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Topological expansion of the 2-matrix model correlation functions: diagrammatic rules for a residue formula

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Abstract: We solve the loop equations of the hermitian 2-matrix model to all orders in the topological $1/N^2$ expansion, i.e. we obtain all non-mixed correlation functions, in terms of residues on an algebraic curve. We give two representations of those residues as Feynman-like graphs, one of them involving only cubic vertices.

Keywords: Matrix Models, Differential and Algebraic Geometry.

Contents

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1. Introduction

The purpose of this article is to generalize the method invented in [13], for the 2-matrix model. The method of [13] is a diagrammatic technique for computing correlation functions of the 1-matrix model in terms of residues on some algebraic curve.

Random matrix models play an important role in physics and mathematics [32], and have a wealth of applications which are too long to list here. In this article, we consider "formal" random matrix integrals, which are known to be generating functions for counting some classes of discrete surfaces [7, 10, 21, 22, 34].

The partition function, free energy and correlation functions are all generating functions enumerating some kinds of graphs (respectively closed graphs, connected closed graphs, open graphs), which graphs can be seen as discrete surfaces.

In the formal model, the size N of matrices, is just a complex parameter, it needs not be an integer, and all observables (free energy, correlation functions) always have a $1/N$ expansion, because for each power of the expansion parameters, there is only a finite number of graphs with a given power of N. The power of N in a graph is its Euler characteristic, and thus the $1/N$ expansion is known as the "topological expansion" discovered by 't Hooft [34]. In the formal model, N is thus an expansion parameter, and working order by order in N enumerates only discrete surfaces of a given topology [7]. An efficient method for dealing with this formal model is to consider the Schwinger-Dyson equations, called loop equations in this context [10,33].

To large N limit (i.e. planar topologies), the solution of loop equations is known to be related to Toda hierarchy [9, 28, 35, 36]. For this reason, the large N expansion of matrix models plays an important role in integrable systems, and in many areas of physics [29]. It was understood by [11] that the low energy effective action of some string theory models is also described by matrix models.

In the beginning, formal matrix models were considered omlyin their 1-cut phase, because a potential which is a small deformation of a quadratic one, must have only one well, i.e. the variables perturbatively explore only one well. However, a $N \times N$ matrix has N eigenvalues, and even though each of them can explore perturbatively only one well, they do not need explore all the same well. That gives "multicut" solutions of matrix models, where the number of eigenvalues near each extremum of the potential is fixed (fixed filling fractions). Multicut solutions play an important role in string theory, as they describe multi-particle states [11,12]. Multicut solutions correspond to enumerating surfaces with contact terms, which can be called "foam of surfaces" as described in [6, 22].

The link between formal matrix models (which always have a $1/N$ expansion) and convergent matrix integrals (which have a $1/N$ expansion only in the 1-cut case under certain assumptions), has been better understood after the work of [6]. We emphasize again, that the results developed in this article concern the formal matrix model with fixed filling fractions, and should not be applied to convergent matrix model directly.

Recently, it has progressively become clear that large N expansion of random matrix models has a strong link with algebraic geometry [27]. The free energy and correlation functions have been computed in terms of properties of an algebraic curve. The large N limit of the 1-point correlation function (called the resolvent) is solution of an algebraic equation, which thus defines an algebraic curve. There have been many works which computed free energy and correlation functions in terms of that algebraic curve. The leading order resolvent and free energy were computed in the 1-cut case (algebraic curve of genus zero) in the pioneering work of [7], then some recursive method for computing correlation functions and free energy to all orders in $1/N$ were invented by [3, 4]. Those methods were first limited to 1-matrix case and 1-cut.

Then for 1-matrix several works have dealt with multicut: Akeman and Ambjørn found the first subleading term for the multicut resolvent and the 2-cut free energy [1, 2], Chekhov [8] and one of the authors together with Kokotov and Korotkin [17] found simultaneously the first subleading term for the multi-cut free energy . Then a (non-recursive) diagrammatic method was invented in [13] to find all correlation functions to all orders, in the multicut case.

The 1-matrix model, corresponds to hyper elliptical curves only. In order to have more general algebraic curves, one needs at least a 2-matrix model. For the 2-matrix models, the loop equations have been known since [33], and have been written in a concise form in [18–20]. They have been used to find the subleading term of the free energy, first in the genus zero case in [15], then in the genus 1 case in [16], and with arbitrary genus in [17]. The purpose of this article is to generalize the diagrammatic method of [13] for the computation of non-mixed correlation functions in the 2-matrix case. We solve the loop equations and present their solutions (the non-mixed correlation function's expansion) under two different diagrammatic forms. We first build a cubic diagrammatic representation before presenting an effective non cubic theory.

Outline of the article:

- In section 2, we introduce the model and our notations.
- Section 3 is dedicated to the derivation of loop equations. We derive the fundamental "master loop equation" before deriving loop equations whose solutions are non-mixed correlation functions
- In section 4, we show how a compact Riemann surface arises from the leading order of the master loop equation and present notations and tools of algebraic geometry needed for the computation of correlation functions.
- In section 5, we present a diagrammatic solution of the loop equations as cubic Feynman-like graphs.
- Section 6 is dedicated to the presentation of another representation of the non-mixed correlation functions as graphs of a non cubic effective theory.
- In section 7, we study the example of the gaussian case corresponding to the 1-matrix model limit.

2. Definitions and notations

2.1 Definition of the formal 2-matrix model with fixed filling fractions

In this article, we are interested in the study of the formal-two-matrix-model and the computation of a whole family of observables. The partition function Z is the formal matrix integral:

$$Z := \int_{H_n \times H_n} dM_1 dM_2 \, e^{-NTr(V_1(M_1)+V_2(M_2)-M_1M_2)} \tag{2.1}$$

where M_1 and M_2 are two $N \times N$ hermitian matrices, dM_1 and dM_2 the products of Lebesgue measures of the real components of M_1 and M_2 respectively, and V_1 and V_2 two polynomial potentials of degree d_1+1 and d_2+1 respectively :

$$V_1(x) = \sum_{k=1}^{d_1+1} \frac{g_k}{k} x^k \qquad , \qquad V_2(y) = \sum_{k=1}^{d_2+1} \frac{\tilde{g}_k}{k} y^k \tag{2.2}$$

Formal integral means it is computed as the formal power series expansion order by order in the g_k's (see [7,10,34]) of a matrix integral, where the non-quadratic terms in the potentials V_1 and V_2 are treated as perturbations near quadratic potentials. Such a perturbative expansion can be performed only near local extrema of $V_1(x) + V_2(y) - xy$, i.e. near points such that:

$$V_1'(\xi_i) = \eta_i \qquad , \qquad V_2'(\eta_i) = \xi_i \tag{2.3}$$

which has $d_1 d_2$ solutions. Therefore, if $\overline{M}_1$ and $\overline{M}_2$ are diagonal matrices, whose diagonal entries are some ξ_i's (resp. η_i's), $(\overline{M}_1, \overline{M}_2)$ is a local extremum of $\mathrm{tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)$ around which we can perform a perturbative expansion.

The choice of such an extremum, around which the perturbative series is computed, is equivalent to the choice of the number of eigenvalues near each pair (ξ_i, η_i), $i = 1, \dots, d_1 d_2$, i.e. the data of $d_1 d_2$ integers n_i such that:

$$\sum_{i=1}^{d_1 d_2} n_i = N \tag{2.4}$$

This means, that we can choose some contours $\mathcal{C}_i$, $i = 1, \dots, d_1 d_2$, such that the following equality holds order by order in the perturbative expansion:

$$\left\langle \frac{1}{2i\pi} \oint_{\mathcal{C}_i} \mathrm{tr} \, \frac{dx}{x - M_1} \right\rangle = -n_i \tag{2.5}$$

The numbers $\frac{n_i}{N}$ are called filling fractions. Thus, in the formal model, filling fractions are fixed parameters.

Fat graphs and discrete random surfaces

Once filling fractions are chosen, we perform the perturbative expansion. Each term of that formal expansion is an expectation value of a gaussian integral, and using Wick's theorem, each term can be represented by a Feynman graph. Because the integration variables are matrices, the graphs are "fat graphs", which have a 2-dimensional structure. The Hermitean matrix models thus enumerate oriented surfaces (other matrix ensembles can enumerate non-oriented surfaces). This Formal expansion equivalent to an enumerating function of Feynman graphs is a standard tool in physics [10, 34]. Random matrices have thus played a role in all theories where one needs to sum over surfaces, i.e. string theory and quantum gravity (i.e. statistical physics on a random lattice).

Following this interpretation, the loop equations [33] can be understood as relationships linking surfaces of different genus and different number of boundaries.

2.2 Notations

2.2.1 Notation for sets of variables

We will consider functions of many variables $x_1, x_2, x_3, \dots, x_k$, or of a subset of those variables. In that purpose we introduce the following notations:

Let K be a $k-$upple of integers:

$$K = (i_1, i_2, \dots, i_k) \tag{2.6}$$

We denote $k = |K|$ the length (or cardinal) of K. For any $j \leq |K|$, we denote K_j the set of all $j-$upples (i.e. subsets of length j) contained in K:

$$K_j := \{J \subset K \ , \ |J| = j\} \tag{2.7}$$

We define the following $k-$upple of complex numbers:

$$\mathbf{x}_K := (x_{i_1}, x_{i_2}, \dots, x_{i_k}) \tag{2.8}$$

2.2.2 Correlation functions

For a given k, we define the correlation function:

$$\overline{w}_k(x_1, \dots, x_k) := N^{k-2} \left\langle \prod_{i=1}^{k} \operatorname{tr} \frac{1}{x_i - M_1} \right\rangle_c \qquad , \tag{2.9}$$

i.e., with the previous notations:

$$\overline{w}_{|K|}(\mathbf{x}_K) := N^{|K|-2} \left\langle \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \qquad , \tag{2.10}$$

where the formal average $\langle . \rangle$ is computed with the measure in eq. (2.1), and the subscript c means connected part (cumulant).

Those correlation functions can be expanded as formal series in $\frac{1}{N^2}$ in the large N limit:

$$\overline{w}_k(\mathbf{x}_K) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} \, \overline{w}_k^{(h)}(\mathbf{x}_K) \tag{2.11}$$

The purpose of this article is to compute $\overline{w}_k^{(h)}(\mathbf{x}_K)$ as residues on an algebraic curve and represent it with Feynman-like graphs of a cubic field theory on the curve.

We also define the following auxiliary functions:

$$\overline{u}_k(x, y; \mathbf{x}_K) := N^{|K|-1} \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \tag{2.12}$$

$$p_k(x, y; \mathbf{x}_K) := N^{|K|-1} \left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \tag{2.13}$$

$$a_k(x; \mathbf{x}_K) := N^{|K|-1} \left\langle \operatorname{tr} \frac{1}{x - M_1} V_2'(M_2) \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \tag{2.14}$$

Notice that $\overline{u}_{k,}(x, y; \mathbf{x}_K)$ is a polynomial in y of degree $d_2 - 1$, and $p_k(x, y; \mathbf{x}_K)$ is a polynomial in x of degree $d_1 - 1$ and in y of degree $d_2 - 1$.

It is convenient to renormalize those functions, and define:

$$u_k(x, y; \mathbf{x}_K) := \overline{u}_k(x, y; \mathbf{x}_K) - \delta_{k,0}(V_2'(y) - x) \tag{2.15}$$

and

$$w_k(\mathbf{x}_K) := \overline{w}_k(\mathbf{x}_K) + \frac{\delta_{k,2}}{(x_1 - x_2)^2} \tag{2.16}$$

Let us remark that all those functions have the same kind of topological expansion as $\overline{w}_k(\mathbf{x}_K)$ and one defines $p_k^{(h)}(x, y; \mathbf{x}_K)$ and $u_k^{(h)}(x, y; \mathbf{x}_K)$ as well like in eq. (2.11).

We define the function:

$$Y(x) := V_1'(x) - w_1(x) \tag{2.17}$$

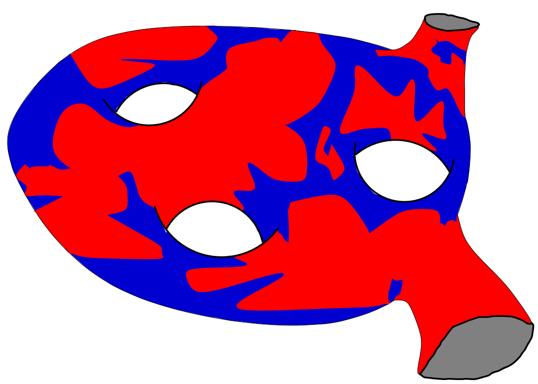
which we see below, describes the algebraic curve.

The $\frac{1}{N^2}$ expansion of such correlation functions is known to enumerate discrete surfaces of a given topology, whose polygons carry a spin + or - (Ising model on a random surface [26, 29]), see [22] for the multicut case i.e. foam of Ising surfaces.

The $\overline{w}_k^{(h)}$ are generating functions enumerating genus h discrete surfaces with k boundaries of spin +.

As an example, $\overline{w}_2^{(3)}$ enumerates surfaces of genus 3 with 2 boundaries:

$$\overline{w}_2^{(3)} =$$



(2.18)

Notice that the question of boundaries with non uniform spin, i.e. with changes of boundary conditions has been solved to leading order only in [14].

3. Loop equations

There exist several methods for computing the free energy and correlation functions, the one we consider here is the "loop equation" method, which is nothing but Schwinger-Dyson, or Ward identities [10, 33]. They implement the Virasoro or

W-algebra constraints on the partition function [27,31], i.e. the fact that the matrix integral is left unchanged under a change of variable. The loop equations are valid in the formal model, order by order in the expansion parameters.

For the 2-matrix model, loop equations have been known since [33], and written in a more systematic way in [18–20, 27].

3.1 The master loop equation

It is well known that in the large N limit, loop equations imply an algebraic equation for the functions w_1, i.e. for the function $Y(x)$, called the master loop equation. Let us briefly recall how to derive it (see [20]):

• the change of variables $M_2 \to M_2 + \epsilon \frac{1}{x-M_1}$ implies:

$$0 = a_0(x) - x\overline{w}_1(x) + 1 \tag{3.1}$$

• the change of variables $M_1 \to M_1 + \epsilon \frac{1}{x-M_1} \frac{V_2'(y)-V_2'(M_2)}{y-M_2}$ implies:

$$\begin{aligned} \overline{w}_1(x)\overline{u}_0(x,y) + \frac{1}{N^2} u_1(x,y;x) &= V_1'(x)\overline{u}_0(x,y) - p_0(x,y) - y\overline{u}_0(x,y) \\ &\quad + V_2'(y)w_1(x) - a_0(x) \end{aligned} \tag{3.2}$$

i.e., putting everything together:

$$(y - Y(x))u_0(x,y) + \frac{1}{N^2}u_1(x,y;x) = (V_2'(y) - x)(V_1'(x) - y) - p_0(x,y) + 1 \tag{3.3}$$

We define:

$$E(x,y) = (V_2'(y) - x)(V_1'(x) - y) - p_0(x,y) + 1 \tag{3.4}$$

The *master loop equation* is thus:

$$\boxed{(y - Y(x))u_0(x,y) + \frac{1}{N^2}u_1(x,y;x) = E(x,y)} \tag{3.5}$$

where $E(x,y)$ is a polynomial of degree $d_1 + 1$ in x and $d_2 + 1$ in y.

3.2 Loop equations for correlation functions

We now derive the loop equations which allow to compute recursively the k-point non-mixed correlation functions.

