma))$ represents one of the generators of the one-dimensional determinant of the set of cycles of σ. The symmetric group $\mathbb{S}_n$ acts naturally on $k[\mathbb{S}_n]\otimes Det(Cycle)$ by conjugation. Let $k[t]$ denotes the space of polynomials in one variable t, $\deg t=-2$. The cyclic $\mathbb{S}-$module underlying our modular operad is the set of graded $\mathbb{S}_n-$modules

$$\mathbb{S}[t]((n))=(\mathbb{S}_n\otimes Det(Cycle)[1])\otimes k[t]$$

where $\mathbb{S}_n$ acts trivially on $k[t]$, and the degree b of the element $\sigma t^g\otimes a_\sigma$ is defined as

$$b=2g+i(\sigma)-1$$

in particular $\mathbb{S}[t]((n,b))$ is a graded $k-$vector space concentrated in degree $(-b)$. We also put $\mathbb{S}[t]((n,0))=0$ for $n\leq 2$. Notice that for $b=0$ we get the underlying $\mathbb{S}-$module of the cyclic operad Ass of associative algebras with invariant scalar products.

Compositions in $\mathbb{S}[t]$ are $k[t]-$linear and defined via sewings and dissections of cycles of permutations. The compositions can be easily described using multiplication on the group of permutations. Let us describe the composition $\mu^{\mathbb{S}[t]}_{G_{(I,J)}}$ along the simplest graph with two vertices (2.2). Let $\sigma t^g\otimes a_\sigma\in\mathbb{S}[t]((I\sqcup\{f\},b))$, $a_\sigma=\sigma_1\wedge\ldots\wedge\sigma_{i(\sigma)}$, $\rho t^{g'}\otimes a_\rho\in\mathbb{S}[t]((J\sqcup\{f'\},b'))$, $a_\rho=\rho_1\wedge\ldots\wedge\rho_{i(\rho)}$ with f belonging to the cycle σ_k and f' belonging to the cycle ρ_l. Let us denote by $\pi_{ff'}$

$$\pi_{ff'}:Aut(\{1,\ldots,n\}\sqcup\{f,f'\})\to\mathbb{S}_n$$

the operation erasing the elements f and f' from the cycles of permutation

$$\pi_{ff'}:(i_1\ldots i_\alpha f j_1\ldots j_\beta f')\to(i_1\ldots i_\alpha j_1\ldots j_\beta)$$

If $b=b'=0$ then we have simply the cyclic permutations and the composition $\mu^{\mathbb{S}[t]}_{G_{(I,J)}}$ coincides with the composition in the cyclic operad Ass, which can be written as $\pi_{ff'}\sigma\rho(ff')$ where (ff') is the transposition $f\leftrightarrow f'$. For general elements of $\mathbb{S}[t]$ we have the following expression

$$\mu^{\mathbb{S}[t]}_{G_{(I,J)}}=(\pi_{ff'}\sigma\rho(ff'))t^{g+g'}\otimes a_\mu$$

where

$$a_\mu=(-1)^{k+l}\pi_{ff'}(\sigma_k\rho_l(ff'))\wedge\sigma_1\wedge\ldots\wedge\widehat{\sigma}_k\wedge\ldots\wedge\sigma_{i(\sigma)}\wedge\rho_1\wedge\ldots\wedge\widehat{\rho}_l\wedge\ldots\wedge\rho_{i(\rho)}$$

The expression for a_μ follows from the natural isomorphisms

$$\begin{aligned} Det(Cycle(\pi_{ff'}\sigma\rho(ff')) &\simeq Det(\{\pi_{ff'}\sigma_k\rho_l(ff')\})\otimes \\ &\otimes Det(Cycle(\sigma)\setminus\{\sigma_k\}\sqcup Cycle(\rho)\setminus\{\rho_l\}), \\ Det(Cycle(\sigma))[-1] &\simeq Det(Cycle(\sigma)\setminus\{\sigma_k\}), \\ Det(Cycle(\rho))[-1] &\simeq Det(Cycle(\rho)\setminus\{\rho_l\}) \end{aligned} \tag{8.1}$$

Let us describe the composition along the graph with one loop (2.3)

$$\mu^{\mathbb{S}[t]}_{G_{2,n}} : Det(H_1(G_{2,n}))\otimes \mathbb{S}[t]((\{1,\ldots,n\}\sqcup\{f,f'\},b))\to \mathbb{S}[t]((n,b+1)).$$

