SPhT-T05/135

Boundary qKZ equation and generalized Razumov-Stroganov sum rules for open IRF models

P. Di Francesco

Service de Physique Théorique de Saclay,
CEA/DSM/SPhT, URA 2306 du CNRS,
F-91191 Gif sur Yvette Cedex, France

We find higher rank generalizations of the Razumov–Stroganov sum rules at $q = -e^{\frac{i\pi}{k+1}}$ for A_{k-1} models with open boundaries, by constructing polynomial solutions of level one boundary quantum Knizhnik–Zamolodchikov equations for $U_q(\mathfrak{sl}(k))$. The result takes the form of a character of the symplectic group, that leads to a generalization of the number of vertically symmetric alternating sign matrices. We also investigate the other combinatorial point $q = -1$, presumably related to the geometry of nilpotent matrix varieties.

AMS Subject Classification (2000): Primary 05A19; Secondary 82B20

09/2005

1. Introduction

Among the many beautiful connections between statistical physics and combinatorics, those involving integrable lattice models are probably the deepest ones. A major advance on the enumeration of various symmetry classes of alternating sign matrices (ASM) was accomplished by Kuperberg [1] by relating these notoriously difficult combinatorial problems to partition functions of the six-vertex model on a square grid with various boundary conditions. Alternatively, Okada [2] reinterpreted these partition functions as characters of various classical Lie groups.

In this note, we address a related combinatorial mystery, the Razumov–Stroganov conjecture [3], that identifies the suitably normalized groundstate vector entries of the O(1) loop model on a semi-infinite cylinder of perimeter $2n$ to numbers of configurations of the fully-packed loop model on an $n \times n$ square grid. A weaker version of this conjecture, the Razumov-Stroganov sum rule [4] identifies the sum of these entries with the number A_n of ASMs of size $n \times n$, and has been proved in [5], in an inhomogeneous version (incorporating position-dependent spectral parameters z_i) making extensive use of the integrability of the loop model. It allows for the identification between the sum of entries of the fully inhomogeneous O(1) loop model and the very same partition function as that used by Kuperberg for the enumeration of ASMs, the so-called Izergin–Korepin (IK) determinant [6]. This proof was adapted to the case of the inhomogeneous O(1) crossing loop model as well [7], and allowed to identify the entries of the groundstate vector with the multidegrees of some matrix varieties [8]. The case of open boundaries for the loop models was also treated in [9] both for non-crossing and crossing loop models respectively, giving rise to other sum rules. In particular, this established the open Razumov–Stroganov sum rule, identifying the sum of suitably normalized groundstate vector entries of the O(1) loop model on a semi-infinite strip of width $2n$ with the number $A_V(2n+1)$ of vertically symmetric alternating sign matrices (VSASM) of size $(2n+1) \times (2n+1)$.

In the context of q-deformations of quantum hall effect wave functions [10], Pasquier recovered the Razumov–Stroganov sum rule (for $q = -e^{\frac{i\pi}{3}}$) by finding the minimal polynomial solution of the quantum Knizhnik–Zamolodchikov (qKZ) equation [11] for $U_q(\mathfrak{sl}(2))$ at level 1. In [12], solving qKZ equations for higher rank algebras $U_q(\mathfrak{sl}(k))$ at level 1 led to a nice generalization of the Razumov–Stroganov sum rule for $q = -e^{\frac{i\pi}{k+1}}$, identifying a weighted sum of entries of the groundstate vector of the inhomogeneous A_{k-1} interaction-round-a-face (IRF) model on a semi-infinite cylinder of perimeter kn with particular Schur functions generalizing the IK determinant and Okada's result. In particular, in the homogeneous limit where all $z_i \to 1$, the sum rule was found to be proportional to the number:

$$A_n^{(k)} = \prod_{j=0}^{n-1} \frac{j! \prod_{i=1}^{k-1} \big((k+1)j+i\big)!}{\prod_{i=0}^{k-1}(kj+i)!} \tag{1.1}$$

reducing to the number of ASMs for $k=2$.

In the present paper, we extend the results of [12] to the case of A_{k-1} IRF models with open boundaries. To this purpose, we introduce in Sect.2 a boundary version of the level 1 qKZ equation which incorporates a parameter r, playing the role of a boundary magnetic field, and such that the ordinary qKZ equation is recovered when $r \to 0$. This extends the boundary qKZ equation of [13] and [14], defined respectively for $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{sl}(k>2))$, both at level 0. We then use the path representation of [12] for the quotients of the Hecke algebra associated to the A_{k-1} models to find minimal polynomial solutions to these boundary qKZ equations in Sect.3. At the generalized Razumov–Stroganov point $q = -e^{\frac{i\pi}{k+1}}$, we compute a weighted sum of entries of this polynomial solution, identified with the groundstate entries of the corresponding inhomogeneous A_{k-1} IRF model on a semi-infinite strip of width kn, with open boundaries (Sect.4.1). This sum rule produces a simple generalization of the Schur functions of [12], in a form very similar to the character formulas of Okada for symplectic groups. In the homogeneous limit where all $z_i \to 1$, all these polynomials collapse to polynomials of r with integer coefficients, and lead at $r=1$ to a natural generalization of the number of VSASMs:

$$A_V^{(k)}(n) = \prod_{j=1}^{n-1} \frac{(2j)!(2nk+2j-1)!}{((k+1)j)!((k+1)(j+n)-1)!} \tag{1.2}$$

which reduces to $A_V(2n+1)$ for $k=2$.

We also investigate the other "combinatorial point" $q=-1$, corresponding to the rational limit of the underlying integrable model (Sect.4.2). There, in the homogeneous limit, we find that all entries of our solution tend to integers, and discuss their possible interpretation as degrees of nilpotent matrix varieties.

2. Quantum Knizhnik-Zamolodchikov equations without and with a boundary

2.1. R matrix and path representation

Throughout this note, we shall use the standard abstract trigonometric solution of the Yang-Baxter equation which reads

$$\check{R}_{i,i+1}(z,w) = \frac{q^{-1}z - qw}{q^{-1}w - qz} I + \frac{z-w}{q^{-1}w - qz} e_i \tag{2.1}$$

where the e_i, $i = 1, 2, ..., N-1$ are the generators of the Hecke algebra $H_N(\tau)$, namely subject to the relations

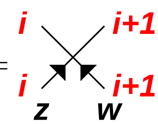
$$e_i e_j = e_j e_i, \ |i-j| > 1, \qquad e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}, \qquad e_i^2 = \tau e_i \tag{2.2}$$

with the parametrization

$$\tau = -(q + q^{-1}) \tag{2.3}$$

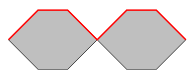
A standard pictorial representation for $\check{R}$ reads

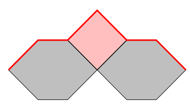
$$\check{R}_{i,i+1}(z, w) = \tag{2.4}$$

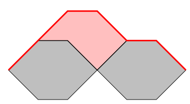


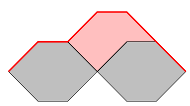
where we have also indicated the action on the spaces i and $i+1$.