• The change of variables $\delta M_2 = \frac{1}{x-M_1} \prod_{i=1}^{k} \operatorname{tr} \frac{1}{x_i - M_1}$ implies (see [20]):

$$a_k(x;\mathbf{x}_K) = x\,\overline{w}_{k+1}(x,\mathbf{x}_K) - N^2\overline{w}_k(\mathbf{x}_K) \tag{3.6}$$

• The change of variables $\delta M_1 = \frac{1}{x-M_1} \frac{V_2'(y)-V_2'(M_2)}{y-M_2} \prod_{i=1}^k \operatorname{tr} \frac{1}{x_i-M_1}$ implies (see [20]):

$$\begin{aligned}
& w_1(x)\,\overline{u}_k(x,y;\mathbf{x}_K) + \sum_{j=0}^{k-1} \sum_{J\in K_j} \overline{u}_j(x,y;\mathbf{x}_J)\,\overline{w}_{k-j+1}(x,\mathbf{x}_{K-J}) \\
& + \frac{1}{N^2}\overline{u}_{k+1}(x,y;x,\mathbf{x}_K) \\
& + \sum_{j=1}^{k} \frac{\partial}{\partial x_j} \frac{\overline{u}_{k-1}(x,y;\mathbf{x}_{K-\{j\}}) - \overline{u}_{k-1}(x_j,y;\mathbf{x}_{K-\{j\}})}{x-x_j} \\
= \; & V_1'(x)\overline{u}_{k,0}(x,y;\mathbf{x}_K) - p_k(x,y;\mathbf{x}_K) \\
& -y\overline{u}_k(x,y;\mathbf{x}_K) + V_2'(y)\overline{w}_{k+1}(x,\mathbf{x}_K) - a_k(x;\mathbf{x}_K)
\end{aligned} \tag{3.7}$$

i.e. for $k \geq 1$:

$$\boxed{\begin{aligned}
(y-Y(x))\,u_k(x,y;\mathbf{x}_K) = & -\sum_{j=0}^{k-1}\sum_{J\in K_j} u_j(x,y;\mathbf{x}_J)\,w_{k-j+1}(x,\mathbf{x}_{K-J}) \\
& -\frac{1}{N^2}u_{k+1}(x,y;x,\mathbf{x}_K) \\
& +\sum_{j=1}^{k} \frac{\partial}{\partial x_j} \frac{u_{k-1}(x_j,y;\mathbf{x}_{K-\{j\}})}{x-x_j} - p_k(x,y;\mathbf{x}_K)
\end{aligned}} \tag{3.8}$$

The purpose of this article is to solve eq. (3.8) and compute $\overline{w}_k^{(h)}$ for all k and h.

4. Leading order and algebraic geometry

4.1 Leading order of the master loop equation

To large N leading order, the master loop equation eq. (3.5) reads:

$$\boxed{(y-Y(x))u_0(x,y) = E(x,y)} \tag{4.1}$$

Since $u_0(x,y)$ is a polynomial in y, it has no singularity for y finite and the LHS vanishes for $y = Y(x)$, i.e.:

$$E(x,Y(x)) = 0 \tag{4.2}$$

This defines an algebraic curve $E(x,y) = 0$.

Notice that to leading order we have:

$$u_0(x,y) = \frac{E(x,y)}{y-Y(x)} \tag{4.3}$$

and

$$u_0(x,Y(x)) = E_y(x,Y(x)) \tag{4.4}$$

4.2 Introduction to some algebraic geometry

We use notations similar to [25] or [24]. Some useful hints for understanding this section can be found in *Appendix A*.

Let us parameterize the curve $E(x, y) = 0$ with a running point p of a compact Riemann surface $\mathcal{E}$. It means that we define two meromorphic functions $x(p)$ and $y(p)$ on $\mathcal{E}$ such that:

$$E(x, y) = 0 \Leftrightarrow \exists p \in \mathcal{E} \quad x = x(p)\ ,\ y = y(p) \tag{4.5}$$

The functions x and y are not bijective. Indeed, since $E(x, y)$ is a polynomial of degree $d_2 + 1$ in y, it has $d_2 + 1$ solutions, i.e. for a given x, there exist $d_2 + 1$ points p on $\mathcal{E}$ such that $x(p) = x$. Thus, the Riemann surface is made of $d_2 + 1$ x-sheets, respectively $d_1 + 1$ y-sheets. Hence, from now on, we use these notations:

$$x(p) = x \Leftrightarrow p = p^j(x) \quad \text{for} \quad j = 0, \dots, d_2 \tag{4.6}$$

$$y(p) = y \Leftrightarrow p = \tilde{p}^j(x) \quad \text{for} \quad j = 0, \dots, d_1 \tag{4.7}$$

We will most often omit the exponent 0 corresponding to the physical sheet: $p = p^0$.

For instance, one can write $E(x, y)$ as:

$$\begin{aligned} E(x(p), y(q)) &= -g_{d_1+1} \times \prod_{i=0}^{d_1} (x(p) - x(\tilde{q}^i(y))) \\ &= -\tilde{g}_{d_2+1} \times \prod_{i=0}^{d_2} (y(q) - y(p^i(x))) \end{aligned} \tag{4.8}$$

Considering that the $w_k^{(h)}$'s, $u_k^{(h)}$'s and $p_k^{(h)}$'s are multivalued functions in their arguments x, we now work with differentials monovalued on the Riemann surface. Let us write the differentials:

$$W_{k+1}(p, \mathbf{p}_K) := w_{k+1}(x(p), \mathbf{x}(p_K)) dx(p) \prod_{i=1}^{k} dx(p_i) \tag{4.9}$$

$$U_k(p, y; \mathbf{p}_K) := u_k(x(p), y; \mathbf{x}(p_K)) dx(p) \prod_{i=1}^{k} dx(p_i) \tag{4.10}$$

$$P_k(x, y; \mathbf{p}_K) := p_k(x, y; \mathbf{x}(p_K)) \prod_{i=1}^{k} dx(p_i) \tag{4.11}$$

Note: In the following, the arguments of a function will be called $x(p)$ or $y(r)$ if the function is defined on the basis, and p or r if the function is defined on the Riemann surface - and so multivalued on the basis-.

Let us now review the notations we use in this article to denote some basic objects. For definitions and details, we refer the reader to *Appendix A* and [25] or [24].

•**Canonical cycles:** $\mathcal{A}_i$, $\mathcal{B}_i$ for $i = 1, \dots, g$ where g is the genus of the compact Riemann surface $\mathcal{E}$ $(0 \leq g \leq d_1 d_2 - 1)$, such that:

$$\mathcal{A}_i \cap \mathcal{B}_i = \delta_{i,j} \tag{4.12}$$

•**Branch points in** x**:** They are the zeroes of dx on the surface. We denote them by a_i, $i = 1, \dots, d_2 + 1 + 2g$.

•**Bergmann kernel:** It is the unique bilinear differential with only one double pole at $p = q$ satisfying:

$$B(p,q) \underset{p\to q}{\sim} \frac{dx(p)dx(q)}{(x(p)-x(q))^2} + \text{finite} \quad \text{and} \quad \forall i \ \oint_{p\in\mathcal{A}_i} B(p,q) = 0 \tag{4.13}$$

•**Abelian differential of third kind:** It is the differential defined by $dS_{q,r}(p) = \int_{q'=r}^{q} B(p,q')$. Notice that it has the following properties:

$$\operatorname*{Res}_{p\to q} dS_{q,r}(p) = 1 = -\operatorname*{Res}_{p\to r} dS_{q,r}(p) \quad \text{and} \quad \forall i \ \oint_{\mathcal{A}_i} dS_{q,r}(p) = 0 \tag{4.14}$$

4.3 Fixed filling fractions

To large N leading order, the loop equation eq. (4.2) is an algebraic equation:

$$E(x, Y(x)) = 0 \tag{4.15}$$

The coefficients of E are determined using filling fractions. Since $w_1(x) = V_1'(x) - Y(x)$, eq. (2.5) gives (up to a redefinition of $\mathcal{A}_i$):

$$\frac{1}{2i\pi}\oint_{\mathcal{A}_i} y dx = -\frac{1}{2i\pi}\oint_{\mathcal{A}_i} x dy = \epsilon_i \tag{4.16}$$

Let us recall that (see section 2.1) the ϵ_i's are called filling fractions, and they are given parameters (moduli) of the model. They don't depend on the potential or on any other parameter.

In particular, since all correlation functions $w_k(x_1, \dots, x_k)$ are obtained by derivation of w_1 with respect to the potential V_1 ([4]), we have for $k \geq 2$:

$$\frac{1}{2i\pi}\oint_{\mathcal{A}_i} w_k(x_1, \dots, x_k) dx_1 = 0 \tag{4.17}$$

Equation eq. (4.16) together with the large x and y behaviors eq. (A.2) and eq. (A.1), are sufficient to determine completely all the coefficients of the polynomial $E(x, y)$, and thus the leading large N resolvent $w_1(x)$.

In what follows, we assume that the leading resolvent, i.e. the function $Y(x)$ is known, and we refer the reader to the existing literature on that topic, for instance [5, 20, 27, 30].

5. Diagrammatic solution as cubic graphs

In this section we present a first way of describing the solution of the loop equation eq. (3.8) by trivalent diagrams whose h loop level corresponds to the h-th term $W_k^{(h)}$ of the topological expansion.

5.1 Solution in the planar limit

Before considering the full $\frac{1}{N^2}$ expansion, let us focus on the structure of the leading terms corresponding to planar fat graphs. Thus the $1/N^2$ terms in the loop equations are omited.

From now on and particularly in this paragraph, we drop the genus zero exponent (0) when it is clear that we deal with the planar limit, i.e. $w_k^{(0)}(\mathbf{x}_K) \to w_k(\mathbf{x}_K)$.

Up to now, the loop equations were written in terms of multivalued functions. It is more appropriate to write them in terms of meromorphic differentials on the Riemann surface. Thus, one writes eq. (3.8) in the planar limit as follows:

$$\begin{aligned}(y(r)-y(p))U_k(p,y(r);\mathbf{p}_K) = & -\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{U_j(p,y(r);\mathbf{p}_J)\,W_{k-j+1}(p,\mathbf{p}_{K-J})}{dx(p)} \\ & +\sum_{j=1}^{k} d_{p_j}\left(\frac{U_{k-1}(p_j,y(r);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\,\frac{dx(p)}{dx(p_j)}\right) \\ & -P_k(x(p),y(r);\mathbf{p}_K)dx(p)\end{aligned} \tag{5.1}$$

Starting from eq. (5.1), we determine W_k and U_k for any k by recursion on k.

Let us assume that one knows $W_j(\mathbf{p}_J)$ for $j \leq k$ and $U_j(p,\mathbf{p}_J)$ for $j \leq k-1$. The first step consists in the determination of $W_{k+1}(p,\mathbf{p}_K)$ as a function of the lower order correlation functions. The second step leads to the computation of $U_k(p,\mathbf{p}_K)$. Once this is done, one knows the correlation functions one order upper. The initial terms W_2 and U_1 can be found in the literature [5, 20, 27] and are rederived in *Appendix B*.

5.1.1 Determination of W_{k+1} for $k \geq 2$

If one chooses $r = p$ in eq. (5.1), one gets (using eq. (4.3) and eq. (4.4)):

$$\begin{aligned}E_y(x(p),y(p))W_{k+1}(p,\mathbf{p}_K) = & -P_k(x(p),y(p);\mathbf{p}_K)\,dx(p) \\ & -\sum_{j=1}^{k-1}\sum_{J\in K_j}\frac{U_j(p,y(p);\mathbf{p}_J)\,W_{k-j+1}(p,\mathbf{p}_{K-J})}{dx(p)} \\ & +\sum_{j=1}^{k} d_{p_j}\left(\frac{U_{k-1}(p_j,y(p);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\,\frac{dx(p)}{dx(p_j)}\right)\end{aligned} \tag{5.2}$$

Notice that the two equations eq. (5.1) and eq. (5.2) imply by recursion, that W_k and U_k are indeed meromorphic differentials on the curve, in all their variables.

We define:

$$\forall(i,j) \qquad R_k^i(p^j,p_K) := \frac{U_k(p^j,y(p^i);p_K)}{E_y(x(p^j),y(p^i))dx(p^j)} \tag{5.3}$$

Note that we have already obtained (see eq. (4.3)) that:

$$R_0^i(p^l) = \delta_{i,l} \tag{5.4}$$

Using eq. (4.14), the Cauchy formula gives:

$$W_{k+1}(p, \mathbf{p}_K) = - \mathop{\mathrm{Res}}_{p' \to p} W_{k+1}(p', \mathbf{p}_K) dS_{p',o}(p) \tag{5.5}$$

where $o \in \mathcal{E}$ is an arbitrary point on the Riemann surface.

The integrand has poles in p' only at $p' = p$ and the branch points $p' = a_s$ (this can be proven recursively by differentiating wrt the potential $\frac{\partial}{\partial V_1}$). Using Riemann bilinear identity eq. (A.10), we can then move the integration contour and get:

$$W_{k+1}(p, \mathbf{p}_K) = \sum_s \mathop{\mathrm{Res}}_{p' \to a_s} W_{k+1}(p', \mathbf{p}_K) dS_{p',o}(p) \tag{5.6}$$

We now introduce the loop equation eq. (5.2) inside this expression and remark that only one term has poles when $p' \to a_s$. Thus $W_{k+1}(p, \mathbf{p}_K)$ can be written:

$$\begin{aligned} W_{k+1}(p, \mathbf{p}_K) &= -\sum_s \mathop{\mathrm{Res}}_{p' \to a_s} \sum_{j=1}^{k-1} \sum_{J \in K_j} \frac{U_j(p', y(p'); \mathbf{p}_J)}{E_y(x(p'), y(p'))} \frac{W_{k-j+1}(p', \mathbf{p}_{K-J})}{dx(p')} dS_{p',o}(p) \\ &= -\sum_s \mathop{\mathrm{Res}}_{p' \to a_s} \sum_{j=1}^{k-1} \sum_{J \in K_j} R_j^0(p', \mathbf{p}_J) W_{k-j+1}(p', \mathbf{p}_{K-J}) dS_{p',o}(p) \end{aligned} \tag{5.7}$$

Notice that $U_k(p, y; \mathbf{p}_K)$ is a polynomial in y whose degree is equal to $d_2 - 1$. Considering its d_2 values for $y = y(p^i)$ with $i \in [1, d_2]$, the interpolation formula reads:

$$\forall y \quad \frac{(y - y(p))U_k(p, y; \mathbf{p}_K)}{E(x(p), y)} = -\sum_{i=1}^{d_2} \frac{U_k(p, y(p^i); \mathbf{p}_K)(y(p) - y(p^i))}{(y - y(p^i))E_y(x(p), y(p^i))} \tag{5.8}$$

for $y = y(p)$, this gives:

$$R_k^0(p, \mathbf{p}_K) = -\sum_{i=1}^{d_2} R_k^i(p, \mathbf{p}_K) \tag{5.9}$$

So, in eq. (5.7), one obtains the recursive formula for $W_k(\mathbf{p}_K)$:

$$\boxed{W_{k+1}(p, \mathbf{p}_K) = \sum_{i=1}^{d_2} \sum_{j=1}^{k-1} \sum_{J \in K_j} \sum_s \mathop{\mathrm{Res}}_{p' \to a_s} R_j^i(p'; \mathbf{p}_J) W_{k-j+1}(p', \mathbf{p}_{K-J}) dS_{p',o}(p)} \tag{5.10}$$

The sum over j represents the summation over all partitions of K into two subsets J and $K - J$.

5.1.2 Determination of R^i_k

In this section, we find a recursion formula for R^i_k.

For this purpose, one needs to know an intermediate expression defining the different U_k's as well as a relation linking the value of

$$\sum_{j=0}^{k-1} U_j(p^i, y(p); \mathbf{p}_J) W_{k-j+1}(p^i, \mathbf{p}_{K-J}) \tag{5.11}$$

for different i's.

Let us rewrite here eq. (5.1):

$$\begin{aligned}(y(r)-y(q))U_k(q,y(r);\mathbf{p}_K) = &-\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{1}{dx(q)}\,U_j(q,y(r);\mathbf{p}_J)\,W_{k-j+1}(q,\mathbf{p}_{K-J})\\ &+\sum_{j=1}^{k} d_{p_j}\left(\frac{U_{k-1}(p_j,y(r);\mathbf{p}_{K-\{j\}})}{x(q)-x(p_j)}\,\frac{dx(q)}{dx(p_j)}\right)\\ &-P_k(x(q),y(r);\mathbf{p}_K)dx(q)\end{aligned} \tag{5.12}$$

In what follows, we use the properties of rational functions defined on the basis and not on the Riemann surface (for some more details, see the case $k=1$ in *Appendix B*).

For $r=q=p^i$, eq. (5.12) reads:

$$\begin{aligned}0 = &-\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{1}{dx(p^i)}\,U_j(p^i,y(p^i);\mathbf{p}_J)\,W_{k-j+1}(p^i,\mathbf{p}_{K-J})\\ &+\sum_{j=1}^{k} d_{p_j}\left(\frac{U_{k-1}(p_j,y(p^i);\mathbf{p}_{K-\{j\}})}{x(p^i)-x(p_j)}\,\frac{dx(p^i)}{dx(p_j)}\right)\\ &-P_k(x(p^i),y(p^i);\mathbf{p}_K)dx(p^i)\\ = &-\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{1}{dx(p)}\,U_j(p^i,y(p^i);\mathbf{p}_J)\,W_{k-j+1}(p^i,\mathbf{p}_{K-J})\\ &+\sum_{j=1}^{k} d_{p_j}\left(\frac{U_{k-1}(p_j,y(p^i);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\,\frac{dx(p)}{dx(p_j)}\right)\\ &-P_k(x(p),y(p^i);\mathbf{p}_K)dx(p)\end{aligned} \tag{5.13}$$

where we have used that $x(p)=x(p^i)$.

