HIDDEN GRASSMANN STRUCTURE IN THE XXZ MODEL

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Abstract. For the critical XXZ model, we consider the space $\mathcal{W}_{[\alpha]}$ of operators which are products of local operators with a disorder operator. We introduce two anti-commutative family of operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ which act on $\mathcal{W}_{[\alpha]}$. These operators are constructed as traces over representations of the q-oscillator algebra, in close analogy with Baxter's Q-operators. We show that the vacuum expectation values of operators in $\mathcal{W}_{[\alpha]}$ can be expressed in terms of an exponential of a quadratic form of $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$.

1. Introduction

This paper continues our study of correlation functions in lattice integrable models [1, 2, 3, 4, 5]. Consider the infinite XXZ spin chain with the Hamiltonian

$$H_{\mathrm{XXZ}} = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} \left(\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3 \right), \tag{1.1}$$

where σ^a $(a = 1, 2, 3)$ are the Pauli matrices and

$$\Delta = \cos \pi\nu$$

is a real parameter. We use the usual notation

$$q = e^{\pi i \nu}.$$

In our previous work [5], we obtained an algebraic representation for general correlation functions of the XXZ model. Here we generalize this result to the situation when a disorder operator is present. In the course we find a new interesting structure behind the model. We consider only the massless regime $|\Delta| < 1$, $0 < \nu < 1$, since it is more important for physics because of its relation to conformal field theory (CFT) discussed below. Explanation about the massive regime $\Delta > 1$ will be given elsewhere.

Let us introduce

$$S(k) = \tfrac{1}{2} \sum_{j=-\infty}^{k} \sigma_j^3 .$$

Denote by $|\mathrm{vac}\rangle$ the ground state of the Hamiltonian, and let α be a parameter. We consider the normalized vacuum expectation values:

$$\frac{\langle \mathrm{vac} | q^{2\alpha S(0)} \mathcal{O} | \mathrm{vac} \rangle}{\langle \mathrm{vac} | q^{2\alpha S(0)} | \mathrm{vac} \rangle} \tag{1.2}$$

where $\mathcal{O}$ is a local operator.

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Locality of $\mathcal{O}$ implies that the operator $q^{2\alpha S(0)}\mathcal{O}$ stabilizes: there exist integers k, l such that for all $j > l$ (resp. $j < k$) this operator acts on the j-th lattice site as 1 (resp. $q^{\alpha\sigma^3}$). If k (resp. l) is the maximal (resp. minimal) integer with this property, $l - k + 1$ will be called the length of the operator $q^{2\alpha S(0)}\mathcal{O}$. The very formulation of the problem implies that we are interested only in local operators $\mathcal{O}$ of total spin 0 (otherwise the correlation function vanishes). Nevertheless, for the sake of convenience we introduce the spaces $\mathcal{W}_{\alpha,s}$ of operators $q^{2\alpha S(0)}\mathcal{O}$ of spin s:

$$\left[S\ ,\ q^{2\alpha S(0)}\mathcal{O}\right] = s\, q^{2\alpha S(0)}\mathcal{O}, \quad S = S(\infty)\,.$$

Also we set

$$\mathcal{W}_\alpha = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha,s}\,.$$

The leading long distance asymptotics of the XXZ spin chain is described by CFT with $c = 1$: that of free bosons $\phi, \bar\phi$ with compactification radius $\beta = \sqrt{1-\nu}$. For an extensive discussion about the XXZ model as an irrelevant perturbation of CFT, we refer the reader to [6]. The space $\mathcal{W}_{\alpha,s}$ corresponds to the space of descendants of the operator

$$e^{\frac{i}{2}\left(\alpha(\beta^{-1}-\beta)(\phi+\bar\phi)+s\beta(\phi-\bar\phi)\right)}\,.$$

Similarly to the conformal case [10, 11, 12], introduction of the disorder parameter α regularizes the problem, and allows to write much nicer formulae than in the case $\alpha = 0$ [1]. Another similarity is that it is very convenient to consider, as an intermediate object which does not enter the final formulae, the following space:

$$\mathcal{W}_{[\alpha]} = \bigoplus_{k=-\infty}^{\infty} \mathcal{W}_{\alpha+k}\ .$$

In this paper we shall introduce two anti-commuting families of operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ acting on $\mathcal{W}_{[\alpha]}$:

$$[\mathbf{b}(\zeta_1), \mathbf{b}(\zeta_2)]_+ = [\mathbf{b}(\zeta_1), \mathbf{c}(\zeta_2)]_+ = [\mathbf{c}(\zeta_1), \mathbf{c}(\zeta_2)]_+ = 0\ .$$

The operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ have the following block structure:

$$\mathbf{b}(\zeta):\ \mathcal{W}_{\alpha+k,s} \to \mathcal{W}_{\alpha+k+1,s-1}, \quad \mathbf{c}(\zeta):\ \mathcal{W}_{\alpha+k,s} \to \mathcal{W}_{\alpha+k-1,s+1}\ .$$

Hence the operator $\mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)$ acts from $\mathcal{W}_{\alpha,0}$ to itself.

The operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ are formal series in $(\zeta-1)^{-1}$. When applied to an operator $q^{2\alpha S(0)}\mathcal{O}$ of length L, the singularity is a pole of order L, in other words, the series terminates at $(\zeta-1)^{-L}$. The action of $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ produces operators of the same or smaller length. The coefficients of $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ give rise to an action of the Grassmann algebra with $2L$ generators. In particular

$$\mathbf{b}(\zeta_1)\cdots\mathbf{b}(\zeta_{L+1})\left(q^{2\alpha S(0)}\mathcal{O}\right) = 0, \quad \mathbf{c}(\zeta_1)\cdots\mathbf{c}(\zeta_{L+1})\left(q^{2\alpha S(0)}\mathcal{O}\right) = 0\,.$$

[1]The formulae are written initially for $|q^\alpha| < 1$ and continued analytically in α, but $\alpha = 0$ is one of singular points where l'Hôpital's rule should be applied.

We introduce also the linear functional on $\mathrm{End}\,(\mathbb{C}^2)$:

$$\mathrm{tr}^{\alpha}(x)=\frac{1}{q^{\frac{\alpha}{2}}+q^{-\frac{\alpha}{2}}}\mathrm{tr}\left(q^{-\frac{1}{2}\alpha\sigma^3}x\right) \tag{1.3}$$

with the obvious properties:

$$\mathrm{tr}^{\alpha}(1)=\mathrm{tr}^{\alpha}(q^{\alpha\sigma^3})=1\,.$$

This gives rise to a linear functional on $\mathcal{W}_{\alpha}$

$$\mathbf{tr}^{\alpha}(X)=\cdots\mathrm{tr}^{\alpha}_1\ \mathrm{tr}^{\alpha}_2\ \mathrm{tr}^{\alpha}_3\cdots(X)\,.$$

Our main result is:

$$\frac{\langle\mathrm{vac}|q^{2\alpha S(0)}\mathcal{O}|\mathrm{vac}\rangle}{\langle\mathrm{vac}|q^{2\alpha S(0)}|\mathrm{vac}\rangle}=\mathbf{tr}^{\alpha}\left(e^{\mathbf{\Omega}}\left(q^{2\alpha S(0)}\mathcal{O}\right)\right)\,, \tag{1.4}$$

where[2] the operator $\mathbf{\Omega}$ acts on $\mathcal{W}_{[\alpha]}$:

$$\mathbf{\Omega}=-\mathrm{res}_{\zeta_1=1}\mathrm{res}_{\zeta_2=1}\left(\boldsymbol{\omega}\left(\zeta_1/\zeta_2\right)\mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}\right)\,,$$

and $\boldsymbol{\omega}(\zeta)$ is a scalar operator on each $\mathcal{W}_{\alpha}$,

$$\boldsymbol{\omega}(\zeta)|_{\mathcal{W}_{\alpha}}=\omega(\zeta,\alpha)1_{\mathcal{W}_{\alpha}}, \tag{1.5}$$

the scalar being

$$\omega(\zeta,\alpha)$$
$$=\frac{4(q\zeta)^{\alpha}}{(1+q^{\alpha})^2}\left(\frac{q^{-\alpha}}{1-q^{-2}\zeta^2}-\frac{q^{\alpha}}{1-q^{2}\zeta^2}\right)+\int\limits_{-i\infty-0}^{i\infty-0}\zeta^{u+\alpha}\frac{\sin\frac{\pi}{2}\left(u-\nu(u+\alpha)\right)}{\sin\frac{\pi}{2}u\cos\frac{\pi\nu}{2}\left(u+\alpha\right)}du\,.$$

For any local operator of length L, the trace is effectively taken over $(\mathbb{C}^2)^{\otimes L}$.

Comments are in order about the meaning of (1.4). In [7, 8], in the setting of inhomogeneous chains, it was conjectured that the thermodynamic limit of the ground state averages in the finite XXZ chain are certain specific solutions of the reduced qKZ (rqKZ) equation given by multiple integrals. Subsequently these integral formulas were also derived from the viewpoint of algebraic Bethe Ansatz [9]. We take these formulas as the definition of the left hand side of (1.4). Following our previous works [2, 5], we present here another formula for solutions of rqKZ equations. The right hand side of (1.4) is its specialization to the homogeneous case. We have no doubt that these two solutions coincide [3]. Since a mathematical proof is lacking at the moment, we propose (1.4) as conjecture. The function $\omega(\zeta,\alpha)$ and $\mathbf{tr}^{\alpha}$ develop singularities at $\alpha=\pm1/\nu$. In view of this, we presume that the formula holds true throughout the range $|\mathrm{Re}\,\alpha|<1/\nu$.