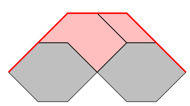
More precisely, we let $\check{R}$ act on the so-called path representation of the A_{k-1} IRF models. To this purpose, the e_i are further constrained to generate the $U_q(\mathfrak{sl}(k))$ (also called A_{k-1} or $SU(k)$) quotient $H_N^{(k)}(\tau)$ of $H_N(\tau)$ obtained by imposing the extra vanishing conditions $Y_k(e_i, e_{i+1}, \ldots, e_{i+k-1}) = 0$, $i = 1, 2, ..., N-k$, where the Young q-symmetrizers Y_k are defined inductively by $Y_1(e_i) = e_i$ and $Y_{m+1}(e_i, \ldots, e_{i+m}) = Y_m(e_i, \ldots, e_{i+m-1})(e_{i+m} - \mu_m) Y_m(e_i, \ldots, e_{i+m-1})$, where $\mu_m = U_{m-1}(\tau)/U_m(\tau)$, U the Chebyshev polynomials subject to $U_m(2\cos x) = \sin(m+1)x/\sin x$. Note that $k = 2$ corresponds to the Temperley-Lieb algebra $TL_N(\tau)$. The path representation is only briefly sketched here and will be described in detail in [15]. The representation basis is indexed by closed paths of length $N = nk$, from and to the origin, on the oriented-link Weyl chamber of $SU(k)$, generated by the k vectors $u_1 = \omega_1$, $u_2 = \omega_2 - \omega_1$, ..., $u_{k-1} = \omega_{k-1} - \omega_{k-2}$, and $u_k = -\omega_{k-1}$, in terms of the fundamental weights ω_i. The paths are allowed to visit only points $\lambda = \sum \lambda_i \omega_i$ with all $\lambda_i \geq 0$, and may be indexed by their sequence of steps, and by a slight abuse of notation we write $\pi = (i_1 i_2 \ldots i_N)$ for the sequence $u_{i_1}, u_{i_2}, \ldots, u_{i_N}$; we also note $\pi_m = i_m$, the m-th step of the path π. For instance, the path of length nk closest to the origin is $(12\ldots k)^n$, namely n repetitions of the sequence $1, 2, \ldots, k$, and we denote it by π_f. Likewise, the path farthest from the origin is $(1)^n(2)^n \ldots (k)^n$, and we denote it by π_0. A useful notation consists in representing each step j by a unit segment forming an angle of $\frac{\pi(k+2-2j)}{2(k+1)}$ with the horizontal direction: each π becomes a broken line touching the x axis at its ends and staying above it. There are exactly $c_{nk}^{(k)} = (kn)! \prod_{0 \leq j \leq k-1} j!/(n+j)!$ such paths. The number $c_{nk}^{(k)}$ is the dimension of the path representation and generalizes the Catalan numbers $c_n = c_{2n}^{(2)}$. The next step consists in decomposing each path in the broken line representation into "tiles" made of (possibly glued) lozenges $L_{i,j}$ with edges corresponding to pairs $1 \leq i < j \leq k$ of steps. To each tile we then associate a specific polynomial of the e's. For instance, each single lozenge with steps at positions $i, i+1$ along the paths corresponds to e_i. The tile decomposition of each path is then unique up to elementary moves between tiles allowed by the algebra relations, and we may therefore associate a unique element of $H_N^{(k)}$ to each path π, thus forming a representation basis $|\pi\rangle$. Actually the map is onto the left ideal $H_N^{(k)}\Omega$, $\Omega = Y_{k-1}(e_1, ..., e_{k-1})Y_{k-1}(e_{k+1}, ..., e_{2k-1})...Y_{k-1}(e_{(n-1)k+1}, ..., e_{nk-1})$, isomorphic to $H_N^{(k)}$. For illustration, let us detail the case $k = 3$, $n = 2$. There are five paths, respectively decomposed as follows: defining $Y_{i,i+1} \equiv Y_2(e_i, e_{i+1}) = e_i e_{i+1} e_i - e_i$, $Z_{i,j} = e_i e_j - 1$, and $\Omega = Y_{1,2}Y_{4,5}$, we have

$|\pi_f\rangle =$  $= \Omega$

$|\pi_3\rangle =$  $= e_3\Omega$

$|\pi_2\rangle =$  $= Z_{2,3}\Omega$ (2.5)

$|\pi_1\rangle =$  $= Z_{4,3}\Omega$

$|\pi_0\rangle =$  $= e_4 Z_{2,3}\Omega$

and the e_i read in this basis:

$$e_1 = \begin{pmatrix} \tau & 0 & 0 & 0 & 1 \\ 0 & \tau & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} \tau & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \tau \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \tau & 0 & 0 & 0 \\ 0 & 0 & \tau & 0 & 1 \\ 0 & 0 & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$e_4 = \begin{pmatrix} \tau & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \tau & 0 \\ 0 & 0 & 1 & 0 & \tau \end{pmatrix} \quad e_5 = \begin{pmatrix} \tau & 0 & 0 & 0 & 1 \\ 0 & \tau & 0 & 1 & 0 \\ 0 & 0 & \tau & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6}$$

Let us now simply list a few basic properties of this representation, of crucial importance for the following:

(P1) $e_i\,|\pi\rangle = \tau\,|\pi\rangle$ if $\pi_i < \pi_{i+1}$ (π locally convex)

(P2) $e_i\,|\pi\rangle = \sum_{\pi'} C_{i,\pi,\pi'}\,|\pi'\rangle$ if $\pi_i \geq \pi_{i+1}$ (π locally flat or concave), for some $C_{i,\pi,\pi'} \in \{0,1\}$

(P3) If $C_{i,\pi,\pi'} = 1$ then π' is locally convex between steps i and $i+1$, namely $\pi'_i < \pi'_{i+1}$

(P4) If $C_{i,\pi,\pi'} = 1$ then either $(\pi_i, \pi_{i+1}) = (\pi'_{i+1}, \pi'_i)$ and $\pi_m = \pi'_m$ for all $m \neq i, i+1$, i.e. π' exceeds π by the unit lozenge L_{π_{i+1},π_i} in the broken line representation, or $\pi' \subset \pi$, namely the broken line representation of π' lies below that of π (or equivalently π' may be completed into π upon adding a number of lozenges).

We conclude with another important property of this representation. There is a duality between the $SU(k)$ and $SU(n)$ representations for $N = kn$, confirmed by the identity of dimensions $c^{(k)}_{nk} = c^{(n)}_{kn}$. Indeed, there exists an involution φ from A_{k-1} to A_{n-1} paths of same length kn, defined as follows. First of all, φ takes π_0 of A_{k-1} to π_f of A_{n-1}. The bijection is then constructed iteratively by simultaneously subtracting (resp. adding) lozenges at the same positions from the A_{k-1} (resp. to the A_{n-1}) paths, so that π is locally convex iff $\varphi(\pi)$ is locally concave. A simpler way of picturing this involution uses yet another expression of A_{k-1} paths, as rectangular standard Young tableaux with k row of n boxes, in which successive steps along the path are recorded by their positions in the corresponding row of the tableau, i.e. if the m-th step is j, then we write the number m in the first available box from the left in the j-th row. In this language, φ is simply the transposition of the tableaux. Moreover, this duality extends to the path representations, namely

$$C^{(n)}_{i,\varphi(\pi'),\varphi(\pi)} = C^{(k)}_{i,\pi,\pi'} \tag{2.7}$$

In practice, this relation is extremely useful: it allows for instance to express the representation (2.5)-(2.6) of $H^{(3)}_6(\tau)$ entirely in terms of the standard Dyck path representation of $TL_6(\tau)$.

Once we pick this path representation, the R-matrix (2.1) may be interpreted as encoding face weights of some IRF model based on $SU(k)$, with edge degrees of freedom in the set of steps, and acting locally at steps $i, i+1$ of the paths. This is the natural generalization of the $O(n)$ loop model in the Dyck path representation, corresponding to $k = 2$ and $n = \tau$.