Now, write eq. (5.12) with $r=p^i$ and $q=p$:

$$\begin{aligned}&(y(p^i)-y(p))U_k(p,y(p^i);\mathbf{p}_K)\\ &= -\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{1}{dx(p)}\,U_j(p,y(p^i);\mathbf{p}_J)\,W_{k-j+1}(p,\mathbf{p}_{K-J})\end{aligned}$$

$$+\sum_{j=1}^{k} d_{p_j}\left(\frac{U_{k-1}(p_j,y(p^i);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\,\frac{dx(p)}{dx(p_j)}\right)$$
$$-P_k(x(p),y(p^i);\mathbf{p}_K)dx(p) \qquad (5.14)$$

and inserting eq. (5.13) we get:

$$\begin{aligned}&(y(p^i)-y(p))U_k(p,y(p^i);\mathbf{p}_K)\\ =& -\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{1}{dx(p)}\,U_j(p,y(p^i);\mathbf{p}_J)\,W_{k-j+1}(p,\mathbf{p}_{K-J})\\ &+\sum_{j=0}^{k-1}\sum_{J\in K_j}\frac{1}{dx(p)}\,U_j(p^i,y(p^i);\mathbf{p}_J)\,W_{k-j+1}(p^i,\mathbf{p}_{K-J}) \qquad (5.15)\end{aligned}$$

This formula is in principle sufficient to compute the U_k's recursively, and then, one can compute the R^i_k's. However, what we need in order to get diagrammatic rules, is a closed recursion relation for the R^i_k's themselves. In order to achieve this aim, we show that:

Lemma: for any $k\geq 1$, one has:

$$U_k(p,y;\mathbf{p}_K) = \frac{E(x(p),y)dx(p)}{y-y(p)}\sum_{r=1}^{d_2}\sum_{K_1\cup\ldots\cup K_r=K}\sum_{j_1\neq j_2\neq\ldots\neq j_r=1}^{d_2}\prod_{t=1}^{r}\frac{W_{|K_t|+1}(p^{j_t},\mathbf{p}_{K_t})}{(y-y(p^{(j_t)}))\,dx(p)} \qquad (5.16)$$

where the sum over $K_1\cup\ldots\cup K_r=K$ is a sum over all partitions of K into r subsets.

Proof: It can be proven easily by recursive action of $\partial/\partial V_1$, as in [4], however, in order to have a self-contained method, we want to derive it here only from the loop equations eq. (3.8).

The proof works by recursion on k. It is proven in *Appendix B* for $k=1$. Let us assume that, it holds for any $l\leq k-1$.

Notice, that since both sides of eq. (5.16) are polynomials of y, of degree d_2-1, it is sufficient to prove that the equality holds for d_2 values of y, namely, it is sufficient to prove it for $y=y(p^i)$, $i=1,\ldots,d_2$. Therefore, one has to prove that:

$$\frac{U_k(p,y(p^i);\mathbf{p}_K)}{dx(p)} = \frac{E_y(x(p^i),y(p^i))}{y(p^i)-y(p)}\sum_{r=1}^{d_2}\sum_{K_1\cup\ldots\cup K_r=K}\sum_{j_1\neq j_2\neq\ldots\neq j_{r-1}\neq 0,i}\frac{W_{|K_r|+1}(p^i,\mathbf{p}_{K_r})}{dx(p)}\prod_{t=1}^{r-1}\frac{W_{|K_t|+1}(p^{j_t},\mathbf{p}_{K_t})}{(y-y(p^{j_t}))\,dx(p)} \qquad (5.17)$$

where only the sums in which one of the j_t's is equal to i contribute.

The recursion hypothesis for $j \leq k-1$, and any $J \in K_j$ gives:

$$\frac{U_j(p^i,y(p^i);\mathbf{p}_J)}{dx(p)} = E_y(x(p^i),y(p^i)) \sum_{r=1}^{d_2} \sum_{J_1\cup\ldots\cup J_r=J} \sum_{j_1\neq j_2\neq\ldots\neq j_r\neq i} \prod_{t=1}^{r} \frac{W_{|J_t|+1}(p^{j_t},\mathbf{p}_{J_t})}{(y(p^i)-y(p^{j_t}))\,dx(p)} \quad (5.18)$$

In order to compute $U_j(p,y(p^i);\mathbf{p}_J)$, one has to keep only terms in the sum such that there exists a t such that $j_t=i$, i.e.

$$\frac{U_j(p,y(p^i);\mathbf{p}_J)}{dx(p)} = E_y(x(p^i),y(p^i)) \sum_{r=1}^{d_2} \sum_{J_1\cup\ldots\cup J_r=J} \sum_{j_1\neq j_2\neq\ldots\neq j_{r-1}\neq 0,i} \frac{W_{|J_r|+1}(p^i,\mathbf{p}_{J_r})}{(y(p^i)-y(p))\,dx(p)} \prod_{t=1}^{r-1} \frac{W_{|J_t|+1}(p^{j_t},\mathbf{p}_{J_t})}{(y(p^i)-y(p^{j_t}))\,dx(p)} \quad (5.19)$$

Insert that into eq. (5.15):

$$\begin{aligned} & (y(p^i)-y(p))U_k(p,y(p^i);\mathbf{p}_K) \\ = \; & -E_y(x(p^i),y(p^i)) \sum_{j=0}^{k-1}\sum_{J\in K_j}\sum_{r=1}^{d_2}\sum_{J_1\cup\ldots\cup J_r=J}\sum_{j_1\neq j_2\neq\ldots\neq j_{r-1}\neq 0,i} \\ & \quad W_{k-j+1}(p,\mathbf{p}_{K-J}) \frac{W_{|J_r|+1}(p^i,\mathbf{p}_{J_r})}{(y(p^i)-y(p))\,dx(p)} \prod_{t=1}^{r-1} \frac{W_{|J_t|+1}(p^{j_t},\mathbf{p}_{J_t})}{(y(p^i)-y(p^{j_t}))\,dx(p)} \\ & +E_y(x(p^i),y(p^i)) \sum_{j=0}^{k-1}\sum_{J\in K_j}\sum_{r=1}^{d_2}\sum_{J_1\cup\ldots\cup J_r=J}\sum_{j_1\neq j_2\neq\ldots\neq j_r\neq i} \\ & \quad W_{k-j+1}(p^i,\mathbf{p}_{K-J}) \prod_{t=1}^{r} \frac{W_{|J_t|+1}(p^{j_t},\mathbf{p}_{J_t})}{(y(p^i)-y(p^{j_t}))\,dx(p)} \end{aligned} \quad (5.20)$$

The difference between these two summation, keeps only $j_t \neq 0, i$, thus:

$$\begin{aligned} & U_k(p,y(p^i);\mathbf{p}_K) \\ = \; & E_y(x(p^i),y(p^i))\,dx(p) \sum_{j=0}^{k-1}\sum_{J\in K_j}\sum_{r=1}^{d_2}\sum_{J_1\cup\ldots\cup J_r=J}\sum_{j_1\neq j_2\neq\ldots\neq j_r\neq i,0} \\ & \quad \frac{W_{k-j+1}(p^i,\mathbf{p}_{K-J})}{(y(p^i)-y(p))\,dx(p)} \prod_{t=1}^{r} \frac{W_{|J_t|+1}(p^{j_t},\mathbf{p}_{J_t})}{(y(p^i)-y(p^{j_t}))\,dx(p)} \end{aligned} \quad (5.21)$$

i.e. we have proven the lemma for k, for $y = y(p^i)$, and since both sides are polynomials in y of degree $d_2 - 1$, the equality holds for all y.

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Theorem: For all $k \geq 1$, one has:

$$\begin{aligned} &\sum_{i=1}^{d_2} \sum_{j=0}^{k-1} \sum_{J\in K_j} U_j(p^i, y(p); \mathbf{p}_J) W_{k-j+1}(p^i, \mathbf{p}_{K-J}) \\ &= \sum_{j=1}^{k-1} \sum_{J\in K_j} U_j(p, y(p); \mathbf{p}_J) W_{k-j+1}(p, \mathbf{p}_{K-J}) \end{aligned} \tag{5.22}$$

Proof of the theorem: Let us simply perform some basic rearrangements:

$$\begin{aligned} & \sum_{i=1}^{d_2} \sum_{j=0}^{k-1} \sum_{J\in K_j} U_j(p^i, y(p); \mathbf{p}_J) W_{k-j+1}(p^i, \mathbf{p}_{K-J}) \\ = \quad & \sum_{K_1 \bigcup L = K} \sum_{j_1=1}^{d_2} W_{|K_1|+1}(p^{j_1}, \mathbf{p}_{K_1}) U_{|L|+1}(p^{j_1}, y(p); \mathbf{p}_L) \\ = \quad & E_y(x(p), y(p)) dx(p) \sum_{K_1 \bigcup L = K} \sum_{j_1=1}^{d_2} \sum_{r=1}^{d_2} \sum_{K_2 \cup \ldots \cup K_{r+1} = L} \sum_{j_2 \neq j_3 \neq \ldots \neq j_r \in [1,d_2]-\{j_1\}} \\ & W_{|K_1|+1}(p^{j_1}, \mathbf{p}_{K_1}) \frac{W_{|K_{r+1}|+1}(p, \mathbf{p}_{K_{r+1}})}{(y(p) - y(p^{j_1}))} \prod_{a=2}^{r} \frac{W_{|K_a|+1}(p^{j_a}, \mathbf{p}_{K_a})}{(y(p) - y(p^{j_a})) dx(p)} \\ = \quad & E_y(x(p), y(p)) dx(p) \sum_{r=1}^{d_2} \sum_{K_1 \cup \ldots \cup K_{r+1} = K} \sum_{j_1 \neq j_2 \neq \ldots \neq j_r = 1}^{d_2} \\ & \prod_{a=1}^{r} \frac{W_{|K_a|+1}(p^{j_a}, \mathbf{p}_{K_a}) W_{|K_{r+1}|+1}(p, \mathbf{p}_{K_{r+1}})}{(y(p) - y(p^{j_a})) dx(p)} \\ = \quad & \sum_{K_{r+1} \bigcup J = K} W_{|K_{r+1}|+1}(p, \mathbf{p}_{K_{r+1}}) U_{|J|}(p, y(p); \mathbf{p}_J) \end{aligned} \tag{5.23}$$

•

This identity simplifies eq. (5.15) which becomes now:

$$\begin{aligned} & (y(p^i) - y(p)) R_k^i(p, \mathbf{p}_K) dx(p) = \\ & W_{k+1}(p^i, \mathbf{p}_K) + \sum_{j=1}^{k-1} \sum_{J\in K_j} \sum_{l\neq 0,i} \frac{U_j(p^l, y(p^i); \mathbf{p}_J) W_{k-j+1}(p^l, \mathbf{p}_{K-J})}{E_y(x(p), y(p^i)) dx(p)} \end{aligned} \tag{5.24}$$

One can now write down the final recursion formula for $R_k^i(p, \mathbf{p}_K)$ in these terms:

$$\boxed{\begin{aligned} R_k^i(p, \mathbf{p}_K) = & \frac{W_{k+1}(p^i, \mathbf{p}_K)}{(y(p^i) - y(p)) dx(p)} \\ & + \sum_{j=1}^{k-1} \sum_{J\in K_j} \sum_{l\neq 0,i} \frac{R_j^i(p^l, \mathbf{p}_J) W_{k-j+1}(p^l, \mathbf{p}_{K-J})}{(y(p^i) - y(p)) dx(p)} \end{aligned}} \tag{5.25}$$

The relations eq. (5.7) and eq. (5.25) allow to compute recursively W_k for any k. This solution can be represented by binary trees as it is presented in section (5.3).

5.2 Solution for any genus

In the previous paragraph, one has kept only the leading terms when performing the changes of variables to obtain the Schwinger-Dyson equations. Let us now write the $\frac{1}{N^2}$ corrective term for the same changes of variables so that we write a system of equations giving the whole $\frac{1}{N^2}$ expansion. One obtains the following loop equations :

$$\begin{aligned} &(y(r)-y(p))U_k(p,y(r);\mathbf{p}_K) \\ &= -P_k(x(p),y(r);\mathbf{p}_K)dx(p) - \sum_{j=0}^{k-1}\frac{1}{dx(p)}U_j(p,y(r);\mathbf{p}_J)W_{k-j+1}(p,\mathbf{p}_{K-J}) \\ &-\frac{1}{N^2}\frac{U_{k+1}(p,y(r);p,\mathbf{p}_k)}{dx(p)} + \sum_j d_{p_j}\left(\frac{U_{k-1}(p_j,y(r);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\frac{dx(p)}{dx(p_j)}\right)\end{aligned} \tag{5.26}$$

For the following, one should remind the expression of the function $Y(x(p))$:

$$Y(x) := V_1'(x) - \overline{w}_1(x) \tag{5.27}$$

Then, for $h \geq 1$:

$$Y^{(h)}(x(p)) = -\frac{W_1^{(h)}(p)}{dx(p)} \tag{5.28}$$

Consider now the $\frac{1}{N^2}$ expansion of this equation order by order. The genus h term (corresponding to the $\frac{1}{N^{2h}}$ term) gives:

$$\begin{aligned} &(y(r)-y(p))U_k^{(h)}(p,y(r);\mathbf{p}_K) - \sum_{m=1}^{h} Y^{(m)}(x(p))U_k^{(h-m)}(p,y(r);\mathbf{p}_K) \\ &= -P_k^{(h)}(x(p),y(r);\mathbf{p}_K)dx(p) \\ &- \sum_{m=0}^{h}\sum_{j=0}^{k-1}\frac{1}{dx(p)}U_j^{(m)}(p,y(r);\mathbf{p}_J)W_{k-j+1}^{(h-m)}(p,\mathbf{p}_{K-J}) \\ &-\frac{U_{k+1}^{(h-1)}(p,y(r);p,\mathbf{p}_k)}{dx(p)} + \sum_j d_{p_j}\left(\frac{U_{k-1}^{(h)}(p_j,y(r);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\frac{dx(p)}{dx(p_j)}\right)\end{aligned} \tag{5.29}$$

When $y(r) = y(p)$:

$$\begin{aligned} &\sum_{m=1}^{h} Y^{(m)}(x(p))U_k^{(h-m)}(p,y(p);\mathbf{p}_K) \\ &= P_k^{(h)}(x(p),y(p);\mathbf{p}_K)dx(p) + \sum_{m=0}^{h}\sum_{j=0}^{k-1}\frac{1}{dx(p)}U_j^{(m)}(p,y(p);\mathbf{p}_J)W_{k-j+1}^{(h-m)}(p,\mathbf{p}_{K-J}) \\ &+\frac{U_{k+1}^{(h-1)}(p,y(p);p,\mathbf{p}_k)}{dx(p)} - \sum_j d_{p_j}\left(\frac{U_{k-1}^{(h)}(p_j,y(p);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\frac{dx(p)}{dx(p_j)}\right)\end{aligned} \tag{5.30}$$

These two equations are the generalization of eq. (5.1) and eq. (5.2) for any genus in the topological expansion. With all these tools, we are now able to compute all the terms of the $\frac{1}{N^2}$ expansion of non mixed traces.

In this section, we proceed in two steps to compute the correlation function $W_k^{(h)}$ for any k and any h, and represent it as a Feynman graph with h loops. The first step consists in the determination of a recursive relation for $W_k^{(h)}$, whereas the second one gives $R_k^{i,(h)} := \frac{U_k^{(h)}(p^j,y(p^i);p_K)}{E_y(x(p^j),y(p^i))dx(p^j)}$ considered the lower order terms known.

For the following, let h and k be two given positive integers. Let us consider $W_j^{(m)}$ known for any j if $m < h$ and any $j \leq k$ if $m = h$. One also assume that $R_j^{i,(m)}$ is known for any i and any j if $m < h$ and any $j < k$ if $m = h$. Starting from these assumptions, one computes $W_{k+1}^{(h)}$ and $R_k^{i,(h)}$, what will allow to know any term recursively.

