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Sum rules for the ground states of the $O(1)$ loop model on a cylinder and the XXZ spin chain

P. Di Francesco #, P. Zinn-Justin ⋆ and J.-B. Zuber •#

The sums of components of the ground states of the $O(1)$ loop model on a cylinder or of the XXZ quantum spin chain at $\Delta = -\frac{1}{2}$ of size L are expressed in terms of combinatorial numbers. The methods include the introduction of spectral parameters and the use of integrability, a mapping from size L to $L+1$, and knot-theoretic skein relations.

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Service de Physique Théorique de Saclay, CEA/DSM/SPhT, URA 2306 du CNRS, C.E.A.-Saclay, F-91191 Gif sur Yvette Cedex, France, Philippe.Di-Francesco@cea.fr

⋆ Laboratoire de Physique Théorique et Modèles Statistiques, UMR 8626 du CNRS, Université Paris-Sud, Bâtiment 100, F-91405 Orsay Cedex, France, pzinn@lptms.u-psud.fr

•# Laboratoire de Physique Théorique et Hautes Énergies, Tour 24-25 5ème étage, Université Pierre et Marie Curie-Paris6; UMR 7589 du CNRS; Université Denis Diderot-Paris7, Boîte 126, 4 pl Jussieu, 75252 Paris Cedex 05, France, zuber@lpthe.jussieu.fr

1. Introduction and summary of results

The observation by Razumov and Stroganov [1] that the components of the ground state of integrable quantum spin chains in some adequate basis enjoy integrality conditions and are connected to known combinatorial problems, to wit, alternating sign matrices (ASM) and their avatars, has been the source of an amazing burst of new developments. It has been soon realized that in many cases these integers in fact count some configurations of other lattice models [2,3]. To be more explicit, the components of the ground state of the Temperley–Lieb Hamiltonian describing the $O(1)$ loop model on an even number of sites with periodic boundary conditions are conjectured to count the numbers of configurations of so-called fully packed loop (FPL) models: this is the now celebrated Razumov–Stroganov conjecture [3]. Other types of boundary conditions have also been considered [4,5,6].

After these original observations and conjectures, a major progress has been the introduction of inhomogeneities in the original problem, in the form of spectral parameters, thus enabling one to use the full machinery of integrable models [7,8]. In particular, this has led to recursion formulae between components of the ground state and ultimately to a proof of the ground state sum rule, a weak but non trivial version of the RS conjecture. These recursion formulae have been shown to follow from an underlying algebraic structure rooted in the Affine Hecke Algebra (AHA) [9], or alternatively, related to the so-called quantum Knizhnik–Zamolodchikov equation ([10] and further references therein). This has then been extended into several distinct directions. Loop models with crossings [11] have been shown to display an amazing relationship to the algebraic geometry of matrix varieties [12,13]. Extensions to other types of boundary conditions [14], to higher rank algebras [10], or to both [15], have been considered in turn.

In the present paper, we return to a case first tackled in the original papers in this domain, that of the $O(1)$ loop model on a square lattice wrapped on a semi-infinite cylinder of odd integer perimeter [2,5], or of the associated periodic XXZ spin chain [1]. In the loop model, the sites of the boundary of the cylinder are pairwise connected via non-intersecting links. The oddness of the perimeter implies that one site remains unmatched, hence giving rise to a defect line, connecting it to the point at infinity on the cylinder. In the spin chain on $L = 2n + 1$ sites, likewise, the ground state is made of $n + 1$ spins pointing upward and n downward (or vice-versa). It is also interesting to consider the case of the $O(1)$ loop model on a cylinder of even perimeter, when the loops wrapping around the cylinder are not allowed to contract, or equivalently, when the corresponding link pattern is drawn on a punctured disk. This case was called periodic with "distinct connectivities" in [6]. It must be distinguished from the more usual loop model with "identified connectivities" which was the subject of [8], which will be also discussed in what follows since we need it to analyze the other cases. The related XXZ spin chain of even size $L = 2n$ has twisted boundary conditions [2], as we recall below in sect. 4. See also [16] for more data on these different boundary conditions.

Let $A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}$ denote the number of alternating sign matrices of size n ([17] and further references therein) and $\mathcal{N}_n$ stand for

$$\mathcal{N}_n = \frac{3^{n/2}}{2^n} \frac{2 \times 5 \times \cdots \times (3n-1)}{1 \times 3 \cdots \times (2n-1)} A_n = 3^{n/2} \frac{\prod_{j=1}^{n} (3j-1)!}{\prod_{j=1}^{n} (n+j)!} \ . \tag{1.1}$$

Let us also introduce the number $A_{\rm HT}(L)$ of "half-turn symmetric" alternating sign matrices [17,18,19,20]

$$A_{\rm HT}(L) = \begin{cases} \prod_{j=0}^{n-1} \frac{3j+2}{3j+1} \left(\frac{(3j+1))!}{(n+j)!} \right)^2 & \text{if } L = 2n \text{ is even} \\ \prod_{j=1}^{n} \frac{4}{3} \left(\frac{(3j)!j!}{(2j)!^2} \right)^2 & \text{if } L = 2n+1 \text{ is odd} \ . \end{cases} \tag{1.2}$$

Okada [21] showed that this number $A_{\rm HT}(L)$ may be expressed in terms of the dimension of certain representations of GL(L)

$$\begin{aligned} A_{\rm HT}(2n+1) &= 3^{-n^2} (\dim_{Y_n'}^{{\rm GL}(2n+1)})^2 \\ A_{\rm HT}(2n) &= 3^{-n(n-1)} \dim_{Y_n}^{{\rm GL}(2n)} \dim_{Y_n'}^{{\rm GL}(2n)} \end{aligned} \ . \tag{1.3}$$

The Young diagram Y_n' which defines that representation is made of one row of length n, two rows of length $n-1$, two of length $n-2$, ..., two of length 1, while the Young diagram Y_n results from the ablation of the first row of Y_n'.

In [2], it was conjectured that for the $O(1)$ loop model on a cylinder of perimeter $L = 2n+1$, all components of the Perron–Frobenius eigenvector Ψ are integers if the smallest one is normalized to be 1, that the sum of components of Ψ over all loop configurations π is

$$\sum_{\pi} \Psi_\pi = A_{\rm HT}(2n+1) = \mathcal{N}_n^2 = 1, 3, 25, 588, 39204 \ldots \qquad L = 2n+1 = 1, 3, \ldots \tag{1.4}$$

and that the largest component is

$$\Psi_{max} = A_n^2 = 1, 1, 4, 49, 1764, \ldots \qquad L = 2n+1 = 1, 3, \ldots \tag{1.5}$$

As for the case of even $L = 2n$ with distinct connectivities, it was conjectured in [6] that the sum of components of the Perron–Frobenius eigenvector Φ^* over all loop configurations π is

$$\sum_{\pi} \Phi^*_\pi = A_{\rm HT}(2n) = 2, 10, 140, 5544 \ldots \qquad L = 2n = 2, 4, 6, 8, \ldots \tag{1.6}$$

The largest component was also conjectured to be

$$\Phi^*_{max} = A_{\rm HT}(2n-1) = 1, 3, 25, 588, \ldots \qquad L = 2n = 2, 4, 6, 8, \ldots \tag{1.7}$$

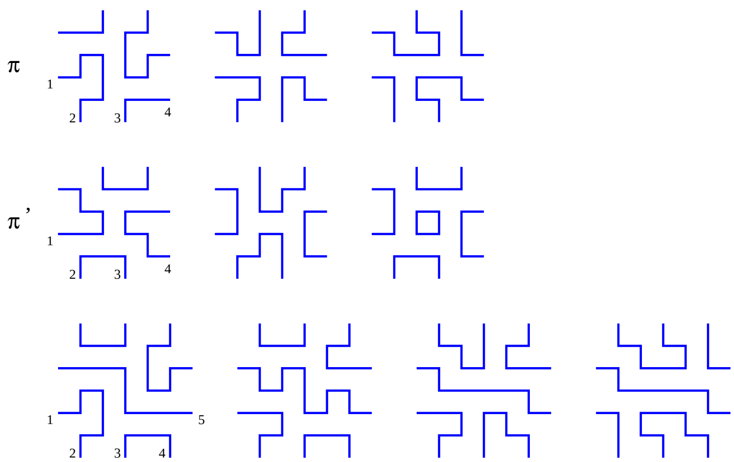
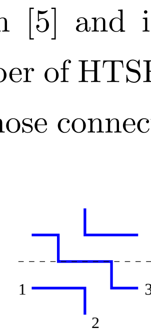


Fig. 1: A sample of half turn symmetric FPL configurations of size 4 and 5. The lower half of these configurations has a link pattern described by, respectively, {⌒ ⌒}, {╮ ⌒ ╭} and {⌒ ⌒ ·}.