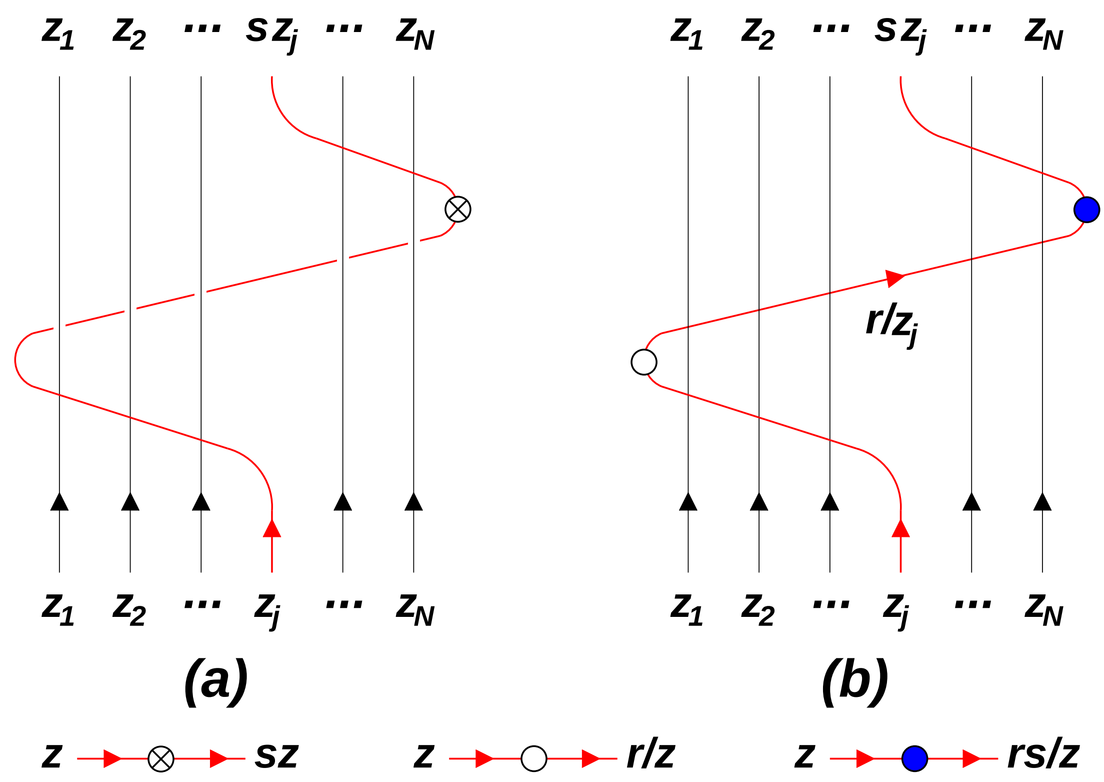


Fig. 1: The regular- (a) and boundary- (b) qKZ equations. Each oriented line carries a spectral parameter. Each intersection between two such lines corresponds to a $\check{R}$ operator. On the second line, we have represented the diagonal operators of respectively multiplication of the spectral parameter by s (shift), inversion of the spectral parameter with weight r ($K^{(r)}(z)$) and weight rs ($K^{(rs)}(z)$), repectively pictured as a crossed dot, empty dot, filled dot. The former is nothing but the composition of the two latter.

2.2. qKZ equations

The qKZ equation [11] reads

$$\begin{aligned}\Psi(z_1, ..., z_{j-1}, sz_j, z_{j+1}, ..., z_N) &= Q\Psi(z_1, ..., z_{j-1}, z_j, z_{j+1}, ..., z_N) \\ Q &= \check{R}_{j+1,j}(z_{j+1}, sz_j) \dots \check{R}_{N,j}(z_N, sz_j) S_j(z)\sigma \check{R}_{1,j}(z_1, z_j) \dots \check{R}_{j-1,j}(z_{j-1}, z_j)\end{aligned} \tag{2.8}$$

with the clear pictorial representation of Fig.1-(a). Here the operator σ rotates cyclically the spectral parameters and the "shift" operator $S_j(z)$ acts as the identity on the space j while it multiplies the spectral parameter z by some scalar s (it is represented by a crossed dot in Fig.1-(a)). In the pictorial representation of Fig.1-(a), each intersection between two oriented lines stands for an $\check{R}$ operator (2.4). Note that the "return" z_j line (from left to right) passes *below* the other (vertical) lines, hence there are no intersections, i.e. no $\check{R}$ matrices involved: the only effect is the operator σ, which rotates the spaces and their spectral parameters according to $(z_j, z_1, ..., z_{j-1}, z_{j+1}, ..., z_N) \to (z_1, ..., z_{j-1}, z_{j+1}, ..., z_N, z_j)$.

In the presence of boundaries, the qKZ equation is modified as depicted in Fig.1-(b). The "return line" now crosses the vertical ones, and the shift of spectral parameter is now replaced by two boundary operators at the far left and right of the return line. The corresponding qKZ equation now reads

$$\begin{aligned}
&\Psi(z_1,...,z_{j-1},sz_j,z_{j+1},...,z_N)=\bar{Q}\Psi(z_1,...,z_{j-1},z_j,z_{j+1},...,z_N)\\
&\bar{Q}=\check{R}_{j+1,j}(z_{j+1},sz_j)\ldots\check{R}_{N,j}(z_N,sz_j)K_j^{(rs)}(\frac{r}{z_j})\check{R}_{j,N}(\frac{r}{z_j},z_N)\ldots\check{R}_{j,j+1}(\frac{r}{z_j},z_{j+1})\\
&\quad\times\check{R}_{j,j-1}(\frac{r}{z_j},z_{j-1})\ldots\check{R}_{j,1}(\frac{r}{z_j},z_1)K_j^{(r)}(z_j)\check{R}_{1,j}(z_1,z_j)\ldots\check{R}_{j-1,j}(z_{j-1},z_j)
\end{aligned}\tag{2.9}$$

The boundary operator $K_j^{(r)}(z)$ acts as the identity on the space j, and changes the spectral parameter z into r/z. Together with $\check{R}$, it satisfies the boundary Yang-Baxter equation of Ref.[16]. In Fig.1b, we have represented respectively $K^{(r)}$ and $K^{(rs)}$ by an empty and a filled dot. Here r is an additional free parameter, which allows to recover in the limit $r\to 0$ the qKZ equation without boundary (2.8) from that with boundaries (2.9). Indeed, in this limit, the spectral parameter along the return line tends to 0: consequently, all $\check{R}$ matrices along the return line degenerate into pure undercrossings, and we recover the picture of Fig.1-(a), upon noting that $S(z)=K^{(rs)}(r/z)K^{(r)}(z)$ is nothing but the shift operator that multiplies the spectral parameter z by s.