5.2.1 A recursive formula for $W_{k+1}^{(h)}$

Let us remind eq. (5.30) in a more suitable way to emphasize that it allows us to compute $W_{k+1}^{(h)}(p, p_K)$ with our assumption:

$$\begin{array}{l} W_{k+1}^{(h)}(p,\mathbf{p}_K)U_0(p,y(p)) = \\ -\sum_{m=0}^{h-1} W_1^{(h-m)}(p)U_k^{(m)}(p,y(p);\mathbf{p}_K) \\ -P_k^{(h)}(p,y(p);\mathbf{p}_K)dx(p)^2 \\ -\sum_{m=0}^{h}\sum_{j=0,m+j\neq 0}^{k-1} U_j^{(m)}(p,y(p);\mathbf{p}_J)W_{k-j+1}^{(h-m)}(p,\mathbf{p}_{K-J}) \\ -U_{k+1}^{(h-1)}(p,y(p);p,\mathbf{p}_k) + \sum_j\sum_j d_{p_j}\left(\frac{U_{k-1}^{(h)}(p_j,y(p);\mathbf{p}_{K-\{j\}})}{x(p)-x(p_j)}\frac{dx(p)}{dx(p_j)}\right)dx(p) \end{array} \tag{5.31}$$

Remark that the RHS contains only known terms except $P_k^{(h)}(p, y(p); \mathbf{p}_K)$. Fortunately, it plays no role in Cauchy formula.

Indeed, we write the Cauchy formula, move the integration contour and vanish integrals around the cycles thanks to the Riemann bilinear identity eq. (A.10). This gives:

$$\begin{aligned} W_{k+1}^{(h)}(p,\mathbf{p}_K) &= -\operatorname*{Res}_{p'\to p} W_{k+1}^{(h)}(p',\mathbf{p}_K)dS_{p',o}(p) \\ &= \sum_s \operatorname*{Res}_{p'\to a_s} W_{k+1}^{(h)}(p',\mathbf{p}_K)dS_{p',o}(p) \end{aligned} \tag{5.32}$$

We now introduce eq. (5.31) inside this formula and keep only terms which have poles at the branch points:

$$\begin{array}{l} W_{k+1}^{(h)}(p,\mathbf{p}_K) = \\ -\sum_{m=0}^{h-1}\sum_s \operatorname{Res}_{p'\to a_s} W_1^{(h-m)}(p')R_k^{(m)}(p';\mathbf{p}_K)dS_{p',o}(p) \\ -\sum_{m=0}^{h}\sum_{j=0,m+j\neq 0}^{k-1}\sum_s \operatorname{Res}_{p'\to a_s} R_j^{(m)}(p';\mathbf{p}_J)W_{k-j+1}^{(h-m)}(p',\mathbf{p}_{K-J})dS_{p',o}(p) \\ -\sum_s \operatorname{Res}_{p'\to a_s} R_{k+1}^{(h-1)}(p';p',\mathbf{p}_k)dS_{p',o}(p) \end{array} \tag{5.33}$$

For convenience, let us note:

$$W_1^{(0)}(p) \equiv 0 \tag{5.34}$$

Then, the recursive definition of $W_{k+1}^{(h)}(p, p_K)$ reads:

$$\boxed{\begin{array}{l} W_{k+1}^{(h)}(p, \mathbf{p}_K) = \\ \sum_{i=1}^{d_2} \sum_{m=0}^{h} \sum_{j=0, m+j\neq 0}^{k} \sum_s \mathrm{Res}_{\, p' \to a_s} R_j^{i,(m)}(p'; \mathbf{p}_J) W_{k-j+1}^{(h-m)}(p', \mathbf{p}_{K-J}) dS_{p',o}(p) \\ + \sum_{i=1}^{d_2} \sum_s \mathrm{Res}_{\, p' \to a_s} R_{k+1}^{i,(h-1)}(p'; p', \mathbf{p}_K) dS_{p',o}(p) \end{array}}$$

(5.35)

5.2.2 A recursive formula for $R_k^{i,(h)}$

The second step consists in the derivation of an equivalent formula for $R_k^{i,(h)}$. We proceed in the same way as for the genus 0 case: we use the rational properties of some of the correlation functions to write the recursive formula, with the aid of a relation similar to eq. (5.22).

Let $G_k^{(h)}(x(q), y(r))$ be :

$$\begin{aligned} G_k^{(h)}(x(q), y(r)) &= (y(r) - y(q)) U_k^{(h)}(q, y(r); \mathbf{p}_K) + \frac{U_{k+1}^{(h-1)}(q, y(r); q, p_k)}{dx(q)} \\ &+ \sum_{m=1}^{h} \sum_{j=0}^{k} \frac{1}{dx(q)} U_j^{(m)}(q, y(r); \mathbf{p}_J) W_{k-j+1}^{(h-m)}(q, \mathbf{p}_{K-J}) \\ &+ \sum_{j=0}^{k-1} \frac{1}{dx(q)} U_j(q, y(r); \mathbf{p}_J) W_{k-j+1}^{(h)}(q, \mathbf{p}_{K-J}) \end{aligned}$$

(5.36)

The loop equation eq. (5.29) shows that $G_k^{(h)}(x(q), y(r))$ is a rational function in $x(q)$ and a polynomial in $y(r)$.

Thus, one has:

$$G_k^{(h)}(x(p^i), y(p^i)) = G_k^{(h)}(x(p), y(p^i)) \tag{5.37}$$

which can be written:

$$\begin{aligned} (y(p^i) - y(p)) U_k^{(h)}(p, y(p^i); \mathbf{p}_K) = & \sum_{m=0}^{h} \sum_{j=0}^{k} \frac{W_{j+1}^{(m)}(p^i, \mathbf{p}_J) U_{k-j}^{(h-m)}(p^i, y(p^i); \mathbf{p}_{K-J})}{dx(p)} \\ & + \frac{U_{k+1}^{(h-1)}(p^i, y(p^i); p^i, \mathbf{p}_k)}{dx} \\ & - \sum_{m=0}^{h} \sum_{j=0}^{k} \frac{W_{j+1}^{(m)}(p, \mathbf{p}_J) U_{k-j}^{(h-m)}(p, y(p^i); \mathbf{p}_{K-J})}{dx(p)} \\ & - \frac{U_{k+1}^{(h-1)}(p, y(p^i); p, \mathbf{p}_k)}{dx} \end{aligned} \tag{5.38}$$

We now establish a relation similar to eq. (5.22) in order to present our recursive formula in such a way that it can be graphically interpreted.

In order to achieve this aim, one has to determine an explicit intermediate formula for $U_k^{(h)}(p, y; p_K)$. Let us assume that (for the proof, see *Appendix C*):

$$\begin{array}{l} U_k^{(h)}(p, y(p^i); \mathbf{p}_K) = \\ \frac{E_y(x,y(p^i))}{y(p^i)-y(p)} \sum_{r=1}^{min(d_2,k+h)} \sum_{K_1 \bigcup \ldots \bigcup K_r = K} \sum_{h_\alpha=0}^{h} \sum_{k_\alpha=|K_\alpha|}^{k+h} \sum_{j_{\alpha,\beta} \neq j_{\alpha',\beta'} \in [1,d_2]-\{i\}} \frac{1}{\Omega} \\ \frac{W_{k_1+1}^{(h_1)}(p^i, \mathbf{p}_{K_1}, p^{j_{1,1}}, \ldots, p^{j_{1,k_1-|K_1|}}) \left(\prod_{\alpha=2}^{r} W_{k_\alpha+1}^{(h_\alpha)}(p^{j_{\alpha,0}}, \mathbf{p}_{K_\alpha}, p^{j_{\alpha,1}}, \ldots, p^{j_{\alpha,k_\alpha-|K_\alpha|}}) \right)}{dx(p)^{r-k-1+\sum k_\alpha} \prod_{\alpha,\beta} y(p^i)-y(p^{j_{\alpha,\beta}})} \end{array} \tag{5.39}$$

where $\Omega = \prod_\alpha (k_\alpha - |K_\alpha|)!$ is a symmetry factor and one has the following constraints:

- $\sum_o (h_\alpha + k_\alpha) = h + k$;
- $0 \leq |K_\alpha| \leq k_\alpha$

One should note that the only external parameter entering these constraints is $k + h$.

It is now possible to derive an equality equivalent to eq. (5.22). One shows – in *Appendix D* – that:

$$\begin{array}{rl} & \sum_{m=0}^{h} \sum_{j=0;mj\neq kh}^{k} W_{j+1}^{(m)}(p, \mathbf{p}_J) U_{k-j}^{(h-m)}(p, y(p); \mathbf{p}_{K-J}) + U_{k+1}^{(h-1)}(p, y(p); p, \mathbf{p}_k) \\ = & \sum_{i=1}^{d_2} \sum_{m=0}^{h} \sum_{j=0;mj\neq kh}^{k} W_{j+1}^{(m)}(p^i, \mathbf{p}_J) U_{k-j}^{(h-m)}(p^i, y(p); \mathbf{p}_{K-J}) \\ & + \sum_{i=1}^{d_2} U_{k+1}^{(h-1)}(p^i, y(p); p^i, \mathbf{p}_k) \end{array} \tag{5.40}$$

This equality allows us to write:

$$\begin{array}{l} (y(p^i) - y(p)) U_k^{(h)}(p, y(p^i); \mathbf{p}_K) = \\ \sum_{m=0}^{h} \sum_{j=0;mj\neq kh}^{k} \sum_{l\neq 0,i} \frac{W_{j+1}^{(m)}(p^l, \mathbf{p}_J) U_{k-j}^{(h-m)}(p^l, y(p^i); \mathbf{p}_{K-J})}{dx(p)} \\ + \sum_{l\neq 0,i} \frac{U_{k+1}^{(h-1)}(p^l, y(p^i); p^l, \mathbf{p}_k)}{dx(p)} + W_{k+1}^{(h)}(p^i, \mathbf{p}_K) E_y(x, y(p^i)) \end{array} \tag{5.41}$$

That is to say:

$$\boxed{\begin{array}{l} R_k^{i,(h)}(p, \mathbf{p}_K) = \sum_{m=0}^{h} \sum_{j=0;mj\neq kh}^{k} \sum_{l\neq 0,i} \frac{W_{j+1}^{(m)}(p^l, \mathbf{p}_J) R_{k-j}^{i,(h-m)}(p^l; \mathbf{p}_{K-J})}{(y(p^i)-y(p))dx(p)} \\ \qquad + \sum_{l\neq 0,i} \frac{R_{k+1}^{i,(h-1)}(p^l; p^l, \mathbf{p}_k)}{(y(p^i)-y(p))dx(p)} + \frac{W_{k+1}^{(h)}(p^i, \mathbf{p}_K)}{(y(p^i)-y(p))dx(p)} \end{array}} \tag{5.42}$$

5.3 Diagrammatic solution: a cubic theory

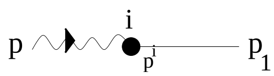
This section is the principal part of the article. We define a correspondence between the correlation functions and a system of Feynman-like graphs. To every k-point function of genus h, we associate a graph with k external legs and h loops and eq. (5.35) and eq. (5.42) become two relations describing these graphs as functions of graphs with less legs or loops thanks to some rules we introduce in this part.

First of all, let us represent diagrammatically eq. (B.6) and eq. (B.16) as the propagators of the theory:

$$W_2(p,q) = \tag{5.43}$$

and

$$R_1^i(p,p_1) = \tag{5.44}$$



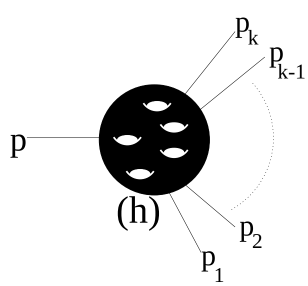
These two diagrams represent the basis of the whole representation: they allow to draw the $k > 2$ correlation functions.

Note that the second propagator can also be seen has a vertex of valence 2, and this is the way it will be presented in the diagrammatic rules.

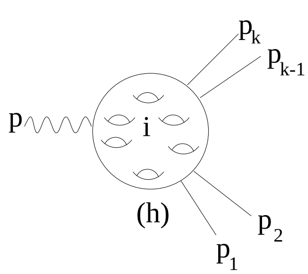
Let us now introduce the whole diagrammatic representation:

Let $R_k^{i,(h)}$, and $W_{k+1}^{(h)}$ respectively, be represented as white and black disks with h holes and k external legs (remember that $W_{k+1}^{(h)}$ is the generating function of discrete surfaces with $k+1$ boundaries and h holes):

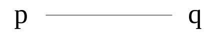
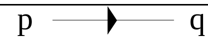
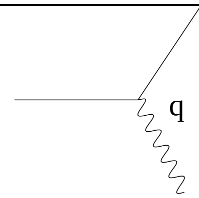
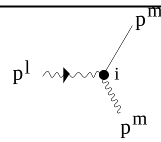
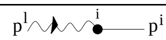
$$W_{k+1}^{(h)}(p, p_K) := \tag{5.45}$$



$$R_k^{i,(h)}(p, p_K) := \tag{5.46}$$

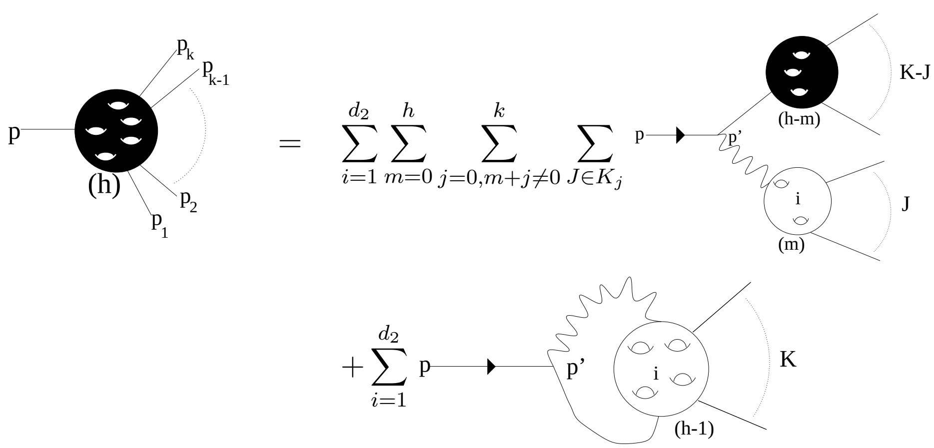


Let us introduce also the following propagators and vertices:

non-arrowed propagator:	 $:= W_2(p,q)$
arrowed propagator:	 $:= dS_{q,o}(p)$
Residue cubic-vertex:	 $:= \sum_s \text{Res}_{\,q\to a_s}$
colored cubic-vertices:	 $:= \frac{(1-\delta_{l,m})(1-\delta_{m,i})(1-\delta_{i,l})}{(y(p^i)-y(p^l))dx(p)}$
2-valent vertex:	 $:= \frac{1}{(y(p^i)-y(p^l))dx(p)}(1-\delta_{i,l})$

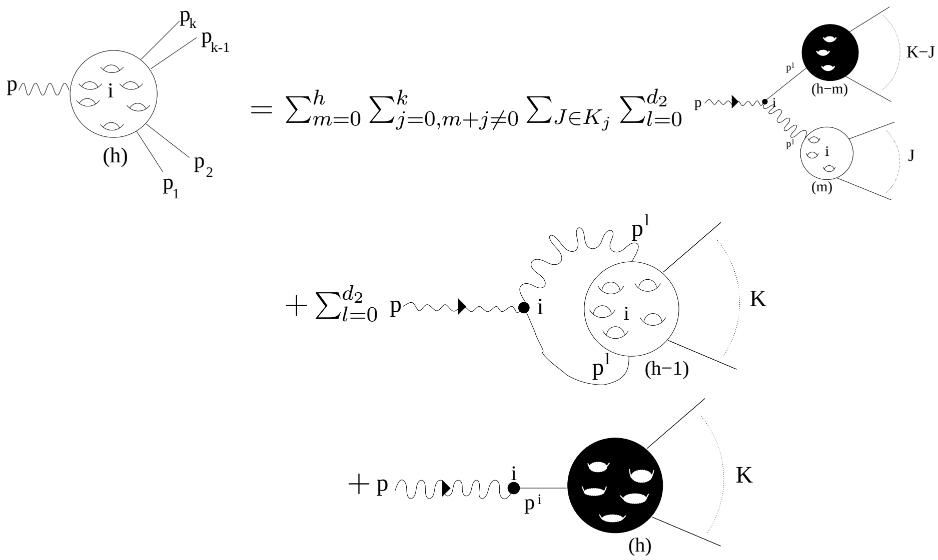
One can now simply interpret the recursion relations eq. (5.35) and eq. (5.42) in terms of diagrams.