In both the odd and the even cases, it was further conjectured in [5] and in [6], respectively, that the individual components Ψ_π, resp. Φ^*_π, count the number of HTSFPLs, that is, half-turn symmetric FPL configurations drawn on a $L \times L$ grid, whose connectivity pattern is described by the arch pattern π. For example, the unique HTSFPL pertaining to $L = 3$ and the link pattern $\pi = \{⌒ \ \cdot\}$ is as shown here on the right:



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while further examples for $L = 4, 5$ are depicted in Fig. 1.

For completeness and future use in this paper we also recall here the sum rule written in [2] for the loop model of even size $L = 2n$ with "identified connectivities". The Perron–Frobenius eigenvector denoted Φ satisfies

$$\sum \Phi_\pi = A_n \ , \tag{1.8}$$

together with $\Phi_{max} = A_{n-1}$. While the latter remains a conjecture, (1.8) has now been established [8] as we recall below.

Turning now to the XXZ spin chain of odd size, the parallel (and prior!) conjectures read [1]

$$\sum_\alpha \tilde\Psi_\alpha = 3^{n/2}\mathcal{N}_n = 1, 3, 15, 126, 1782\ldots \qquad L = 2n+1 = 1, 3\ldots \tag{1.9}$$

for the sum over all spin configurations of total spin 1/2, the largest component is

$$\tilde\Psi_{max} = A_n = 1, 1, 2, 7, 42\ldots \qquad L = 2n+1 = 1, 3, \ldots \tag{1.10}$$

and the square norm of Ψ is

$$\sum_\alpha \tilde\Psi_\alpha^2 = \mathcal{N}_n^2 = A_{\rm HT}(2n+1) \tag{1.11}$$

when the normalization is such that $\tilde\Psi_{+\cdots+-\cdots-} = 1$. For an even size $L = 2n$, (with twisted boundary conditions, see Sect. 4), the ground state wave function is complex, and the parallel conjectures of [2] and [22] read

$$\sum_\alpha \tilde\Phi_\alpha = 3^{n/2}A_n \tag{1.12a}$$

$$\sum_\alpha \tilde\Phi_\alpha^2 = A_n^2 \tag{1.12b}$$

$$\sum_\alpha |\tilde\Phi_\alpha|^2 = A_{\rm HT}(2n) \tag{1.12c}$$

$$\tilde\Phi_{max} = \tilde\Phi_{(+-)^n} = \mathcal{N}_{n-1}{\rm e}^{i\pi/6} \tag{1.12d}$$

with the normalization that $\tilde\Phi_{min} = \tilde\Phi_{++\cdots+--\cdots-} = {\rm e}^{i\pi n/6}$. The first of these conjectures is now established, since it is a corollary of the sum rule (1.8), as will be reexplained below.

Examples: $L = 5$ $(n = 2)$: $\Psi = (1,1,1,1,1,4,4,4,4,4)$ in the loop basis, $\tilde\Psi = (1,1,1,1,1,2,2,2,2,2)$ after change to the spin basis. Note that for L odd, there are always L repeats due to the breaking of rotational symmetry by the defect. Indicating these repeats with a superscript, at $L = 7$, $\Psi = (1^{(7)}, 6^{(7)}, 14^{(14)}, 49^{(7)})$ for link patterns, $\tilde\Psi = (1^{(7)}, 3^{(14)}, 4^{(7)}, 7^{(7)})$ for spins. In even sizes, for $L = 4$, $\Phi^* = (3^{(2)}, 1^{(4)})$ and $\tilde\Phi = (\sqrt{3}e^{\pm i\pi/6}, e^{\pm i\pi/3}, 1^{(2)})$; for $L = 6$, $\Phi^* = (25^{(2)}, 9^{(6)}, 5^{(6)}, 1^{(6)})$, $\tilde\Phi = (5e^{\pm i\pi/6}, (\sqrt{7}e^{\pm i\alpha})^{(3)}, (\sqrt{7}e^{\pm i\beta})^{(2)}, \sqrt{7}e^{\pm i\gamma}, (e^{\pm i\pi/6})^{(2)}, e^{\pm i\pi/2})$ with $\alpha = \mathrm{Arctan}\, 1/3\sqrt{3}$, $\beta = \frac{\pi}{3} - \alpha$, $\gamma = \frac{\pi}{3} + \alpha$.

In view of previous experience, it is very natural to extend the discussion to the inhomogeneous version of the loop model (whose precise definition is recalled below in Sect. 2). Indeed the main result of [8] (Theorem 5) is that for the even size (IC) loop model,

$$\sum_{\pi\in LP_{2n}} \Phi_\pi(z_1,\ldots,z_{2n}) = 3^{-n(n-1)/2} s_{Y_n}(z_1,\ldots,z_{2n}) \ , \tag{1.13}$$

so that in the homogeneous limit $z_i \to 1$, (1.8) follows. The main result of *this* paper is that

$$\begin{cases} \sum_{\pi\in LP_{2n+1}} \Psi_\pi(z_1,\ldots,z_{2n+1}) = 3^{-n^2} s_{Y_{n+1}}(z) s_{Y'_n}(z) & \text{if } L=2n+1 \\ \sum_{\pi\in LP^*_{2n}} \Phi^*_\pi(z_1,\ldots,z_{2n}) = 3^{-n(n-1)}\, s_{Y'_n}(z) s_{Y_n}(z) & \text{if } L=2n \end{cases} \ , \tag{1.14}$$

as we will show in Sect. 2.5 and 3. In these expressions, $s_Y(z)$ stands for the Schur function labelled by a Young diagram Y, a symmetric function of $z_1,\ldots,z_L$. The diagrams Y'_n and Y_n have been defined above, after (1.3), thus Y_{n+1} has two rows of length n, two of length $n-1$, etc, two of length 1. From this follow the proofs of the sum rules (1.4) and (1.6). The sum rule (1.9) for the odd spin chains will be also derived in Sect. 4.5, and (1.11) and (1.12*b*) in Sect. 4.6. On the conceptual level, it may be interesting to notice that our derivation makes use not only of the previously mentionned techniques, introduction of spectral parameters, recursion equations and qKZ equation, etc, but also of a new idea borrowed from knot theory, namely the use of skein relations.

The educated reader will recognize in (1.14) formulae equivalent to those written by Razumov and Stroganov [20] for the partition function of the square-ice model with boundary conditions appropriate to the enumeration of half-turn symmetric ASMs. We return to this in our Conclusion.

For the sake of the reader, we summarize in Table 1 and Table 2 some notations and data for the various situations that we consider in this paper.

Size L	Model	Space	dimension	Perron–Frobenius eigenvector	Transfer matrix
$2n+1$	loop	LP_{2n+1}	$\binom{2n+1}{n}$	Ψ_π	T
$2n$	loop ("I.C.")	LP_{2n}	$C_n = \frac{(2n)!}{n!(n+1)!}$	Φ_π	T
$2n$	loop ("D.C.")	LP^*_{2n}	$\binom{2n}{n}$	Φ^*_π	T^*
$2n+1$	spin	$\{s_z=\frac{1}{2}\}$	C_{n+1}	$\tilde\Psi_\alpha$	$\tilde T$
$2n$	spin	$\{s_z=0\}$	C_n	$\tilde\Phi_\alpha$	$\tilde T$

Table 1. Notations for the different models and boundary conditions considered in this paper

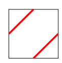
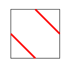
B	equ. (4.10)	$B_0,\ B_\infty$	equ. (3.2)
L_{2n+1}	§ 2.1	L_{2n+2}	§ 2.3
$P_0,\ P_\infty$	§ 2.3	$Q,\ P_\pm$	§ 4.3
S	§ 4.2	$s_{Y_n}\ ,\ s_{Y'_n},\ Y_n\ ,\ Y'_n$	equ. (1.3), (1.14)

Table 2. Some other notations with the section or equation number where they first appear.

2. The inhomogeneous $O(1)$ loop model in odd size

2.1. Link patterns, transfer matrix and R-matrices

The model is defined on a semi-infinite cylinder of square lattice of odd perimeter $2n+1$ whose faces are covered by either of the two following face configurations

 or  . (2.1)

In a given configuration, labelling cyclically $i = 1, 2, \dots 2n+1$ the centers of the boundary edges (and with the convention that $i+2n+1 \equiv i$), we note that these points are connected among themselves or to the point at infinity via nonintersecting curves. Actually, drawing on each face the two configurations (2.1) with probabilities p and $1-p$, $p \in (0,1)$, leads only to situations where *one* boundary point is connected to infinity, while the remaining $2n$ are pairwise connected. Forgetting about the underlying lattice, the connection pattern, also called link pattern, is simply a chord diagram, namely a configuration of $2n+1$ points on a circle, pairwise connected by nonintersecting arcs within the interior disc, with one unmatched point that we connect to the center of the disc. The set of such link patterns is denoted by LP_{2n+1} and has cardinality $d_n = \binom{2n+1}{n}$. Later on, we regard this set as a linear space, spanned by the previous link patterns, thus of dimension d_n.