The equation (2.8) is equivalently standardly reduced to a system of equations for Ψ, namely

$$\begin{aligned}
\tau_i\Psi(z_1,\ldots,z_N)&=\check{R}_{i,i+1}(z_{i+1},z_i)\Psi(z_1,\ldots,z_N),\quad i=1,2,\ldots,N-1\\
\sigma\Psi(z_2,...,z_N,sz_1)&=c\Psi(z_1,...,z_N)
\end{aligned}\tag{2.10}$$

where τ_i acts on functions of the z's by interchanging z_i and z_{i+1}, c is an irrelevant constant, and σ now acts as a rotation of states in the representation picked for the $\check{R}$'s. Similarly, we may reduce (2.9) to the following system:

$$\tau_i\Psi(z_1,\ldots,z_N)=\check{R}_{i,i+1}(z_{i+1},z_i)\Psi(z_1,\ldots,z_N),\ i=1,2,\ldots,N-1\tag{2.11a}$$

$$\Psi(\frac{r}{z_1},z_2,...,z_{N-1},z_N)=c_1(z_1)\Psi(z_1,...,z_N)\tag{2.11b}$$

$$\Psi(z_1,z_2,...,z_{N-1},\frac{rs}{z_N})=c_N(z_N)\Psi(z_1,...,z_N)\tag{2.11c}$$

for some functions of a single spectral parameter c_1 and c_N. Note that Eqs.(2.11b,c) express the covariance of Ψ under the action of $K_1^{(r)}(z_1)$ and $K_N^{(rs)}(z_N)$. The latter are obtained by showing that the operator $\bar{Q}$ in (2.9) is intertwined by $K_1^{(r)}$ and $K_N^{(rs)}$, as a direct consequence of the boundary Yang-Baxter equation. The boundary qKZ equation (2.11) has the advantage of being ridden of the cyclicity condition (second line of (2.10)), replaced by reflective boundary conditions, somewhat easier to deal with. We now restrict ourselves to vectors Ψ in the path representation above, namely look for $\Psi = \sum_\pi \Psi_\pi |\pi\rangle$, π running over the set of A_{k-1} paths of length $N = kn$. When written in components, Eqs.(2.11) become

$$(q^{-1}z_{i+1} - qz_i)\partial_i \Psi_\pi(z_1, \dots, z_N) = \sum_{\pi'} C_{i,\pi',\pi} \Psi_{\pi'}(z_1, \dots, z_N), \quad 1 \le i \le N-1 \tag{2.12a}$$

$$\Psi_\pi(\frac{r}{z_1}, z_2, ..., z_{N-1}, z_N) = c_1(z_1)\Psi_\pi(z_1, ..., z_N) \tag{2.12b}$$

$$\Psi_\pi(z_1, z_2, ..., z_{N-1}, \frac{rs}{z_N}) = c_N(z_N)\Psi_\pi(z_1, ..., z_N) \tag{2.12c}$$

where the divided difference operator ∂_i acts on functions of the z's as

$$\partial_i f = \frac{\tau_i f - f}{z_i - z_{i+1}} \tag{2.13}$$

We also denote by $t_i = (q^{-1}z_{i+1} - qz_i)\partial_i$. The equations (2.12$a$) express nothing but $t_i\Psi = (e_i - \tau)\Psi$.

By analogy with the ordinary qKZ case, when dealing with the A_{k-1} path representation above, we write

$$s = q^{2(k+l)} \tag{2.14}$$

and call l the level of the (boundary) qKZ equation (2.11). The parameter r is free, and has the dimension of the square of a spectral parameter. In the next section, we construct polynomial solutions of the system (2.11) for level $l = 1$.

3. Polynomial solutions for the level 1 boundary qKZ equation

We now look for polynomial solutions of level 1 boundary qKZ equations (2.11) with minimal degree.

3.1. Solutions

We have the remarkable property for all k, that the equations $(2.12a)$ may all be solved for Ψ_π in terms solely of Ψ_{π_0}. This relies on the property (**P4**) above of the path representation. Indeed, we may order the paths by inclusion. Then each path π locally convex at $i, i+1$ may be viewed as the addition of a lozenge $L_{\pi_i,\pi_{i+1}}$ to a smaller path π^-, such that $\pi^-_m = \pi_m$ for $m \neq i, i+1$ and $(\pi^-_i, \pi^-_{i+1}) = (\pi_{i+1}, \pi_i)$. Using property (**P4**), we see that $e_i\pi^- = \pi + \sum \pi''$, where the sum runs over paths $\pi'' \subset \pi^-$. Turning now to Eq.$(2.12a)$, we see that, as all the π' involved in the sum on the r.h.s. are such that $C_{i,\pi',\pi} = 1$, only one of them is strictly included in π, namely π^-, while all others contain π. Stripping iteratively lozenges from paths, we may therefore express each Ψ_{π^-} as $t_i\Psi_\pi - \sum_{\pi''}\Psi_{\pi''}$, with already known π'' containing π. This allows for expressing iteratively every Ψ_π in terms of Ψ_{π_0}, in a triangular manner w.r.t. inclusion of paths.

The "fundamental" component Ψ_{π_0} is fixed by (i) some highest weight conditions $(t_i + \tau)\Psi_{\pi_0} = 0$ for all i not multiple of n (expressing that π_0 is locally flat except at the convex points multiple of n, see below), (ii) the reflective boundary conditions $(2.12b, c)$, and finally the requirement that it be of smallest possible degree.

As a direct consequence of Eq.$(2.12a)$, Ψ_π vanishes whenever $z_{i+1} = q^2 z_i$ and π is concave or flat at $i, i+1$, namely $\pi_i \geq \pi_{i+1}$. Indeed, from property (**P3**), the vector $|\pi\rangle$ cannot be the image of anything under e_i; but precisely when $z_{i+1} = q^2 z_i$, the matrix $\check{R}$ becomes proportional to e_i (c.f. the definition (2.1)), hence Ψ_π must vanish. This is straightforwardly extended to any portion of π flat or concave say between positions i and j, upon considering the appropriate product or $\check{R}$ matrices: if $z_l = q^2 z_k$ for some $i \leq k < l \leq j$, then Ψ_π vanishes. Applying this to π_0 which is flat everywhere except at points multiple of n where it is convex, we find that Ψ_{π_0} must factor out the terms $\prod_{(m-1)n+1\leq i<j\leq mn}(q^2 z_i - z_j)$ for all $m = 1, 2, ..., k$. Alternatively, wherever π_0 is locally flat, i.e. for all i not multiple of n, we have $(t_i + \tau)\Psi_{\pi_0} = 0$, as $|\pi_0\rangle$ has no antecedent under e_i: this "highest weight condition" is analogous to that of [12]. It implies that once the above factors are taken out of Ψ, the remaining is a polynomial symmetric in the variables $\{z_{(m-1)n+1}, z_{(m-1)n+2}, ..., z_{mn}\}$ separately for all $m = 1, 2, ..., k$. Finally, by an inductive argument similar to that used in Ref.[9], we can show that Ψ_{π_0} factors out terms $(q^{2m}r - z_k z_l)$ for all pairs k, l such that $(m-1)n+1 \leq k \leq l \leq mn$, $m = 1, 2, ..., k$. Indeed, the left boundary condition $(2.12b)$ allows to show that $\prod_{2\leq j\leq n}(q^2 r - z_1 z_j)$ divides Ψ_{π_0}. Next, we let $z_1 \to \infty$, namely pick in Ψ the coefficient of top degree in z_1: in this case the

space 1 decouples from the picture of Fig.1(a), as its crossings with the j line are undone and replaced by undercrossings, and we are left with the same boundary problem, but with line 1 erased, and now a reflective boundary condition with parameter r in terms of z_2. Repeating iteratively this process, namely taking successively $z_2, z_3, ..., z_{n-1} \to \infty$ exhausts all the announced extra factors for $m = 1$. Assuming each variable occurs in Ψ with maximal degree $2(n-1)$, we find that the leftover dependence on z_n must be proportional to z_n^{n-1}, and we may write $\Psi_{\pi_0}(\infty, ..., \infty, z_n, z_{n+1}, ..., z_N) = C z_n^{n-1} F(z_{n+1}, ..., z_N)$. The full vector $\Psi(\infty, ..., \infty, z_n, z_{n+1}, ..., z_N)$ now obeys a left reflective boundary condition of the form

$$\Psi(\infty, ..., \infty, \frac{r}{z_n}, z_{n+1}, ..., z_N) = c_1(z_n)\Psi(\infty, ..., \infty, z_n, z_{n+1}, ..., z_N) \tag{3.1}$$