The relation eq. (5.35) reads:



(5.47)

And given lower order $R_l^{i,(m)}$'s and $W_l^{(m)}$'s, one can obtain $R_k^{i,(h)}$ diagrammatically by writing eq. (5.42):



(5.48)

From these diagrammatic relations, one can see that $W_{k+1}^{(h)}$ is obtained by *the summation over all diagrams with 1 root, k leaves and h loops* following the rules:

- *The vertices have valence 2 or 3; there are* $2h+k-1$ *trivalent vertices;*
- *The edges, are arrowed or not, the arrowed edges are waved or not;*
- *The subgraph made of arrowed edges forms a skeleton tree (i.e. a tree whose vertices have valence up to 3);*
- *from each trivalent vertex comes one waved and one non-waved propagator;*
- *two vertices linked with a waved propagator have different indices;*
- *the k leaves are non-arrowed propagators finishing at* p_j*'s (i.e.* $B(.,p_j)$*);*
- *the root is an arrowed non waved propagator starting from p.*

A practical way to draw these graphs is to draw every skeleton tree of arrows, put k non arrowed propagators as leaves, close it with h non arrowed propagators linking one vertex to one of its descendents in order to obtain h loops and then put waves so that from each trivalent vertex comes one waved and one non-waved arrow with the possibility that every waved arrow leads to a bivalent vertex.

Remarks:

- The order for computing the residues is following the arrows backwards from leaves to root.
- W_{k+1} is symmetric in its $k+1$ variables, although it is not obvious from this representation.
- There is no symmetry factor arising in this representation unlike [13].

5.4 Examples

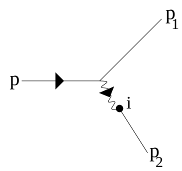
Let us briefly show some diagrams for small h and small k.

5.4.1 Leading terms: tree level

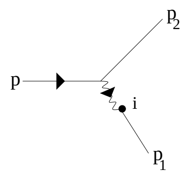
We begin by the leading terms of the first correlation functions, i.e. for $h = 0$.

- $k = 3$:

$$W_3^{(0)}(p, p_1, p_2) = \sum_{i=1}^{d_2}$$



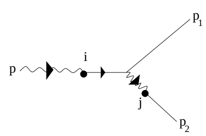
$+$



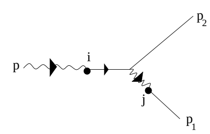
$$= \sum_{i=1}^{d_2} \sum_s \text{Res}_{\,p' \to a_s} \left[\frac{B(p'^i, p_1) B(p', p_2)}{(y(p'^i) - y(p')) dx(p')} + \frac{B(p'^i, p_2) B(p', p_1)}{(y(p'^i) - y(p')) dx(p')} \right] dS_{p',o}(p) \tag{5.49}$$

and

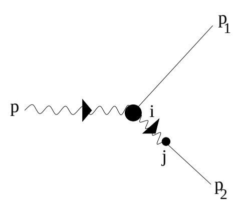
$$R_2^{i,(0)}(p, p_1, p_2) = \sum_{j=1}^{d_2}$$



$+$



$$+ \sum_{j \neq i} \tag{5.50}$$



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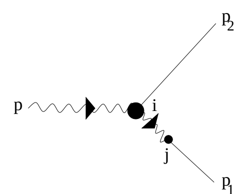


Let us show that $W_3^{(0)}(p, p_1, p_2)$ is indeed symmetric in p_1, p_2 and p_3.

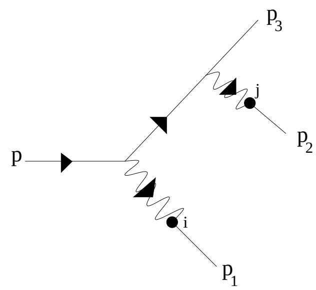
For every branch point a, let $\overline{q}$ be the only q^i such that $dx(\overline{q}) \to 0$ when $q \to a$.

$$\begin{aligned}
W_3^{(0)}(p, p_1, p_2) &= \sum_{i=1}^{d_2} \sum_s \text{Res}_{\,q \to a_s} \frac{B(q^i, p_1) B(q, p_2) + B(q^i, p_2) B(q, p_1)}{(y(q^i) - y(q)) dx(q)} dS_{q,o}(p) \\
&= \sum_{i=1}^{d_2} \sum_s \text{Res}_{\,q \to a_s} \text{Res}_{\,r \to q^i} \frac{B(r, p_1) B(q, p_2) + B(r, p_2) B(q, p_1)}{(y(r) - y(q))(x(r) - x(q)) dx(q)} dS_{q,o}(p) \\
&= \sum_s \text{Res}_{\,q \to a_s} \text{Res}_{\,r \to \overline{q}} \frac{B(r, p_1) B(q, p_2) + B(r, p_2) B(q, p_1)}{(y(r) - y(q))(x(r) - x(q)) dx(q)} dS_{q,o}(p) \\
&= \sum_s \text{Res}_{\,q \to a_s} \frac{B(q, p_1) B(\overline{q}, p_2) dS_{q,\overline{q}}(p)}{(y(\overline{q}) - y(q)) dx(q)} \\
&= - \sum_s \text{Res}_{\,q \to a_s} \frac{B(q, p_1) B(q, p_2) dS_{q,\overline{q}}(p)}{(y(\overline{q}) - y(q)) dx(q)} \\
&= \sum_s \text{Res}_{\,q \to a_s} \frac{B(q, p_1) B(q, p_2) B(q, p)}{dx(q) dy(q)}
\end{aligned} \tag{5.51}$$

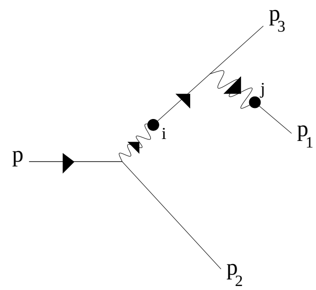
which is nothing but the formula found in [30] and is a way of writing Rauch's variational formula.

- $k=4$:

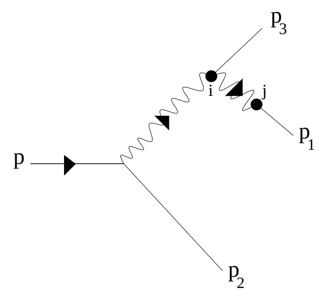
$$W_4^{(0)}(p,p_1,p_2,p_3) = \sum_{i=1}^{d_2}\sum_{j=1}^{d_2}$$



$$+\sum_{i=1}^{d_2}\sum_{j=1}^{d_2}$$



$$+\sum_{i=1}^{d_2}\sum_{j\neq i=1}^{d_2}$$



$$+(\text{ permutations of } \{p_1,p_2,p_3\}\ ) \tag{5.52}$$

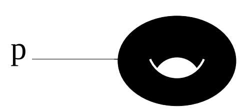
One has to consider all the permutations on the external legs. Thus, $W_4^{(0)}$ is the sum over 18 different diagrams.

5.4.2 Topological expansion: one and two loops level

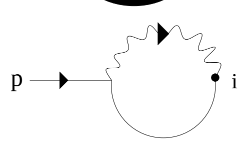
Consider now the first non planar examples beginning by the simplest one, the one loop correction to the one point function.

- $k=1$ and $h=1$:

$$W_1^{(1)}(x(p))dx(p) =$$



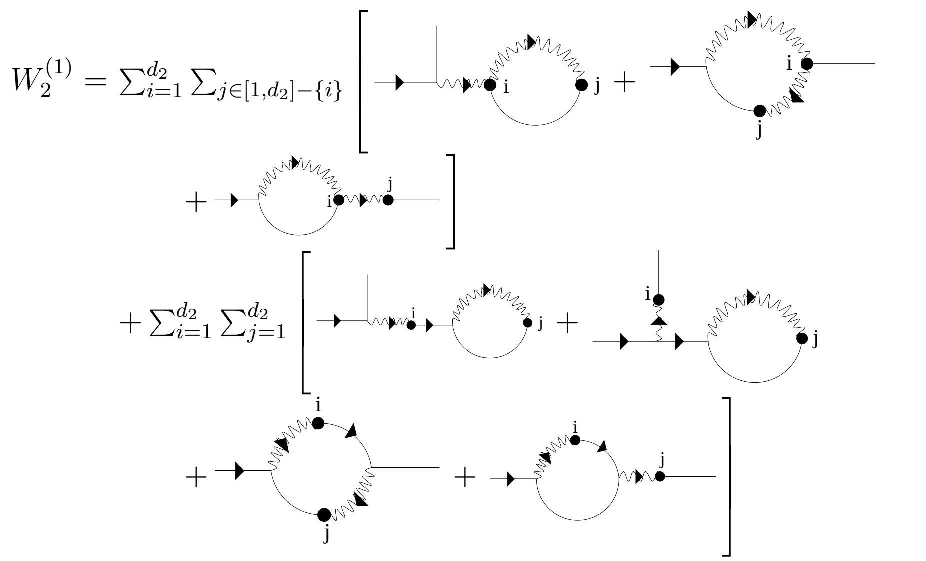
$$= \sum_{i=1}^{d_2}$$



$$= \sum_{i=1}^{d_2}\sum_s \operatorname*{Res}_{q\to a_s} dS_{q,o}(p)\frac{B(q,q^i)}{y(q^i)-y(q)} \tag{5.53}$$

One can check that this is identical to the result of [17].

- $k = 2$ and $h = 1$:

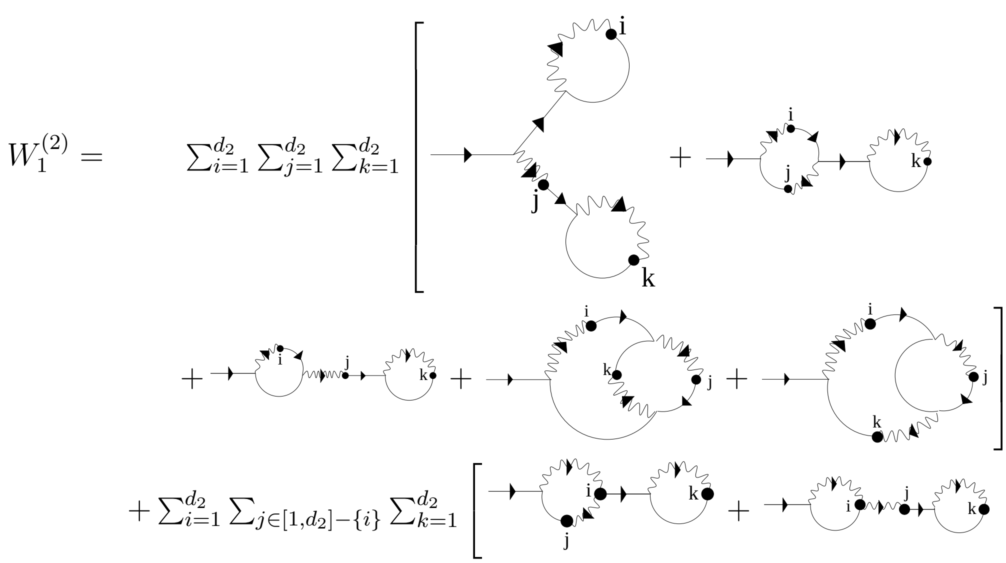


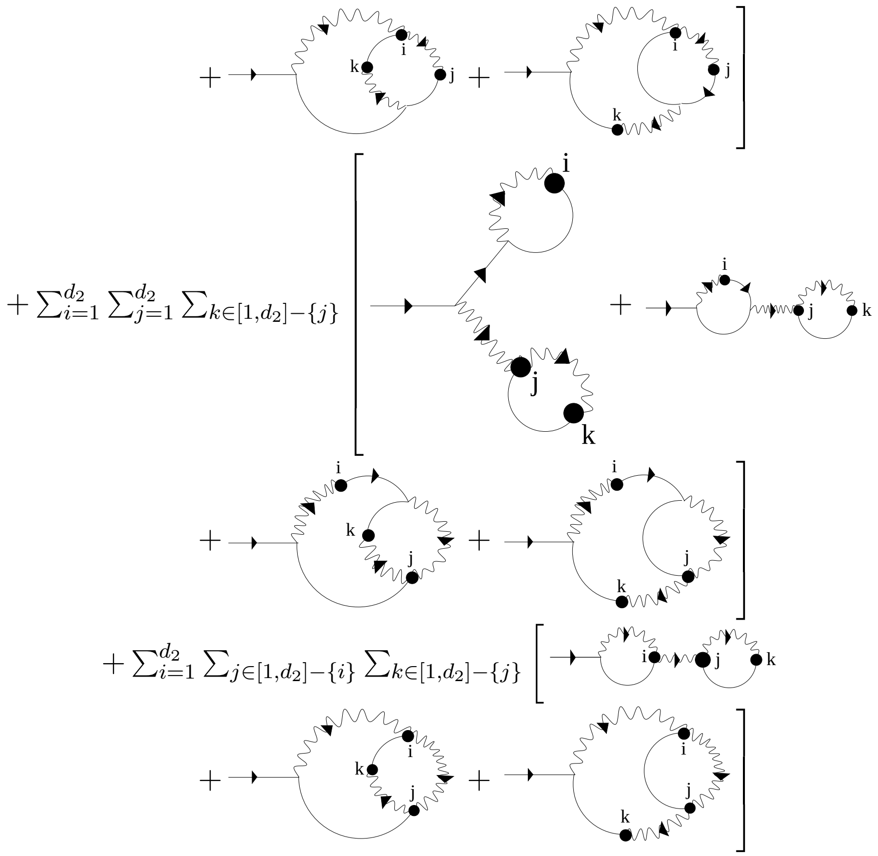
(5.54)

Analytically, this reads:

$$
\begin{aligned}
&W_2^{(1)}(p,p_1) = \\
&\sum_{i=1}^{d_2} \sum_{j\in[1,d_2]-\{i\}} \sum_s \operatorname{Res}_{p'\to a_s} \frac{dS_{p',o}(p)}{(y(p'^i)-y(p'))(y(p'^i)-y(p'^j))dx^2(p')} \\
&\quad [B(p',p_1)B(p'^i,p'^j) + B(p'^i,p_1)B(p',p'^j) + B(p',p'^i)B(p_1,p'^j)] \\
&+ \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \sum_{s,t} \operatorname{Res}_{p'\to a_s} \operatorname{Res}_{p''\to a_t} \frac{dS_{p',o}(p)}{(y(p'^i)-y(p'))(y(p''^j)-y(p''))dx(p')dx(p'')} \\
&\quad [B(p',p_1)B(p'',p''^j)dS_{p'',o}(p'^i) + B(p'^i,p_1)B(p'',p''^j)dS_{p'',o}(p') \\
&\qquad + B(p'',p')B(p_1,p''^j)dS_{p'',o}(p'^i) + B(p_1,p'')B(p',p''^j)dS_{p'',o}(p'^i)]
\end{aligned}
\tag{5.55}
$$

- $k = 1$ and $h = 2$:

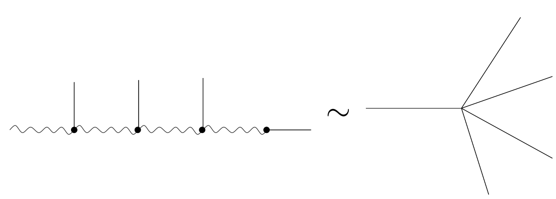




(5.56)

6. An effective non cubic theory

The Feynman-like graphs described up to now correspond to cubic vertices only, but the price to pay is the introduction of auxiliary functions $R_k^{i,(h)}$. Nevertheless, in order to study some problems, this property is not needed and one may prefer an effective diagrammatic representation for only $W_k^{(h)}$ but vertices with valence up to $d_2 - 1$. This section is dedicated to building such a diagrammatic representation. It consists in resumming the linked waved vertices into one multivalent vertex:



(6.1)

6.1 Leading order: Genus 0

We have already written the equations necessary to define this effective theory. Let us consider eq. (5.7) and eq. (5.16):

$$W_{k+1}(p,\mathbf{p}_K) = -\sum_s \mathop{\mathrm{Res}}_{p'\to a_s} \sum_{j=1}^{k-1} \sum_{J\in K_j} \frac{1}{dx'} \frac{U_j(p',y(p');\mathbf{p}_J)}{E_y(x(p'),y(p'))} W_{k-j+1}(p',\mathbf{p}_{K-J}) dS_{p',o}(p) \tag{6.2}$$

$$U_k(p,y;\mathbf{p}_K) = \frac{E(x(p),y)dx(p)}{y-y(p)} \sum_{r=1}^{d_2} \sum_{K_1\cup\ldots\cup K_r=K} \sum_{j_1\neq j_2\neq\ldots\neq j_r=1}^{d_2} \prod_{t=1}^{r} \frac{W_{|K_t|+1}(p^{j_t},\mathbf{p}_{K_t})}{(y-y(p^{j_t}))\,dx(p)} \tag{6.3}$$

