Higher genus hyperelliptic reductions of the Benney equations

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Abstract. It was shown by Gibbons and Tsarev (1996 *Phys. Lett.* **A 211** 19, 1999 *Phys. Lett.* **A 258** 263) that N-parameter reductions of the Benney equations correspond to N-parameter families of conformal maps. Here, we consider a specific set of these, the hyperelliptic reductions. The mapping function for this is calculated explicitly by inverting a second-kind Abelian integral on the stratum Θ_1 of the Jacobi variety of a genus $g\,(g \geq 3)$ hyperelliptic curve. This is done using a method based on the result of Jorgenson (1992 *Israel Journal of Mathematics* **77** 273).

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1. Introduction

1.1. Reductions of the Benney Moment Equations

The Benney equations [3] are an example of an infinite system of hydrodynamic type. These can be written as a Vlasov equation [7], [15]

$$\frac{\partial f}{\partial t_2} + p\,\frac{\partial f}{\partial x} - \frac{\partial A^0}{\partial x}\frac{\partial f}{\partial p} = 0.$$

Here $f = f(x,p,t)$ is a distribution function and the moments are defined by

$$A^n = \int_{-\infty}^{\infty} p^n f\,\mathrm{d}p.$$

Benney showed that this system has infinitely many conserved densities, polynomial in the moments A^n.

Following [14] and [1], we will now consider reductions of the moment equations; that is the case where only a finite number, N, of the A^n are independent. Here, the moment equations can be reduced to a diagonal system of hydrodynamic type with N Riemann invariants, $\hat{\lambda}_i$ say, dependent on N characteristic speeds, $\hat{p}_i$. We will assume that the characteristic speeds are real and distinct.

It was shown by Tsarev and one of the authors that in such a case the reductions correspond to $N-$parameter families of conformal mappings of slit domains. For details of the properties of these maps and the general construction of such a domain see [8] and [9]. We will now consider a specific set of these reductions which we will call the hyperelliptic reductions.

1.2. Hyperelliptic reductions

For this set of reductions the conformal mapping $\lambda(p):\Gamma_1\to\Gamma_2$ is defined as follows. Let Γ_1 be the upper half $p-$plane with $3n$ real points marked on it, $p_i\,(i=1,\ldots,2n)$ and the set of characteristic speeds $\hat{p}_j\,(j=1,\ldots n)$. These satisfy

$$p_1<\hat{p}_2<p_3<p_4<\hat{p}_3<p_5<\cdots<p_{2n-1}<\hat{p}_n<p_{2n}.$$

The domain Γ_2 is the upper half $\lambda-$plane with n vertical slits going from the fixed real points λ_i^0 to the variable points $\hat{\lambda}_i$ $(i=1,\ldots,n)$. Here, $\hat{\lambda}_i$ is the Riemann invariant associated with the characteristic speed $\hat{p}_i$ and it satisfies the relation

$$\mathrm{Re}\left(\hat{\lambda}_i\right)=\lambda_i^0.$$

We now impose the conditions

$$\lambda(p)=p+\mathrm{O}\left(\frac{1}{p}\right)\qquad\text{as}\quad p\to\infty \tag{1}$$

and

$$\lambda(p_{2i-1})=\lambda(p_{2i})=\lambda_i^0\quad(i=1,\ldots,n). \tag{2}$$

It follows that $\lambda(p)$ is a function of n independent parameters which may be taken to be $\mathrm{Im}(\hat{\lambda}_i)\,(i=1,\ldots,n)$, the varying heights of the slits ‡ and that Γ_2 is a polygonal domain. The map $p\to\lambda(p)$ is thus of Schwarz-Christoffel type:

$$\lambda(p)=p+\int_\infty^p\left[\varphi(p')-1\right]\,\mathrm{d}p' \tag{3}$$

where $\varphi(p)$ is given by

$$\varphi(p)=\frac{\prod_{i=1}^n(p-\hat{p}_i)}{\sqrt{\prod_{i=1}^{2n}(p-p_i)}}.$$

Figure 1. (The n parameter reduction) The p-plane with n branch cuts.

‡ Note that since $\mathrm{Im}(\lambda)\geq 0\,\forall\,p$ and the distribution function $f=-\pi\mathrm{Im}(\lambda)$, the distribution function is negative.

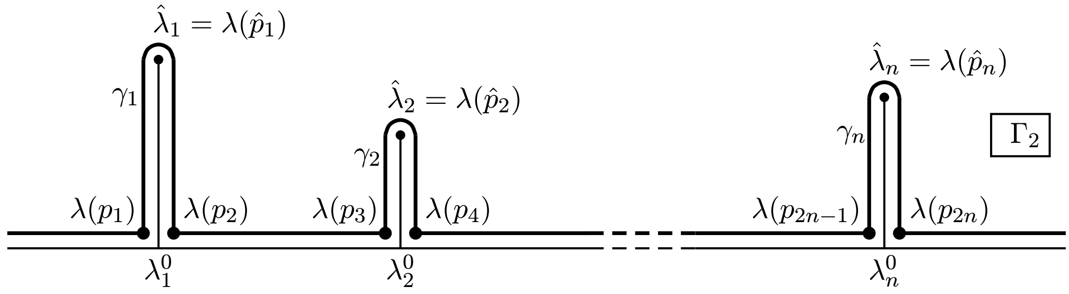

Figure 2. The λ-plane associated with figure 1.