We now note that $\pi_0 = e_n \pi^-$, for the unique π^- such that $\pi_i^- = \pi_i$ for $i \neq n, n+1$, and $(\pi_n^-, \pi_{n+1}^-) = (\pi_{n+1}, \pi_n)$. According to (2.12$a$), we have $t_n \Psi_{\pi_0} = \Psi_{\pi^-}$. Expressing that the component Ψ_{π^-} obeys the new left reflection condition (3.1) solely in terms of Ψ_{π_0} yields the following relation for F:

$$\begin{aligned} c_1(z_n)\,\frac{q^{-1}z_{n+1} - q z_n}{z_n - z_{n+1}} &\big((z_{n+1})^{n-1} F(z_n, ...) - (z_n)^{n-1} F(z_{n+1}, ...)\big) \\ &= \frac{q^{-1} z_n z_{n+1} - qr}{qr - z_n z_{n+1}} \Big((z_{n+1})^{n-1} F(\frac{r}{z_n}, ...) - \Big(\frac{r}{z_n}\Big)^{n-1} F(z_{n+1}, ...)\Big) \end{aligned} \tag{3.2}$$

Picking $z_n = q^{-2} z_{n+1}$ finally yields $F(\frac{q^2 r}{z_{n+1}}, ...) = (q^2 r / z_{n+1}^2)^{n-1} F(z_{n+1}, ...)$ and therefore Ψ_{π_0} now has all the announced extra factors for $m = 2$ (everything works now as if the left boundary parameter had been rescaled as $r \to r q^2$). Proceeding iteratively, we finally get all the announced factors for $m = 1, 2, ..., k$. We claim that the solution of minimal degree has no extra factors in its Ψ_{π_0}, namely that

$$\Psi_{\pi_0} = \prod_{m=1}^{k} \prod_{(m-1)n+1 \leq i < j \leq mn} (q^2 z_i - z_j)(r q^{2m} - z_i z_j) \tag{3.3}$$

and that Eqs.(2.12) are satisfied for $s = q^{2(k+1)}$ and

$$c_1(z) = \Big(\frac{r}{z^2}\Big)^{n-1}, \qquad c_N(z) = \Big(\frac{r q^{2(k+1)}}{z^2}\Big)^{n-1} \tag{3.4}$$

This fixes the solution entirely.

As in the case without boundaries of [12], the solution may alternatively be fixed by some global requirement that it vanishes for some specific choices of the spectral parameters. It was found indeed that in the case without boundary Ψ must vanish for any ordered sequence of $k+1$ spectral parameters of the form $\{z, q^2 z, ..., q^{2k} z\}$, the so-called "quantum incompressibility" condition, first noticed by Pasquier [10] in the case $k = 2$, and allowing for interpreting Ψ as q-deformed wave function of fractional quantum Hall effect. In the present case with boundaries, we find that Ψ must vanish for any ordered $k + 1$-tuple of spectral parameters of the form

$$\{z, q^2 z, ..., q^{2(m-1)} z, \frac{rq^2}{z}, \frac{rq^4}{z}, ..., \frac{rq^{2(k-m)}}{z}\} \tag{3.5}$$

for $m = 2, 3, ..., k + 1$. This generalizes the quantum incompressibility condition, in that the wave function must vanish whenever $k+1$ electrons come into contact with one another or now with their *reflections*.

As in the case of [12], we may also derive recursion relations for the components, relating size $N = kn$ to size $N' = k(n - 1)$. This is done by simply taking $z_1 = z, z_2 = q^2 z, ..., z_k = q^{2(k-1)} z$, in which case only paths with $\pi_i = i$ for $i = 1, 2, ..., k$ have non-vanishing components in Ψ. The rest of the path, $\pi' = (\pi_{k+1}, ..., \pi_N)$ is an A_{k-1} path of length $N' = N - k$, and we have

$$\Psi_\pi(z_1, ..., z_N)\Big|_{z_1=q^{-2}z_2=...=q^{-2(k-1)}z_k=z} = \left(\prod_{j=k+1}^{N} (q^{2k} z - z_j)(rq^2 - z z_j)\right) \Psi_{\pi'}(z_{k+1}, ..., z_N) \tag{3.6}$$

where the prefactor is fixed by the degree and vanishing requirements.

3.2. Other sizes

As already touched-upon in the previous section, there is an easy way of extending our polynomial solutions to all sizes N, not necessarily a multiple of k. Starting from a solution at $N = kn$, we may indeed send successively the first j spectral parameters to 0 (or equivalently to ∞), leaving us with only non-vanishing components of Ψ for paths π starting with the (unique) convex sequence $1, 2, 3, ..., j$. This provides us with a solution to the boundary qKZ equation of $z_{j+1}, ..., z_N$, hence in size $N - j$. Note that now in the corresponding representation the paths start at the point $u_1 + u_2 + ... + u_j$ in the Weyl chamber of $SU(k)$ and end at its origin. This construction is valid for all $j = 1, 2, ..., k - 1$ hence allows to exhaust all possible sizes.

4. Sum rules

4.1. Generalizations of the Razumov–Stroganov sum rule at $q = -e^{\frac{i\pi}{k+1}}$

We proceed as in [12]. We introduce a covector v such that

$$e_i v = \tau v, \qquad i = 1, 2, ..., N-1 \tag{4.1}$$

This covector is unique under the condition that all its entries be strictly positive and that $v_{\pi_f} = 1$, and exists only if $q = -e^{\frac{i\pi}{k+1}}$ or its conjugate, namely $\tau = 2\cos\frac{\pi}{k+1}$. The entries of v are directly constructed from the path representation and may be expressed as products of ratios of the Chebyshev polynomials $U_m(\tau)$ with $m < k$, associated to the tiles in the decomposition of the paths, and all of which are positive for the above choice of τ. For illustration, for $k = 3$, $n = 2$, we take $\tau = \sqrt{2}$, and associate the factor $U_1(\tau) = \sqrt{2}$ to every lozenge and the factor $U_2(\tau) = 1$ to every tile made of two lozenges in the path decomposition (2.5), corresponding respectively to the left eigenvalues of v on e_i and on $Z_{i,j}$, resulting in $v = \{1, \sqrt{2}, 1, 1, \sqrt{2}\}$ in the path basis.

The condition (4.1) immediately implies that $v\check{R}_{i,i+1}(z, w) = v$, readily checked from the definition (2.1). Henceforth, the combination $v \cdot \Psi$ is a symmetric polynomial of the z's, as a direct consequence of Eq.(2.11a). Introducing

$$I_{kn}^{(k)}(z_1, z_2, ..., z_{kn}|r) \equiv (-i/q)^{kn(n-1)/2} v \cdot \Psi(z_1, ..., z_{kn}) \tag{4.2}$$

with $i = \sqrt{-1}$, we find that $I_{kn}^{(k)}$ is a polynomial of degree $kn(n-1)/2$ of r, whose coefficients are all symmetric polynomials of the z's, and the top degree and constant coefficients are respectively

$$\begin{aligned} I_{kn}^{(k)}\big|_{r^{kn(n-1)/2}} &= s_Y(z_1, ..., z_{kn}) \\ I_{kn}^{(k)}\big|_{r^0} &= (z_1 z_2 ... z_{kn})^{n-1}\, s_Y(z_1, ..., z_{kn}) \end{aligned} \tag{4.3}$$

where s_Y is the Schur function associated to the Young diagram Y with k rows of $n-1$ boxes, k rows of $n-2$ boxes, ..., k rows of 1 box. The second line of (4.3) is nothing but the generalized Razumov-Stroganov sum rule of [12], recovered in the limit $r \to 0$ as explained above. The polynomial $I_{kn}^{(k)}(z_1, ..., z_{kn}|r)$ has only non-negative integer coefficients in r and the z's. Note also the covariance under $z_i \to r/z_i$ for each i, inherited from the boundary reflections (2.11b, c) and the overall symmetry, namely

$$I_{kn}^{(k)}(z_1, ..., z_{j-1}, \frac{r}{z_j}, z_{j+1}, ..., z_{kn}|r) = \left(\frac{r}{z_j^2}\right)^{n-1} I_{kn}^{(k)}(z_1, ..., z_{j-1}, z_j, z_{j+1}, ..., z_{kn}|r) \tag{4.4}$$

which allows to relate the coefficients of $I_{kn}^{(k)}$ for different powers of r. In particular, taking simultaneously all $z_i \to r/z_i$ relates the top and constant coefficients, and the first line of (4.3) follows from the second. The polynomial $I_{kn}^{(k)}$ (4.2) is entirely determined up to global normalization by the following properties:

(i) it is a symmetric polynomial of the z's quasi-homogeneous under $z_i \to \lambda z_i$ for all i and $r \to \lambda^2 r$, with total degree $3kn(n-1)/2$ and partial degree $2(n-1)$ in each of the z's and $kn(n-1)/2$ in r

(ii) it vanishes for all the choices of ordered k-tuples of spectral parameters (3.5), with $q = -e^{\frac{i\pi}{k+1}}$

and the normalization is fixed by (4.3).

Let us first discuss the Temperley-Lieb case $k = 2$ and n arbitrary, where drastic simplifications occur. At the Razumov–Stroganov point $q = -e^{\frac{i\pi}{3}}$, $\tau = 1$, we have $v_\pi = 1$ for all π, hence $I_{2n}^{(2)}$ is simply the sum of components of Ψ. The quantity $I_{2n}^{(2)}$ reduces to the result of [9] at $r = 1$. For generic r, the determinantal formulas of [9] are straightforwardly generalized to

$$\begin{aligned} I_{2n}^{(2)}(z_1,...,z_{2n}|r) &= \frac{\prod_{i,j=1}^n (z_i^2 + z_i z_{j+n} + z_{j+n}^2)(r^2 + r z_i z_{j+n} + z_i^2 z_{j+n}^2)}{\prod_{1\le i<j\le n}(z_i - z_j)(r - z_i z_j)(z_{i+n} - z_{j+n})(r - z_{i+n} z_{j+n})} \\ &\times \det_{1\le i,j\le n}\left(\frac{1}{z_i^2 + z_i z_{j+n} + z_{j+n}^2}\frac{1}{r^2 + r z_i z_{j+n} + z_i^2 z_{j+n}^2}\right) \end{aligned} \tag{4.5}$$

which is an open version of the Izergin–Korepin determinant [6], and the Pfaffian formula becomes

$$\begin{aligned} I_{2n}^{(2)}(z_1,...,z_{2n}|r)^2 &= \prod_{1\le i<j\le 2n}^{n} \frac{(z_i^2 + z_i z_j + z_j^2)(r^2 + r z_i z_j + z_i^2 z_j^2)}{(z_i - z_j)(r - z_i z_j)} \\ &\times \mathrm{Pf}_{1\le i<j\le 2n}\left(\frac{(z_i - z_j)(r - z_i z_j)}{(z_i^2 + z_i z_j + z_j^2)(r^2 + r z_i z_j + z_i^2 z_j^2)}\right) \end{aligned} \tag{4.6}$$

These are proved for instance by using the symmetry and the recursion relations obeyed by $v \cdot \Psi$ as a consequence of Eq.(3.6), and showing that the r.h.s. is the unique symmetric polynomial obeying the same constraints. In the homogeneous case where all $z_i = 1$, $I_{2n}^{(2)}$ reduces to a reciprocal polynomial of r with non-negative integer coefficients, equal respectively to $3^{n(n-1)/2}$ times the number A_n of alternating sign matrices (ASM) of size $n \times n$ at $r = 0$ (a result proved in [5]), and to $3^{n(n-1)}$ times the number $A_V(2n+1)$ of vertically symmetric alternating sign matrices (VSASM) of size $(2n+1)\times(2n+1)$ at $r=1$ (a result proved in [9]), with respective values

$$A_n=\prod_{j=0}^{n-1}\frac{(3j+1)!j!}{(2j+1)!(2j)!},\qquad \text{and}\qquad A_V(2n+1)=\prod_{j=0}^{n-1}\frac{(6j+4)!(2j+2)!}{(4j+4)!(4j+2)!}\tag{4.7}$$

For general k, we have the following expression for the sum rule $I_{kn}^{(k)}$:

$$I_{nk}^{(k)}(z_1,...,z_{nk}|r)=\tilde{s}_Y(z_1,...,z_{nk}|r)\tag{4.8}$$

where $\tilde{s}_Y$ are "symplectic" Schur functions depending on the extra parameter r, and defined as

$$\tilde{s}_Y(z_1,...,z_{nk}|r)=z_1z_2...z_{kn}\ \frac{\det_{1\le i,j\le kn}\left(z_i^{\ell_{kn+1-j}+j}-(r/z_i)^{\ell_{kn+1-j}+j}\right)}{\det_{1\le i,j\le kn}\left(z_i^j-(r/z_i)^j\right)}\tag{4.9}$$

with Y as in (4.3), and $\ell_i=n-1-[(i-1)/k]$, is the number of boxes in the i-th row, $i=1,2,...,kn$ (we have added k bottom rows of 0 box for convenience). The sum rule (4.8) is proved as usual by checking that the r.h.s. fulfills all requirements of degree and vanishings above, and is therefore fixed by uniqueness. This formula is a nice multi-parameter generalization of that written by Okada [2] for VSASMs, in terms of specialized characters of the symplectic group.

In the particular case $n=2$ and k arbitrary, It is easy to show that (4.8) reduces to

$$I_{2k}^{(k)}(z_1,...,z_{2k}|r)=\sum_{m=0}^{k}r^m s_{Y_m}(z_1,...,z_{2k})\tag{4.10}$$

expressed in terms of ordinary Schur functions for the young diagrams Y_m with $k-m$ rows of 2 boxes and k rows of 1 box. Consequently, in the homogeneous limit where all $z_i\to 1$, we have

$$I_{2k}^{(k)}(1,1,...,1|r)=\sum_{m=0}^{k}r^m\frac{k+1}{2k+1}\binom{2k+1}{m}\binom{2k+1}{k-m}\tag{4.11}$$

and we get

$$A_2^{(k)}=\frac{1}{k+1}I_{2k}^{(k)}(1,1,...,1|0)=c_k\tag{4.12}$$

the k-th Catalan number, while at $r=1$ we obtain

$$A_V^{(k)}(2)=\frac{1}{(k+1)^2}I_{2k}^{(k)}(1,1,...,1|1)=2\frac{(4k+1)!}{(3k+2)!(k+1)!}\tag{4.13}$$

$n\backslash k$	1	2	3	4	5
1	1	1	1	1	1
2	1	3	13	68	399
3	1	26	1938	246675	43475640
4	1	646	3251625	46278146640	1238084585216726
5	1	45885	61003011480	444316978958627636	9080679253196247653297250

Table I: The numbers $A_V^{(k)}(n)$ for $1 \leq n, k \leq 5$.