Here we study the inhomogeneous version of this model, in which plaquettes above the boundary point i are picked among the two faces (2.1) with respective probabilities p_i and $1-p_i$, $p_i \in (0,1)$. We wish to compute the probabilities P_π that random configurations connect the boundary point according to a given $\pi \in LP_{2n+1}$. Clearly, these probabilities are invariant under the addition of a row of $2n+1$ plaquettes to the original semi-infinite cylinder, which amounts to an equation of the form

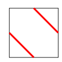
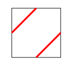
$$\sum_{\pi' \in LP_{2n+1}} T(p_1, \dots, p_{2n+1})_{\pi,\pi'} \mathrm{P}_{\pi'} = \mathrm{P}_\pi \tag{2.2}$$

where the transfer matrix T acts on link patterns in an obvious way, by concatenation. The vector $\mathrm{P} = \{\mathrm{P}_\pi\}_{\pi\in LP_{2n+1}}$ is determined as the properly normalized Perron–Frobenius eigenvector of T, with eigenvalue 1.

This sytem is known to be integrable, as T may be constructed by multiplying and then tracing plaquette operators that satisfy the Yang–Baxter equation. Indeed, parametrizing the probabilities as

$$p_i = \frac{q\, z_i - q^{-1}t}{q\, t - q^{-1}z_i} \tag{2.3}$$

where q is a complex cubic root of unity $q = -e^{i\pi/3}$, we may identify the i-th plaquette operator (or R-matrix) as

$$R_i(z_i,t) = \frac{q\, z_i - q^{-1}t}{q\, t - q^{-1}z_i}$$  $$+\ \frac{z_i - t}{q\, t - q^{-1}z_i}$$  (2.4)

where z_i and t are spectral parameters attached respectively to the vertical line above point i, and to the horizontal one running around the cylinder. With the parametrization (2.3), the probability vector P is clearly a rational fraction of the z's. In the sequel, we will use a different normalization $\Psi \propto \mathrm{P}$ in which the entries of Ψ are coprime polynomials of the z's.

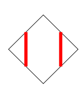
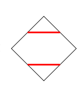
In [8,12], it was shown that the transfer matrix relation (2.2) may be equivalently replaced by a system of relations of the form

$$\check{R}_{i,i+1}(z_i, z_{i+1})\Psi = \tau_i \Psi,\ \ i = 1, 2, \ldots 2n+1 \tag{2.5}$$

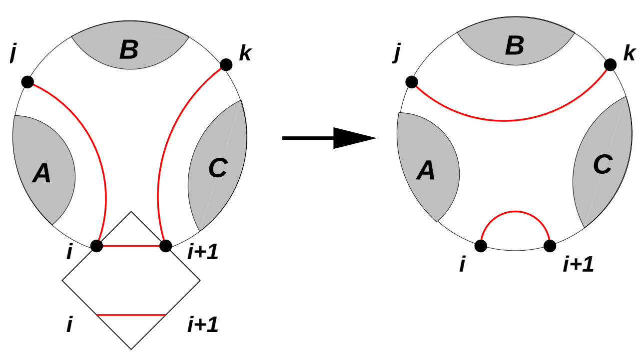
where τ_i is simply the interchange of spectral parameters $z_i \leftrightarrow z_{i+1}$ and $\check{R}$ is the tilted plaquette operator

$$\check{R}_{i,i+1}(z,w) = \frac{q\, z - q^{-1}w}{q\, w - q^{-1}z} I + \frac{z - w}{q\, w - q^{-1}z} e_i \tag{2.6}$$

in terms of the local identity I and Temperley–Lieb operators e_i, $i = 1, 2, \ldots, 2n+1$ defined pictorially as

$I =$  and $e_i =$ 

and acting at points i and $i+1$ on link patterns by concatenation as indicated schematically below:



(2.7)

Note that if $k = i$ and $j = i+1$, e_i acts by creating a loop which we allow ourselves to erase, and therefore this action leaves the pattern invariant.

The e_i obey the usual conditions $e_i e_{i\pm1} e_i = e_i$ and $e_i^2 = e_i$ that define the (cyclic) Temperley–Lieb algebra $TL(1)$.

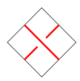
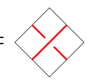
The last equation $i = 2n+1$ of (2.5) may be replaced by a cyclic invariance condition

$$\Psi(z_2, z_3, \ldots, z_{2n+1}, z_1) = \Psi(z_1, z_2, \ldots, z_{2n+1}) \ . \tag{2.8}$$

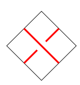
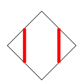
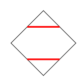
Finally, we will also be using $\check{R}$ matrices with the second spectral parameter sent either to zero or to infinity. Up to a multiplicative redefinition by $-q^{\pm 3/2}$, the two corresponding plaquette operators[1]

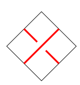
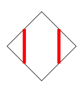
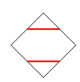
$$\begin{aligned} t_i^{-1} &\equiv -q^{3/2} \check{R}_{i,i+1}(z,0) = q^{-1/2} I + q^{1/2} e_i \\ t_i &= -q^{-3/2} \check{R}_{i,i+1}(z,\infty) = q^{1/2} I + q^{-1/2} e_i \end{aligned} \tag{2.9}$$

may be interpreted respectively as under- and over-crossings of links, with pictorial representation

$t_i =$  and $t_i^{-1} =$ 

and the two pieces of Eq. (2.9) are nothing else than the celebrated skein relations for knots, which read pictorially

 $= q^{-1/2}$  $+ q^{1/2}$ 

 $= q^{1/2}$  $+ q^{-1/2}$  . (2.10)

2.2. The quantum Knikhnik–Zamolodchikov equation

More generally, it was noted in [9] that the condition that q be a cubic root of unity could be relaxed and that equations (2.5) and (2.8) are a particular case of quantum Knizhnik–Zamolodchikov (qKZ) equation [10,15]. The latter amounts to the system

$$\begin{aligned} \check{R}_{i,i+1}(z_i, z_{i+1}) \Psi &= \tau_i \Psi \qquad i = 1, 2, \ldots 2n \\ \sigma \Psi(z_2, \ldots, z_{2n+1}, s z_1) &= c \Psi(z_1, \ldots, z_{2n+1}) \end{aligned} \tag{2.11}$$

[1] Note that the following choice of normalization for t_i is ad-hoc to ensure rotational invariance of the crossing move, and coincides with the standard crossing operators of knot theory. However, the prefactors $-q^{\pm 3/2}$ will be irrelevant at the particular point $q^{1/2} = e^{-i\pi/3}$ to which we will restrict later on.

where s, c are scalars and the operator σ acts on link patterns via the cyclic rotation of labels $i \to i+1$. Here $\check{R}$ denotes the same plaquette operator as before, except that its definition involves the generators e_i of the Temperley–Lieb algebra $TL(\tau)$, (*i.e.* satisfying the relation $e_i^2 = \tau e_i$), with $\tau = -q - q^{-1}$. Note that we may replace the second line of (2.11) by an equation of the form

$$\check{R}_{2n+1,1}(z_{2n+1}, sz_1)\Psi(z_1, \ldots, z_{2n+1}) = \Psi(s^{-1}z_{2n+1}, \ldots, sz_1) \tag{2.12}$$

where the omitted z's are left unchanged.

We now look for polynomial solutions $\Psi(z_1, \ldots, z_{2n+1})$ of this system, with *minimal* degree. From the first line of Eq. (2.11) we learn that whenever points i and $i+1$ are not connected in π, Ψ_π factors out a term $qz_i - q^{-1}z_{i+1}$, and more generally if no two points between i and $j > i$ are connected, then Ψ_π factors out $qz_i - q^{-1}z_j$. For the link pattern π_0 connecting points $i \leftrightarrow 2n+2-i$, $i = 1, 2, \ldots, n$ while $n+1$ is unmatched, this fixes the base component Ψ_{π_0} to be

$$\Psi_{\pi_0} = \prod_{1 \le i < j \le 2n+1} \frac{q\, z_i - q^{-1} z_j}{q - q^{-1}} \tag{2.13}$$

up to multiplication by a symmetric polynomial, which we pick to be 1 for the sake of minimality. This in turn determines both scalars in (2.11) to be $s = q^3$ and $c = q^{3n}$. The other entries of Ψ are then determined from Eq. (2.11), which reads in components

$$\begin{aligned} -(q^{-1}z_{i+1} - qz_i)\partial_i \Psi_\pi &= \sum_{\substack{\pi' \neq \pi \\ e_i \pi' = \pi}} \Psi_{\pi'} \\ \Psi_\pi(z_2, \ldots, z_{2n+1}, sz_1) &= c\Psi_{\sigma^{-1}\pi}(z_1, \ldots, z_{2n+1}) \ , \end{aligned} \tag{2.14}$$

where

$$\partial_i f(z_1, \ldots, z_i, z_{i+1}, \ldots, z_{2n+1}) := \frac{f(z_1, \ldots, z_{i+1}, z_i, \ldots, z_{2n+1}) - f(z_1, \ldots, z_i, z_{i+1}, \ldots, z_{2n+1})}{z_{i+1} - z_i} \ . \tag{2.15}$$

Also true for any q is the recursion relation:

$$\begin{aligned} \Psi_{\phi_i(\pi)}(z_1, \ldots, z_L)|_{z_{i+1} = q^2 z_i} &= \frac{1}{(q-q^{-1})^{2L-3}} z_i \prod_{j=1}^{i-1} (qz_j - q^{-1}z_i)(z_j - qz_i) \\ &\times \prod_{j=i+2}^{L} (qz_{i+1} - q^{-1}z_j)(z_{i+1} - qz_j)\Psi_\pi(z_1 \ldots z_{i-1}, z_{i+1} \ldots z_L) \end{aligned} \tag{2.16}$$

where ϕ_i inserts two consecutive points connected via a "little arch" between points $i-1$ and i in any link pattern π of size $2n-1$; all other components vanish when $z_{i+1} = q^2 z_i$.