One of the conditions in (1) and (2) may be replaced by the constraint that the residue of $\varphi(p)$, as $p \to \infty$ on either sheet, is zero. This provides a relation between the set of points p_i and the set of characteristic speeds $\hat{p}_j$. Rewriting

$$\varphi(p) = \frac{p^n - \alpha_{n-1}\, p^{n-1} - \alpha_{n-2}\, p^{n-2} - \cdots - \alpha_1\, p - \alpha_0}{\sqrt{\prod_{i=1}^{2n}(p - p_i)}},$$

we find that the expansion of $\varphi(p)$ near infinity is

$$1 + \frac{\left(\frac{1}{2}\sum_{i=1}^{2n} p_i - \alpha_{n-1}\right)}{p} + \mathrm{O}\left(\frac{1}{p^2}\right).$$

The condition on the residue is therefore satisfied when

$$\alpha_{n-1} = \frac{1}{2}\sum_{i=1}^{2n} p_i$$

that is,

$$\sum_{i=1}^{n} \hat{p}_i = \frac{1}{2}\sum_{i=1}^{2n} p_i. \tag{4}$$

It follows that $\varphi(p)\,\mathrm{d}p$ is a *second* kind Abelian differential on the Riemann surface

$$R_g = \left\{ (p, v) : v^2 = \prod_{i=1}^{2n} (p - p_i) \right\}.$$

where $g = n - 1$. That is, the differential 1-form $\varphi(p)\,\mathrm{d}p$ is meromorphic on R_g with zero residue at each singular point.

This surface may be constructed from two copies of the complex $p-$plane joined along the closed intervals

$$[p_{2i-1}, p_{2i}] \qquad (i = 1, 2, \ldots, g+1).$$

A homology basis $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_g; \mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_g)$ for R_g is given in figure 3.

The first three examples of these maps, $g = 0, 1, 2,$ have been worked out in detail. For $g = 0$ the mapping may be calculated directly. The case of the $n = 2$ elliptic reduction was evaluated in [14] by Yu and Gibbons. The $n = 3$ genus 2 hyperelliptic reduction was studied in [1] by the authors. We now consider the case for $g \geq 3$. All such maps, once known explicitly, correspond to reductions of Benney's equations to systems

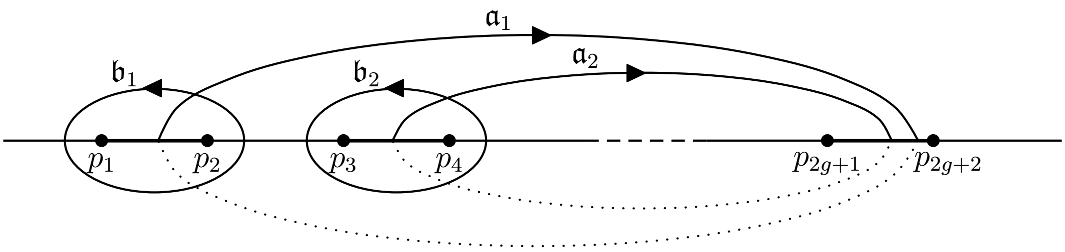

Figure 3. A homology basis on the genus g Riemann surface, R_g. The $\mathfrak{b}$-cycles are closed loops on the first sheet and the $\mathfrak{a}$-cycles are completed on the second sheet (broken line). These have intersection index given by $\mathfrak{a}_i \circ \mathfrak{a}_j = \mathfrak{b}_i \circ \mathfrak{b}_j = 0$, $\mathfrak{a}_i \circ \mathfrak{b}_j = -\mathfrak{a}_j \circ \mathfrak{b}_i = \delta_{ij}$.

of hydrodynamic type with finitely many Riemann invariants. Tsarev's generalised hodograph transformation [13] leads to solutions of these, in terms of the solution of an over-determined system of linear equations. The construction of n-parameter families of such maps is thus an important step towards understanding the solutions of these equations.

2. Transformation of the integral

Following [1], the integral we need to evaluate is (3):

$$\lambda(p) = p + \int_{\infty}^{p} \left[\frac{\prod_{i=1}^{g+1} (p' - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2g+2} (p' - p_i)}} - 1 \right] \mathrm{d}p'.$$

Setting $p = p_{2g+2} - 1/t$ in the integrand $(\varphi(p) - 1)\,\mathrm{d}p$, we find

$$(\varphi(p) - 1)\,\mathrm{d}p = \left(\frac{\mathrm{A}_{g+1}\, t^{g+1} + \mathrm{A}_g\, t^g + \cdots + \mathrm{A}_2\, t^2 + \mathrm{A}_1\, t + (-1)^{g+1}}{\sqrt{\prod_{i=1}^{2g+2} [(p_{2g+2} - p_i)t - 1]}} - 1 \right) \frac{\mathrm{d}t}{t^2} \tag{5}$$

for some constants A_i $(i = 1, 2, \ldots, g+1)$. We note here that

$$\mathrm{A}_1 = (-1)^g \sum_{i=1}^{g+1} (p_{2g+2} - \hat{p}_i).$$

This may be expressed in terms of just the p_i using identity (4):