It will be shown that the operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ commute with the adjoint action of the shift operator U and of local integrals of motion I_p on $\mathcal{W}_{[\alpha]}$. Since $q^{-\alpha S}$ commutes with U, I_p, one immediately concludes that the vacuum expectation values

[2] In [5] the operator $\mathbf{\Omega}$ was denoted by Ω^{*}.

[3] It is known to be the case in the massive regime, see [2]. We also confirm the coincidence at the free fermion point, see section 3.

of $U\left(q^{2\alpha S(0)}\mathcal{O}\right)U^{-1}-q^{2\alpha S(0)}\mathcal{O}$ and $\left[I_p,q^{2\alpha S(0)}\mathcal{O}\right]$ given by (1.4) vanish, as it should be.

In our opinion the appearance of anti-commuting operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ is quite remarkable. In the next section we explain how these operators are constructed using the q-oscillators. We explain their relation to the Jordan-Wigner fermions in the XX case in Section 3. In Appendix we briefly discuss the generalization of our previous formulae [5] to the case when the disorder operator is present.

For the sake of simplicity we consider the homogeneous chain only. We give brief explanations about the inhomogeneous case when needed. We do not give complete proofs, but just sketch the derivation of the main statements. We tried to make this paper as brief as possible, leaving the details to a separate publication.

2. Operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$

First we prepare our notation for the L-operators. Consider the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The universal R-matrix of this algebra belongs to the tensor product $\mathfrak{b}_+\otimes\mathfrak{b}_-$ of its two Borel subalgebras. By an L-operator we mean its image under an algebra map $\mathfrak{b}_+\otimes\mathfrak{b}_-\longrightarrow N_1\otimes N_2$, where N_1,N_2 are some algebras. In this paper we always take N_2 to be the algebra $M=Mat(2,\mathbb{C})$ of 2×2 matrices. As for N_1 we make several choices: $U_q(\mathfrak{sl}_2)$, M, the q-oscillator algebra Osc (see below) or $Osc\otimes M^{\pm}$, where $M^{\pm}\subset M$ are the subalgebras of upper and lower triangular matrices. For economy of symbols, we use the same letter L to designate these various L-operators. We always put indices, indicating to which tensor product of algebras they belong. We use $j,k,\cdots$ as labels for the lattice sites, and $a,b,\cdots$ as those for the 'auxiliary' two-dimensional space. Accordingly we write the matrix algebra as M_j or M_a. Capital letters $A,B,\cdots$ will indicate the q-oscillator algebra Osc. Finally, for $Osc\otimes M^{\pm}$ we use pairs of indices such as $\{A,a\}$.

The first case of L-operators is when $N_1=U_q(\mathfrak{sl}_2)$:

$$L_j(\zeta)=\begin{pmatrix}\zeta q^{\frac{H+1}{2}}-\zeta^{-1}q^{-\frac{H+1}{2}} & (q-q^{-1})Fq^{\frac{H-1}{2}}\\ (q-q^{-1})q^{-\frac{H-1}{2}}E & \zeta q^{-\frac{H-1}{2}}-\zeta^{-1}q^{\frac{H-1}{2}}\end{pmatrix}_j\in U_q(\mathfrak{sl}_2)\otimes M_j. \tag{2.1}$$

Here $E,F,q^{\pm H/2}$ are the standard generators of $U_q(\mathfrak{sl}_2)$. The suffix j in the right hand side means that it is considered as a 2×2 matrix in M_j. This is an exceptional case when we do not put any index for the first ('auxiliary') tensor factor; we shall never use several copies of $U_q(\mathfrak{sl}_2)$. Mapping further $U_q(\mathfrak{sl}_2)$ to M_a, we obtain the second L-operator

$$L_{a,j}(\zeta)\in M_a\otimes M_j\,,$$

which actually coincides with the standard 4×4 R-matrix.

The next case is due originally to Bazhanov, Lukyanov and Zamolodchikov [12]. Let us consider the q-oscillators a, a^* satisfying

$$aa^*-q^2a^*a=1-q^2.$$

It is convenient to introduce one more element q^D such that

$$q^D a^* = a^* q^{D+1}, \qquad q^D a = a q^{D-1}\,,$$
$$a^* a = 1 - q^{2D}, \quad a a^* = 1 - q^{2D+2}\,.$$

Denote by Osc the algebra generated by $a, a^*, q^{\pm D}$ with the above relations. We consider the following two representations of Osc,

$$W^+ = \bigoplus_{k=0}^{\infty} \mathbb{C}|k\rangle, \quad a^*|k-1\rangle = |k\rangle, \quad D|k\rangle = k|k\rangle, \quad a|0\rangle = 0;$$
$$W^- = \bigoplus_{k=-\infty}^{-1} \mathbb{C}|k\rangle, \quad a|k+1\rangle = |k\rangle, \quad D|k\rangle = k|k\rangle, \quad a^*|-1\rangle = 0\,.$$

In the root of unity case, if r is the smallest positive integer such that $q^{2r} = 1$, we consider the r-dimensional quotient of $W^\pm$ generated by $|0\rangle$ or $|-1\rangle$.

The L-operator associated with Osc is given by

$$L^+_{A,j}(\zeta) = i\zeta^{-\frac{1}{2}} q^{-\frac{1}{4}} \begin{pmatrix} 1 & -\zeta a^*_A \\ -\zeta a_A & 1 - \zeta^2 q^{2D_A+2} \end{pmatrix}_j \begin{pmatrix} q^{D_A} & 0 \\ 0 & q^{-D_A} \end{pmatrix}_j \in Osc_A \otimes M_j\,.$$

This L-operator satisfies the crossing symmetry relation:

$$L^+_{A,j}(\zeta)^{-1} = \frac{1}{\zeta - \zeta^{-1}} \overline{L}^+_{A,j}(\zeta)\,,$$

where we have set

$$\overline{L}^+_{A,j}(\zeta) = \sigma^2_j L^+_{A,j}(\zeta q^{-1})^{t_j} \sigma^2_j\,,$$

and t_j stands for the transposition in M_j. We use also another L-operator

$$L^-_{A,j}(\zeta) = \sigma^1_j L^+_{A,j}(\zeta) \sigma^1_j\,.$$

Consider the product $L^+_{A,j}(\zeta) L_{a,j}(\zeta)$. It is well known that this product can be brought to a triangular form, giving rise in particular to Baxter's 'TQ-equation' for transfer matrices. Namely, introducing

$$G^+_{A,a} = \begin{pmatrix} q^{-D_A} & 0 \\ 0 & q^{D_A} \end{pmatrix}_a \begin{pmatrix} 1 & a^*_A \\ 0 & 1 \end{pmatrix}_a, \quad G^-_{A,a} = \sigma^1_a G^+_{A,a} \sigma^1_a\,,$$

one easily finds that

(2.2)

$$L^+_{\{A,a\},j}(\zeta) = \left(G^+_{A,a}\right)^{-1} L^+_{A,j}(\zeta) L_{a,j}(\zeta) G^+_{A,a}$$
$$= \begin{pmatrix} (\zeta q - \zeta^{-1} q^{-1}) L^+_{A,j}(\zeta q^{-1}) q^{-\frac{\sigma^3_j}{2}} & 0 \\ (q - q^{-1}) L^+_{A,j}(\zeta q) \sigma^+_j q^{-2D_A + \frac{1}{2}} & (\zeta - \zeta^{-1}) L^+_{A,j}(\zeta q) q^{\frac{\sigma^3_j}{2}} \end{pmatrix}_a \in \left(Osc_A \otimes M^-_a\right) \otimes M_j\,.$$

For the inverse matrix one has:

(2.3)

$$L^{+}_{\{A,a\},j}(\zeta)^{-1}=\frac{1}{(\zeta-\zeta^{-1})(\zeta q-\zeta^{-1}q^{-1})(\zeta q^{-1}-\zeta^{-1}q)}$$
$$\times\begin{pmatrix}(\zeta-\zeta^{-1})q^{\frac{\sigma^3_j}{2}}\ \overline{L}^{+}_{A,j}(\zeta q^{-1}) & 0\\ -(q-q^{-1})\sigma^{+}_{j}\ \overline{L}^{+}_{A,j}(\zeta q)q^{-2D_A+\frac{1}{2}} & (\zeta q^{-1}-\zeta^{-1}q)q^{-\frac{\sigma^3_j}{2}}\ \overline{L}^{+}_{A,j}(\zeta q)\end{pmatrix}_a .$$

Again we shall use another L-operator:

$$L^{-}_{\{A,a\},j}(\zeta)=\sigma^1_a\sigma^1_jL^{+}_{\{A,a\},j}(\zeta)\sigma^1_a\sigma^1_j\in\left(Osc_A\otimes M^{+}_a\right)\otimes M_j\,.$$

Some information will be needed about R-matrices which intertwine these L-operators. First, consider the Yang-Baxter equation:

$$R_{A,B}(\zeta_1/\zeta_2)L^{\pm}_{A,j}(\zeta_1)L^{\pm}_{B,j}(\zeta_2)=L^{\pm}_{B,j}(\zeta_2)L^{\pm}_{A,j}(\zeta_1)R_{A,B}(\zeta_1/\zeta_2)\,. \tag{2.4}$$

The R-matrix appearing in (2.4) is given by

$$R_{A,B}(\zeta)=P_{A,B}h(\zeta,u_{A,B})\zeta^{D_A+D_B}\,,$$

where $P_{A,B}$ is the permutation,

$$u_{A,B}=a^{*}_{A}q^{-2D_A}a_B,$$

and the function $h(\zeta,u)$ is given by

$$h(\zeta,u)=\sum_{n=0}^{\infty}\left(-uq^{-1}\right)^n\prod_{j=1}^{n}\frac{q^{j-1}\zeta^{-1}-q^{-j+1}\zeta}{q^j-q^{-j}}\,.$$

When q is not a root of unity, the series for $R_{A,B}(\zeta)$ is well defined because the action of $u_{A,B}$ on $W^{\pm}\otimes W^{\pm}$ is locally nilpotent. Otherwise we replace the right hand side by the sum $\sum_{n=0}^{r-1}$, if r is the smallest positive integer such that $q^{2r}=1$.