2.3. Projections to link patterns of larger size

In this section, we introduce two projection operators P_0 and P_∞ from LP_{2n+1} into the vector space generated by the link patterns with $2n+2$ points without unmatched points, denoted LP_{2n+2}. These two projections would allow us to relate the solutions of the qKZ equation for punctured discs with $2n+1$ points on their perimeter to those for unpunctured discs of perimeter $2n+2$. We decide from now on to restrict ourselves to the particular value $q = -e^{i\pi/3}$ (RS point), as we will be mostly reasoning on transfer matrices for the loop model on a punctured disk, for which the periodicity is requested.

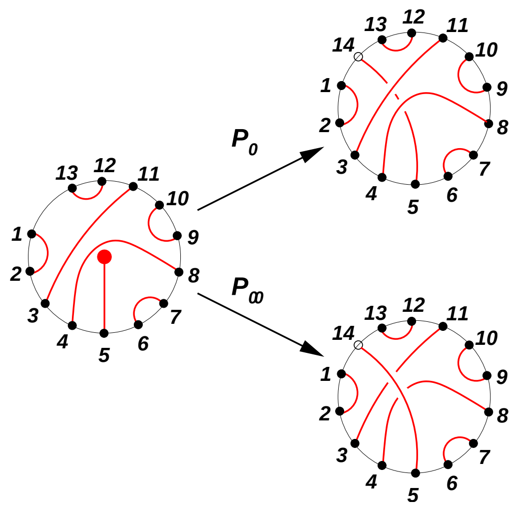


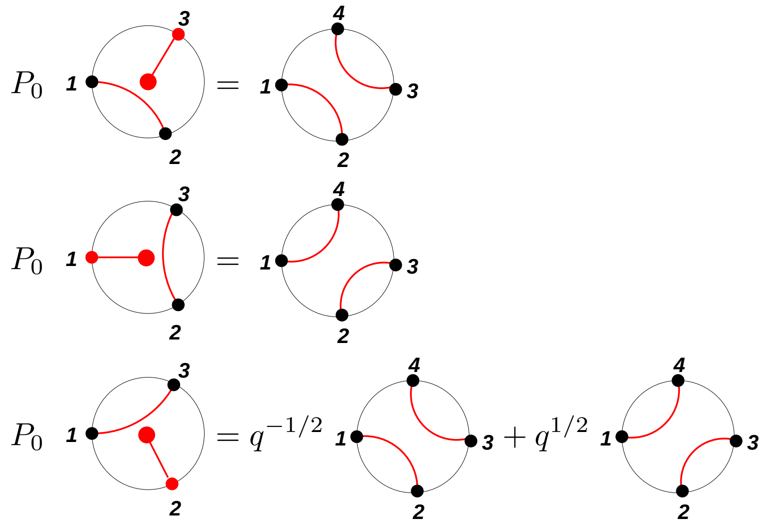
Fig. 2: Example of projections P_0 and P_∞ of a link pattern of size 13. We first add a point 14 between 1 and 13 on the boundary of the link pattern, and then connect the latter to the center, itself linked to the unmatched point (labelled 5 here), via a link passing under (P_0) or over (P_∞) those separating the two points. Finally, we must use the skein relations (2.9) to express the latter as linear combinations of non-crossing link patterns of size 14.

The projections P_0 and P_∞ are defined as follows. For any link pattern of size $2n+1$, let us first add an extra point labelled $2n+2$ on the boundary of the disc, between points 1 and $2n+1$. Next this point is linked to the center of the disc, which is itself connected to the (unique) unmatched point of the link pattern. But by doing so, the added link may have to cross existing links. As illustrated in Fig.2, we define P_0 by imposing that all these crossings be undercrossings, and P_∞ by imposing that they all be overcrossings. We then simply have to use the skein relations (2.10) at each crossing to obtain a linear combination of non-crossing link patterns of size $2n+2$.

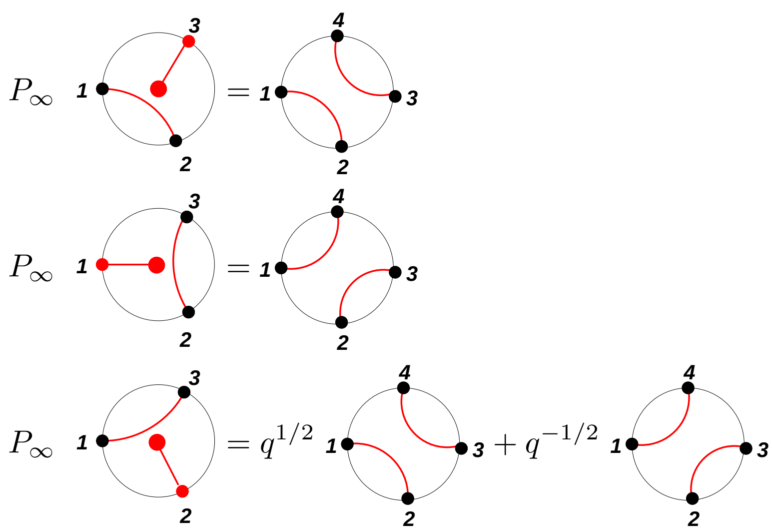
It is also clear that the mappings P_0 and P_∞ are surjective: every π' in LP_{2n+2} in which the point $2n+2$ is matched with some i is the image by P_0 or P_∞ of a $\pi \in LP_{2n+1}$

in which i is unmatched and the other points form the same pairs. This justifies to call P_0 and P_∞ "projections" from LP_{2n+1} to LP_{2n+2}.

For illustration, there are 3 link patterns of size 3, and their projections in the vector space of non-crossing link patterns of size 4 read:



(2.17)



(2.18)

2.4. Projections and Transfer Matrices

Let $T(t|z_1,\ldots,z_{2n+1})$ denote the transfer matrix for the loop model on a *punctured disk* of perimeter $2n+1$, obtained from (2.2) via the parametrization (2.3), and likewise $T(t|z_1,\ldots,z_{2n+2})$ for a *non-punctured* disk of perimeter $2n+2$. Then the projection operators P_0 and P_∞ of previous section act as intertwiners between odd and even transfer matrices, where in the latter the additional spectral parameter is taken to 0 or ∞

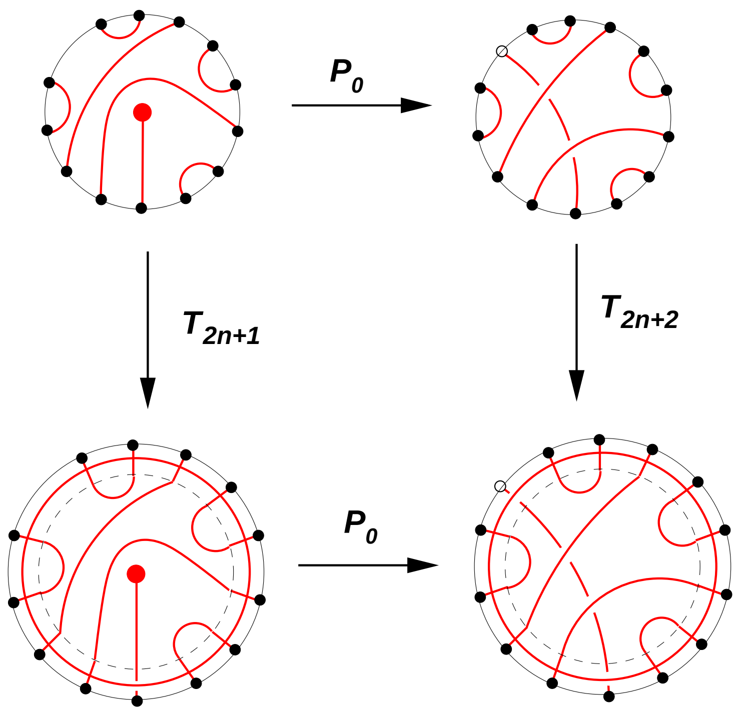


Fig. 3: The projection P_0 intertwines the transfer matrices for size $2n+1$ and $2n+2$, the latter with $z_{2n+2} = 0$. We have represented the transfer matrices pictorially as adding an extra row to the disk. Each intersection between lines stands for an $\check{R}$ matrix operator. That $z_{2n+2} = 0$ imposes here that the added intersection be an undercrossing, in agreement with the action of P_0.

respectively, namely:

$$\begin{aligned} P_0 T(t|z_1, \ldots, z_{2n+1}) &= T(t|z_1, \ldots, z_{2n+1}, 0) P_0 \\ P_\infty T(t|z_1, \ldots, z_{2n+1}) &= T(t|z_1, \ldots, z_{2n+1}, \infty) P_\infty \; . \end{aligned} \tag{2.19}$$

This equality is easily proved graphically, see Fig. 3.