$$\mathrm{A}_1 = \frac{(-1)^g}{2} \sum_{i=1}^{2g+1} (p_{2g+2} - p_i). \tag{6}$$

If we now remove the constant imaginary factor

$$k = \left(\frac{-4}{\prod_{i=1}^{2g+1} (p_{2g+2} - p_i)} \right)^{\frac{1}{2}}$$

from (5), then we obtain a standardized form for the irrational denominator,

$$\begin{aligned}\varphi(p)\,\mathrm{d}p &= k\left(\frac{\mathrm{A}_{g+1}\,t^{g+1}+\mathrm{A}_g\,t^g+\cdots+\mathrm{A}_2\,t^2+\mathrm{A}_1\,t+(-1)^{g+1}}{s}\right)\frac{\mathrm{d}t}{t^2}\\ &= k\left(\mathrm{A}_{g+1}\,t^{g-1}+\mathrm{A}_g\,t^{g-2}\cdots+\mathrm{A}_2+\frac{\mathrm{A}_1}{t}+\frac{(-1)^{g+1}}{t^2}\right)\frac{\mathrm{d}t}{s} \qquad (7)\end{aligned}$$

where

$$\begin{aligned}s^2 &= -k^2+\left[k^2\sum_{i=1}^{2g+1}(p_{2g+2}-p_i)\right]t+\cdots+\mu_{2g}\,t^{2g}+4\,t^{2g+1}\\ &= \mu_0+\mu_1\,t+\cdots+\mu_{2g}\,t^{2g}+4\,t^{2g+1}. \qquad (8)\end{aligned}$$

The term

$$\varphi_1(p)\,\mathrm{d}p = k\left(A_{g+1}\,t^{g-1}+A_g\,t^{g-2}+\cdots+A_2\right)\frac{\mathrm{d}t}{s}$$

in (7) may be evaluated directly since the set

$$\mathrm{d}u_i = t^{i-1}\frac{\mathrm{d}t}{s} \qquad (i=1,2,\ldots,g)$$

forms a basis of holomorphic Abelian differentials. The last two terms in $\varphi(p)\,\mathrm{d}p$ can be rewritten using (6) and the definitions of μ_0 and μ_1 in (8). We have

$$\begin{aligned}\varphi_2(p)\,\mathrm{d}p &= k\left[\frac{(-1)^{g+1}}{t^2}+\frac{A_1}{t}\right]\frac{\mathrm{d}t}{s}\\ &= (-1)^{g+1}\,k\left[\frac{1}{t^2}-\frac{1}{2}\left(\sum_{i=1}^{2g+1}(p_{2g+2}-p_i)\right)\frac{1}{t}\right]\frac{\mathrm{d}t}{s}\\ &= (-1)^{g+1}\,k\left[\frac{1}{t^2}+\frac{1}{2}\frac{\mu_1}{\mu_0}\frac{1}{t}\right]\frac{\mathrm{d}t}{s}. \qquad (9)\end{aligned}$$

This is a second kind differential on R_g. As in the genus 2 case, we can evaluate $\varphi_2(p)\,\mathrm{d}p$ using a restriction of the Jacobi inversion theorem to a one complex dimensional subspace of the Jacobi variety, the one-dimensional stratum of the theta divisor, Θ_1.

3. The Θ divisor

Following Enolski [4], [5], let $R_g(s,t)$ be the hyperelliptic curve where s and t satisfy

$$s^2 = 4\prod_{i=1}^{2g+1}(t-t_i) = \sum_{i=0}^{2g}\mu_i t^i + 4t^{2g+1}.$$

We define a set of holomorphic and their associated set of second kind differentials on R_g to be, respectively,

$$\mathrm{d}u_i = t^{i-1}\frac{\mathrm{d}t}{s} \qquad (i=1,2,\ldots,g) \qquad (10)$$

and

$$\mathrm{d}r_i = \sum_{k=i}^{2g+1-i}(1+k-i)\,\mu_{1+i+k}\,\frac{t^k\mathrm{d}t}{4s} \qquad (i=1,2,\ldots,g). \qquad (11)$$

From the period integrals of these differentials we form the matrices $\omega, \omega', \eta, \eta'$:

$$2\omega = \left(\oint_{\mathfrak{a}_i} \mathrm{d}u_j \right) \qquad 2\omega' = \left(\oint_{\mathfrak{b}_i} \mathrm{d}u_j \right)$$
$$2\eta = \left(-\oint_{\mathfrak{a}_i} \mathrm{d}r_j \right) \qquad 2\eta' = \left(-\oint_{\mathfrak{b}_i} \mathrm{d}r_j \right) \qquad (i, j = 1, 2, \ldots, g).$$

These matrices satisfy the generalized Legendre relation

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^{\mathrm{T}} = -\frac{i\pi}{2} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix.

Letting $\Lambda = 2\omega \oplus 2\omega'$ be the lattice generated by the periods of the holomorphic differentials, the Jacobi variety, $\mathrm{Jac}(R_g)$, is the g-dimensional complex torus $\mathbb{C}^g/\Lambda$. The Jacobi variety can be subdivided into $k-$dimensional strata, Θ_k, defined by

$$\Theta_k = \sum_{i=1}^{k} \int_{(t_0,s_0)}^{(t_i,s_i)} \mathrm{d}\mathbf{u} + 2\omega \boldsymbol{K}_{(t_0,s_0)} \qquad (k = 1, 2, \ldots, g)$$

where $\boldsymbol{K}_{(t_0,s_0)}$ is the vector of Riemann constants with base point (t_0, s_0). These have the structure $\mathrm{Jac}(R_g) = \Theta_g \supset \Theta_{g-1} \supset \cdots \supset \Theta_2 \supset \Theta_1$. Such stratifications have been studied by Ônishi [12] and others.