Second, consider the Yang-Baxter equation for the L-operators $L^{+}_{\{A,a\},j}$:

(2.5)

$$R^{+}_{\{A,a\},\{B,b\}}(\zeta_1/\zeta_2)L^{+}_{\{A,a\},j}(\zeta_1)L^{+}_{\{B,b\},j}(\zeta_2)=L^{+}_{\{B,b\},j}(\zeta_2)L^{+}_{\{A,a\},j}(\zeta_1)R^{+}_{\{A,a\},\{B,b\}}(\zeta_1/\zeta_2)\,.$$

The corresponding R-matrix has the form

$$R^{+}_{\{A,a\},\{B,b\}}(\zeta)=\begin{pmatrix}\mathcal{R}_{1,1}(\zeta)&0&0&0\\ \mathcal{R}_{2,1}(\zeta)&\mathcal{R}_{2,2}(\zeta)&0&0\\ \mathcal{R}_{3,1}(\zeta)&0&\mathcal{R}_{3,3}(\zeta)&0\\ \mathcal{R}_{4,1}(\zeta)&\mathcal{R}_{4,2}(\zeta)&\mathcal{R}_{4,3}(\zeta)&\mathcal{R}_{4,4}(\zeta)\end{pmatrix}_{a,b}. \tag{2.6}$$

The entries $\mathcal{R}_{i,j}(\zeta)$ can be found by a direct calculation. In this paper we shall need only two of them:

$$\mathcal{R}_{1,1}(\zeta)=q^{-D_A}R_{A,B}(\zeta)q^{D_B},\quad \mathcal{R}_{4,4}(\zeta)=-\zeta^2q^{D_A}R_{A,B}(\zeta)q^{-D_B}\,.$$

Up to scalar coefficients depending on ζ, these operators can be guessed immediately, but the coefficient, especially the sign, in $\mathcal{R}_{4,4}(\zeta)$ is important for us. As usual we define:

$$R^{-}_{\{A,a\},\{B,b\}}(\zeta) = \sigma^1_a\sigma^1_b R^{+}_{\{A,a\}\{B,b\}}(\zeta)\sigma^1_a\sigma^1_b\,.$$

Now we have everything necessary for the definition of the operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$. For two integers $k \le l$ we set

$$M_{[k,l]} = M_k \otimes \cdots \otimes M_l\,.$$

This is the algebra of linear operators on the 'quantum space' on the interval $[k,l]$. Our main object is the monodromy matrix

$$T^{\pm}_{\{A,a\},[k,l]}(\zeta) = L^{\pm}_{\{A,a\},l}(\zeta)\cdots L^{\pm}_{\{A,a\},k}(\zeta) \in Osc_A \otimes M^{\mp}_a \otimes M_{[k,l]}\,. \tag{2.7}$$

Define further an element $\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta) \in Osc_A \otimes M^{\mp}_a \otimes \mathrm{End}(M_{[k,l]})$ by setting

$$\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta)(X_{[k,l]}) = T^{\pm}_{\{A,a\},[k,l]}(\zeta)\cdot(1_{A,a}\otimes X_{[k,l]})\cdot T^{\pm}_{\{A,a\},[k,l]}(\zeta)^{-1}\,, \tag{2.8}$$

where $1_{A,a} = 1_A \otimes 1_a$ and $X_{[k,l]} \in M_{[k,l]}$. To illustrate the definition, we have, for $x_{\{A,a\}} \in Osc_A \otimes M^{\mp}_a$ and $X_{[k,l]} \in M_{[k,l]}$, an equality in $Osc_A \otimes M^{\mp}_a \otimes M_{[k,l]}$

$$\begin{aligned}&\Bigl(\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta)\cdot x_{\{A,a\}}\otimes id\Bigr)\left(X_{[k,l]}\right)\\ &= T^{\pm}_{\{A,a\},[k,l]}(\zeta)\cdot\left(1_{\{A,a\}}\otimes X_{[k,l]}\right)\cdot T^{\pm}_{\{A,a\},[k,l]}(\zeta)^{-1}\cdot(x_{\{A,a\}}\otimes 1_{[k,l]})\,,\end{aligned}$$

where id is the identity operator in $\mathrm{End}(M_{[k,l]})$.

We define $\mathbb{T}^{\pm}_{A,[k,l]}(\zeta) \in Osc_A \otimes \mathrm{End}(M_{[k,l]})$ and $\mathbb{T}_{a,[k,l]}(\zeta) \in M_a \otimes \mathrm{End}(M_{[k,l]})$ in a similar manner.

In the following we shall use only $\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta)^{-1}$. We understand certain inconvenience in using the inverse operators, but it has for us a historical reason: once we define the transfer-matrix as in [7], the order of multipliers is fixed everywhere.

We have the Yang-Baxter equation

$$\begin{aligned}&\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta_1)^{-1}\mathbb{T}^{\pm}_{\{B,b\},[k,l]}(\zeta_2)^{-1}R^{\pm}_{\{A,a\},\{B,b\}}(\zeta_1/\zeta_2)\\ &\quad= R^{\pm}_{\{A,a\},\{B,b\}}(\zeta_1/\zeta_2)\mathbb{T}^{\pm}_{\{B,b\},[k,l]}(\zeta_2)^{-1}\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta_1)^{-1}\,,\end{aligned} \tag{2.9}$$

where the identity is in $Osc_A \otimes M^{\mp}_a \otimes Osc_B \otimes M^{\mp}_b \otimes \mathrm{End}(M_{[k,l]})$.

Of particular importance are the Q-operators acting on local operators. They are defined as

$$\begin{aligned}\mathbf{Q}^{+}_{[k,l]}(\zeta,\alpha) &= \mathrm{tr}^{+}_{A}\left(q^{2\alpha D_A}\ \mathbb{T}^{+}_{A,[k,l]}(\zeta)^{-1}\right)(1-q^{2(\alpha-\mathbf{S})}),\\ \mathbf{Q}^{-}_{[k,l]}(\zeta,\alpha) &= \mathrm{tr}^{-}_{A}\left(q^{-2\alpha(D_A+1)}\ \mathbb{T}^{-}_{A,[k,l]}(\zeta)^{-1}\right)q^{2\mathbf{S}}(1-q^{2(\alpha-\mathbf{S})})\,,\end{aligned} \tag{2.10}$$

where $\mathbf{S}$ stands for the adjoint action of the total spin operator

$$\mathbf{S}(X_{[k,l]}) = \left[S(l)-S(k-1)\ ,\ X_{[k,l]}\right], \qquad X_{[k,l]} \in M_{[k,l]}\,.$$

The trace functionals $\mathrm{tr}^{+}_{A}\bigl(q^{2\alpha D_A}Y_A\bigr)$ and $\mathrm{tr}^{-}_{A}\bigl(q^{-2\alpha(D_A+1)}Y_A\bigr)$ for $Y_A \in Osc_A$ are defined as analytic continuations with respect to α of traces over W^+ and W^- from the region $|q^\alpha| < 1$. The Q-operators (2.10) are mutually commuting families of operators. They are so normalized that $\mathbf{Q}^{\pm}_{[k,l]}(0,\alpha) = 1$.

Regarding $\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta)^{-1}$ as a matrix in $M^{\mp}_a$, let us write its entries as

$$\mathbb{T}^{+}_{\{A,a\},[k,l]}(\zeta)^{-1} = \begin{pmatrix} \mathbb{A}^{+}_{A,[k,l]}(\zeta) & 0 \\ \mathbb{C}^{+}_{A,[k,l]}(\zeta) & \mathbb{D}^{+}_{A,[k,l]}(\zeta) \end{pmatrix}_a ,$$

$$\mathbb{T}^{-}_{\{A,a\},[k,l]}(\zeta)^{-1} = \begin{pmatrix} \mathbb{A}^{-}_{A,[k,l]}(\zeta) & \mathbb{B}^{-}_{A,[k,l]}(\zeta) \\ 0 & \mathbb{D}^{-}_{A,[k,l]}(\zeta) \end{pmatrix}_a ,$$

where $\mathbb{A}^{+}_{A,[k,l]}(\zeta)$, etc., are elements of $Osc_A \otimes \mathrm{End}(M_{[k,l]})$. It follows from the definition that $\mathbb{T}^{\pm}_{\{A,a\},[k,l]}(\zeta)^{-1}$ have poles of order $l-k+1$ at the points $\zeta^2 = 1, q^{\pm 2}$. However, looking at the formulae (2.2)– (2.3), one realizes that at the pole $\zeta^2 = 1$ only $\mathbb{C}^{+}_{A,[k,l]}(\zeta)$ and $\mathbb{B}^{-}_{A,[k,l]}(\zeta)$ are singular. This motivates, at least partly, the following definition:

$$\mathbf{c}_{[k,l]}(\zeta,\alpha) = q^{\alpha-\mathbf{S}}\left(1-q^{2(\alpha-\mathbf{S})}\right)\mathrm{sing}_{\zeta=1}\left[\zeta^{\alpha-\mathbf{S}}\mathrm{tr}^{+}_{A}\left(q^{2\alpha D_A}\ \mathbb{C}^{+}_{A,[k,l]}(\zeta)\right)\right], \tag{2.11}$$

$$\mathbf{b}_{[k,l]}(\zeta,\alpha) = q^{2\mathbf{S}}\,\mathrm{sing}_{\zeta=1}\left[\zeta^{-\alpha+\mathbf{S}}\mathrm{tr}^{-}_{A}\left(q^{-2\alpha(D_A+1)}\ \mathbb{B}^{-}_{A,[k,l]}(\zeta)\right)\right]. \tag{2.12}$$

Here and after, $\mathrm{sing}_{\zeta=1}[f(\zeta)]$ signifies the singular part of $f(\zeta)$ at $\zeta = 1$:

$$\mathrm{sing}_{\zeta=1}\left[f(\zeta)\right] = \frac{1}{2\pi i}\int \frac{f(\xi)}{\zeta-\xi}d\xi\,, \tag{2.13}$$

where the integral is taken over a simple closed curve containing $\xi = 1$ inside, while $\xi = \zeta$ and other singular points of $f(\xi)$ are outside. We note that

$$[\mathbf{S}, \mathbf{c}_{[k,l]}(\zeta,\alpha)] = \mathbf{c}_{[k,l]}(\zeta,\alpha), \quad [\mathbf{S}, \mathbf{b}_{[k,l]}(\zeta,\alpha)] = -\mathbf{b}_{[k,l]}(\zeta,\alpha)\,.$$

There are several important properties of operators $\mathbf{c}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{b}_{[k,l]}(\zeta,\alpha)$ which we formulate as Lemmas.