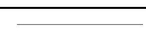
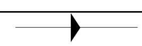
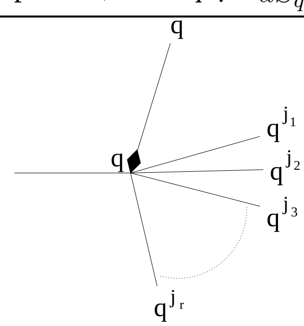
This second equation taken for $y=y(p)$ reads:

$$\frac{U_k(p,y(p);\mathbf{p}_K)}{E_y(x(p),y(p))dx(p)} = \sum_{r=1}^{d_2} \sum_{K_1\cup\ldots\cup K_r=K} \sum_{j_1\neq j_2\neq\ldots\neq j_r=1}^{d_2} \prod_{t=1}^{r} \frac{W_{|K_t|+1}(p^{j_t},\mathbf{p}_{K_t})}{(y(p)-y(p^{j_t}))\,dx(p)} \tag{6.4}$$

Introduce it in eq. (6.2) and get a closed recursive formula for the W_k's:

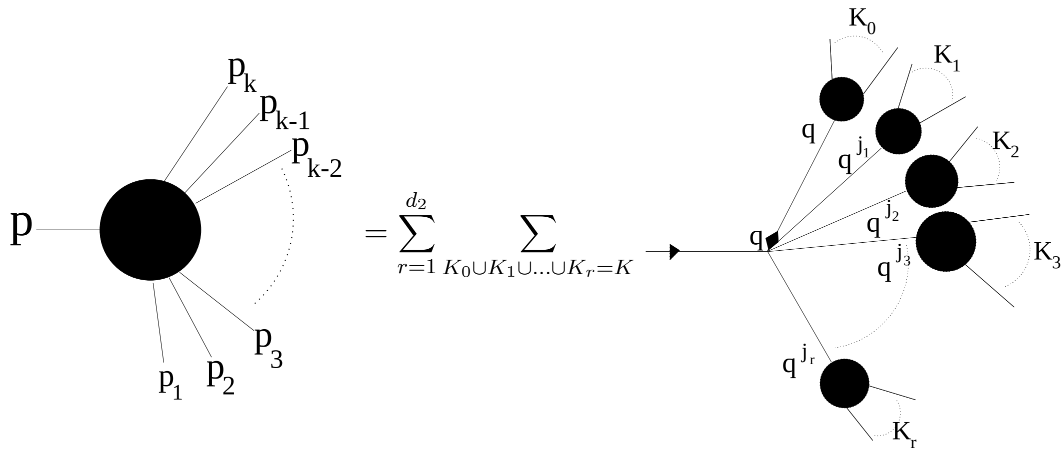
$$\boxed{\begin{array}{l} W_{k+1}(p,\mathbf{p}_K) = -\sum_s \mathrm{Res}_{\,p'\to a_s} \sum_{r=1}^{d_2} \sum_{K_0\cup K_1\cup\ldots\cup K_r=K} \sum_{j_1\neq j_2\neq\ldots\neq j_r=1}^{d_2} \\ \qquad W_{|K_0|+1}(p',\mathbf{p}_{K_0}) \prod_{t=1}^{r} \frac{W_{|K_t|+1}(p'^{j_t},\mathbf{p}_{K_t})}{(y(p')-y(p'^{j_t}))\,dx(p')} dS_{p',o}(p) \end{array}} \tag{6.5}$$

Let us introduce the following Feynman rules:

non-arrowed propagator:	p  q $:= W_2(p,q)$
arrowed propagator:	p  q $:= dS_{q,o}(p)$
r+2 - vertex ($1\leq r\leq d_2$) with one marked edge:	 $:= -\sum_s \sum_{j_1\neq\ldots\neq j_r\neq 0} \mathrm{Res}_{\,q\to a_s} \prod_{t=1}^{r} \frac{1}{(y(q)-y(q^{j_t}))dx(q)}$

Remark that one leg of the multiple vertex is marked: on this leg, there is no summation over the different sheets.

Using these rules, one can diagrammatically write the recursive relation as follows:



(6.6)

From this relation, one can see that $W_{k+1}(p, \mathbf{p}_K)$ is obtained as the *summation over all trees with* $k+1$ *external legs* and following the rules:

- *The vertices have valence r+2 such as* $1 \leq r \leq min(k-1, d_2)$*;*
- *The edges are arrowed;*
- *One of the legs of each vertex is marked*
- *The k leaves are non arrowed propagators ending at* p_j*'s;*
- *The root is an arrowed propagator starting from p.*

The drawbacks of these effective rules induced by the existence of multivalent vertices is balanced by the simplicity of the vertices and the absence of different propagators.

6.2 Any genus h

Let us now study the extension of this theory to any genus.

Once again, the fundamental equations have already been written. Let us recall to mind eq. (5.35) and eq. (5.39):

$$\begin{array}{l} W_{k+1}^{(h)}(p, \mathbf{p}_K) = \\ - \sum_{m=0}^{h} \sum_{j=0, m+j \neq 0}^{k} \sum_s \mathrm{Res}_{p' \to a_s} \frac{U_j^{(m)}(p', y(p'); \mathbf{p}_J)}{E_y(x(p'), y(p'))} W_{k-j+1}^{(h-m)}(p', \mathbf{p}_{K-J}) dS_{p',\alpha}(p) \\ - \sum_s \mathrm{Res}_{p' \to a_s} \frac{U_{k+1}^{(h-1)}(p', y(p'); p', \mathbf{p}_K)}{E_y(x(p'), y(p'))} dS_{p',\alpha}(p) \end{array} \qquad (6.7)$$

and, for $i \neq 0$:

$$U_k^{(h)}(p,y(p^i);\mathbf{p}_K) = \frac{E_y(x,y(p^i))}{y(p^i)-y(p)} \sum_{r=1}^{min(d_2,k+h)} \sum_{K_1\bigcup\ldots\bigcup K_r=K} \sum_{h_\alpha=0}^{h} \sum_{k_\alpha=|K_\alpha|}^{k+h} \sum_{j_{\alpha,\beta}\neq j_{\alpha',\beta'}\in[1,d_2]-\{i\}} \frac{1}{\Omega} \frac{W_{k_1+1}^{(h_1)}(p^i,\mathbf{p}_{K_1},p^{j_{1,1}},\ldots,p^{(j_{1,k_1-|K_1|})})\left(\prod_{\alpha=2}^{r} W_{k_\alpha+1}^{(h_\alpha)}(p^{j_{\alpha,0}},\mathbf{p}_{K_\alpha},p^{j_{\alpha,1}},\ldots,p^{j_{\alpha,k_\alpha-|K_\alpha|}})\right)}{dx(p)^{r-k-1+\sum k_\alpha}\prod_{\alpha,\beta} y(p^i)-y(p^{j_{\alpha,\beta}})} \tag{6.8}$$

In order to introduce this second formula inside the first one, one has to use the interpolation formula to consider the case where $i=0$:

$$\frac{U_l^{(m)}(p,y(p);\mathbf{p}_L)}{E_y(x(p),y(p))} = -\sum_{r=1}^{min(d_2,l+m)} \sum_{L_1\bigcup\ldots\bigcup L_r=L} \sum_{m_\alpha=0}^{m} \sum_{l_\alpha=|L_\alpha|}^{l+m} \sum_{j_1\neq\ldots\neq j_r\in[1,d_2]} \frac{1}{\Omega} \frac{W_{l_1+1}^{(m_1)}(p^{j_{1,0}},\mathbf{p}_{L_1},p^{j_{1,1}},\ldots,p^{j_{1,l_1-|L_1|}})\prod_{\alpha=2}^{r} W_{l_\alpha+1}^{(m_\alpha)}(p^{j_{\alpha,0}},\mathbf{p}_{L_\alpha},p^{j_{\alpha,1}},\ldots,p^{j_{\alpha,l_\alpha-|L_\alpha|}})}{dx(p)^{r-l-1+\sum l_\alpha}(y(p^{j_{1,0}})-y(p))\prod_{\alpha,\beta}(y(p^{j_{1,0}})-y(p^{j_{\alpha,\beta}})} \tag{6.9}$$

Recursively, it is easy to check that it can be written:

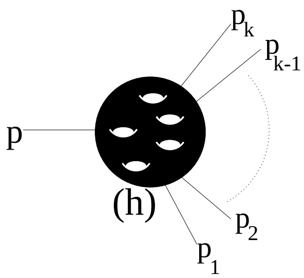
$$\frac{U_l^{(m)}(p,y(p);\mathbf{p}_L)}{E_y(x(p),y(p))dx(p)} = \sum_{r=1}^{min(d_2,l+m)} \sum_{L_1\bigcup\ldots\bigcup L_r=L} \sum_{m_\alpha=0}^{m} \sum_{l_\alpha=|L_\alpha|}^{l+m} \sum_{j_{\alpha,\beta}\neq j_{\alpha',\beta'}\in[1,d_2]} \frac{1}{\Omega'} \prod_{\alpha=1}^{r} \frac{W_{l_\alpha+1}^{(m_\alpha)}(p^{j_{\alpha,0}},\mathbf{p}_{L_\alpha},p^{j_{\alpha,1}},\ldots,p^{j_{\alpha,l_\alpha-|L_\alpha|}})}{dx(p)^{l_\alpha-|L_\alpha|+1}\prod_{\beta=0}^{l_\alpha-|L_\alpha|}(y(p)-y(p^{j_{\alpha,\beta}}))} \tag{6.10}$$

where Ω' is some other symmetry factor depending only on the same parameters as Ω.

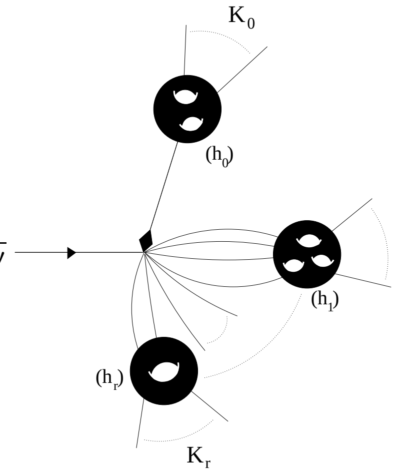
One is now able to write an explicit recursion formula for the $W_k^{(h)}$'s that can be graphically represented with the Feynman rules introduced in this section. The introduction of eq. (6.10) in eq. (6.7) gives:

$$\begin{aligned} W_{k+1}^{(h)}(p,\mathbf{p}_K) = & -\sum_s \operatorname*{Res}_{p'\to a_s} \sum_{r=1}^{d_2} \sum_{K_0\bigcup K_1\bigcup\ldots\bigcup K_r=K} \sum_{h_\alpha=0}^{h} \sum_{k_\alpha=|K_\alpha|}^{k+h} \sum_{j_{\alpha,\beta}\neq j_{\alpha',\beta'}\in[1,d_2]} \frac{1}{\Omega'} \\ & dS_{p',o}(p) W_{|K_0|+1}^{(h_0)}(p',\mathbf{p}_{K_0}) \prod_{\alpha=1}^{r} \frac{W_{k_\alpha+1}^{(h_\alpha)}(p'^{j_{\alpha,0}},\mathbf{p}_{K_\alpha},p'^{j_{\alpha,1}},\ldots,p'^{j_{\alpha,k_\alpha-|K_\alpha|}})}{dx(p')^{k_\alpha-|K_\alpha|+1}\prod_{\beta=0}^{k_\alpha-|K_\alpha|}(y(p')-y(p'^{j_{\alpha,\beta}}))} \\ & -\sum_s \operatorname*{Res}_{p'\to a_s} \frac{U_{k+1}^{(h-1)}(p',y(p');p',\mathbf{p}_K)}{E_y(x(p'),y(p'))} dS_{p',\alpha}(p) \end{aligned} \tag{6.11}$$

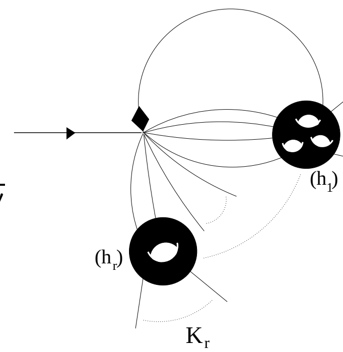
That is to say:



$$= \sum_{r=1}^{d_2} \sum_{h_\alpha} \sum_{K_0\cup K_1\cup\ldots\cup K_r=K} \frac{1}{\Omega'}$$



$$+\sum_{r=1}^{d_2}\sum_{h_\alpha}\sum_{K_1\cup\ldots\cup K_r=K}\frac{1}{\Omega'} \tag{6.12}$$



Remark that we have splitted the diagrams in the RHS in order to reproduce the recursion relation. Nevertheless, the first term in the RHS is nothing else but a particular case of the second term where the marked leg of the vertex is left alone inside one of the W's.

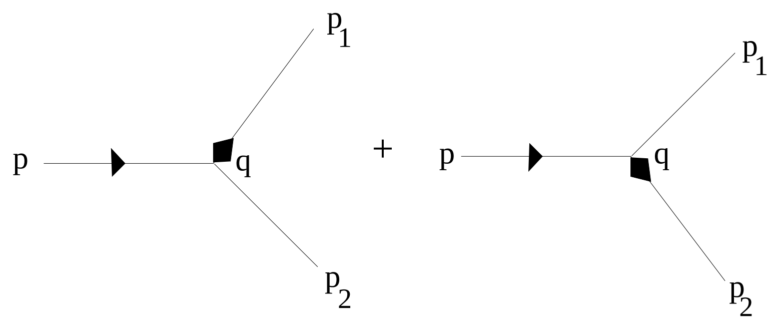
Hence, the h-th order expansion term of the correlation function $W_{k+1}^{(h)}$ is obtained as the *summation over all Feynman diagrams with* $k+1$ *external legs and* h *loops* following the same rules as exposed in the genus 0 case, i.e.:

- *The vertices have valence r+2 such as* $1 \leq r \leq d_2$*;*
- *The edges are arrowed or not;*
- *One of the legs of each vertex is marked;*
- *The subgraph made of arrowed edges forms a skeleton tree;*
- *The k leaves are non arrowed propagators ending at* p_j*'s;*
- *The root is an arrowed propagator starting from p;*
- *a non arrowed edge links a vertex to one of its descendants along the tree.*

6.3 Examples

Let us review some simple examples of this description.

$W_3^{(0)}(p, p_1, p_2) =$

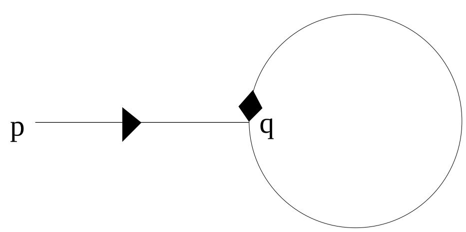


(6.13)

Analytically, this reads:

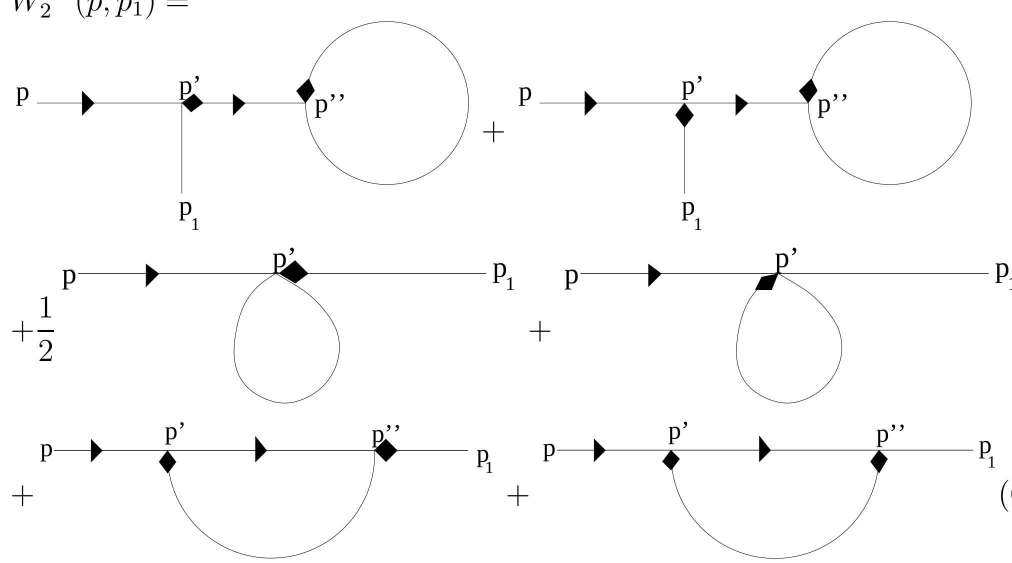
$$W_3^{(0)}(p, p_1, p_2) = \sum_{i=1}^{d_2} \sum_s \operatorname{Res}_{q \to a_s} \left[B(q^i, p_1)B(q, p_2) + B(q^i, p_2)B(q, p_1)\right] \frac{dS_{q,o}(p)}{(y(q^i) - y(q))dx(q)} \tag{6.14}$$

$W_1^{(1)}(p) =$



$$= \sum_s \sum_{i=1}^{d_2} \operatorname{Res}_{q \to a_s} dS_{q,o}(p) \frac{B(q, q^i)}{(y(q^i) - y(q))dx(q)} \tag{6.15}$$

$W_2^{(1)}(p, p_1) =$



(6.16)

7. The gaussian case: the 1-matrix model limit.

In this section, we are interested in the special case where $d_2 = 1$, i.e. one has a gaussian potential in M_2. This situation is very important because it links our results to the 1-matrix model studied in [13]. Indeed, when one of the potentials is gaussian – V_2 for example –, the integration over one of the variables – M_2 in this case – is gaussian and can be straightforwardly performed without giving any contribution to the formal expansion. Then, the 2-matrix model with one gaussian potential $V_2(y) = \frac{g_2}{2} y^2$ is equivalent to the 1-matrix model with a potential $V = V_1 - \frac{x^2}{2g_2}$. We check in this part that our results coincide with the ones obtained directly from the 1-matrix model in [13]. Actually, it is a good way to better understand the structure obtained.