Apply now Eqs. (2.19) to Ψ: we find that $P_\bullet \Psi$ is an eigenvector of $T(t|z_1, \ldots, z_{2n+1}, \bullet)$ with eigenvalue 1, $\bullet = 0, \infty$. This eigenvalue being generically non-degenerate, we conclude that $P_\bullet \Psi$ is proportional to the vector $\Phi(z_1, \ldots, z_{2n+1}, \bullet)$, defined as the ground state eigenvector of the system on a disk of even perimeter $2n+2$, as introduced in [8]:

$$\begin{aligned} P_0 \Psi &= A_0(z_1, \ldots, z_{2n+1})\, \Phi(z_1, \ldots, z_{2n+1}, 0) \\ P_\infty \Psi &= A_\infty(z_1, \ldots, z_{2n+1})\, \Phi(z_1, \ldots, z_{2n+1}, \infty) \; . \end{aligned} \tag{2.20}$$

Here the notation $\Phi(z_1, \ldots, z_{2n+1}, \infty)$ is slightly abusive, and stands for the limit $\lim_{z_{2n+2} \to \infty} \Phi(z_1, \ldots, z_{2n+1}, z_{2n+2})/z_{2n+2}^n$. The polynomial character of the quantities A_0 and A_∞ may be deduced from the recursion relations (2.16) by sending spectral parameters to 0 or ∞; these polynomials will be determined in the next section.

2.5. Ground state sum rule

The fundamental remark of Razumov and Stroganov is that at the special point $q = -e^{i\pi/3}$ (RS point) the problem simplifies drastically, and leads to all sorts of combinatorial wonders. This point may be characterized as the unique one (up to conjugation) where there exists a non-trivial *common* left eigenvector to all operators e_i, or equivalently to all plaquette operators $\check{R}$. In view of the action (2.7) of e_i on link patterns, it is clear that, under e_i, each link pattern gives rise to exactly one link pattern. However, if i is already connected to $i+1$ in the link pattern, the action of e_i creates a loop, which may be removed at the expense of a multiplicative factor τ. The existence of a common left eigenvector to all e_i imposes therefore that $\tau = 1$, for which we pick the root $q = -e^{i\pi/3}$, $q^{1/2} = e^{-i\pi/3}$. This eigenvector is then simply the sum of components over all link patterns, namely the covector v with all entries equal to 1 in the link pattern basis.

As a consequence of this definition, we have $v\check{R} = v$ for all arguments, including when the second spectral parameter is zero or the infinity, hence $vt_i = vt_i^{-1} = v$ as well (as $q^{3/2} = -1$ at the RS point). Moreover, let v' be the vector with all entries 1 in the link pattern basis of LP_{2n+2}. This implies that

$$v' P_0 = v \quad \text{and} \quad v' P_\infty = v \tag{2.21}$$

hence we finally get

$$\begin{aligned} v \cdot \Psi &= A_0(z_1, \ldots, z_{2n+1}) v' \cdot \Phi(z_1, \ldots, z_{2n+1}, 0) \\ v \cdot \Psi &= A_\infty(z_1, \ldots, z_{2n+1}) v' \cdot \Phi(z_1, \ldots, z_{2n+1}, \infty) \ . \end{aligned} \tag{2.22}$$

In [8], the quantity $v' \cdot \Phi(z_1, \ldots, z_{2n+2})$ has been identified with the so-called Izergin–Korepin determinant, which, at the RS point, is equal to the GL$(2n+2)$ Schur function $s_{Y_{n+1}}(z_1, \ldots, z_{2n+2})$ for the Young diagram Y_{n+1} with two rows of n boxes, two rows of $n-1$ boxes, etc..., two rows of one box, as defined in Sect. 1. Sending z_{2n+2} to 0 amounts to restricting the Schur function to GL$(2n+1)$, while keeping the same tableau Y_{n+1}. Sending z_{2n+2} to ∞ amounts to restricting the Schur function to GL$(2n+1)$, while truncating the Young tableau into Y'_n equal to Y_{n+1} with the first row removed. Hence $v \cdot \Psi = A_0 s_{Y_{n+1}}(z_1, \ldots, z_{2n+1}) = A_\infty s_{Y'_n}(z_1, \ldots, z_{2n+1})$. Finally, we make the hypothesis that the Schur functions $s_{Y_{n+1}}(z_1, \ldots, z_{2n+1})$ and $s_{Y'_n}(z_1, \ldots, z_{2n+1})$ have no common factor. From this hypothesis, which is supported by the examination of the first cases but would require a complete proof, we deduce that $v \cdot \Psi$ is a polynomial multiple of

$s_{Y_{n+1}} s_{Y'_n}$. Now the degree of $s_{Y_{n+1}}$ is the total number of boxes in Y_{n+1}, hence $s_{Y_{n+1}} s_{Y'_n}$ has degree $n(n+1)+n^2 = n(2n+1)$. This coincides with the degree of Ψ as given by that of Ψ_{π_0} in Eq.. (2.13), hence we conclude that $v \cdot \Psi$ is proportional to $s_{Y_{n+1}} s_{Y'_n}$ by a scalar factor, fixed to be 3^{-n^2} by the recursion relation (2.16).

We therefore identify the polynomials $A_0 \propto s_{Y'_n}$ and $A_\infty \propto s_{Y_{n+1}}$, and get the multi-parameter sum rule

$$\sum_{\pi \in LP_{2n+1}} \Psi_\pi(z_1, \ldots, z_{2n+1}) = 3^{-n^2} s_{Y_{n+1}}(z_1, \ldots, z_{2n+1}) s_{Y'_n}(z_1, \ldots, z_{2n+1}) \; . \tag{2.23}$$

In the homogeneous limit where all the z_i tend to 1, the Schur functions $s_{Y_{n+1}}$ and $s_{Y'_n}$ reduce to the dimensions of the corresponding representations of $\mathrm{GL}(2n+1)$, and we note that $\dim_{Y_{n+1}} = \dim_{Y'_n}$ because the two Young diagrams form together a rectangle of size $(2n+1) \times n$, (and thus $s_{Y'_n}(\frac{1}{z}) = \prod_j z_j^{3n} s_{Y_{n+1}}(z)$). This leads to (1.4) via (1.3).

3. The inhomogeneous $O(1)$ loop model in even size

As discussed in Sect. 1, the inhomogeneous $O(1)$ loop model may also be considered on a punctured disk of even perimeter $2n$: link patterns now have an isolated puncture in the center of the disk, and we denote by LP^*_{2n} their set, of cardinality $\binom{2n}{n}$. Applying the same line of thought as in the previous section, we now define mappings P_0 and P_∞ from LP^*_{2n} to LP_{2n+1}, by simply creating an extra point between the points 1 and $2n$, and connecting it to the puncture via a link passing *below* or *above* all the crossed ones. Comparing the dimensions it is clear that P_0 and P_∞ cannot be surjective, though they are presumably injective. Using the same type of arguments as in Sect. 2, we find that there is an intertwining relation

$$\begin{aligned} P_0 T^*(t|z_1, \ldots, z_{2n}) &= T(t|z_1, \ldots, z_{2n}, 0) P_0 \\ P_\infty T^*(t|z_1, \ldots, z_{2n}) &= T(t|z_1, \ldots, z_{2n}, \infty) P_\infty \; . \end{aligned} \tag{3.1}$$

where T^* is the inhomogeneous transfer matrix acting on the span of LP^*_{2n}. This now leads to

$$\begin{aligned} B_0(z_1, \ldots, z_{2n}) P_0 \Phi^*(z_1, \ldots, z_{2n}) &= \Psi(z_1, \ldots, z_{2n}, 0) \\ B_\infty(z_1, \ldots, z_{2n}) P_\infty \Phi^*(z_1, \ldots, z_{2n}) &= \Psi(z_1, \ldots, z_{2n}, \infty) \end{aligned} \tag{3.2}$$

where once again $\Psi(z_1, \ldots, z_{2n}, \infty)$ denotes the highest degree ($= 2n$) terms in z_{2n+1} of Ψ and B_0 and B_∞ will be determined soon.