The Abel map, $\mathfrak{A} : R_g \to \mathrm{Jac}(R_g)$, is given by $\mathbf{u}(z)$:

$$u_i(z) = \int_{z_0}^{z} \mathrm{d}u_i, \qquad (i = 1, 2, \ldots, g)$$

where the $u_i(z)$ are taken modulo Λ and the base point $z_0 = (t_0, s_0)$ is any fixed point in R_g. These create a one-dimensional image of the hyperelliptic curve in the Jacobi variety. For the inversion theorem we require an extension of this map to a set of points.

Definition 3.1 *A divisor $\mathcal{D}$ on the Riemann surface R_g is defined by the finite formal sum*

$$\mathcal{D} = \sum_{i=1}^{M} n_i \, z_i$$

where $n_i \in \mathbb{Z}$ and $z_i = (s_i, t_i) \in R_g$.

We define the Abel mapping of $\mathcal{D}$ into $\mathrm{Jac}(R_g)$ by

$$\mathfrak{A}(\mathcal{D}) = \sum_{i=1}^{M} n_i \int_{z_0}^{z_i} \mathrm{d}\boldsymbol{u} \qquad \mathrm{mod}\, \Lambda.$$

The lower limit of integration, here the point z_0, is called the base point of the Abel map. From now we shall set this to be (∞, ∞).

3.1. Hyperelliptic functions

Definition 3.2 *The theta function is defined by the Fourier series*

$$\theta((2\omega)^{-1}\boldsymbol{u}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left\{i\pi\left[\boldsymbol{m}^{\mathrm{T}}\tau\,\boldsymbol{m} + \boldsymbol{m}^{\mathrm{T}}(\omega^{-1})\,\boldsymbol{u}\right]\right\},$$

where $\tau = \omega^{-1}\omega'$ *is a symmetric matrix with positive definite imaginary part.*

One important property of this function is that it is zero when $\boldsymbol{u} = 2\omega\boldsymbol{K}$, the vector of Riemann constants associated with the point (∞,∞). For further properties see [4].

From the $\theta-$function we define the Kleinian σ-function of the curve R_g to be

$$\sigma(\boldsymbol{u}) = C\,\exp\left(\boldsymbol{u}^{\mathrm{T}}\chi\boldsymbol{u}\right)\theta((2\omega)^{-1}\boldsymbol{u} - \boldsymbol{K})$$

where

$$C = \sqrt{\frac{\pi^3}{\det 2\omega}}\left(\frac{1}{\prod_{1\le i<j\le 2g+1}(t_i - t_j)}\right)^{\frac{1}{4}}$$

and $\chi = \eta\,(2\omega)^{-1}$ is a symmetric matrix.

In analogy to the Weierstrass $\wp-$function, the Kleinian $\wp-$function is defined as [4]

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j}\ln\left[\sigma(\boldsymbol{u})\right] = \left(\frac{\sigma_i\,\sigma_j - \sigma_{ij}\,\sigma}{\sigma^2}\right)(\boldsymbol{u})$$

where

$$\sigma_i = \frac{\partial}{\partial u_i}\sigma(\mathbf{u}), \quad \sigma_{ij} = \frac{\partial^2}{\partial u_j \partial u_i}\sigma(\mathbf{u}).$$

Higher logarithmic derivatives of σ are expressed similarly. For example

$$\wp_{ijkl} = -\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l}\ln\left[\sigma(\boldsymbol{u})\right].$$

3.2. Jacobi Inversion formula

Theorem 1 *(Jacobi inversion theorem) [4] The Abel preimage of the point* $\boldsymbol{u} \in \mathrm{Jac}(R_g)$ *is given by the set* $S = \{(t_1,s_1),(t_2,s_2),\ldots,(t_g,s_g)\} \in (R_g)^g$, *where* t_k *are the zeros of the polynomial*

$$\mathcal{P}(t;\boldsymbol{u}) = t^g - t^{g-1}\wp_{g,g}(\boldsymbol{u}) - t^{g-2}\wp_{g,g-1}(\boldsymbol{u}) - \ldots - \wp_{g,1}(\boldsymbol{u})$$

and the s_k *are given by*

$$s_k = -\left.\frac{\partial\mathcal{P}(t;\boldsymbol{u})}{\partial u_g}\right|_{t=t_k}.$$

For the integral of the differential (9), we need the preimage of $\boldsymbol{u}$ when the points $t_i \to \infty\,(i = 2,\ldots,g)$. That is, for the case when $S = \{(t_1,s_1)\}$ and so $\boldsymbol{u} \in \Theta_1$:

$$\mathfrak{A}(S) = \int_{\infty}^{t_1} \mathrm{d}\boldsymbol{u}.$$

This relation has been calculated from the results of Jorgenson [11] by Enolski (see Appendix A). We obtain

$$t_1 = -\left.\frac{\sigma_1}{\sigma_2}(\boldsymbol{u})\right|_{\boldsymbol{u}\in\Theta_1} \tag{12}$$

where the one-dimensional stratum Θ_1 may be defined as

$$\Theta_1 = \{\boldsymbol{u} : \sigma(\boldsymbol{u}) = 0,\, \sigma_k(\boldsymbol{u}) = 0 \ (k = 3, \ldots, g)\}\,.$$

This useful result (12) was first given by Grant in [10].