In this case, the Riemann surface is an hyperelliptical surface with only two x-sheets. The equation $x(p) = x$ has only two solutions. Let us call them p and $\overline{p}$, i.e. $p^0 = p$ and $p^1 = \overline{p}$. They obey the following relations:

$$x(p) = x(\overline{p}) y(p) = -y(\overline{p}) \tag{7.1}$$

The algebraic equation generating the Riemann surface reads:

$$E(x(p), y(r)) = -g_2(y(r) - y(p))(y(r) - y(\overline{p})) = -g_2(y(r)^2 - y(p)^2) \tag{7.2}$$

One can also remark that:

$$U_k(p, y; \mathbf{p}_K) = g_2 W_{k+1}(p, \mathbf{p}_K) \tag{7.3}$$

That is to say:

$$R_k^0(p, \mathbf{p}_K) = \frac{U_k(p, y(p); \mathbf{p}_K)}{E_y(x(p), y(p)) dx(p)} = -\frac{W_{k+1}(p, \mathbf{p}_K)}{2y(p)dx(p)} \tag{7.4}$$

So that:

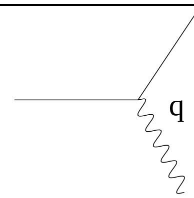
$$R_k^0(\overline{p}, \mathbf{p}_K) = R_k^1(p, \mathbf{p}_K) = \frac{W_{k+1}(p, \mathbf{p}_K)}{2y(p)dx(p)} \tag{7.5}$$

Diagrammatic rules.

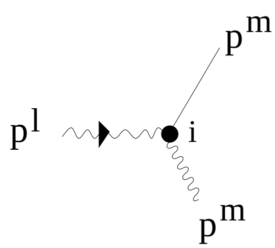
One can now study how the diagrammatic rules introduced earlier behave in this limit.

- **The cubic rules**

  Because V_2 is gaussian, the Feynman rules become:

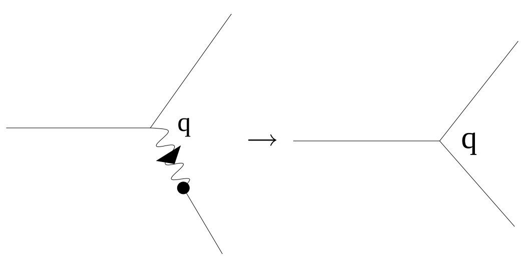
non-arrowed propagator:	p ——— q $:= W_2(p,q)$
arrowed propagator:	p —▶— q $:= dS_{q,o}(p)$
Residue cubic-vertex:	 $:= \sum_s Res_{q\to a_s}$
simple vertex:	 $:= -\frac{1}{2y(p)dx(p)}$

The last component of the Feynman diagrams, the colored cubic-vertex, implies three different x-sheets. Because there exists only two such sheets in the gaussian case, this vertex vanishes:



$$\equiv 0 \tag{7.6}$$

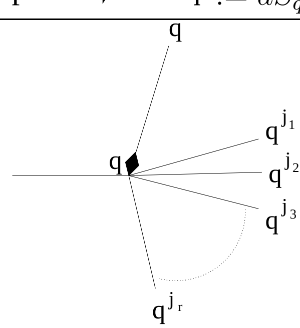
Considered that the bivalent and trivalent vertices only appear together, one can merge them into one whose value is equal to $-\sum_s \text{Res}_{\,q\to a_s} \frac{1}{2y(q)dx(q)}$, and one recovers [13]:



(7.7)

- **The effective theory**

The effect of the gaussian limit on the effective theory is to make it cubic. One obtains the following rules:

non-arrowed propagator:	p ——— q $:= W_2(p,q)$
arrowed propagator:	p —▶— q $:= dS_{q,o}(p)$
cubic vertex (only for r=1):	 $:= -\sum_s \text{Res}_{\,q\to a_s} \frac{1}{2y(q)dx(q)}$

Hence, the two theories turn into only one cubic theory in this limit which is the one derived in [13]. Indeed, the corresponding recursive relation appears to be:

$$\begin{aligned} W_{k+1}^{(h)}(p,p_K) &= -\sum_l \operatorname*{Res}_{q\to a_l} \frac{W_{k+2}^{(h-1)}(q,q,p_K) dS_{q,o}(p)}{2y(q)dx(q)} \\ &- \sum_{m=0}^{h} \sum_{j=0, j+m\neq 0}^{k} \sum_l \operatorname*{Res}_{q\to a_l} \frac{W_{j+1}^{(m)}(q,p_J) W_{k-j+1}^{(h-m)}(q,p_{K-J}) dS_{q,o}(p)}{2y(q)dx(q)} \end{aligned} \tag{7.8}$$

Remark:

Diagrammatically, this limit can be easily interpreted. Starting from the general cubic theory, in order, to obtain the 1-matrix model graphs from the 2-matrix model ones, one only has to take the length of the waved propagators to 0. In this case, the graphs containing at least one colored vertex vanish.

Everything works as if the waved propagators of the 2-matrix model were unstable particles which decay into stable ones represented by non-waved propagators. Then the 1-matrix limit is obtained by taking the life time of these particles to 0.

One shall also note that there is no symmetry factor in the 2-matrix model graphs of the cubic theory whereas there are not well understood ones in the 1-matrix case. The derivation of the 1-matrix model as a limit exhibits how these factors arise. They come from the same contribution given by different diagrams in this limit. This observation exhibits how the 2-matrix model seems more fundamental.

8. Conclusion

In this article, we have generalized the diagrammatic technique of [13] to compute all non-mixed correlation functions of the 2-matrix model, to all orders in the topological expansion.

The result can be represented diagrammatically, with some cubic Feynman rules, which are just convenient notations for writing residues on an algebraic curve and it is not clear whether there exists a field theory giving rise to these graphs or not.

This shows that the method discovered in [13] is very universal, i.e. it works for all algebraic curves, not only hyper elliptical curves.

The future prospects of that work are to find the diagrammatic rules for computing the free energy to all order in the topological expansion, and also all mixed correlation functions (using the result of [14]). Another possible extension is to work out the multimatrix model, i.e. the chain of matrices as in [20], and in particular the limit of matrix quantum mechanics. We believe that this technique could apply to many other integrable models.

Another question, is to understand the limit of critical points, i.e. when some branch points and double points start to coalesce. It seems that the diagrammatic technique should just reduce to consider only residues at branch points which become critical. One may expect to recover some relation with the Kontsevich integral, in relationship with KP integrable hierarchies.

Acknowledgments

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Appendix A Needed tools of algebraic geometry

We review here some definitions and properties all along this article.

Behaviors at ∞. We see from eq. (2.17), that at large x, we have $y \sim V_1'(x) - \frac{1}{x} + O(1/x^2)$ in the x-physical sheet. (resp. at large y, we have $x \sim V_2'(y) - \frac{1}{y} + O(1/y^2)$ in the y-physical sheet). This means that the functions $x(p)$ and $y(p)$ have two poles, ∞_+ and ∞_- on $\mathcal{E}$. The function $x(p)$ has a simple pole at ∞_+ and a pole of degree d_2 at ∞_-, while the function $y(p)$ has a simple pole at ∞_- and a pole of degree d_1 at ∞_+. We have:

$$y(p) \underset{p\to\infty_+}{\sim} V_1'(x(p)) - \frac{1}{x(p)} + O(1/x(p)^2) \tag{A.1}$$

$$x(p) \underset{p\to\infty_-}{\sim} V_2'(y(p)) - \frac{1}{y(p)} + O(1/y(p)^2) \tag{A.2}$$

In particular:

$$\mathop{\mathrm{Res}}_{\infty_+} y\,dx = \mathop{\mathrm{Res}}_{\infty_-} x\,dy = 1 \tag{A.3}$$

Genus and cycles. The curve $\mathcal{E}$ is a compact Riemann surface with a finite genus $g \leq d_1 d_2 - 1$. If $g = 0$, $\mathcal{E}$ is simply connected, and if $g \neq 0$, there exist $2g$ linearly independent irreducible cycles on $\mathcal{E}$, such that by removing those $2g$ cycles we get a simply connected domain. It is possible to choose canonically the $2g$ cycles as $\mathcal{A}_i$, $\mathcal{B}_i$, $i = 1, \dots, g$, such that:

$$\mathcal{A}_i \cap \mathcal{A}_j = 0 \qquad , \qquad \mathcal{B}_i \cap \mathcal{B}_j = 0 \qquad , \qquad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij} \tag{A.4}$$

Branch points. The x-branch points a_i, $i = 1, \dots, d_2 + 1 + 2g$, are the zeroes of the differential dx, respectively, the y-branch points b_i, $i = 1, \dots, d_1 + 1 + 2g$, are the zeroes of dy. We assume here, that all branch points are simple and distinct, i.e. that the potentials are not critical. Notice also, that $E_y(x(p), y(p))$ vanishes (simple zeroes) at the branch points (it vanishes in other points too).

Bergmann kernel. On the Riemann surface $\mathcal{E}$, there exists a unique Abelian bilinear differential $B(p,q)$, with one double pole at $p = q$, such that:

$$B(p,q) \underset{p\to q}{\sim} \frac{dx(p)dx(q)}{(x(p) - x(q))^2} + \text{finite} \quad \text{and} \quad \forall i \;\; \oint_{p\in\mathcal{A}_i} B(p,q) = 0 \tag{A.5}$$

It is symmetric:

$$B(p,q) = B(q,p) \tag{A.6}$$

Its expression in terms of theta-functions can be found in [24,25], it depends only on the complex structure of $\mathcal{E}$.

Abelian differential of third kind.

On the Riemann surface $\mathcal{E}$, there exists a unique abelian differential of the third kind $dS_{q,r}(p)$, with two simple poles at $p = q$ and at $p = r$, such that:

$$\operatorname*{Res}_{p\to q} dS_{q,r}(p) = 1 = -\operatorname*{Res}_{p\to r} dS_{q,r}(p) \quad \text{and} \quad \forall i \ \oint_{\mathcal{A}_i} dS_{q,r}(p) = 0 \tag{A.7}$$

We have:

$$dS_{q,r}(p) = \int_{q'=r}^{q} B(p,q') \tag{A.8}$$

where the integration path does not intersect any $\mathcal{A}_i$ or $\mathcal{B}_i$.

$dS_{q,r}(p)$ is a differential on $\mathcal{E}$ in terms of p, but it is a multivalued function of q (and of r). After crossing a cycle $\mathcal{B}_i$, it has no discontinuity, and after crossing a cycle $\mathcal{A}_i$, it has a discontinuity:

$$\operatorname{disc}\left(dS_{q,r}(p)\right) = dS_{q_+,r}(p) - dS_{q_-,r}(p) = \oint_{q'\in\mathcal{B}_i} B(p,q') \tag{A.9}$$

Note that the discontinuity is independent of q.

Riemann bilinear identity.

If ω is a differential form on $\mathcal{E}$, such that $\oint_{q\in\mathcal{A}_i}\omega(q) = 0$, we have:

$$\begin{aligned}
\sum_i \operatorname*{Res}_{q\to z_i} \omega(q) dS_{q,r}(p) &= \sum_{i=1}^{g} \oint_{q\in\mathcal{A}_i} \operatorname{disc}_{\mathcal{A}_i}(\omega(q) dS_{q,r}(p)) \\
&\quad - \sum_{i=1}^{g} \oint_{q\in\mathcal{B}_i} \operatorname{disc}_{\mathcal{B}_i}(\omega(q) dS_{q,r}(p)) \\
&= \sum_{i=1}^{g} \oint_{q\in\mathcal{A}_i} \omega(q) \operatorname{disc}_{\mathcal{A}_i}(dS_{q,r}(p)) \\
&= \sum_{i=1}^{g} \operatorname{disc}_{\mathcal{A}_i}(dS_{q,r}(p)) \oint_{q\in\mathcal{A}_i} \omega(q) \\
&= 0
\end{aligned} \tag{A.10}$$

where the LHS is the sum over all residues on a fundamental domain, the poles z_i are all the poles of ω as well as the pole at $q = p$. This identity is obtained by moving the integration contours on the surface, and taking carefully into account discontinuities along the nontrivial cycles (see [24, 25]).

Appendix B Two points function in the planar limit

We present here a new derivation leading term of the 2-point function's leading term $W_2(p_1, p_2)$.

This case is of special interest because it represents some initial condition for the diagrammatic rules. In fact, the two correlation functions $W_2(p_1, p_2)$ and $U_1(p_1, y; p_2)$, are the basis of the whole structure of the $W_k^{(h)}$'s. Moreover, it allows us to show through a simple example the way we proceed further for the general case.

We first rederive the well known result that the two point function is nothing else but the Bergmann Kernel (see [5] for instance).

Let $o \in \mathcal{E}$ be an arbitrary point on the Riemann surface. Since the Abelian differential of the 3rd kind defined in eq. (4.14) $dS_{q,o}(p)$ behaves as $\frac{dx(p)}{x(p)-x(q)}$ when $q \to p$, one can write the Cauchy formula under the form:

$$W_2(p,p_1) = -\operatorname*{Res}_{q\to p} dS_{q,o}(p) W_2(q;p_1) \tag{B.1}$$

One can see from eq. (5.2) with $k=1$, and from eq. (4.17), that the integrand in the RHS has poles only for $q \to p$ and $q \to p_1$, Since W_2 has vanishing $\mathcal{A}$-cycles due to eq. (4.17), we can use the Riemann bilinear identity eq. (A.10), and get:

$$W_2(p,p_1) = \operatorname*{Res}_{q\to p_1} dS_{q,o}(p) W_2(q;p_1) \tag{B.2}$$

For $k=1$, eq. (5.2) reads:

$$\begin{aligned} E_y(x(p),y(p))\, W_2(p,p_1) &= -P_1(x(p),y(p);p_1)dx(p) \\ &\quad + d_{p_1}\left(\frac{U_0(p_1,y(p))}{x(p)-x(p_1)}\,\frac{dx(p)}{dx(p_1)}\right) \end{aligned} \tag{B.3}$$

and thus we have:

$$\begin{aligned} W_2(p;p_1) &= \operatorname*{Res}_{q\to p_1} dS_{q,o}(p) W_2(q;p_1) \\ &= -\operatorname*{Res}_{q\to p_1} dS_{q,o}(p) \frac{P_1(x(q),y(q);p_1)dx(q)}{E_y(x(q),y(q))} \\ &\quad + \operatorname*{Res}_{q\to p_1} dS_{q,o}(p) \frac{d_{p_1}\left(\frac{U_0(p_1,y(q))}{x(q)-x(p_1)}\,\frac{dx(q)}{dx(p_1)}\right)}{E_y(x(q),y(q))} \end{aligned} \tag{B.4}$$

Since $P_1(x(q),y(q);p_1)$ is a polynomial in $x(q)$ and $y(q)$, it has no pole at $q=p_1$. For the second term we use eq. (4.3):

$$\begin{aligned} W_2(p;p_1) &= \operatorname*{Res}_{q\to p_1} dS_{q,o}(p) W_2(q;p_1) \\ &= d_{p_1} \operatorname*{Res}_{q\to p_1} dS_{q,o}(p) \frac{E(x(p_1),y(q))\,dx(q)}{(x(q)-x(p_1))(y(q)-y(p_1))E_y(x(q),y(q))} \\ &= d_{p_1}\, dS_{p_1,o}(p) \\ &= B(p_1,p) \end{aligned} \tag{B.5}$$

We thus recover the well known result: the two-points function is equal to the Bergmann kernel on the Riemann surface corresponding to the algebraic equation $E(x,y)=0$ (cf [5, 20, 27, 30]).