As a consequence, we find that the even case sum rule $B_0 v \cdot \Psi(z_1, \ldots, z_{2n})$ must be equal to the quantities (2.23) taken for $z_{2n+1} = 0$ while $B_\infty v \cdot \Psi(z)$ equals that for $z_{2n+1} = \infty$ (with the abovementioned division by an appropriate power of z_{2n+1}). These restrictions on the last spectral parameter have the effect of truncating the Schur functions. More precisely, when $z_{2n+1} \to 0$, the factor $s_{Y_{n+1}}(z_1, \ldots, z_{2n+1})$ tends to $s_{Y_{n+1}}(z_1, \ldots, z_{2n}) = (z_1 \ldots z_{2n}) s_{Y_n}(z_1, \ldots, z_{2n})$ where Y_n now has two rows of length $n-1$, two rows of length $n-2$, etc. two rows of length 1. Meanwhile, the factor $s_{Y_n'}(z_1, \ldots, z_{2n+1})$ tends to $s_{Y_n'}(z_1, \ldots, z_{2n})$. When $z_{2n+1} \to \infty$, the leading coefficient in $s_{Y_{n+1}}(z_1, \ldots, z_{2n+1})$ truncates to $s_{Y_n'}(z_1, \ldots, z_{2n+1})$ as explained before, while that in $s_{Y_n'}(z_1, \ldots, z_{2n+1})$ truncates to $s_{Y_n}(z_1, \ldots, z_{2n})$. This is consistent for $B_0 \propto z_1 \ldots z_{2n}$ and B_∞ a constant. In both limits, we reach the same sum rule

$$\sum_{\pi \in LP_{2n}^*} \Phi_\pi^*(z_1, \ldots, z_{2n}) = 3^{-n(n-1)} s_{Y_n}(z_1, \ldots, z_{2n})\, s_{Y_n'}(z_1, \ldots, z_{2n}) \, . \tag{3.3}$$

The proportionality factor is fixed by the even analogue of the recursion relation (2.16), together with the normalization of the component

$$\Phi_{\pi_0}^*(z_1, \ldots, z_{2n}) = \prod_{1 \le i < j \le 2n} \frac{q\, z_i - q^{-1} z_j}{q - q^{-1}} \tag{3.4}$$

for the fully nested link pattern π_0 connecting points i and $2n+1-i$, $i = 1, 2, \ldots, n$, while the puncture sits in the face delimited by the little arch connecting n to $n+1$ and the disk boundary.

4. The XXZ spin chain and six-vertex model at $\Delta = -1/2$

We present here results concerning another closely related model: the XXZ spin chain at the value $\Delta = \frac{q+q^{-1}}{2} = -1/2$ of the anisotropy. As above we shall need an inhomogeneous version of the model, which is the inhomogeneous six-vertex model. New results are found mostly in the case of an odd size chain, since the even case is already covered (if a little implicitly) by [8], but we shall need to introduce the model for arbitrary size anyway.

4.1. Definition of the XXZ spin chain

The XXZ spin chain is given by the Hamiltonian

$$\tilde{H} = -\frac{1}{2}\sum_{i=1}^{L}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z) \tag{4.1}$$

acting on $(\mathbb{C}^2)^{\otimes L}$, each $\mathbb{C}^2$ being a single spin space: $|+\rangle \equiv \binom{1}{0}$, $|-\rangle \equiv \binom{0}{1}$. We first take Δ and q generic, $\Delta = \frac{q+q^{-1}}{2}$. We choose here periodic boundary conditions for L odd: $\sigma_{L+1} \equiv \sigma_1$, and twisted periodic boundary conditions for L even: $\sigma^z_{L+1} \equiv \sigma^z_1$, $\sigma^\pm_{L+1} \equiv q^{\pm 2}\sigma^\pm_1$ where $\sigma^\pm = \sigma^x \pm i\sigma^y$. We can also write these as $\sigma_{L+1} \equiv \Omega\,\sigma_1\,\Omega^{-1}$ with $\Omega = \begin{pmatrix} -q & 0 \\ 0 & -q^{-1}\end{pmatrix}$ for L even and $\Omega = 1$ for L odd.

We also define the transfer matrix of the inhomogeneous six-vertex model (see also appendix B of [8] where the case L even is treated). For spectral parameters $z_1, \ldots, z_L$, it is given by

$$\tilde{T} \equiv \tilde{T}(t|z_1, \ldots, z_L) := \mathrm{tr}_0\left(R_{L,0}(z_L, t)\cdots R_{1,0}(z_1, t)\Omega\right) \tag{4.2}$$

where the R matrices act on the tensor product of the physical space $(\mathbb{C}^2)^{\otimes L}$, each factor being labelled by $i = 1, \ldots, L$, and of another $\mathbb{C}^2$, labelled by 0. The expression of R acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$ reads (in the so-called homogeneous gradation)

$$R(z,t) = \frac{1}{q\,t - q^{-1}z}\begin{pmatrix} q\,z - q^{-1}t & 0 & 0 & 0 \\ 0 & z-t & (q-q^{-1})t & 0 \\ 0 & (q-q^{-1})z & z-t & 0 \\ 0 & 0 & 0 & q\,z - q^{-1}t \end{pmatrix}. \tag{4.3}$$

The twist Ω acts on the auxiliary space, which is consistent with the chosen boundary conditions for the spin chain since $\Omega^{-1}R_{1,0}\Omega = R(\sigma_1, \Omega^{-1}\sigma_0\Omega) = R(\Omega\sigma_1\Omega^{-1}, \sigma_0) = R_{L+1,0}$ where the second equality follows from $U(1)$ invariance of the R-matrix.

When all z's are equal, $\tilde{T}$ commutes with the Hamiltonian $\tilde{H}$. In particular their ground states are identical.

4.2. Equivalence with the $O(1)$ loop model

For any q there is a mapping from the $O(1)$ loop model to the XXZ/six-vertex model. See also [16] for a similar construction. Call $\omega^{1/2}$ a square root of $\omega := -q$. To a link pattern π in LP_L associate the tensor product over the set of arches of π, of the vectors $\omega^{1/2}\,|+\rangle_j \otimes |-\rangle_k + \omega^{-1/2}\,|-\rangle_j \otimes |+\rangle_k$, where the indices j, k are the endpoints of the arch (and indicate the labels of the two spaces $\mathbb{C}^2$ in which these vectors live) ordered in the following way: $j < k$ for L even, while for L odd, $j < k < \ell$ or $\ell < j < k$ or $k < \ell < j$, ℓ being the label of the unmatched point. In the case L odd we also choose the spin of the unmatched point to be $|+\rangle_\ell$. The result is a vector in $(\mathbb{C}^2)^{\otimes L}$ which satisfies that its total z spin $s^z = 0$ or $s^z = 1/2$ depending on parity. For generic q this mapping, which we call S, is injective. In the case L odd its image is in fact the whole $s^z = 1/2$ subspace, of dimension $\binom{L}{(L-1)/2}$. In the case L even it is a subspace ($U_q(\mathfrak{sl}(2))$ singlets) of dimension $\frac{L!}{(L/2)!(L/2+1)!}$. Furthermore, it is easy to show that the transfer matrices of the $O(1)$ loop model and of the six-vertex model are intertwined by S:

$$\tilde{T}(t|z_1,\ldots,z_L)\,S = S\,T(t|z_1,\ldots,z_L)\ . \tag{4.4}$$

This is essentially a consequence of the fact that the R-matrix (4.3) involves just another representation of the Temperley–Lieb algebra; explicitly, if $\check{R} = R\mathcal{P}$ where $\mathcal{P}$ switches factors of tensor products, then the formula (2.6) holds with the Temperley–Lieb generator $e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ 0 & 1 & \omega^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The twist, necessary for L even to take care of the arbitrariness in the ordering $j < k$ of the indices, disappears for L odd thanks to the unmatched point. Henceforth, the loop model is equivalent to a sector of the six-vertex model.

At $q = -\mathrm{e}^{i\pi/3}$, however, the mapping S is no longer injective for L odd. In fact, one can show that the kernel of S is exactly the same as the kernel of the projector P_0 of Sect. 2.3, so that the dimension of the image is nothing else than $\frac{(2n+2)!}{(n+1)!(n+2)!}$, the dimension of the space of link patterns of size $2n+2$. So at this special point, the *odd* six-vertex model is equivalent to the *even* loop model of size one more.