Let $\sigma t^g\otimes a_\sigma\in\mathbb{S}[t]((\{1,\ldots,n\}\sqcup\{f,f'\},b))$, $a_\sigma=\sigma_1\wedge\ldots\wedge\sigma_{i(\sigma)}$ and $f\in\sigma_k$ and $f'\in\sigma_l$ with $k<l$. The pair ff' defines the oriented loop $e_{ff'}$ and hence an element $e_{ff'}[1]$ of $Det(H_1(G_{2,n}))$. Then the composition $\mu^{\mathbb{S}[t]}_{G_{2,n}}$ on $e_{ff'}[1]\otimes(\sigma t^g\otimes a_\sigma)$ is the sewing of cycles σ_k and σ_l times t

$$\mu^{\mathbb{S}[t]}_{G_{2,n}}=(\pi_{ff'}\sigma(ff'))t^{g+1}\otimes a_{\mu'}$$

where

$$a_{\mu'}=(-1)^{k+l-1}\pi_{ff'}(\sigma_k\sigma_l(ff'))\wedge\sigma_1\wedge\ldots\wedge\widehat{\sigma}_k\wedge\ldots\wedge\widehat{\sigma}_l\wedge\ldots\wedge\sigma_{i(\sigma)}$$

following a sequence of natural isomorphisms analogous to (8.1). If the elements f and f' belong to the same cycle $f,f'\in\sigma_k$ then composition $\mu^{\mathbb{S}[t]}_{G_{2,n}}$ on $e_{ff'}[1]\otimes(\sigma t^g\otimes a_\sigma)$ is the dissection of the cycle σ_k into two cycles whose relative order in $Det(Cycle)$ is determined by the orientation of the edge $e_{ff'}$:

$$\mu^{\mathbb{S}[t]}_{G_{2,n}}=(\pi_{ff'}\sigma(ff'))t^{g}\otimes a_{\mu''}$$

where, if we denote by σ_k^f and $\sigma_k^{f'}$ the two cycles of $\sigma_k(ff')$ containing f and f' correspondingly, then we have

$$a_{\mu''}=(-1)^{k-1}(\pi_f\sigma_k^f)\wedge(\pi_{f'}\sigma_k^{f'})\wedge\sigma_1\wedge\ldots\wedge\widehat{\sigma}_k\wedge\ldots\wedge\sigma_{i(\sigma)}$$

which follows from natural isomorphisms analogous to (8.1). Remark that if one of the cycles σ_k^f, $\sigma_k^{f'}$ consist of just one element f or f' correspondingly, which happens precisely when f and $f'$are neighbours in the cycle σ_k, then the composition is zero in such case $((\pi_f\sigma_k^f)=0$ or $(\pi_{f'}\sigma_k^{f'})=0)$.

One can check that these compositions define on $\mathbb{S}[t]$ the structure of twisted modular $Det-$operad. Namely, contraction of several edges corresponds, forgetting the elements from $k[t]\otimes Det(Cycle)$, to successive operators of multiplications by transpositions followed by erasing operators. But operators corresponding to different edges commute $[\pi_{ff'},\pi_{gg'}]=0$, $[\pi_{ff'},(gg')]=0$. It follows that the composition on the level of permutations is associative with respect to the morphisms of stable graphs. It follows from commutativity of the diagrams of natural isomorphisms that the rules for compositions of decorations from $k[t]\otimes Det(Cycle)$ are also compatible with morphisms of stable graphs.

8.2. **Feynman transform of $\mathbb{S}[t]$.** Let us consider the Feynman transform of $\mathbb{S}[t]$. Notice that $\mathbb{S}[t]((n))$ has a basis labeled by partitions of n into i subsets with cyclic orders on the subsets, plus the nonnegative integer g,such that $2(2g+i-2)+n>0$ and plus the choice of an element from $Det(Cycle)$. It follows that the Feynman transform $\mathcal{F}_{Det}\mathbb{S}[t]$ has the basis labeled by the stable ribbon graphs plus a choice of generator of $\mathcal{K}Det^{-1}(G)$. If (G,α_G) is such an element, let $\nu(G)$ be the number of punctures of the Riemann surface $\Sigma(G)$ associated with G. Then it belongs to the subspace $\mathcal{F}_{Det}\mathbb{S}[t]((n,b))$ where n is the number of legs of G and