$$\boxed{W_2(p;p_1) = B(p,p_1)} \tag{B.6}$$

Let us now compute $U_1(p,y;p_1)$. For $k=1$, eq. (5.1) reads:

$$\begin{aligned} \frac{(y(r)-y(q))U_1(q,y(r);p_1)}{dx(q)} &= -\frac{W_2(q;p_1)U_0(q,y(r))}{dx(q)^2} - P_1(x(q),y(r);p_1) \\ &\quad +d_{p_1}\left(\frac{U_0(p_1,y(r))}{(x(q)-x(p_1))\,dx(p_1)}\right) \end{aligned} \tag{B.7}$$

take it for $q=r=p^i$:

$$\begin{aligned} 0 &= -\frac{W_2(p^i;p_1)U_0(p^i,y(p^i))}{dx(p^i)^2} - P_1(x(p^i),y(p^i);p_1) \\ &\quad +d_{p_1}\left(\frac{U_0(p_1,y(p^i))}{(x(p^i)-x(p_1))\,dx(p_1)}\right) \end{aligned} \tag{B.8}$$

using that $x(p)=x(p^i)$, we have:

$$\begin{aligned} 0 &= -\frac{W_2(p^i;p_1)U_0(p^i,y(p^i))}{dx(p)^2} - P_1(x(p),y(p^i);p_1) \\ &\quad +d_{p_1}\left(\frac{U_0(p_1,y(p^i))}{(x(p)-x(p_1))\,dx(p_1)}\right) \end{aligned} \tag{B.9}$$

Now, write eq. (B.7) with $q=p$ and $r=p^i$:

$$\begin{aligned} \frac{(y(p^i)-y(p))U_1(p,y(p^i);p_1)}{dx(p)} &= -\frac{W_2(p;p_1)U_0(p,y(p^i))}{dx(p)^2} - P_1(x(p),y(p^i);p_1) \\ &\quad +d_{p_1}\left(\frac{U_0(p_1,y(p^i))}{(x(p)-x(p_1))\,dx(p_1)}\right) \end{aligned} \tag{B.10}$$

and insert eq. (B.9), you get:

$$\begin{aligned} (y(p^i)-y(p))U_1(p,y(p^i);p_1) &= \frac{W_2(p^i;p_1)U_0(p^i,y(p^i))}{dx(p)} \\ &\quad -\frac{W_2(p;p_1)U_0(p,y(p^i))}{dx(p)} \end{aligned} \tag{B.11}$$

Using eq. (4.3), i.e. $U_0(p,y)=\frac{E(x(p),y)}{y-y(p)}dx(p)$, this implies:

$$(y(p^i)-y(p))U_1(p,y(p^i);p_1) = W_2(p^i;p_1)E_y(x(p^i),y(p^i)) \tag{B.12}$$

Since $U_1(p, y; p_1)$ is a polynomial of degree $d_2 - 1$ in y, we can reconstruct it through the interpolation formula:

$$U_1(p, y; p_1) \quad = \quad \frac{E(x(p), y)}{(y - y(p))} \sum_{i=1}^{d_2} \frac{1}{y - y(p^i)} \frac{(y(p^i) - y(p))U_1(p, y(p^i); p_1)}{E_y(x(p^i), y(p^i))} \tag{B.13}$$

i.e.

$$U_1(p, y; p_1) = \frac{E(x(p), y)}{(y - y(p))} \sum_{i=1}^{d_2} \frac{W_2(p^i, p_1)}{y - y(p^i)} \tag{B.14}$$

and in particular, at $y = y(p)$, we have:

$$R_1^0(p, p_1)dx(p) = \frac{U_1(p, y(p); p_1)}{E_y(x(p), y(p))} = \sum_{i=1}^{d_2} \frac{W_2(p^i, p_1)}{y(p) - y(p^i)} \tag{B.15}$$

and for $i \neq 0$, we have:

$$R_1^i(p, p_1)dx(p) = \frac{U_1(p, y(p^i); p_1)}{E_y(x(p), y(p^i))} = \frac{W_2(p^i, p_1)}{(y(p^i) - y(p))} \tag{B.16}$$

Appendix C Computation of Eq. (5.39)

In this appendix one proves recursively eq. (5.39) for any k and h.

Let us suppose that this formula is know for any $U_l^{(m)}$ with $m \leq h - 1$ and for any $U_l^{(h)}$ with $l \leq k - 1$. One writes it:

$$\begin{aligned} &U_l^{(m)}(p, y(p^i); \mathbf{p}_L) = \\ &\frac{E_y(x,y(p^i))}{y(p^i)-y(p)} \sum_{r=1}^{min(d_2,k+h)} \sum_{L_1 \bigcup \ldots \bigcup L_r = L} \sum_{m_\alpha=0}^{m} \sum_{l_\alpha=|L_\alpha|}^{l+m} \sum_{j_{\alpha,\beta} \neq j_{\alpha',\beta'} \in [1,d_2]-\{i\}} \frac{1}{\Omega} \\ &\frac{W_{l_1+1}^{(m_1)}(p^i, \mathbf{p}_{L_1}, p^{j_{1,1}}, \ldots, p^{j_{1,l_1-|L_1|}}) \left(\prod_{\alpha=2}^{r} W_{l_\alpha+1}^{(m_\alpha)}(p^{j_{\alpha,0}}, \mathbf{p}_{L_\alpha}, p^{j_{\alpha,1}}, \ldots, p^{j_{\alpha,l_\alpha-|L_\alpha|}}) \right)}{dx(p)^{r-l-1+\sum L_\alpha} \prod_{\alpha,\beta} y(p^i) - y(p^{j_{\alpha,\beta}})} \end{aligned} \tag{C.1}$$

Let us introduce some shortened notations so that one can write this proof in a few pages.

Considering the sum on the RHS of eq. (C.1), one can see that there are two different kinds of terms:

- If $l_1 = |L_1|$, one can factorise the term $W_{|L_1|+1}^{(m_1)}(p^i, \mathbf{p}_{L_1})$. Let us note the sum of these terms $W(p^i, p_L)W(p_L, p^j)$ where we have noted W instead of $W_{|L_1|+1}^{(m_1)}$ to indicate that these are formal notations;

- the other terms correspond to the sum over all $l_1 \neq |L_1|$. Let us denote them by $W(p^i, p_L, p^j)W(p_L, p^j)$.

Using these notations, one can shortly write eq. (C.1):

$$U_l^{(m)}(p, y(p^i); \mathbf{p}_L) = W(p^i, p_L)W(p_L, p^j) + W(p^i, p_L, p^j)W(p_L, p^j) \tag{C.2}$$

Thus the interpolation formula gives:

$$U_l^{(m)}(p^i, y(p^i); p_L) = W(p_L, p^j) + W(p, p_L)W(p_L, p^j) + W(p, p_L, p^j)W(p_L, p^j) \tag{C.3}$$

where the first term corresponds to the sum where all j_β's are different from i and 0 and there is no W_{l_i} whose argument is p or p^i.

On the other hand, one knows the relation 5.38:

$$\begin{aligned} U_k^{(h)}(p, y(p^i); \mathbf{p}_K) = & \sum_{m=0}^h \sum_{j=0}^k \frac{W_{j+1}^{(m)}(p^i, \mathbf{p}_J) U_{k-j}^{(h-m)}(p^i, y(p^i); \mathbf{p}_{K-J})}{(y(p^i)-y(p))dx(p)} \\ & - \sum_{m=0}^h \sum_{j=0}^k \frac{W_{j+1}^{(m)}(p, \mathbf{p}_J) U_{k-j}^{(h-m)}(p, y(p^i); \mathbf{p}_{K-J})}{(y(p^i)-y(p))dx(p)} \\ & - \frac{U_{k+1}^{(h-1)}(p, y(p^i); p, \mathbf{p}_k)}{(y(p^i)-y(p))dx} + \frac{U_{k+1}^{(h-1)}(p^i, y(p^i); p^i, \mathbf{p}_k)}{(y(p^i)-y(p))dx} \end{aligned} \tag{C.4}$$

Remark that the terms in the RHS of this equation correspond to the criterion of the hypothesis and one can then express them as a product of W's following the notations introduced earlier. This reads:

$$\begin{aligned} & U_k^{(h)}(p, y(p^i); p_K) = \\ & W(p^i, p_K)W(p_K, p^j) + W(p^i, p_K)W(p, p_K)W(p_K, p^j) \\ & + W(p^i, p_K)W(p, p_K, p^j)W(p_K, p^j) - W(p, p_K)W(p^i, p_K)W(p_K, p^j) \\ & - W(p, p_K)W(p^i, p_K, p^j)W(p_K, p^j) + W(p^i, p_K, p^j)W(p_K, p^j) \\ & + W(p^i, p, p_K)W(p_K, p^j) + W(p, p_K)W(p^i, p_K, p^j)W(p_K, p^j) \\ & + W(p^i, p, p_K, p^j)W(p_K, p^j) + W(p, p_K, p^j)W(p^i, p_K, p^j)W(p_K, p^j) \\ & - W(p, p^i, p_K)W(p_K, p^j) - W(p^i, p_K)W(p, p_K, p^j)W(p_K, p^j) \\ & - W(p, p^i, p_K, p^j)W(p_K, p^j) - W(p^i, p_K, p^j)W(p, p_K, p^j)W(p_K, p^j) \\ & = W(p^i, p_K)W(p_K, p^j) + W(p^i, p_K, p^j)W(p_K, p^j) \end{aligned} \tag{C.5}$$

So one has proven the formula for $U_k^{(h)}$.

Because this formula is true for h=0, it is true for any k and h.

Appendix D Derivation of Eq. (5.40)

One wants to show that:

$$\begin{aligned}&\sum_{m=0}^{h}\sum_{j=0;mj\neq kh}^{k} W_{j+1}^{(m)}(p,p_J)U_{k-j}^{(h-m)}(p,y(p);p_{K-J}) + \frac{U_{k+1}^{(h-1)}(p,y(p);p,p_k)}{dx} = \\ &\sum_{i=1}^{d_2}\sum_{m=0}^{h}\sum_{j=0;mj\neq kh}^{k} W_{j+1}^{(m)}(p^i,p_J)U_{k-j}^{(h-m)}(p^i,y(p);p_{K-J}) \\ &+\sum_{i=1}^{d_2}\frac{U_{k+1}^{(h-1)}(p^i,y(p);p^i,p_k)}{dx}\end{aligned} \tag{D.1}$$

Let us compute the difference D between the two sides of the equation by the introduction of eq. (5.39) written with some few different notations which are defined as follows:

- $l = r + h - \sum_\alpha h_\alpha$;
- $u_\beta = \sum_{\epsilon=1}^{\beta}(k_\epsilon - |K_\epsilon|) - \beta$.

One can then write:

$$\begin{aligned}
D = &\sum_{m=0}^{h}\sum_{j=0;mj\neq kh}^{k} W_{j+1}^{(m)}(p,p_J)E_y(x,y(p))\sum_{i=1}^{d_2}\frac{1}{y(p)-y(p^i)} \\
&\times \sum_{r=1}^{d_2}\sum_{o=1}^{r}\sum_{h_o=0}^{h-m}\sum_{k_o=0}^{k+h-j-m}\sum_{j_2\neq\ldots\neq j_l\in[1,d_2]-\{i\}}\sum_{K_1\bigcup\ldots\bigcup K_r=K}\frac{1}{\Omega} \\
&\times\frac{W_{k_1+1}^{(h_1)}(p^i,p_{K_1},p^{j_{r+1}},\ldots,p^{j_{r+u_1}})\left(\prod_{\beta=2}^{r}W_{k_\beta+1}^{(h_\beta)}(p^{j_\beta},p_{K_\beta},p^{j_{r+u_{\beta-1}+1}},\ldots,p^{j_{r+u_\beta}})\right)}{\prod_{\gamma=2}^{l}y(p^i)-y(p^{j_\gamma})} \\
&-\sum_{i=1}^{d_2}\sum_{m=0}^{h}\sum_{j=0;mj\neq kh}^{k} W_{j+1}^{(m)}(p^i,p_J)\frac{E_y(x,y(p))}{y(p)-y(p^i)} \\
&\times \sum_{r=1}^{d_2}\sum_{o=1}^{r}\sum_{h_o=0}^{h-m}\sum_{k_o=0}^{k+h-j-m}\sum_{j_2\neq\ldots\neq j_l\in[1,d_2]-\{i\}}\sum_{K_1\bigcup\ldots\bigcup K_r=K}\frac{1}{\Omega} \\
&\times\frac{W_{k_1+1}^{(h_1)}(p,p_{K_1},p^{j_{r+1}},\ldots,p^{j_{r+u_1}})\left(\prod_{\beta=2}^{r}W_{k_\beta+1}^{(h_\beta)}(p^{j_\beta},p_{K_\beta},p^{j_{r+u_{\beta-1}+1}},\ldots,p^{j_{r+u_\beta}})\right)}{\prod_{\gamma=2}^{l}y(p)-y(p^{j_\gamma})} \\
&+E_y(x,y(p))\sum_{i=1}^{d_2}\frac{1}{y(p)-y(p^i)} \\
&\times \sum_{r=1}^{d_2}\sum_{o=1}^{r}\sum_{h_o=0}^{h-1}\sum_{k_o=0}^{k+h}\sum_{j_2\neq\ldots\neq j_l\in[1,d_2]-\{i\}}\sum_{K_1\bigcup\ldots\bigcup K_r=K}\frac{1}{\Omega} \\
&\times\left[\frac{W_{k_1+1}^{(h_1)}(p^i,p,p_{K_1},p^{j_{r+1}},\ldots,p^{j_{r+u_1}})\left(\prod_{\beta=2}^{r}W_{k_\beta+1}^{(h_\beta)}(p^{j_\beta},p_{K_\beta},p^{j_{r+u_{\beta-1}+1}},\ldots,p^{j_{r+u_\beta}})\right)}{\prod_{\gamma=2}^{l}y(p^i)-y(p^{j_\gamma})}\right. \\
&+\frac{W_{k_1+1}^{(h_1)}(p^i,p_{K_1},p^{j_{r+1}},\ldots,p^{j_{r+u_1}})W_{k_2+1}^{(h_2)}(p,p^{j_2},p_{K_2},p^{j_{r+u_1+1}},\ldots,p^{j_{r+u_2}})}{\prod_{\gamma=2}^{l}y(p^i)-y(p^{j_\gamma})} \\
&\left.\times\prod_{\beta=3}^{r}W_{k_\beta+1}^{(h_\beta)}(p^{j_\beta},p_{K_\beta},p^{j_{r+u_{\beta-1}+1}},\ldots,p^{j_{r+u_\beta}})\right] \\
&-\sum_{i=1}^{d_2}\frac{E_y(x,y(p))}{y(p)-y(p^i)} \\
&\times \sum_{r=1}^{d_2}\sum_{o=1}^{r}\sum_{h_o=0}^{h-1}\sum_{k_o=0}^{k+h}\sum_{j_2\neq\ldots\neq j_l\in[1,d_2]-\{i\}}\sum_{K_1\bigcup\ldots\bigcup K_r=K}\frac{1}{\Omega} \\
&\times\left[\frac{W_{k_1+1}^{(h_1)}(p^i,p,p_{K_1},p^{j_{r+1}},\ldots,p^{j_{r+u_1}})\left(\prod_{\beta=2}^{r}W_{k_\beta+1}^{(h_\beta)}(p^{j_\beta},p_{K_\beta},p^{j_{r+u_{\beta-1}+1}},\ldots,p^{j_{r+u_\beta}})\right)}{\prod_{\gamma=2}^{l}y(p)-y(p^{j_\gamma})}\right. \\
&+\frac{W_{k_1+1}^{(h_1)}(p,p_{K_1},p^{j_{r+1}},\ldots,p^{j_{r+u_1}})W_{k_2+1}^{(h_2)}(p^i,p^{j_2},p_{K_2},p^{j_{r+u_1+1}},\ldots,p^{j_{r+u_2}})}{\prod_{\gamma=2}^{l}y(p^i)-y(p^{j_\gamma})} \\
&\left.\times\prod_{\beta=3}^{r}W_{k_\beta+1}^{(h_\beta)}(p^{j_\beta},p_{K_\beta},p^{j_{r+u_{\beta-1}+1}},\ldots,p^{j_{r+u_\beta}})\right]
\end{aligned} \tag{D.2}$$

The difference between the two first terms leaves only the terms corresponding to $u_1 \neq 0$ in the first one minus $u_1 \neq 0$ in the second one.

The difference between two last terms will allow us to compensate the preceding ones. Indeed, the terms with p^i and p together in the same correlation function straightforwardly vanish and one gets the exact opposite to the two first terms remaining.

Thus D=0 and the equality D.1 is proven.

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