4.3. Mapping of odd size to even size

In analogy with the loop model, we now define a mapping of the six-vertex model from $L = 2n+1$ to $L = 2n+2$. It is very simple: to a basis element α, that is a sequence of $2n+1$ spins, we associate the new vector $\alpha-$ obtained by concatenating α and an extra minus spin. Call Q this mapping. We also need the projections $P_\pm$ within the model of size $2n+2$ which project onto the subspaces where the last spin is $\pm$, orthogonally to the subspace where it is $\mp$, so that $P_+ + P_- = 1$, $P_+P_- = P_-P_+ = 0$.

We now have the two following properties of the $P_\pm$ and Q operators:

(i) The subspace $\mathrm{Im}(P_+)$ is stable under $\tilde{T}(z_1,\ldots,z_{2n+1},\infty)$:

$$P_+\tilde{T}(t|z_1,\ldots,z_{2n+1},0) = \tilde{T}(t|z_1,\ldots,z_{2n+1},0)P_+\ . \tag{4.5}$$

Indeed, if the last spectral parameter is zero, the corresponding matrix $R_{2n+2,0}$ becomes

$$R(0,t) = -q^{-1} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \tag{4.6}$$

which means if the spin $2n+2$ is $+$ it will stay so.

(ii) At $q^3=1$, Q intertwines the odd and even spin-chain transfer matrices, up to the projector P_-:

$$Q\tilde{T}(t|z_1,\ldots,z_{2n+1}) = P_-\tilde{T}(t|z_1,\ldots,z_{2n+1},0)Q \; . \tag{4.7}$$

This time we assume that the last spin is $-$. According to Eq. (4.6), it might become $+$ after action of $\tilde{T}$; so we project again with P_-. Now the action of $R_{2n+2,0}(0,t)$ becomes simply the twist of the auxiliary space with its lower right submatrix, that is the diagonal matrix $\left(\begin{smallmatrix} -q^{-1} & 0 \\ 0 & -q^{-2} \end{smallmatrix}\right)$: at $q^3=1$ this compensates exactly the twist $\Omega = \left(\begin{smallmatrix} -q & 0 \\ 0 & -q^{-1} \end{smallmatrix}\right)$ and we find that the r.h.s. of Eq. (4.7) is simply the transfer matrix of odd size with periodic boundary conditions and an additional spin at site $2n+2$ which is $-$.

4.4. Polynomial eigenvector

From now on we always set $q = -\mathrm{e}^{i\pi/3}$, $\omega^{1/2} = \mathrm{e}^{i\pi/6}$. It is easy to check that $\tilde{T}$, just like T, possesses the eigenvalue 1; we denote by $\tilde{\Psi}$ (resp. $\tilde{\Phi}$) the corresponding polynomial eigenvector for L odd (resp. even), normalized so that its entries are coprime. This leaves an arbitrary constant in the normalization which will be fixed below.

The various maps defined above allow us to write several relations between the various eigenvectors. First and foremost, using properties (i) and (ii) of Sect. 4.3 we find that

$$Q\tilde{\Psi}(z_1,\ldots,z_{2n+1}) = CP_-\tilde{\Phi}(z_1,\ldots,z_{2n+1},0) \tag{4.8}$$

or in components, $\tilde{\Psi}_\alpha(z_1,\ldots,z_{2n+1}) = C\tilde{\Phi}_{\alpha-}(z_1,\ldots,z_{2n+1},0)$, with C a normalization constant. Let us prove this. In this paragraph parameters are omitted with the assumption that $z_{2n+2}=0$. On the one hand, one can apply $\tilde{\Psi}$ to Eq. (4.7). We find: $Q\tilde{\Psi} = P_-\tilde{T}Q\tilde{\Psi}$. On the other hand, decompose $\tilde{\Phi} = P_+\tilde{\Phi} + P_-\tilde{\Phi}$ and apply $P_-\tilde{T}$: using Eq. (4.5) we find $P_-\tilde{T}P_-\tilde{\Phi} = P_-\tilde{\Phi}$. Comparing, we find that $Q\tilde{\Psi}$ and $P_-\tilde{\Phi}$ must be proportional. The proportionality factor must be a constant because components on both sides of the equation are coprime polynomials. Its numerical value will be determined below.

We also have the connection with loop models. For L even this was already discussed in appendix B of [8] and we find simply

$$S\Phi = \tilde{\Phi} \tag{4.9}$$

(up to a constant, which we fix to be 1). However for L odd, we only have

$$S\Psi = B(z_1, \ldots, z_{2n+1})\tilde{\Psi} \tag{4.10}$$

where B is a polynomial to be determined in next section. B appears because S is not injective (at $q^3 = 1$), so that the components of $S\Psi$ can have a non-trivial GCD. (Such a situation does not arise for L even since S is then injective).

4.5. Sum rule

We now wish to compute the sum of entries of $\tilde{\Psi}$. In order to do so we can rely on the mapping to the even sized system:

$$\sum_\alpha \tilde{\Psi}_\alpha(z_1, \ldots, z_{2n+1}) = C \sum_\alpha \tilde{\Phi}_{\alpha-}(z_1, \ldots, z_{2n+1}, 0) \tag{4.11}$$

and then use the mapping S to the loop model. The normalization constant C will be adjusted at the end of the computation. Via S, each link pattern of size $2n+2$ contributes $\omega^{1/2}(\omega^{1/2}+\omega^{-1/2})^n$ to the r.h.s. of Eq. (4.11), so $\sum_\alpha \tilde{\Phi}_{\alpha-} = \omega^{1/2}(\omega^{1/2}+\omega^{-1/2})^n \sum_\pi \Phi_\pi = \omega^{1/2} 3^{n/2} 3^{-n(n+1)/2} s_{Y_{n+1}}(z_1, \ldots, z_{2n+2})$ according to (1.13). So we have

$$\sum_\alpha \tilde{\Psi}_\alpha(z_1, \ldots, z_{2n+1}) = 3^{-n^2/2}\omega^{1/2} C s_{Y_{n+1}}(z_1, \ldots, z_{2n+1}) \ . \tag{4.12}$$

To adjust the constant C in (4.11), we finally impose that the "base component" of $\tilde{\Psi}$, *i.e.* its smallest component in the homogeneous limit, be 1. By (4.8), it is given by that of $\tilde{\Phi}$, itself proportional to that of Φ,

$$\tilde{\Psi}_{+\cdots+-\cdots-} = \omega^{(n+1)/2}\ C \prod_{\substack{1\le i<j\le n+1 \\ \text{or}\ n+2\le i<j\le 2n+1}} \frac{q\, z_i - q^{-1} z_j}{q - q^{-1}} \prod_{i=n+2}^{2n+1} \frac{q\, z_i}{q - q^{-1}}\ , \tag{4.13}$$

so at $z_i = 1$, $C^{-1} = (-i\sqrt{3})^{-n}\omega^{(n+1)/2} q^n = 3^{-n/2}\omega^{1/2}$ and the homogeneous sum rule is

$$\sum_\alpha \tilde{\Psi}_\alpha = 3^{-n(n-1)/2} s_{Y_{n+1}}(1^{2n+1}) = 3^{n/2}\mathcal{N}_n\ . \tag{4.14}$$

As a side-product, by using S directly on an odd-sized system, we can also compute the proportionality polynomial factor B:

$$B \sum_\alpha \tilde{\Psi}_\alpha = (\omega^{1/2} + \omega^{-1/2})^n \sum_\pi \Psi_\pi = 3^{n/2} 3^{-n^2} s_{Y_{n+1}} s_{Y'_n} \tag{4.15}$$

so that $B = 3^{-n(n-1)/2}\omega^{-1/2} C^{-1} s_{Y'_n} = 3^{-n^2/2} s_{Y'_n}(z)$.

4.6. Bilinear form and sum of squares

These sum rules possess obvious corollaries dealing with the sum of squares of components. Indeed, for generic q there is a bilinear form which to a pair of link patterns π and π' of LP_{2n+2} associates $\langle\pi|\pi'\rangle = (-q-1/q)^{\#}$ where # is the number of loops obtained by pasting together the two diagrams along their common boundary. Via S, this bilinear form becomes simply diagonal in the spin basis:

$$\langle\pi|\pi'\rangle = \sum_{\alpha}(S\pi)_\alpha(S\pi')_\alpha \ . \tag{4.16}$$

We now specialize to $q = -\mathrm{e}^{i\pi/3}$. Since $-q-1/q = 1$, we have the identity $v{\cdot}\Phi = \langle\Phi|\pi_0\rangle$ where π_0 is here any fixed link pattern. More generally, the bilinear form is degenerate of rank 1 so that we can write (in size $L = 2n+2$)

$$\sum_{\alpha}\tilde{\Phi}_\alpha^2 = \langle\Phi|\Phi\rangle = \langle\Phi|\pi_0\rangle\,\langle\pi_0|\Phi\rangle = (v\cdot\Phi)^2 = 3^{-n(n+1)}s_{Y_{n+1}}^2 \tag{4.17}$$

and in particular in the homogeneous limit, $\sum_\alpha \tilde{\Phi}_\alpha^2 = A_{n+1}^2$, which is (1.12$b$).