4. Evaluation of the integral

We now further transform the integrand $(\varphi_1(p) + \varphi_2(p))\,\mathrm{d}p$ using the substitution $t = (-\sigma_1/\sigma_2)(\boldsymbol{u})$ (12) and the definitions of the holomorphic differentials, $\mathrm{d}u_i$ $(i = 1, 2, \ldots, g)$ (10).

Table 1. A list of branch points (p_i) and poles $(\infty_\pm)$ of $\lambda(p)$ with the corresponding points in the t and $\mathbf{u}$ variables.

(p)	p_1	p_2	$\ldots$	p_{2g+1}	p_{2g+2}	$\infty_\pm$
(t)	t_1	t_2	$\ldots$	t_{2g+1}	∞	$0_\pm$
$(\mathbf{u})$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\ldots$	$\mathbf{u}_{2g+1}$	$\mathbf{0}$	$\pm\mathbf{u}_0$

Lemma 1 *Let* $t = (-\sigma_1/\sigma_2)(\boldsymbol{u})$ *where* $\boldsymbol{u} \in \Theta_1$ *and define* $\mathrm{d}u_i = t^{i-1}\mathrm{d}t/s$, *a set of holomorphic differentials on* R_g. *Then*

$$\varphi(p)\,\mathrm{d}p = k\left(\boldsymbol{A}^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{u}\right) + (-1)^{g+1}\,k\,\left(\frac{{\sigma_2}^2}{{\sigma_1}^2}(\boldsymbol{u}) - \frac{1}{2}\frac{\mu_1}{\mu_0}\frac{\sigma_2}{\sigma_1}(\boldsymbol{u})\right)\frac{\mathrm{d}t}{s}$$

where $\boldsymbol{A}^{\mathrm{T}} = (A_2, A_3, \ldots, A_{g+1})\,.$

The term

$$\varphi_2(\boldsymbol{u})\,\mathrm{d}u_1 = \left(\frac{{\sigma_2}^2}{{\sigma_1}^2}(\boldsymbol{u}) - \frac{1}{2}\frac{\mu_1}{\mu_0}\frac{\sigma_2}{\sigma_1}(\boldsymbol{u})\right)\mathrm{d}u_1$$

is a second kind differential with a pole of order 2 at $\boldsymbol{u} = \pm\boldsymbol{u}_0$ (see Table 1). This can be verified as follows.

Since $\boldsymbol{u}_0$ is a regular point on the hyperelliptic curve R_g, we can evaluate the expansion of φ_2 near $\boldsymbol{u}_0$ in terms of the local parameter t. Setting $v_k = {\boldsymbol{e}_k}^{\mathrm{T}} \cdot (\boldsymbol{u} - \boldsymbol{u}_0)$ where $(\boldsymbol{e}_k)_j = \delta_{kj}$, we have

$$\begin{aligned} v_k &= \int_\infty^t \mathrm{d}u_k - \int_\infty^0 \mathrm{d}u_k \\ &= \int_0^t \frac{t^{k-1}}{\sqrt{4\,t^{2g+1} + \mu_{2g}\,t^{2g} + \ldots + \mu_1\,t + \mu_0}}\,\mathrm{d}t. \end{aligned}$$

This gives

$$v_k = \left(\frac{1}{k}\frac{1}{\sqrt{\mu_0}}\right) t^k - \left(\frac{1}{2(k+1)}\frac{\mu_1}{\mu_0{}^{\frac{3}{2}}}\right) t^{k+1} + \mathrm{O}(t^{k+2}) \qquad (k = 1, 2, \ldots, g)$$

and so for $k > 1$

$$v_k = \left(\frac{1}{k}\mu_0{}^{(k-1)/2}\right) v_1{}^k + \mathrm{O}\left(v_1{}^{k+1}\right). \tag{13}$$

The Taylor series of φ_2 near $\boldsymbol{u}_0$ can thus be expressed in terms of the single parameter $v_1 = \boldsymbol{e}_1{}^{\mathrm{T}} \cdot (\boldsymbol{u} - \boldsymbol{u}_0)$. We have

$$\frac{\sigma_2}{\sigma_1}(\boldsymbol{u}_0 - (\boldsymbol{u}_0 - \boldsymbol{u})) = \frac{(\sigma_2) + (\sigma_{12})\, v_1 + \cdots}{(\sigma_{11})\, v_1 + \ldots} = \left(\frac{\sigma_2}{\sigma_{11}}\right) v_1^{-1} + \mathrm{O}(1)$$

and

$$\begin{aligned}\frac{\sigma_2{}^2}{\sigma_1{}^2}(\boldsymbol{u}_0 - (\boldsymbol{u}_0 - \boldsymbol{u})) &= \frac{\sigma_2{}^2 + (2\,\sigma_2\,\sigma_{12})\, v_1 + \ldots}{\sigma_{11}{}^2\, v_1^2 + (\sigma_{11}\,\sigma_{111})\, v_1^3 + (2\,\sigma_{11}\,\sigma_{12})\, v_1\, v_2 + \ldots} \\ &= \left(\frac{\sigma_2{}^2}{\sigma_{11}{}^2}\right) v_1^{-2} + \left(2\frac{\sigma_2\,\sigma_{12}}{\sigma_{11}{}^2} - \frac{\sigma_2{}^2\,\sigma_{111}}{\sigma_{11}{}^3} - \sqrt{\mu_0}\frac{\sigma_2{}^2\,\sigma_{12}}{\sigma_{11}{}^3}\right) v_1^{-1} + \mathrm{O}(1)\end{aligned}$$

(using (13)).