Lemma 2.1. *The operators $\mathbf{c}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{b}_{[k,l]}(\zeta,\alpha)$ satisfy the following anti-commutation relations:*

$$\mathbf{c}_{[k,l]}(\zeta_1,\alpha-1)\mathbf{c}_{[k,l]}(\zeta_2,\alpha) = -\mathbf{c}_{[k,l]}(\zeta_2,\alpha-1)\mathbf{c}_{[k,l]}(\zeta_1,\alpha)\,, \tag{2.14}$$

$$\mathbf{b}_{[k,l]}(\zeta_1,\alpha+1)\mathbf{b}_{[k,l]}(\zeta_2,\alpha) = -\mathbf{b}_{[k,l]}(\zeta_2,\alpha+1)\mathbf{b}_{[k,l]}(\zeta_1,\alpha)\,. \tag{2.15}$$

Proof. Consider the Yang-Baxter equations (2.9) for $+$. Using the R-matrix (2.6) one finds:

(2.16)

$$\zeta_1^{-2}\mathbb{C}^{+}_{A,[k,l]}(\zeta_1)\mathbb{C}^{+}_{B,[k,l]}(\zeta_2)$$
$$+\,\zeta_2{}^{-2}q^{D_A}R_{A,B}(\zeta_1/\zeta_2)q^{-D_B}\cdot\mathbb{C}^{+}_{B,[k,l]}(\zeta_2)\mathbb{C}^{+}_{A,[k,l]}(\zeta_1)\cdot q^{-D_B}R_{A,B}(\zeta_1/\zeta_2)^{-1}q^{D_A} = \cdots$$

where $\cdots$ stands for a sum of terms which contain at least one $\mathbb{A}^{+}_{[k,l]}(\zeta_i)$ or $\mathbb{D}^{+}_{[k,l]}(\zeta_i)$, and hence have vanishing singular parts at $\zeta_i = 1$. Multiplying (2.16) by $q^{2(\alpha-1)D_A+2\alpha D_B}$, taking the trace and the singular part, one immediately gets (2.14). Similarly one proves (2.15) using (2.9) for $-$. $\square$

Lemma 2.2. *We have the following reduction relations:*

$$\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l-1]}\cdot 1_l\right)=\mathbf{c}_{[k,l-1]}(\zeta,\alpha)\left(X_{[k,l-1]}\right)\cdot 1_l\,, \tag{2.17}$$

$$\mathbf{b}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l-1]}\cdot 1_l\right)=\mathbf{b}_{[k,l-1]}(\zeta,\alpha)\left(X_{[k,l-1]}\right)\cdot 1_l\,, \tag{2.18}$$

$$\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(q^{\alpha\sigma^3_k}\cdot X_{[k+1,l]}\right)=q^{(\alpha-1)\sigma^3_k}\cdot\mathbf{c}_{[k+1,l]}(\zeta,\alpha)\left(X_{[k+1,l]}\right)\,, \tag{2.19}$$

$$\mathbf{b}_{[k,l]}(\zeta,\alpha)\left(q^{\alpha\sigma^3_k}\cdot X_{[k+1,l]}\right)=q^{(\alpha+1)\sigma^3_k}\cdot\mathbf{b}_{[k+1,l]}(\zeta,\alpha)\left(X_{[k+1,l]}\right)\,. \tag{2.20}$$

Proof. The equations (2.17), (2.18) are trivial consequences of the definition. In contrast, eqs. (2.19), (2.20) are far from being obvious.

Consider the first of them. By definition we have:

$$\begin{aligned}&\frac{1}{q^{\alpha-\mathbf{S}}\left(1-q^{2(\alpha-\mathbf{S})}\right)}\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(q^{\alpha\sigma^3_k}\cdot X_{[k+1,l]}\right)\\&=\ \mathrm{sing}_{\zeta=1}\left[\mathrm{tr}^+_A\left(q^{2\alpha D_A}\ \mathbb{C}^+_{A,[k,l]}(\zeta)\left(q^{\alpha\sigma^3_k}\cdot X_{[k+1,l]}\right)\right)\zeta^{\alpha-s-1}\right]\,,\end{aligned}$$

where s is the spin of $X_{[k+1,l]}$.

Let us simplify the trace. We will use the crossing symmetry

$$\mathcal{P}^-_{j,\bar{j}}L^+_{A,j}(\zeta q^{-1})L^+_{A,\bar{j}}(\zeta)=(\zeta-\zeta^{-1})\mathcal{P}^-_{j,\bar{j}}\,, \tag{2.21}$$

$$\mathcal{P}^-_{j,\bar{j}}L^+_{\{A,a\},j}(\zeta q^{-1})L^+_{\{A,a\},\bar{j}}(\zeta)=(\zeta q-\zeta^{-1}q^{-1})(\zeta-\zeta^{-1})(\zeta q^{-1}-\zeta^{-1}q)\mathcal{P}^-_{j,\bar{j}}\,, \tag{2.22}$$

where $\mathcal{P}^-_{j,\bar{j}}$ is the anti-symmetrizer. Introducing consecutively some additional two-dimensional spaces, we have

(2.23)

$$\begin{aligned}&\mathrm{tr}^+_A\left(q^{2\alpha D_A}\ \mathbb{C}^+_{A,[k,l]}(\zeta)\left(q^{\alpha\sigma^3_k}\cdot X_{[k+1,l]}\right)\right)\\&=\mathrm{tr}_a\mathrm{tr}^+_A\left(\sigma^+_a L^+_{\{A,a\},k}(\zeta)^{-1}\cdot\mathbb{T}^+_{\{A,a\},[k+1,l]}(\zeta)^{-1}\left(X_{[k+1,l]}\right)\cdot q^{\alpha\sigma^3_k}L^+_{\{A,a\},k}(\zeta)q^{2\alpha D_A}\right)\\&=\frac{1}{(\zeta-\zeta^{-1})(\zeta q-\zeta^{-1}q^{-1})(\zeta q^{-1}-\zeta^{-1}q)}\\&\times\mathrm{tr}_{\bar{k}}\mathrm{tr}_a\mathrm{tr}^+_A\left(\sigma^+_a L^+_{\{A,a\},\bar{k}}(\zeta q^{-1})\cdot 2\mathcal{P}^-_{k,\bar{k}}\cdot\mathbb{T}^+_{\{A,a\},[k+1,l]}(\zeta)^{-1}\left(X_{[k+1,l]}\right)\cdot q^{\alpha\sigma^3_k}\cdot L^+_{\{A,a\},k}(\zeta)q^{2\alpha D_A}\right)\,.\end{aligned}$$

Now use

$$q^{\alpha\sigma^3_k}L^+_{A,k}(\zeta)q^{2\alpha D_A}=q^{2\alpha D_A}L^+_{A,k}(\zeta)q^{\alpha\sigma^3_k}$$

and the cyclicity of the trace to simplify (2.23) further:

$$\begin{aligned}&\mathrm{tr}^+_A\left(q^{2\alpha D_A}\ \mathbb{C}^+_{A,[k,l]}(\zeta)\left(q^{\alpha\sigma^3_k}\cdot X_{[k+1,l]}\right)\right)\\&=\frac{1}{(\zeta-\zeta^{-1})(\zeta q-\zeta^{-1}q^{-1})(\zeta q^{-1}-\zeta^{-1}q)}\\&\times\mathrm{tr}_{\bar{k}}\mathrm{tr}_a\mathrm{tr}^+_A\left(\mathbb{T}^+_{\{A,a\},[k+1,l]}(\zeta)^{-1}\left(X_{[k+1,l]}\right)q^{2\alpha D_A}\mathcal{L}(\zeta)q^{\alpha\sigma^3_k}\right)\,.\end{aligned} \tag{2.24}$$

It is easy to see that

$$\begin{aligned}\mathcal{L}(\zeta) &= 2\mathcal{P}^-_{k,\bar{k}} L^+_{\{A,a\},k}(\zeta)\sigma^+_a L^+_{\{A,a\},\bar{k}}(\zeta q^{-1}) \\ &= \begin{pmatrix} \zeta q - \zeta^{-1}q^{-1} & 0 \\ 0 & (q-q^{-1})q^{-2D_A-\frac{1}{2}} \end{pmatrix}_a 2\mathcal{P}^-_{k,\bar{k}} L^+_{A,k}(\zeta q^{-1}) L^+_{A,\bar{k}}(\zeta) \\ &\times \begin{pmatrix} q^{-\frac{\sigma^3_k}{2}}\sigma^+_{\bar{k}} & q^{\frac{\sigma^3_{\bar{k}}-\sigma^3_k}{2}} \\ \sigma^+_k\sigma^+_{\bar{k}} & \sigma^+_k q^{\frac{\sigma^3_{\bar{k}}}{2}} \end{pmatrix}_a \begin{pmatrix} (q-q^{-1})q^{-2D_A+\frac{1}{2}} & 0 \\ 0 & \zeta q^{-1} - \zeta^{-1} q \end{pmatrix}_a . \end{aligned} \tag{2.25}$$

where we used

$$L^+_{A,j}(\zeta q)\sigma^+_j q^{-2D_A+\frac{1}{2}} = q^{-2D_A-\frac{1}{2}} L^+_{A,j}(\zeta q^{-1})\sigma^+_j \,.$$

In view of (2.21), $\mathcal{L}(\zeta)$ is divisible by $\zeta-\zeta^{-1}$, and in (2.24) we can drop the diagonal elements of $\mathbb{T}^+_{\{A,a\},[k+1,l]}(\zeta)^{-1}$, arriving immediately at (2.19).