As noticed in [22], the even-size twisted XXZ Hamiltonian is invariant under simultaneous complex conjugation and spin reversal, and the ground state $\tilde{\Phi}$ is chosen to be invariant under this operation. Thus $\tilde{\Phi}_{\alpha+} = \overline{\tilde{\Phi}_{\overline{\alpha}-}}$, and (4.17) may be rewritten as

$$\langle\Phi|\Phi\rangle = (v\cdot\Phi)^2 = 3^{-n(n+1)}s_{Y_{n+1}}^2 = \sum_{\alpha}\tilde{\Phi}_\alpha^2 = \sum_{\alpha'}(\tilde{\Phi}_{\alpha'-}^2 + \tilde{\Phi}_{\alpha'+}^2) = 2\Re\Big(\sum_{\alpha'}\tilde{\Phi}_{\alpha'-}^2\Big)\ .$$

Using Eq. (4.8), we can then derive the sum rule for the squares of the $\tilde{\Psi}$, in odd size. By taking the real part of $C^{-2}\sum_{\alpha'}\tilde{\Psi}_{\alpha'}^2(z) = \sum_{\alpha'}\tilde{\Phi}_{\alpha'-}^2(z,0)$, (the $\tilde{\Psi}$ are real), and using the value of C determined above, we get

$$\sum_{\alpha'}\tilde{\Psi}_{\alpha'}^2 = \frac{\Re\sum_{\alpha'}\tilde{\Phi}_{\alpha'-}^2}{\Re C^{-2}} = 3^{-n^2}s_{Y_{n+1}}^2(z_1,\ldots,z_{2n+1},0)\ . \tag{4.18}$$

In the homogeneous limit, we get

$$\sum_{\alpha'}\tilde{\Psi}_{\alpha'}^2 = \mathcal{N}_n^2 = A_{\mathrm{HT}}(2n+1) \tag{4.19}$$

as announced in (1.11).

In contrast, the alledged sum rule (1.12c) seems to be of a different nature, since its deformation by spectral parameters does not involve symmetric functions of $z_1,\ldots,z_{2n}$.

5. Conclusion

In this paper, we have proved multi-parameter sum rules for the components of the groundstate vector of the inhomogeneous $O(1)$ loop model on a punctured disk of odd or even perimeter, or of its XXZ chain counterpart. Our strategy has relied on the construction of projection operators onto components of the groundstate vector of the same model, but on a non-punctured disk of perimeter one more, in which the extra inhomogeneity (spectral parameter) is taken to either 0 or ∞. The new ingredient here is the use of knot-theoretic crossing operators to relate the non-crossing link patterns of both models. Admittedly, our proof in the odd case still relies on a technical assumption that we have been unable to prove in general, namely the fact that the two Schur functions $s_{Y_{n+1}}$ and $s_{Y'_n}$ of Sect. 2.5 have no common factor. It would be highly desirable to fill this gap.

In the derivations of Sect. 2 and 3 we made an assumption of minimal degree, *viz* that the base component of Ψ or Φ is given by (2.13), (3.4). The justification of this assumption could be done as in [8], through the use of the Bethe Ansatz, or by appealing to the representation theory of the qKZ equation.

As already mentionned in Sect. 1, our formulae for the sums of components of the $O(1)$ loop model lead to the same expressions as those derived by Razumov and Stroganov for the partition functions of the square-ice model with adequate boundary conditions. The latter are designed so as to make a bijection between the states of the lattice model and half turn symmetric ASMs of size $L \times L$. In [20], the partition function of the six-vertex model on a $L \times \lfloor \frac{1}{2}(L+1) \rfloor$ grid with such "half-turn symmetric" boundary conditions [19] was computed as a function of spectral parameters x_i and y_j, $i, j = 1, \dots, \lfloor \frac{L+1}{2} \rfloor$. It is a symmetric function of the x's and the y's, thanks to the Yang–Baxter equation, and it was shown to be a product of two factors. For the special value of the crossing parameter corresponding to our $q = -e^{i\pi/3}$ (their $a = -q$), and in the odd size case ($L = 2n + 1$), upon specialization of the last spectral parameter $y_{n+1} = x_{n+1}$, it becomes a completely symmetric function of all its spectral parameters and it reads (in our notations)

$$\begin{aligned} &Z_{\mathrm{HT}}(x_1, \dots, x_n, y_1, \dots, y_n) && L = 2n \\ &\quad = (-3)^n \prod_{i=1}^{n} (x_i y_i)^{1-2n} s_{Y_n}(x_1^2, \dots, y_n^2) s_{Y'_n}(x_1^2, \dots, y_n^2) \\ &Z_{\mathrm{HT}}(x_1, \dots, x_{n+1}, y_1, \dots, y_n, y_{n+1} = x_{n+1}) && L = 2n+1 \\ &\quad = 3^n \prod_{i=1}^{n+1} x_i^{-2n} \prod_{i=1}^{n} y_i^{-2n} s_{Y_{n+1}}(x_1^2, \dots, x_{n+1}^2, y_1^2, \dots, y_n^2) s_{Y'_n}(x_1^2, \cdots, y_n^2) \end{aligned} \tag{5.1}$$

that is, up to a trivial factor $((-3)^{n^2}\prod(x_j y_j)^{1-2n}$, resp. $3^{n(n+1)}\prod x_j^{-2n}\prod y_j^{-2n})$, the same expression as $\sum_\pi \Phi^*$, resp. $\sum_\pi \Psi$. Recall that the partition function, upon some specialization of its spectral parameters, yields the "refined x-enumerations" of HTSASMs. The agreement between $Z_{\rm HT}(z)$ and the sum of the components is a further indication of the detailed connection between the two sets of problems: the determination of the Perron–Frobenius eigenvectors of the $O(1)$ loop models and the counting of HTSASMs. More precisely it gives support to the strong RS conjecture that Ψ_π or Φ^*_π count the number of HTSFPL configurations of link pattern π (see Sect. 1).

At this stage, our results do not, however, provide any direct way to verify the conjectures about the largest components of the Perron–Frobenius eigenvectors (see Sect. 1), nor to even confirm the empirical observation that in all cases (but the twisted $\tilde\Phi$) all components are positive integers once the smallest one is normalized to 1.

Another interesting question is what survives of these constructions for generic values of q (not at the RS point). One expects in particular from previous experience [10,23] that the point $q=-1$ have some geometrical interpretation, involving (multi-)degrees of some matrix schemes, and we indeed checked for odd sizes that all components of Ψ become (possibly vanishing) non-negative integers in the homogeneous limit $z_i \to 1$. It is not too difficult to convince oneself that the correct definition of the projection operators P_0 and P_∞ of Sect. 2 requires switching to different conventions for the crossing operators $T_i \equiv \check R_{i,i+1}(z,0)$. Via these projections, we may relate the vector Ψ at generic q to solutions of a modified qKZ equation for size one more, but with the extra spectral parameter sent to 0 or ∞. In the case of size $2n+1$, we have found that P_0 creates a solution of this modified qKZ equation in size $2n+2$ with $z_{2n+2}\to 0$, whose base component Θ_{π_0}, corresponding to the fully nested link pattern π_0 relating i to $2n+3-i$, $i=1,2,\ldots,n+1$, simply reads:

$$\Theta_{\pi_0}(z_1,\ldots,z_{2n+1}) = \Phi_{\pi_0}(z_1,\ldots,z_{2n+1},0)\; G(z_1,\ldots,z_{2n+1},\infty) \tag{5.2}$$

where Φ_{π_0} stands as before for the (completely factorized) base component of the loop model on an unpunctured disk of size $2n+2$, while G stands for the so-called Gaudin determinant

$$\begin{aligned} G(z_1,\ldots,z_{2n+2}) = {} & \frac{\prod_{1\le i,j\le n+1}(qz_i - q^{-1}z_{n+j+1})(q^{1/2}z_i - q^{-1/2}z_{n+j+1})}{\prod_{1\le i<j\le n+1}(z_i-z_j)(z_{n+i+1}-z_{n+j+1})} \\ & \times \det_{1\le i,j\le n+1}\left(\frac{1}{qz_i - q^{-1}z_{n+j+1}}\,\frac{1}{q^{1/2}z_i - q^{-1/2}z_{n+j+1}}\right) \end{aligned} \tag{5.3}$$

and the symbol ∞ in place of z_{2n+2} stands as usual for the suitably normalized large $z_{2n+2} \to \infty$ limit. Note that P_∞ does just the opposite, namely interchanges 0 and ∞ in Eq. (5.2). This is remarkably reminiscent of the higher degree solution to the qKZ equation found by Pasquier [9] on the even size unpunctured disks, for which the cyclicity condition analogous to that of the second line of eq.(2.14) has the *same* shift $s = q^3$ (as opposed to $s = q^6$ for the minimal degree solution).

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