These expansions may be simplified by using the substitutions for $\sigma_{11}(\boldsymbol{u}_0)$ and $\sigma_{111}(\boldsymbol{u}_0)$ calculated in Appendix B. This gives

$$\left(\frac{\sigma_2{}^2}{\sigma_1{}^2} - \frac{1}{2}\frac{\mu_1}{\mu_0}\frac{\sigma_2}{\sigma_1}\right)(\boldsymbol{u}_0 - (\boldsymbol{u}_0 - \boldsymbol{u})) = \left(\frac{1}{\mu_0}\right) v_1^{-2} + \mathrm{O}(1) \qquad (\forall\, g \geq 3). \tag{14}$$

In analogy to the genus 2 case, we now consider the function

$$\Psi(\boldsymbol{u}) = -\frac{1}{\mu_0}\frac{\sigma_{11}}{\sigma_1}(\boldsymbol{u})$$

for $\boldsymbol{u} \in \Theta_1$. Since $\mathrm{d}u_i = (-\sigma_1/\sigma_2)^{(i-1)}\,\mathrm{d}u_1$, the derivative of Ψ with respect to u_1 along $\Theta_1 = \{\boldsymbol{u} : \sigma = 0,\ \sigma_k = 0\ (k = 3, \ldots, g)\}$ is

$$\begin{aligned}\psi &= \frac{\mathrm{d}}{\mathrm{d}u_1}\left[-\frac{1}{\mu_0}\frac{\sigma_{11}}{\sigma_1}\right] \\ &= -\frac{1}{\mu_0}\left[\sum_{i=1}^{g}(-1)^{i-1}\left(\frac{\sigma_1}{\sigma_2}\right)^{i-1}\left(\frac{\sigma_{11i}}{\sigma_1} - \frac{\sigma_{11}\,\sigma_{1i}}{\sigma_1{}^2}\right)\right].\end{aligned} \tag{15}$$

This function is only singular when $\sigma_1(\boldsymbol{u}) = 0$, that is when $\boldsymbol{u} = \pm\boldsymbol{u}_0$.

We calculate the Taylor series of ψ near the singular point $\boldsymbol{u}_0$ as follows. Since just the first three terms in the sum contain negative powers of σ_1 we will rewrite $\psi(\boldsymbol{u})$ as

$$\psi = -\frac{1}{\mu_0}\left[\left(-\sigma_{11}{}^2\right)\frac{1}{\sigma_1{}^2} + \left(\sigma_{111} + \frac{\sigma_{11}\,\sigma_{12}}{\sigma_2}\right)\frac{1}{\sigma_1} + \mathrm{O}(1)\right] \qquad (\forall g \geq 3)$$

for $\boldsymbol{u}$ near $\boldsymbol{u}_0$. If we now take the limit $\boldsymbol{u} \to \boldsymbol{u}_0 \Leftrightarrow p \to \infty$, we obtain

$$\begin{aligned}\lim_{\boldsymbol{u}\to\boldsymbol{u}_0}\left[\frac{1}{\mu_0}\frac{\sigma_{11}{}^2}{\sigma_1{}^2}\right] &= \lim_{v_i\to 0}\left[\frac{(\sigma_{11}{}^2) + (2\,\sigma_{11}\,\sigma_{111})\, v_1 + \cdots}{(\mu_0\,\sigma_{11}{}^2)\, v_1^2 + (\mu_0\,\sigma_{11}\,\sigma_{111})\, v_1^3 + (2\mu_0\,\sigma_{11}\,\sigma_{12})\, v_1\, v_2 + \cdots}\right] \\ &= \lim_{v_1\to 0}\left[\left(\frac{1}{\mu_0}\right) v_1^{-2} + \left(\frac{1}{\mu_0}\frac{\sigma_{111}}{\sigma_{11}} - \frac{1}{\sqrt{\mu_0}}\frac{\sigma_{12}}{\sigma_{11}}\right) v_1^{-1} + \mathrm{O}(1)\right]\end{aligned}$$

and

$$\lim_{\boldsymbol{u}\to\boldsymbol{u}_0}\left[-\frac{1}{\mu_0}\left(\frac{\sigma_{111}}{\sigma_1}+\frac{\sigma_{11}\,\sigma_{12}}{\sigma_2\,\sigma_1}\right)\right] = \lim_{v_i\to 0}\left[\frac{-(\sigma_{111}\,\sigma_2+\sigma_{11}\,\sigma_{12})+\cdots}{(\mu_0\,\sigma_2\,\sigma_{11})v_1+\cdots}\right]$$
$$= \lim_{v_1\to 0}\left[\left(-\frac{1}{\mu_0}\frac{\sigma_{111}}{\sigma_{11}}-\frac{1}{\mu_0}\frac{\sigma_{12}}{\sigma_2}\right)v_1^{-1}+\mathrm{O}(1)\right].$$

Combining these gives

$$\lim_{\boldsymbol{u}\to\boldsymbol{u}_0}\psi(\boldsymbol{u}) = \lim_{v_1\to 0}\left[\left(\frac{1}{\mu_0}\right)v_1^{-2}+\left(-\frac{1}{\sqrt{\mu_0}}\frac{\sigma_{12}}{\sigma_{11}}(\boldsymbol{u}_0)-\frac{1}{\mu_0}\frac{\sigma_{12}}{\sigma_2}(\boldsymbol{u}_0)\right)v_1^{-1}+\mathrm{O}(1)\right]$$
$$= \left(\frac{1}{\mu_0}\right)v_1^{-2}+\mathrm{O}(1) \qquad (\forall g\geq 3) \tag{16}$$

(using substitution (B.1)).