The proof of (2.20) is similar. □

Remark. The above construction carries over to inhomogeneous chains where an independent spectral parameter ξ_j is attached to each site j. The operators $\mathbf{c}_{[k,l]}(\zeta;\xi_k,\cdots,\xi_l)$, $\mathbf{b}_{[k,l]}(\zeta;\xi_k,\cdots,\xi_l)$ are defined via the above construction with two modifications:

(i) In the definition (2.7), each $L^{\pm}_{\{A,a\},j}(\zeta)$ is replaced by $L^{\pm}_{\{A,a\},j}(\zeta/\xi_j)$.

(ii) The singular part is understood as an integral (2.13) around the points $\xi_k,\cdots,\xi_l$.

Lemma 2.1 and Lemma 2.2 remain valid. □

Lemma 2.2 allows us to define universal operators $\mathbf{b}(\zeta,\alpha)$, $\mathbf{c}(\zeta,\alpha)$:

Definition 2.3. *For any operator $q^{2\alpha S(0)}\mathcal{O}\in\mathcal{W}_\alpha$, let $\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}$ be its restriction to the finite interval $[k,l]$ of the lattice. We define*

$$\mathbf{b}(\zeta,\alpha):\ \mathcal{W}_{\alpha,s}\to\mathcal{W}_{\alpha+1,s-1}\ , \tag{2.26}$$

$$\mathbf{c}(\zeta,\alpha):\ \mathcal{W}_{\alpha,s}\to\mathcal{W}_{\alpha-1,s+1}\ , \tag{2.27}$$

by setting

$$\mathbf{b}(\zeta,\alpha)\left(q^{2\alpha S(0)}\mathcal{O}\right) = \lim_{k\to-\infty,l\to\infty}\mathbf{b}_{[k,l]}(\zeta,\alpha)\left(\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}\right)\ , \tag{2.28}$$

$$\mathbf{c}(\zeta,\alpha)\left(q^{2\alpha S(0)}\mathcal{O}\right) = \lim_{k\to-\infty,l\to\infty}\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}\right)\ . \tag{2.29}$$

It follows from Lemma 2.2 that for any particular operator $q^{2\alpha S(0)}\mathcal{O}$ the expressions under the limit in (2.28), (2.29) stabilize for sufficiently large interval $[k,l]$. Hence the limit is well-defined. In particular we have, for any k,

$$\mathbf{b}(\zeta,\alpha)(q^{2\alpha S(k)})=0,\quad \mathbf{c}(\zeta,\alpha)(q^{2\alpha S(k)})=0\,. \tag{2.30}$$

Denoting by $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ the operators acting on the direct sum $\mathcal{W}_{[\alpha]}$ we have the anti-commutativity

$$[\mathbf{b}(\zeta_1),\mathbf{b}(\zeta_2)]_+ = [\mathbf{c}(\zeta_1),\mathbf{c}(\zeta_2)]_+ = 0\ .$$

In Appendix, we give a brief summary of the algebraic formula for the correlation functions in the presence of disorder. The result is expressed in terms of the operator

$$\boldsymbol{\Omega} = -\mathrm{res}_{\zeta_1=1}\mathrm{res}_{\zeta_2=1}\left(\mathbf{X}(\zeta_1,\zeta_2)\boldsymbol{\omega}(\zeta_2/\zeta_1)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}\right), \tag{2.31}$$

where $\mathbf{X}(\zeta_1,\zeta_2)|_{\mathcal{W}_\alpha} = \mathbf{X}(\zeta_1,\zeta_2,\alpha)$, the operator $\mathbf{X}(\zeta_1,\zeta_2,\alpha)$ is given in either of the two formulas (A.3), (A.4), $\boldsymbol{\omega}(\zeta)$ is given by (1.5). The following result allows us to express $\boldsymbol{\Omega}$ in terms of $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$. At the same time, the existence of two equivalent representations guarantees the anti-commutativity between the latter.

Lemma 2.4. *The operator* $\mathbf{X}(\zeta_1,\zeta_2)$ *can be evaluated as follows:*

$$\mathbf{X}(\zeta_1,\zeta_2)|_{\mathcal{W}_\alpha} = \mathbf{b}(\zeta_2,\alpha-1)\mathbf{c}(\zeta_1,\alpha) = -\mathbf{c}(\zeta_1,\alpha+1)\mathbf{b}(\zeta_2,\alpha)\,. \tag{2.32}$$

Proof. Consider the formula (A.3). We have:

$$\begin{aligned}
&\mathrm{tr}_{a,b}\left(B^0_{b,a}(\zeta_2/\zeta_1)\mathbb{T}_a(\zeta_1)^{-1}\mathbb{T}_b(\zeta_2)^{-1}\right)\mathbf{Q}^+(\zeta_1,\alpha+1)\mathbf{Q}^-(\zeta_2,\alpha+1)\\
&\quad= \mathrm{tr}_{a,b}\mathrm{tr}^+_A\mathrm{tr}^-_B\left(B^0_{b,a}(\zeta_2/\zeta_1)\mathbb{T}_a(\zeta_1)^{-1}\mathbb{T}_b(\zeta_2)^{-1}\mathbb{T}^+_A(\zeta_1)^{-1}\mathbb{T}^-_B(\zeta_2)^{-1}\right.\\
&\quad\left.\times q^{2(\alpha+1)(D_A-D_B-1)}\right)(1-q^{2(\alpha+1-\mathbf{S})})^2q^{2\mathbf{S}}\,.
\end{aligned}$$

We move $\mathbb{T}_b(\zeta_2)^{-1}$ through $\mathbb{T}^+_A(\zeta_1)^{-1}$ using the Yang-Baxter equation

$$L^+_{A,b}(\zeta_1/\zeta_2)\,\mathbb{T}_b(\zeta_2)^{-1}\mathbb{T}^+_A(\zeta_1)^{-1} = \mathbb{T}^+_A(\zeta_1)^{-1}\mathbb{T}_b(\zeta_2)^{-1}L^+_{A,b}(\zeta_1/\zeta_2)\,.$$

Now $\mathbb{T}_a(\zeta_1)^{-1}\mathbb{T}^+_A(\zeta_1)^{-1}$ and $\mathbb{T}_b(\zeta_2)^{-1}\mathbb{T}^-_B(\zeta_2)^{-1}$ come together. Conjugating by $G^+_{A,a}$, $G^-_{B,b}$, we can combine them into the monodromy matrices $\mathbb{T}^+_{\{A,a\}}(\zeta_1)^{-1}$, $\mathbb{T}^-_{\{B,b\}}(\zeta_2)^{-1}$. In these monodromy matrices we drop diagonal elements because they have no singularities at $\zeta_i=1$. Then by a straightforward calculation we come to

$$\begin{aligned}
&\mathrm{tr}_{a,b}\left(B^0_{b,a}(\zeta_2/\zeta_1)\mathbb{T}_a(\zeta_1)^{-1}\mathbb{T}_b(\zeta_2)^{-1}\right)\mathbf{Q}^+(\zeta_1,\alpha+1)\mathbf{Q}^-(\zeta_2,\alpha+1)\\
&\simeq -\mathrm{tr}^+_A\mathrm{tr}^-_B\left(\mathbb{C}^+_A(\zeta_1)\mathbb{B}^-_B(\zeta_2)q^{2(\alpha+1)D_A-2\alpha(D_B+1)-2}\right)(1-q^{2(\alpha-\mathbf{S}+1)})^2q^{2\mathbf{S}}
\end{aligned} \tag{2.33}$$

where $\simeq$ means that the singular parts are identical. Similarly we have:

$$\begin{aligned}
&\mathrm{tr}_{a,b}\left(B^1_{a,b}(\zeta_1/\zeta_2)\mathbb{T}_b(\zeta_2)^{-1}\mathbb{T}_a(\zeta_1)^{-1}\right)\mathbf{Q}^-(\zeta_2,\alpha-1)\mathbf{Q}^+(\zeta_1,\alpha-1)\\
&\simeq -\mathrm{tr}^+_A\mathrm{tr}^-_B\left(\mathbb{B}^-_B(\zeta_2)\mathbb{C}^+_A(\zeta_1)q^{2\alpha D_A-2(\alpha-1)(D_B+1)}\right)(1-q^{2(\alpha-\mathbf{S}-1)})^2q^{2\mathbf{S}}\,.
\end{aligned} \tag{2.34}$$

Eq. (2.32) follows from (2.33), (2.34) and the definition of $\mathbf{b}(\zeta,\alpha)$, $\mathbf{c}(\zeta,\alpha)$. □

The main formula (1.4) follows from (2.31), (2.32) and (A.1).