$$b(G) = 2\gamma(G) - 1 + \nu(G)$$

where

$$\gamma(G) = g(N\Sigma(G)) + \sum_{v\in Virt(G)} (g(v) + i(v) - 1)$$

where $g(N\Sigma(G))$ is the genus of the normalisation of $\Sigma(G)$, in other words $\gamma(G)$ is the genus of $\Sigma(G)$ taking into account the genus defects associated with vertices of G. It is easy to see that every chain complex $\mathcal{F}_{Det}\mathbb{S}[t]((n,b))$ is in fact the direct sum of complexes

$$\mathcal{F}_{Det}\mathbb{S}[t]((n,b)) = \bigoplus_{b=2\gamma-1+\nu} \mathcal{F}_{Det}\mathbb{S}[t]((n,\gamma,\nu))$$

The moduli space $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}(p)/\mathbb{S}_\nu$ has natural (orbi-)cellular decomposition with cells indexed by the isomorphism classes of stable ribbon graphs with $\gamma = \gamma(G)$, $\nu = \nu(G)$ and $|Leg(G)| = 0$. Moreover, the differential on $\mathcal{F}_{Det}\mathbb{S}[t]$ coincides with the differential of the cochain complex of the (orbi-)cell complex so that we have the identity

Proposition 2. $H^i(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu) \simeq H_{-i}(\mathcal{F}_{Det}\mathbb{S}[t]((0,\gamma,\nu)))[2\gamma-1]$

Proof. This identification is analogous to the proposition 9.5 of [GK]. Firstly, for every stable ribbon graph G the vector space $\mathcal{K}(G)$ is naturally isomorphic to $H^{top}_c(C_G)$ where C_G is the (orbi-)cell corresponding to G. To prove that the complex $\mathcal{F}_{Det}\mathbb{S}[t]((0,\gamma,\nu))[1-2\gamma]$ computes indeed the cohomology $\oplus_i H^i(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu)$ it is sufficient to identify the inverse to the sheaf of cohomology of the fibers of projection $\overline{\mathcal{M}}^{comb}_{\gamma,\nu}/\mathbb{S}_\nu \to \overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu$, i.e. to show that

$$Det(G)^{-1} \bigotimes_{v\in Virt(G)} Det^{-1}(Cycle(\sigma(v))[1-2g(v)] \simeq Det^{-1}(P_{\Sigma(G)})[1-2\gamma] \tag{8.2}$$

In order to prove this let us consider the surface $\widetilde{\Sigma}(G)$ which is obtained topologically as follows. Let us remove a small neighborhoud of every singular point v of $\Sigma(G)$. If v has the genus defect $g(v)$ and $i(v)$ branches are meeting at v, then let us glue instead of this neighborhoud a curve $\Sigma(v)$ of the genus $g(v)$ and with $i(v)$ boundary components. We obtain topologically a curve of genus γ without ν marked points. Then

$$Det(H^*(\widetilde{\Sigma}(G), \sqcup_{v\in Virt(G)}\Sigma(v))) \simeq Det(H^*(|G|, Vert(G)) \simeq Det(G)^{-1}[1-|Vert(G)|]$$

since $(\widetilde{\Sigma}(G), \sqcup_{v\in Virt(G)}\Sigma(v))$ is homotopic to $(|G|, Vert(G))$. Using the Poincare duality for the compact surface $\widetilde{\Sigma}(G) \sqcup P_{\Sigma(G)}$ and the Mayer-Vietoris sequence as in loc.cit. we get

$$Det(H^*(\widetilde{\Sigma}(G))) \simeq Det^{-1}(P_{\Sigma(G)})[2-2\gamma]$$

Also, by similar arguments

$$Det(H^*(\Sigma(v))) \simeq Det^{-1}(Cycle(\sigma(v))[2-2g(v)]$$

Now the equality (8.2) follows from the exact sequence associated with the pair $(\widetilde{\Sigma}(G), \sqcup_{v\in Virt(G)}\Sigma(v))$. □