From the expansion of φ_2 (14) and ψ (16) near their singular points, it follows that $(\varphi_2(\boldsymbol{u})-\psi(\boldsymbol{u}))$ is a holomorphic function on R_g. We thus have that

$$(-1)^{g+1}\,\varphi_2(\boldsymbol{u})\,\mathrm{d}u_1+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{.}\,\mathrm{d}\boldsymbol{u} = (-1)^{g+1}\,\psi(\boldsymbol{u})\,\mathrm{d}u_1+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{.}\,\mathrm{d}\boldsymbol{u} \tag{17}$$

for some g-vector of constants $\boldsymbol{B}=(B_2,B_3,\ldots,B_{g+1})^{\mathrm{T}}$.

5. Evaluation of the vector $\boldsymbol{B}$.

Following [2], let f be a function on the Riemann surface R_g. The divisor of f, (f), is defined as

$$(f)=\sum n_i\,Z_i-\sum m_i\,P_i \qquad n_i,m_i\in\mathbb{Z}^+$$

where Z_i is a zero of f of degree n_i and P_i is a pole of f of order m_i. The degree of the divisor of f is

$$deg\,(f)=\sum n_i-\sum m_i.$$

For any function f and Abelian differential $\mathrm{d}v$ the following hold:

$$\begin{aligned} deg\,(f) &= 0;\\ deg\,(\mathrm{d}v) &= 2g-2. \end{aligned} \tag{18}$$

We will now consider the Abelian differential

$$(-1)^{g+1}\,[\varphi_2(\boldsymbol{u})-\psi(\boldsymbol{u})]\,\mathrm{d}u_1.$$

By construction, $\mathrm{d}u_1$ is a first kind Abelian differential. It therefore has no poles on R_g and zeros of total degree $(2g-2)$. From section 4, we know that the hyperelliptic function $(\varphi_2-\psi)$ has no poles and so, by (18), it cannot have any zeros. Hence, for some constant C_0, we have

$$C_0\,\mathrm{d}u_1=(-1)^{g+1}\,[\varphi_2(\boldsymbol{u})-\psi(\boldsymbol{u})]\,\mathrm{d}u_1.$$

Rewriting this using identity (17) gives

$$C_0\,\mathrm{d}u_1=(\boldsymbol{B}-\boldsymbol{A})^{\mathrm{T}}\boldsymbol{.}\,\mathrm{d}\boldsymbol{u}$$
$$\Rightarrow \qquad C_0\,\frac{\mathrm{d}t}{s}=\left[(B_2-A_2)+(B_3-A_3)\,t+\cdots+(B_{g+1}-A_{g+1})\,t^{g-1}\right]\frac{\mathrm{d}t}{s}.$$

Matching coefficients of t, we see

$$C_0 = B_2 - A_2$$

and so

$$B_i = A_i \qquad (i = 3, \dots, g+1).$$

The value of B_2 may be found by evaluating $(\varphi_2(\boldsymbol{u}) - \psi(\boldsymbol{u}))$ at a specific point. If, for example, we take $\boldsymbol{u} = \boldsymbol{u}_0$, then we obtain

$$C_0 = \lim_{\boldsymbol{u}\to\boldsymbol{u}_0} [\varphi_2(\boldsymbol{u}) - \psi(\boldsymbol{u})] = \left(\frac{1}{\sqrt{\mu_0}}\frac{\sigma_{22}}{\sigma_2}(\boldsymbol{u}_0) + \frac{2}{\mu_0}\frac{\sigma_{112}}{\sigma_2}(\boldsymbol{u}_0) - \frac{2}{\mu_0}\frac{{\sigma_{12}}^2}{{\sigma_2}^2}(\boldsymbol{u}_0)\right) + \mathrm{O}(v_1)$$

(using substitutions (B.1), (B.2) and (B.3) from Appendix B). From this we have

$$B_2 = A_2 + (-1)^{g+1}\left(\frac{1}{\sqrt{\mu_0}}\frac{\sigma_{22}}{\sigma_2}(\boldsymbol{u}_0) + \frac{2}{\mu_0}\frac{\sigma_{112}}{\sigma_2}(\boldsymbol{u}_0) - \frac{2}{\mu_0}\frac{{\sigma_{12}}^2}{{\sigma_2}^2}(\boldsymbol{u}_0)\right).$$

It would be possible to rewrite $\sigma_{112}(\boldsymbol{u}_0)$ in terms of lower order $\sigma-$derivatives using the following procedure. For each $g \geq 1$ there exists a set of PDE of the form

$$\wp_{ijkl} - f(\mu_0, \dots, \mu_{2g+1}; \wp_{mn}) = 0, \tag{19}$$

where $1 \leq i \leq j \leq k \leq l \leq g$ and $1 \leq m \leq n \leq g$ (see [4]). If we expand (19) for $\boldsymbol{u}$ near $\boldsymbol{u}_0$, then we get Taylor series equal to zero. The relations between the $\sigma-$derivatives at the point $\boldsymbol{u}_0 \in \Theta_1$ are then found by setting $\sigma(\boldsymbol{u}_0) = \sigma_1(\boldsymbol{u}_0) = \sigma_k(\boldsymbol{u}_0) = 0 \quad (k = 3, \dots, g)$ and equating each coefficient with zero. This process, however, cannot easily be generalized for all $g \geq 3$.