Let U be the shift operator by one lattice unit, which acts on local operators by adjoint:

$$U\sigma^a_jU^{-1} = \sigma^a_{j+1}.$$

There is also an infinite set of local integrals of motion which commute with U and among themselves. The last important property of $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ is their invariance:

Lemma 2.5. *The operators* $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ *commute with the adjoint action of the shift operator* U *and of the local integrals of motion.*

Proof. For U the statement of this lemma follows immediately from the definition, essentially it is a consequence of Lemma 2.2.

The local integrals of motion are of the form

$$I_p = \sum_{j=-\infty}^{\infty} d_{j,p}\,, \tag{2.35}$$

where $d_{j,p}$ is an operator acting non-trivially on the sites $j, \cdots, j+p$. We shall call operators of the type (2.35) p-local operators.

Let us write the 4×4 R-matrix as $\check{R}_{j,k}(\xi) = P_{j,k}L_{j,k}(\xi)$. We set

$$U_{[k,l]}(\xi) = (q-q^{-1})^{k-l}\check{R}_{l,l-1}(\xi)\cdots\check{R}_{k+1,k}(\xi)\,.$$

Following the remark after Lemma 2.2, consider $\mathbf{c}_{[k,l]}$ with one inhomogeneity:

$$\mathbf{c}_{[k,l]}(\zeta;\xi,1,\cdots,1) \text{ and } \mathbf{c}_{[k,l]}(\zeta;1,\cdots,1,\xi)\,.$$

It is clear from the definition that

$$\begin{aligned} U_{[k,l]}(\xi)\cdot\mathbf{c}_{[k,l]}(\zeta;\xi,1,\cdots,1)\left(\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}\right)\cdot U_{[k,l]}(\xi)^{-1} \\ = \mathbf{c}_{[k,l]}(\zeta;1,\cdots,1,\xi)\left(U_{[k,l]}(\xi)\cdot\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}\cdot U_{[k,l]}(\xi)^{-1}\right)\,. \end{aligned} \tag{2.36}$$

Let $\xi = 1+\epsilon$. Then

$$U_{[k,l]}(\xi) = \exp\left(\sum_{p=1}^{\infty}\epsilon^p I_{[k,l],p}\right)\,.$$

Due to the Campbell-Hausdorff formula, the operators $I_{[k,l],p}$ are p-local. For finite k,l these operators do not commute because of some boundary terms, but in the limit $k\to-\infty$, $l\to\infty$ they coincide with the local integrals of motion I_p which are combined into the generating function:

$$U(\xi) = \exp\Bigl(\sum_{p=1}^{\infty}\epsilon^p I_p\Bigr)\,.$$

In the right hand side of (2.36) we have the expression

$$U_{[k,l]}(\xi)\cdot\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}\cdot U_{[k,l]}(\xi)^{-1} = \sum\epsilon^p\left(q^{2\alpha S(0)}\mathcal{O}\right)^{(p)}_{[k,l]}\,.$$

Here the p-local operators $I_{[k,l],p}$ act by multiple adjoint. It is clear that for every given degree p we can find a large enough interval $[k,l]$ in order that

$$\left(q^{2\alpha S(0)}\mathcal{O}\right)^{(p)}_{[k,l]} = \left(\left(q^{2\alpha S(0)}\mathcal{O}\right)^{(p)}\right)_{[k,l]}\,,$$

where

$$U(\xi)\cdot q^{2\alpha S(0)}\mathcal{O}\cdot U(\xi)^{-1} = \sum\epsilon^p\left(q^{2\alpha S(0)}\mathcal{O}\right)^{(p)}\,.$$

Obviously

$$\mathrm{length}\left(\left(q^{2\alpha S(0)}\mathcal{O}\right)^{(p)}\right) \le \mathrm{length}\left(q^{2\alpha S(0)}\mathcal{O}\right) + 2p\,.$$

Now considering (2.36) order by order in ϵ, choosing for every order sufficiently large interval $[k,l]$ and using the inhomogeneous version of Lemma 2.2 and the definition of $\mathbf{c}(\zeta)$, we get:

$$U(\xi)\cdot\mathbf{c}(\zeta)\left(q^{2\alpha S(0)}\mathcal{O}\right)\cdot U(\xi)^{-1}=\mathbf{c}(\zeta)\left(U(\xi)\cdot q^{2\alpha S(0)}\mathcal{O}\cdot U(\xi)^{-1}\right), \tag{2.37}$$

which is understood as an equality of power series in ϵ. □

3. Free fermion point

Consider the point $\nu=1/2$, $q=i$. For this coupling constant the Hamiltonian turns into

$$H_{XX}=\sum_{j=-\infty}^{\infty}\left(\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+\right),$$

and can be diagonalized by the Jordan-Wigner transformation:

$$\psi_k^{\pm}=\sigma_k^{\pm}e^{\mp\pi i S(k-1)}.$$

The space $\mathcal{W}_{[\alpha]}$ becomes a direct sum of two components:

$$\mathcal{W}_{[\alpha]}=\mathcal{W}_{\alpha}\oplus\mathcal{W}_{\alpha+1}.$$

We set

$$y=e^{\frac{\pi i\alpha}{2}},$$

so that the space $\mathcal{W}_\alpha$ consists of operators of the form $y^{2S(0)}\mathcal{O}$. There are two fermion operators acting in the space of states, so, there are four of them acting on the space of operators by left and right multiplication. It is convenient to introduce the following four operators:

$$\begin{aligned}\Psi_k^{\pm}(X)&=\psi_k^{\pm}X-(-1)^{F(X)}X\psi_k^{\pm},\\ \Phi_{\alpha,k}^{\pm}(X)&=\frac{1}{1-y^{\mp2}}\left(\psi_k^{\pm}X-y^{\mp2}(-1)^{F(X)}X\psi_k^{\pm}\right).\end{aligned} \tag{3.1}$$

where $F(X)$ is the fermionic number of the operator X.

We have $\Phi^{\pm}_{\alpha+2,k}=\Phi^{\pm}_{\alpha,k}$. These operators are natural for us because $\Psi_k^{\pm}$ annihilate 1 while $\Phi^{\pm}_{\alpha,k}$ annihilate y^{2S} (recall that at plus or minus infinity $y^{2S(0)}\mathcal{O}$ stabilizes to 1 or y^{2S}). The operators $\Psi_k^{\pm}$, $\Phi^{\pm}_{\alpha,k}$ satisfy the canonical anti-commutation relations:

$$[\Psi_k^{\epsilon},\Psi_l^{\epsilon'}]_+=[\Phi_{\alpha,k}^{\epsilon},\Phi_{\alpha,l}^{\epsilon'}]_+=0,\quad[\Psi_k^{\epsilon},\Phi_{\alpha,l}^{\epsilon'}]_+=\delta_{\epsilon+\epsilon',0}\delta_{k,l}. \tag{3.2}$$

It is clear, however, that the operators $\mathbf{b}(\zeta,\alpha)$, $\mathbf{c}(\zeta,\alpha)$ cannot be constructed as linear combinations of $\Psi_k^{\pm}$, $\Phi^{\pm}_{\alpha,k}$. Indeed the operators $\mathbf{b}(\zeta,\alpha)$, $\mathbf{c}(\zeta,\alpha)$ are translationally invariant, in particular, they annihilate $y^{2S(k)}$ for any k, see (2.30). Clearly this is impossible for any linear combination of $\Psi_k^{\pm}$, $\Phi^{\pm}_{\alpha,k}$. Our plan in this section is as follows. First, we find a compact expression for $\mathbf{b}(\zeta,\alpha)$ and $\mathbf{c}(\zeta,\alpha)$ in terms of $\Psi_k^{\pm}$, $\Phi^{\pm}_{\alpha,k}$. Then we show that our formula gives the same result for the correlators as the one obtained by a straightforward calculation based on normal ordering.

The calculation of $\mathbf{b}(\zeta,\alpha)$, $\mathbf{c}(\zeta,\alpha)$ at the free fermon point is summarized by

Lemma 3.1. *At the free fermion point, the operators* $\mathbf{b}(\zeta,\alpha)$ *and* $\mathbf{c}(\zeta,\alpha)$ *are given by*

$$\mathbf{b}(\zeta,\alpha)=\frac{2i^{-\mathbf{S}}}{1-(-1)^{\mathbf{S}}y^2}\ \mathrm{sing}_{\zeta=1}\left[\zeta^{-\alpha+\mathbf{S}}\Psi^-(\zeta)E^-(\zeta,\alpha-\mathbf{S})\frac{\zeta}{1+\zeta^2}\right], \tag{3.3}$$
$$\mathbf{c}(\zeta,\alpha)=2y\ \mathrm{sing}_{\zeta=1}\left[\zeta^{\alpha-\mathbf{S}}\Psi^+(\zeta)E^+(\zeta,\alpha-\mathbf{S})\frac{\zeta}{1+\zeta^2}\right],$$

where

$$\Psi^\pm(\zeta)=\sum_{j=-\infty}^{\infty}\Psi_j^\pm\left(\frac{1+\zeta^2}{1-\zeta^2}\right)^j \tag{3.4}$$

and

$$E^\pm(\zeta,\alpha)=\exp\left(\mathcal{N}\left[\Phi_\alpha^\pm\log\left(I-\zeta^2M\right)\Psi^\mp-\Phi_\alpha^\mp\log\left(I+\zeta^2M\right)\Psi^\pm\right]\right). \tag{3.5}$$

In the last formula we consider $\Phi_{\alpha,j}^\pm$ *(resp.* $\Psi_j^\pm$*) as components of a row (resp. column) vector,*

$$M=(1+u)(1-u)^{-1},\qquad \left(u\Psi^\pm\right)_j=\Psi_{j+1}^\pm,$$

and $\log\left(1\pm\zeta^2M\right)$ *are understood as Taylor series in* u. $\mathcal{N}[\cdot]$ *stands for the normal ordering which applies only to operators acting at the same site. For them we set*

$$\mathcal{N}[\Phi_{\alpha,j}^\epsilon\Psi_j^{\epsilon'}]=\begin{cases}\Phi_{\alpha,j}^\epsilon\Psi_j^{\epsilon'} & (j>0),\\ -\Psi_j^{\epsilon'}\Phi_{\alpha,j}^\epsilon & (j\le 0).\end{cases} \tag{3.6}$$

Since the q-oscillators become fermions at $q=i$, Lemma can be shown by manipulations with exponentials of quadratic forms in fermions. Details will be given in another publication.