For arbitrary k, the homogeneous case of all $z_i = 1$ yields again for $I_{kn}^{(k)}$ a reciprocal polynomial of r, with non-negative integer coefficients. The constant coefficient of this polynomial is the result of the homogeneous limit of the generalized Razumov-Stroganov sum rule computed in [12], and reads $(k+1)^{n(n-1)/2}$ times an integer $A_n^{(k)}$ generalizing the number of ASMs $A_n = A_n^{(2)}$, and which may be rewritten as

$$A_n^{(k)} = \prod_{j=0}^{n-1} \frac{j! \prod_{i=1}^{k-1} \big((k+1)j+i\big)!}{\prod_{i=0}^{k-1} (kj+i)!} \tag{4.14}$$

The numbers (4.14) are nothing but the properly normalized dimensions of the GL_{kn} representations Y [17]. At $r = 1$ the homogeneous value of $I_{kn}^{(k)}$ reads $(k+1)^{n(n-1)}$ times a number $A_V^{(k)}(n)$ generalizing the number of $(2n+1) \times (2n+1)$ VSASMs $A_V(2n+1)$. We find that

$$A_V^{(k)}(n) = \prod_{j=1}^{n-1} \frac{(2j)!(2nk+2j-1)!}{((k+1)j)!((k+1)(j+n)-1)!} \tag{4.15}$$

These numbers are the properly normalized dimensions of the SP_{2kn} representations Y [17]. The numbers $A_V^{(k)}(n)$ are easily checked to reduce to $A_V(2n+1)$ for $k = 2$ and all n, and to (4.13) for $n = 2$, and all k. The numbers $A_V^{(k)}(n)$ are displayed in Table I for $n, k = 1, 2, ..., 5$. For illustration, for $k = 3$, $n = 2$ we have $I_6^{(3)}(1, 1, ..., 1|r) = 20+84r+84r^2+20r^3$, from which we read the values of $A_2^{(3)} = 20/4 = 5$ and $A_V^{(3)}(2) = (20+84+84+20)/4^2 = 13$.

As mentioned in Sect.3.2, we have access to analogous sum rules for arbitrary sizes N not necessarily a multiple of k by sending spectral parameters to 0 and reducing the space of path states accordingly. In size $nk - j$, the sum rule simply reads (4.8), with say $z_1 = z_2 = ... = z_j = 0$ and where the first j rows have been deleted from Y. This leads to more integer numbers in the homogeneous limit.

4.2. More combinatorics at $q=-1$

In [12], a connection was established between the "rational limit" $q \to -1, z_i \to 1$ of the $U_q(\mathfrak{sl}(k))$ level 1 minimal polynomial solution of the qKZ equation without boundary and so-called extended Joseph polynomials, related to the geometry of the variety of upper triangular matrices with vanishing k-th power. As a particular corollary of this relation, the entries of Ψ were found to tend to positive integers in the rational homogeneous limit, in which one lets first all $z_i \to 1$, and then $q \to -1$ after dividing out by the appropriate power of $q+1$. These integers were interpreted as the degrees of the components of the abovementioned variety.

In the present case, we may also consider an analogous rational limit. Of course, at $r=0$, we may reproduce the rational limit of [12]. If we want a different result from that of [12], we need to let $r \to 1$ as well. Substituting

$$q = -e^{-\epsilon a/2}, \qquad z_i = e^{-\epsilon w_i}, \qquad r = e^{-\epsilon\rho} \tag{4.16}$$

into our minimal polynomial solution Ψ of the $U_q(\mathfrak{sl}(k))$ level 1 boundary qKZ equation, and taking $\epsilon \to 0$, we obtain a two-parameter (a,ρ) family of polynomials of the w's. At $a=\rho=0$, these are some sorts of reflected versions of the Joseph polynomials [18], which await some algebro-geometric interpretation. A remarkable property is that for $a=1, \rho=0$, and in the homogeneous limit where $w_i \to 0$ for all i, the above polynomials reduce to *non-negative integers*, yet to be interpreted as degrees of components of some matrix variety.

Remarkably, for $k=2$ and n arbitrary, we have found that for a suitable normalization of Ψ these would-be degrees sum to the total degree of the "Brauer scheme" considered in [8], suggesting a possible connection. More precisely, taking $r=1$ and $z_i=1$ for all i in our solution Ψ, we have

$$\lim_{q\to-1} \frac{1}{(q^2-1)^{2n(n-1)}} \sum_\pi \Psi_\pi(1,1,\ldots,1) = \det_{0\le i,j\le n-1} \binom{2i+2j+1}{2i} \tag{4.17}$$

where the numbers on the r.h.s. are the degree of the Brauer scheme $\{M \in M_{2n}(\mathbb{C}), M \bullet M = 0\}$ where $\bullet$ stands for a suitable deformation of the ordinary matrix product [8], also obtained as sum rules in the O(1) crossing loop model on a semi-infinite cylinder [7]. For illustration, in the cases $n=1,2,3$ we find the following limits $\psi_{2n}^{(1)}$ when $q \to -1$ of $\Psi(z_i=1|r=1)/(q^2-1)^{2n(n-1)}$:

$$\psi_2^{(1)} = \{1\}, \quad \psi_4^{(1)} = \{5,2\}, \quad \psi_6^{(1)} = \{149,52,58,40,8\} \tag{4.18}$$

expressed in the standard Dyck path basis of $TL_6(\tau)$, ordered by inclusion. The components of (4.18) sum respectively to $1, 7, 307$, the degrees of the Brauer scheme for matrices of size $2, 4, 6$ respectively. For completeness, the corresponding $r = 0$ solutions of [12] yield the following limits $\psi_{2n}^{(0)}$ when $q \to -1$ of $\Psi(z_i = 1|r = 0)/(q^2 - 1)^{n(n-1)}$:

$$\psi_2^{(0)} = \{1\}, \quad \psi_4^{(0)} = \{2, 1\}, \quad \psi_6^{(0)} = \{10, 4, 4, 4, 1\} \tag{4.19}$$

The above observation is particularly intriguing, as we already know that another particular point ($q = -e^{\frac{i\pi}{3}}$) of the *same* solution Ψ also displays entries leading to non-negative integers for both $r = 0$ and $r = 1$, as mentioned in the previous section. These entries Ψ_π were actually identified via the full Razumov–Stroganov conjecture [3] to fully-packed loop configurations on a $n \times n$ square grid (at $r = 0$) or to vertically symmetric fully-packed loop configurations on a $(2n+1) \times (2n+1)$ square grid (at $r = 1$) connected in both cases according to the same link pattern as that encoded in the Dyck path π.

The case $k > 2$ also seems to lead to non-negative integer entries in the rational homogeneous limit, provided we take $r = 1$ and all $z_i = 1$, and then consider

$$\lim_{q \to -1} \frac{1}{(q^2 - 1)^{kn(n-1)}} \sum_\pi \Psi_\pi(1, 1, ..., 1) \tag{4.20}$$

These are presumably related to the degrees of higher order generalizations of the Brauer loop scheme [15]. For illustration, we have for $k = 3$ and $n = 2$, in the basis (2.5):

$$\psi_6^{(1)} = \{60, 28, 13, 15, 6\} \tag{4.21}$$

whereas the corresponding $r = 0$ limit (with $(q^2 - 1)^{kn(n-1)}$ replaced by $(q^2 - 1)^{kn(n-1)/2}$ in the limit of (4.20)) yields in the basis (2.5):

$$\psi_6^{(0)} = \{6, 3, 2, 2, 1\} \tag{4.22}$$

As mentioned in Sect.3.2, we have access to analogous integer Ψ's and sum rules for arbitrary sizes N not necessarily a multiple of k, presumably relating to generalized Brauer schemes of matrices of corresponding sizes.