8.3. **Operad $\mathcal{F}_{Det}\mathbb{S}[t]$ and higher genus GW-invariants.** The operad $\mathcal{F}_{Det}\mathbb{S}[t]$ arises naturally in the counting of holomorphic curves of arbitrary genus with boundaries in the set of Lagrangian submanifolds of a simplectic manifold. Namely, consider the total number of holomorphic maps of the surfaces of genus g with i boundary components with sets of points lying on these boundaries, such that the boundaries are mapped to the fixed set of graded Lagrangian submanifolds, intersecting transversally, and the points are mapped to the intersection points. This gives naturally the set of elements of $\bigoplus_n (V^{\otimes n}\otimes\mathbb{S}[t]((n,b)))^{\mathbb{S}_n}$ where $b = 2g+i-1$, and V is the graded vector spaces with the basis labeled by points of intersection of Lagrangian submanifolds. Then the standard arguments, involving the degeneration of one-parameter families of such maps, show that this set of elements satisfy the quantum master equation (5.5) of the Batalin-Vilkovisky geometry on the affine $\mathbb{S}[t]-$manifold. We discuss this in more details in [B1] and explain how this leads to the combinatorial description of Gromov-Witten invariants with values in cohomology of Deligne-Mumford moduli spaces via the characteristic class map and the cyclic homology of the twisted modular operad.

8.4. **The map $\chi\mathcal{F}\mathcal{A}ss \to \mathbb{S}[t]$.** Recall that forgetting the differential the Feynman transform $\mathcal{F}\mathcal{A}ss$ of the cyclic operad $\mathcal{A}ss$ is a $\mathcal{K}-$operad generated by the $\mathbb{S}-$module $\mathcal{A}ss^{dual}$. It follows that $\mathcal{F}\mathcal{A}ss$ has a basis labeled by ribbon graphs with a choice of a generator of the one-dimensional vector space $\mathcal{K}(G)$. The complex $\mathcal{F}\mathcal{A}ss((n,b))$ is decomposed as the sum of subcomplexes $\mathcal{F}\mathcal{A}ss((n,\gamma,\nu))$ according to the genus γ and the number of punctures ν of the Riemann surface associated with the ribbon graph, see [GK]. Notice that

$$Det(G) \simeq \mathcal{K}\mathcal{D}_\chi$$

where $\mathcal{D}_\chi$ is the coboundary associated with the $\mathbb{S}-$module $\chi = \mathfrak{s}\Sigma^{-1}$. It follows that $\chi\mathcal{F}\mathcal{A}ss$ is a modular $Det-$operad which as a $k-$vector space consist of linear combinations of elements $[G]\otimes\widehat{\alpha}_G$ where G is a ribbon graph and $\widehat{\alpha}_G$ is an element of the one-dimensional vector space $Det(G)\otimes(\otimes_{v\in Virt(G)}Det(Leg(v)[-3])$. The subset of legs of ribbon graph G adjacent to a given puncture has natural cyclic order. It follows that every ribbon graph G defines naturally a permutation σ_G on the set $Leg(G)$. Notice that for graphs with at least one leg adjacent to every puncture we have

$$Det(G) \simeq Det(Cycle(\sigma_G))[2\gamma - 1] \tag{8.3}$$

see loc.cit, page 117. Let G be a trivalent ribbon graph. The cyclic order on $Leg(v)$ gives a canonical element in $Det(Leg(v))[-3]$ for every vertex v. This is the element $e_1\wedge e_2\wedge e_3$ if $e_1 \to e_2 \to e_3 \to e_1$ denotes the cyclic order on the three flags. Let $\widehat{\alpha}_G^{can}$ denotes the product of an element $\alpha_G \in Det(G)$ with the tensor product of the canonical elements in $\otimes_{v\in Virt(G)}Det(Leg(v))[-3]$. Let $\alpha_{\sigma_G} \in Det(Cycle(\sigma_G))[2\gamma - 1]$ denotes the element corresponding to α_G under the isomorphism (8.3).

Proposition 3. *Let G be a trivalent ribbon graph such that for every puncture of G there is a leg adjacent to it. Let us put $\phi([G]\otimes\widehat{\alpha}_G^{can}) = \sigma_G t^\gamma \otimes \alpha_{\sigma_G}$ for such graph and $\phi([G]\otimes\widehat{\alpha}_G) = 0$ for all other ribbon graphs. Then ϕ defines natural morphism of twisted modular $Det-$operads $\chi\mathcal{F}\mathcal{A}ss \to \mathbb{S}[t]$.*

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