We remark that the exponent of (3.5) is well defined as an operator on $\mathcal{W}_\alpha$. Indeed by definition it consists of $\mathcal{N}\left(\Phi_{\alpha,k}^\pm\Psi_l^\mp\right)$ with $l\ge k$. On a particular operator in $\mathcal{W}_\alpha$ only a finite number of these operators do not vanish.

It has been said that, unlike $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$, formulae containing fermions necessarily break the translational invariance. We choose the point $k=1$ as the origin and consider only operators of the form

$$y^{2S(0)}\mathcal{O}_> \tag{3.7}$$

where $\mathcal{O}_>$ acts only on the interval $[1,\infty)$. Any operator in $\mathcal{W}_\alpha$ can be brought to the form (3.7) by a shift, so we do not really lose generality. In the sequel we need the operators on a half line:

$$\mathbf{b}_>(\zeta,\alpha)=\mathbf{b}_{[1,\infty)}(\zeta,\alpha),\qquad \mathbf{c}_>(\zeta,\alpha)=\mathbf{c}_{[1,\infty)}(\zeta,\alpha).$$

They are defined as in (3.3), replacing $E^\pm(\zeta,\alpha)$, $\Psi^\pm(\zeta)$ and $\Phi_\alpha^\pm(\zeta)$ by $E_>^\pm(\zeta,\alpha)$, $\Psi_>^\pm(\zeta)$ and $\Phi_{\alpha,>}^\pm(\zeta)$, respectively. The latter are given by the same formulae (3.4), (3.5) with non-positive components of fermions removed.

In the free fermion case the function $\omega(\zeta,\alpha)$ can be calculated explicitly. Putting it together with (3.3), we rewrite our main formula in the free fermion case as follows.

(3.8)

$$\frac{\langle \text{vac}|\ y^{2S(0)}\mathcal{O}_>\ |\text{vac}\rangle}{\langle \text{vac}|\ y^{2S(0)}\ |\text{vac}\rangle} = \mathbf{tr}^{\alpha}_{>}\left(e^{\mathbf{\Omega}_>}\left(\mathcal{O}_>\right)\right),$$

$$\mathbf{\Omega}_> = \frac{i}{\sin\frac{\pi\alpha}{2}}\text{res}_{\zeta_1=1}\text{res}_{\zeta_2=1}\left(\frac{\zeta_1^{\alpha}\zeta_2^{-\alpha}-1}{\zeta_1^2+\zeta_2^2}E^-_>(\zeta_2,\alpha)E^+_>(\zeta_1,\alpha)\Psi^-_>(\zeta_2)\Psi^+_>(\zeta_1)\frac{d\zeta_1^2}{1+\zeta_1^2}\frac{d\zeta_2^2}{1+\zeta_2^2}\right),$$

where $\mathbf{tr}^{\alpha}_{>}$ means that the trace is calculated over the positive half of the chain only.

Now notice that

$$\Psi^{\pm}_{>}(\zeta)(I)=0,\quad \mathbf{tr}^{2(\alpha+1)}_{>}\left(\Phi^{\pm}_{\alpha,>}(\zeta)\left(\mathcal{O}_>\right)\right)=0, \tag{3.9}$$

$$\psi^{\pm}_j\mathcal{O}_> = \left(\Phi^{\pm}_{\alpha,j}-\frac{y^{\mp 2}}{1-y^{\mp 2}}\Psi^{\pm}_j\right)\left(\mathcal{O}_>\right).$$

So, by changing $\mathbf{tr}^{\alpha}_{>}$ to $\mathbf{tr}^{2(\alpha+1)}_{>}$, the operators $\Phi^{\mp}_{\alpha,>}$ and $\Psi^{\pm}_{>}$ can be considered as creation-annihilation operators in the space of operators. For efficient application of them we need the following:

Lemma 3.2. *The following identity holds:*

$$\mathbf{tr}^{\alpha}_{>}\left(e^{\mathbf{\Omega}_>}\left(\mathcal{O}_>\right)\right) = \mathbf{tr}^{2(\alpha+1)}_{>}\left(e^{\widetilde{\mathbf{\Omega}}_>}\left(\mathcal{O}_>\right)\right) \tag{3.10}$$

where

$$\widetilde{\mathbf{\Omega}}_> = \frac{i}{\sin\frac{\pi\alpha}{2}}\text{res}_{\zeta_1=1}\text{res}_{\zeta_2=1}\left(\frac{\zeta_1^{\alpha}\zeta_2^{-\alpha}}{\zeta_1^2+\zeta_2^2}\Psi^-_>(\zeta_2)\Psi^+_>(\zeta_1)\frac{d\zeta_1^2}{1+\zeta_1^2}\frac{d\zeta_2^2}{1+\zeta_2^2}\right).$$

The formulae (3.9) and (3.10) allow an explicit calculation of correlators. One easily obtains:

$$\frac{\langle \text{vac}|\ y^{2S(0)}\psi^+_{k_1}\cdots\psi^+_{k_p}\psi^-_{l_p}\cdots\psi^-_{l_1}\ |\text{vac}\rangle}{\langle \text{vac}|y^{2S(0)}\ |\text{vac}\rangle} = \det\left(\langle\psi^+_{k_i}\psi^-_{l_j}\rangle\right)_{i,j=1,\cdots,p} \tag{3.11}$$

where

(3.12)

$$\langle\psi^+_k\psi^-_l\rangle = \frac{i}{\sin\frac{\pi\alpha}{2}}\left(-\frac{y}{2}\delta_{k,l}+\ \text{res}_{\zeta_1=1}\text{res}_{\zeta_2=1}\frac{\zeta_1^{\alpha}\zeta_2^{-\alpha}}{\zeta_1^2+\zeta_2^2}\left(\frac{1+\zeta_1^2}{1-\zeta_1^2}\right)^k\left(\frac{1+\zeta_2^2}{1-\zeta_2^2}\right)^l\frac{d\zeta_1^2}{1+\zeta_1^2}\frac{d\zeta_2^2}{1+\zeta_2^2}\right).$$

On the other hand, one can calculate the correlators (3.11) directly by normal ordering $y^{2S(0)}$. The result is the same: (3.12) is the two-point function while (3.11) is obtained by the Wick theorem.

This calculation is unsatisfactory because we had to pass through the fermions $\Psi^{\pm}_k$, $\Phi^{\pm}_{\alpha,k}$. It would be much better to find a basis in the space of local operators, on which the original operators $\mathbf{b}(\zeta,\alpha)$, $\mathbf{c}(\zeta,\alpha)$ act nicely. Such a construction would have a chance to generalize to an arbitrary coupling constant. For the moment we cannot do that.

4. Conclusion

The main result of this paper can be formulated as follows. We consider the space $\mathcal{W}_{[\alpha]}$ of local operators in the presence of a disorder field. We have shown that the vacuum expectation values of operators in $\mathcal{W}_{[\alpha]}$ can be expressed in terms of two anti-commutative families of operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ acting on $\mathcal{W}_{[\alpha]}$. At present, we do not know how to organize the space $\mathcal{W}_{[\alpha]}$ in order to describe efficiently the action of $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$. The operators $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ should be considered as annihilation operators, as both of them kill the 'vacua', i.e., operators $q^{2\alpha S(k)}$, for all k. What is missing is a construction of creation operators. Even in the free fermion case, we were able rather to make a detour than to actually solve the problem.

In fact, the problem of constructing creation operators cannot be solved literally, because $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ have a large common kernel. Consider the restricted operators $\mathbf{b}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{c}_{[k,l]}(\zeta,\alpha)$ acting on the space of dimension 4^{l-k+1}. In the free fermion case, it can be shown that the dimension of the kernel is 2^{l-k+1}. Numerical experiments indicate that the dimension stays the same generically.

Because of this kernel, we cannot expect operators satisfying the canonical anti-commutation relations with $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$. So the first problem is to understand the meaning of the kernel. Obviously, the difference of any two operators in the kernel has vanishing expectation value. The origin of these operators with zero vacuum expectation values is a mystery to us. The only operators for which this property can be easily explained are the descendants generated by adjoint action of local integrals of motion, but for them the vacuum expectation values vanish for a different reason: $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ commute with the adjoint action of local integrals of motion as is explained by Lemma 2.5.

Understanding the origin of the kernel of $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$, and the construction of creation operators, are the problem which we wish to solve.

Appendix A.

In this appendix, we sketch how the results of [5] can be modified to the situation when disorder is present.