5. Conclusion

In this note, we have constructed the minimal polynomial solutions to the level 1 $U_q(\mathfrak{sl}(k))$ boundary qKZ equation, and investigated its "combinatorial points". We have derived some higher rank generalizations of the Razumov-Stroganov sum rules for the open boundary O(1) spin chain. We have actually found a family of solutions depending on one parameter r playing the role of boundary magnetic field. The closed and open boundary cases of the Razumov–Stroganov sum rules are recovered for $r = 0$ and 1 respectively. Sum rules for the components of the solution à la Razumov–Stroganov are generically obtained at a unique point $\tau = 2\cos\frac{\pi}{k+1}$ for each k, where a covector v with the properties (4.1) and with positive entries can be constructed.

In view of trying to cook up a full conjecture à la Razumov–Stroganov, we may try to consider the vector w with entries $w_\pi = (-i/q)^{kn(n-1)/2} v_\pi \Psi_\pi$: unfortunately, these entries are not real for generic r. For illustration, for $k = 3$, $n = 2$, the corresponding entries read in the basis (2.5):

$$
\begin{gathered}
w_f = 4(1{+}9r + 9r^2 + r^3), w_3 = 4(1 + 7r + 7r^2 + r^3), w_2 = 4(1 + (2 + i)r + (2 - i)r^2 + r^3), \\
w_1 = 4(1 + (2 - i)r + (2 + i)r^2 + r^3), \qquad w_0 = 4(1 + r + r^2 + r^3)
\end{gathered}
\tag{5.1}
$$

and sum to the abovementioned value of $I_6^{(3)}(1, 1, ..., 1|r)$. In this particular case however, the entries of w reduce to *non-negative integers* in both cases $r = 0, 1$, with $w^{(0)} = 4\{1, 1, 1, 1, 1\}$ and $w^{(1)} = 8\{10, 8, 3, 3, 2\}$. In general, it is tempting to conjecture that w has integer entries, and we have indeed observed this fact for $r = 0$ and the cases $n = 2$, $k = 3, 4, 5$, and $n = 3$, $k = 3$. For $r = 1$ however we have found a counterexample for $n = 2$, $k = 4$, in which the 14 entries of w, although positive and summing to $4^2 A_V^{(4)}(2) = 16\times 68$, are not all integers. So if a full Razumov–Stroganov conjecture exists it has to be more subtle in the $r = 1$ case.

Note that, as we have $q^{2(k+1)} = 1$ at the generalized Razumov–Stroganov point, the left and right boundary conditions in (2.11) coincide. When $r = 0$, we are precisely in the case where Ψ is the groundstate vector of the inhomogeneous A_{k-1} IRF model on a semi-infinite cylinder of perimeter N as pointed out in [12], and governed by the cyclic Hamiltonian $H = \sum_{1\le i\le N}(\tau - e_i)$, $e_N = \sigma e_{N-1}\sigma^{-1}$ an additional generator for the cyclic Hecke algebra. For $r = 1$, Ψ is the groundstate vector of the inhomogeneous A_{k-1} IRF model on a semi-infinite strip of width N, with reflecting boundary conditions, and governed by the Hamiltonian $H = \sum_{1\le i\le N-1}(\tau - e_i)$. In both cases, the entries $(-i/q)^{kn(n-1)/2}\Psi_\pi$

and v_π are manifestly positive, as these are respectively the entries of the right and left Perron-Frobenius eigenvectors of the corresponding Hamiltonians. Alternatively, we may view the $r = 1$ model as describing a stochastically "growing interface" like in [19] for the Temperley–Lieb case $k = 2$, upon interpreting the action of $\check{R}_{i,i+1}(z_i, z_j)$ (2.1) on a path as the identity with probability $1 - t_{i,j} = (z_j - q^2 z_i)/(q^2 z_j - z_i)$ and as e_i/τ with probability $t_{i,j}$, which (i) leaves the path unchanged (if π is convex at $i, i+1$), (ii) adds a lozenge to it (if π is concave at $i, i+1$) (iii) or shrinks it to a smaller path π' such that $C_{i,\pi,\pi'} = 1$ (if π is flat or concave at $i, i+1$), the last two operations weighted by $1/\tau$. The combination of entries $\mathcal{P}(\pi) = v_\pi \Psi_\pi/(v \cdot \Psi)$ may then be interpreted as the *invariant probability* for the interface to be equal to the path π. These invariant probabilities read in the case $k = 3$, $n = 2$: $\{\frac{5}{13}, \frac{4}{13}, \frac{3}{26}, \frac{3}{26}, \frac{1}{13}\}$ for the paths of (2.5). The above counterexample to integrality simply shows that these invariant probabilities will not be rational numbers in general. It is still possible nevertheless that generalizations of the observables considered in [19] could lead to simple rational expectation values. For instance, from our numerical data, we conjecture that the probability of convex transitions $\pi_i < \pi_{i+1}$ in the above interfaces reads

$$C(2, k) = \frac{(k-1)(13k+4)}{2(2k-1)(4k+1)} \tag{5.2}$$

for the A_{k-1} growth problem with $n = 2$. This is easily checked for $k = 3$, as the paths of (2.5) have respectively $\{4, 3, 3, 3, 2\}$ convex transitions among their 5 transitions, leading to a probability $\frac{1}{5}\{4, 3, 3, 3, 2\} \cdot \{\frac{5}{13}, \frac{4}{13}, \frac{3}{26}, \frac{3}{26}, \frac{1}{13}\} = \frac{43}{65}$.

The sum rules obtained in this paper produce nice integer sequences, which we expect to be counting generalizations of ASMs and of VSASMs. A first step in understanding what exactly is counted by these numbers could be to find some higher $k > 2$ analogues of the Izergin–Korepin determinant (or its open version (4.5)) for our sum rules $I_{kn}^{(k)}(z_1, ..., z_{kn}|r)$. We may also try to generalize the Pfaffian formula (4.6). The hope would be eventually to be able to define an A_{k-1} vertex model with a particular type of (domain-wall) boundary conditions on a finite domain, whose partition function exactly matches $I_{kn}^{(k)}(z_1, ..., z_{kn}|r)$. Then the configurations of such a model would form the desired generalizations of ASMs or VSASMs. This might then open the route to generalizations of other symmetry classes of such objects, according to the symmetries of the above domain, in the spirit of [1].

As a side remark, the number of different objects enumerated by the ASM number A_n is quite considerable: totally symmetric self-complementary plane partitions (TSSCPP), descending plane partitions (DPP), rhombus tilings of particular domains of the triangular lattice, osculating walks on a square grid, ice configurations on a square with domain wall boundaries etc, and these give rise to many different formulas for A_n, as determinants or Pfaffians. We may wonder which ones of these objects and/or formulas will be amenable to higher k generalizations, maybe all of them?

The other combinatorial point $q = -1$ ($\tau = 2$) seems to be of a different nature. It corresponds to the rational limit of the above A_{k-1} IRF models, namely giving rise to a rational solution of the Yang-Baxter equation. Another rational solution of a similar nature was considered in [7] and led to sum rules for the O(1) model of crossing loops. All these models seem to be related to the geometry of some specific matrix varieties, allowing in particular for interpreting in the homogeneous limit the various integer entries of Ψ at hand as degrees of the components of these varieties. The identification of these varieties for the open boundary cases is under way [15].

Acknowledgments: We thank M. Bauer, P. Zinn-Justin and J.-B. Zuber for useful discussions. This work is partly supported by the Geocomp project (ACI *Masse de Données*) and the European network "ENIGMA", grant MRT-CT-2004-5652.

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