Consider an operator $q^{2\alpha S(0)}\mathcal{O}$ which is stable on $(-\infty, k-1]$ and $[l+1,\infty)$. Consider also an inhomogeneous chain, where spectral parameters $\xi_k,\cdots,\xi_l$ are attached to all the sites of the lattice where $q^{2\alpha S(0)}\mathcal{O}$ acts non-trivially. Then the ground state becomes dependent on $\xi_k,\cdots,\xi_l$, and we have

$$\frac{\langle \mathrm{vac} \mid q^{2\alpha S(0)}\mathcal{O}\ |\mathrm{vac}\rangle}{\langle \mathrm{vac} \mid q^{2\alpha S(0)}|\mathrm{vac}\rangle} = \mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)\left(\left(q^{2\alpha S(0)}\mathcal{O}\right)_{[k,l]}\right)\ .$$

Here $\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)$ is a linear functional on $(\mathbb{C}^2)^{\otimes(l-k+1)}$ subject to several requirements [7]:

$$\begin{aligned}
&\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_{j+1},\xi_j,\cdots,\xi_l,\alpha)(X_{[k,l]})\\
&\qquad=\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_j,\xi_{j+1},\cdots,\xi_l,\alpha)\left(\check{\mathbb{R}}_{j,j+1}(\xi_j/\xi_{j+1})^{-1}(X_{[k,l]})\right),\\
&\mathbf{h}_{[k,l]}(\xi_k q^{-1},\cdots,\xi_l,\alpha)(X_{[k,l]})=\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)\left(\mathbf{A}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)(X_{[k,l]})\right),\\
&\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)\left(X_{[k,l-1]}\cdot 1_l\right)=\mathbf{h}_{[k,l-1]}(\xi_k,\cdots,\xi_{l-1},\alpha)\left(X_{[k,l-1]}\right),\\
&\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)\left(q^{\alpha\sigma_k^3}\cdot X_{[k+1,l]}\right)=\mathbf{h}_{[k+1,l]}(\xi_k,\cdots,\xi_l,\alpha)\left(X_{[k+1,l]}\right)\ ,
\end{aligned}$$

where

$$\begin{aligned}
&\check{\mathbb{R}}_{j,j+1}(\xi_j/\xi_{j+1})(X)=\check{R}_{j,j+1}(\xi_j/\xi_{j+1})X\left(\check{R}_{j,j+1}(\xi_j/\xi_{j+1})\right)^{-1},\\
&\mathbf{A}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)(X)=\left((T^{-1})^{t_k}\cdot\sigma_k^2\cdot Xq^{-\alpha\sigma_k^3}\cdot\sigma_k^2\right)^{t_k}\cdot T\,,\\
&T=R_{k,l}(\xi_k/\xi_l)\cdots R_{k,k+1}(\xi_k/\xi_{k+1})\,.
\end{aligned}$$

The way for solving these equations is absolutely parallel to the one described in [5]. The answer takes the form

$$\mathbf{h}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)\left(X_{[k,l]}\right)=\mathbf{tr}^{\alpha}\left(e^{\mathbf{\Omega}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)}\left(X_{[k,l]}\right)\right)\ . \tag{A.1}$$

In order to describe $\mathbf{\Omega}_{[k,l]}$, let $\mathbb{T}_{[k,l]}(\zeta)\in U_q(\mathfrak{sl}_2)\otimes\mathrm{End}(M_{[k,l]})$ denote the monodromy matrix (2.8) constructed via the L-operator (2.1). (Normally we do not write the dependence on $\xi_k,\cdots,\xi_l$ explicitly.)

Introduce operators $\mathbf{X}^i_{[k,l]}(\zeta_1,\zeta_2,\alpha)$ $(i=0,1)$ depending rationally on ζ_1,ζ_2 by setting

$$\begin{aligned}
&\mathrm{tr}_{a,b}\left(B^0_{a,b}(\zeta_1/\zeta_2)\mathbb{T}_{b,[k,l]}(\zeta_2)^{-1}\mathbb{T}_{a,[k,l]}(\zeta_1)^{-1}\right)\mathrm{Tr}_{d(\zeta_1/\zeta_2)}\left(\mathbb{T}_{[k,l]}\left(\sqrt{\zeta_1\zeta_2}\right)^{-1}q^{-(\alpha+1)H}\right)\\
&=\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha}\mathbf{X}^0_{[k,l]}(\zeta_1,\zeta_2,\alpha)+\left(\frac{\zeta_2}{\zeta_1}\right)^{\alpha}\mathbf{X}^0_{[k,l]}(\zeta_2,\zeta_1,\alpha),\\
&\mathrm{tr}_{a,b}\left(B^1_{a,b}(\zeta_1/\zeta_2)\mathbb{T}_{b,[k,l]}(\zeta_2)^{-1}\mathbb{T}_{a,[k,l]}(\zeta_1)^{-1}\right)\mathrm{Tr}_{d(\zeta_1/\zeta_2)}\left(\mathbb{T}_{[k,l]}\left(\sqrt{\zeta_1\zeta_2}\right)^{-1}q^{-(\alpha-1)H}\right)\\
&=\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha}\mathbf{X}^1_{[k,l]}(\zeta_1,\zeta_2,\alpha)+\left(\frac{\zeta_2}{\zeta_1}\right)^{\alpha}\mathbf{X}^1_{[k,l]}(\zeta_2,\zeta_1,\alpha)\,.
\end{aligned}$$

Here Tr_d stands for the trace functional on $U_q(\mathfrak{sl}_2)$ (analytic continuation of the trace with respect to dimension d) used in [2],

$$d(\zeta)=\frac{\log\zeta}{\log q}\,,$$

and $B^0(\zeta)$, $B^1(\zeta)$ are 4×4 matrices given by

$$B^0(\zeta)=\frac{(\zeta-\zeta^{-1})}{(\zeta q-\zeta^{-1}q^{-1})(\zeta q^{-1}-\zeta q)}\begin{pmatrix}0&0&0&0\\0&q&-\zeta^{-1}&0\\0&-\zeta&q^{-1}&0\\0&0&0&0\end{pmatrix}$$

and

$$B^1(\zeta)=\left(\sigma^1\otimes\sigma^1\right)\cdot B^0(\zeta)\cdot\left(\sigma^1\otimes\sigma^1\right)\,.$$

The notation being as above, the formula for $\mathbf{\Omega}_{[k,l]}$ is given as follows.

$$\mathbf{\Omega}_{[k,l]}(\xi_k,\cdots,\xi_l,\alpha)=-\frac{1}{2\pi i}\iint \mathbf{X}_{[k,l]}(\zeta_1,\zeta_2,\alpha)\omega(\zeta_2/\zeta_1,\alpha)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2} \tag{A.2}$$

where integrals are taken around $\xi_k,\cdots,\xi_l$. The operator $\mathbf{X}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$ is presented in either of the following two equivalent forms:

$$\begin{aligned}\mathbf{X}_{[k,l]}(\zeta_1,\zeta_2,\alpha)&=\mathrm{sing}_{\zeta_1,\zeta_2=\xi_k,\cdots,\xi_l}\left[\mathbf{X}^0_{[k,l]}(\zeta_1,\zeta_2,\alpha)\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha-\mathbf{S}}\right]\\&=\mathrm{sing}_{\zeta_1,\zeta_2=\xi_k,\cdots,\xi_l}\left[\mathbf{X}^1_{[k,l]}(\zeta_1,\zeta_2,\alpha)\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha-\mathbf{S}}\right]\,.\end{aligned}$$

In the formulas (A.1), (A.2) only operators of spin 0 are considered. Here we have introduced $\mathbf{S}$ for later convenience.

Now we give the only part of the construction which has no analogues in [5]. Let us consider the homogeneous case. In particular, we replace $\frac{1}{(2\pi i)^2}\int\int$ by $\mathrm{res}_{\zeta_1=1}\mathrm{res}_{\zeta_2=1}$ and drop the index $[k,l]$. From [12] we learn:

$$\begin{aligned}&\mathrm{Tr}_{d(\zeta_1/\zeta_2)}\left(\mathbb{T}\left(\sqrt{\zeta_1\zeta_2}\right)^{-1}q^{-\alpha H}\right)=\\&=\left(\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha}\mathbf{Q}^+(\zeta_1,\alpha)\mathbf{Q}^-(\zeta_2,\alpha)-\left(\frac{\zeta_2}{\zeta_1}\right)^{\alpha}\mathbf{Q}^-(\zeta_1,\alpha)\mathbf{Q}^+(\zeta_2,\alpha)\right)\frac{1}{q^{\alpha-\mathbf{S}}-q^{-\alpha+\mathbf{S}}}\,.\end{aligned}$$

This implies

$$\mathbf{X}(\zeta_1,\zeta_2,\alpha)$$

$$\begin{aligned}&=\mathrm{sing}_{\zeta_1=1}\,\mathrm{sing}_{\zeta_2=1}\left[\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha+1-\mathbf{S}}\mathrm{tr}_{a,b}\left(B^0_{b,a}(\zeta_2/\zeta_1)\mathbb{T}_a(\zeta_1)^{-1}\mathbb{T}_b(\zeta_2)^{-1}\right)\right.\\&\qquad\left.\times\,\mathbf{Q}^+(\zeta_1,\alpha+1)\mathbf{Q}^-(\zeta_2,\alpha+1)\right]\frac{-1}{q^{\alpha+1-\mathbf{S}}-q^{-\alpha-1+\mathbf{S}}}\end{aligned} \tag{A.3}$$

$$\begin{aligned}&=\mathrm{sing}_{\zeta_1=1}\,\mathrm{sing}_{\zeta_2=1}\left[\left(\frac{\zeta_1}{\zeta_2}\right)^{\alpha-1-\mathbf{S}}\mathrm{tr}_{a,b}\left(B^1_{a,b}(\zeta_1/\zeta_2)\mathbb{T}_b(\zeta_2)^{-1}\mathbb{T}_a(\zeta_1)^{-1}\right)\right.\\&\qquad\left.\times\,\mathbf{Q}^-(\zeta_2,\alpha-1)\mathbf{Q}^+(\zeta_1,\alpha-1)\right]\frac{1}{q^{\alpha-1-\mathbf{S}}-q^{-\alpha+1+\mathbf{S}}}\,.\end{aligned} \tag{A.4}$$

These two formulae will be used in Section 2 to derive a new expression for $\mathbf{\Omega}$ (see Lemma 2.4).

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