6. Result

Setting

$$\begin{aligned} k &= \pm\sqrt{\mu_0} = \pm\left(\frac{-4}{\prod_{i=1}^{2g+1}(p_{2g+2} - p_i)}\right)^{\frac{1}{2}}, \\ \widetilde{B}_2 &= (-1)^{g+1}\left(\frac{1}{\sqrt{\mu_0}}\frac{\sigma_{22}}{\sigma_2}(\boldsymbol{u}_0) + \frac{2}{\mu_0}\frac{\sigma_{112}}{\sigma_2}(\boldsymbol{u}_0) - \frac{2}{\mu_0}\frac{{\sigma_{12}}^2}{{\sigma_2}^2}(\boldsymbol{u}_0)\right) \end{aligned}$$

and substituting

$$p = p_{2g+2} - \frac{1}{t} = p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\boldsymbol{u})$$

into (3) we have

$$\begin{aligned} \lambda(p) &= p + \int_\infty^p [\varphi(p') - 1]\,\mathrm{d}p' \\ &= \left(p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\boldsymbol{u})\right) + \int_0^{\frac{1}{(p_{2g+2}-p)}} \left[k\,\boldsymbol{A}^{\mathrm{T}} \boldsymbol{.}\, \mathrm{d}\boldsymbol{u} + k\,\widetilde{B}_2\,\mathrm{d}u_1 + (-1)^{g+1} k\left(\frac{\mathrm{d}}{\mathrm{d}u_1}\Psi(\boldsymbol{u})\right)\mathrm{d}u_1 - \frac{\mathrm{d}t}{t^2}\right] \\ &= \left(p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\boldsymbol{u})\right) + \left[k\left(\boldsymbol{A} + \widetilde{B}_2\,\boldsymbol{e}_1\right)^{\mathrm{T}} \boldsymbol{.}\, \boldsymbol{u} + (-1)^g \frac{k}{\mu_0}\frac{\sigma_{11}}{\sigma_1} - \frac{\sigma_2}{\sigma_1}(\boldsymbol{u})\right] + \widetilde{C}. \end{aligned}$$

The value of the constant $\widetilde{C}$ can be found by considering the limit of $(\lambda(p) - p)$ as $p \to \infty_+ \Leftrightarrow \boldsymbol{u} \to +\boldsymbol{u}_0$. Since

$$\lim_{p\to\infty} [\,\lambda(p) - p\,] = 0,$$

we have that

$$\widetilde{C} = -k\,\left(\boldsymbol{A} + \widetilde{B}_2\,\boldsymbol{e}_1\right)^{\mathrm{T}} \boldsymbol{.}\, \boldsymbol{u}_0 + \lim_{\boldsymbol{u}\to\boldsymbol{u}_0}\left[(-1)^{\,g+1}\frac{k}{\mu_0}\frac{\sigma_{11}}{\sigma_1}(\boldsymbol{u}) + \frac{\sigma_2}{\sigma_1}(\boldsymbol{u})\right].$$

Expanding the terms in this limit we obtain

$$\begin{aligned}
\lim_{\boldsymbol{u}\to\boldsymbol{u}_0}\left[(-1)^{\,g+1}\frac{k}{\mu_0}\frac{\sigma_{11}}{\sigma_1}\right] &= (-1)^{\,g+1}\left(\frac{k}{\mu_0}\right)\lim_{v_i\to 0}\left[\frac{(\sigma_{11}) + (\sigma_{111})\,v_1 + \cdots}{(\sigma_{11})\,v_1 + (\frac{1}{2}\sigma_{111})\,v_1^2 + (\sigma_{12})\,v_2 + \cdots}\right]\\
&= (-1)^{\,g+1}\left(\frac{k}{\mu_0}\right)\lim_{v_1\to 0}\left[v_1^{-1} + \left(\frac{1}{2}\frac{\sigma_{111}}{\sigma_{11}} - \frac{\sqrt{\mu_0}}{2}\frac{\sigma_{12}}{\sigma_{11}}\right) + \mathrm{O}(v_1)\right]\\
&= (-1)^{\,g+1}\left(\frac{k}{\mu_0}\right)\lim_{v_1\to 0}\left[v_1^{-1} + \left(2\frac{\sigma_{12}}{\sigma_2} + \frac{1}{4}\frac{\mu_1}{\sqrt{\mu_0}}\right) + \mathrm{O}(v_1)\right]
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\boldsymbol{u}\to\boldsymbol{u}_0}\left[\frac{\sigma_2}{\sigma_1}\right] &= \lim_{v_i\to 0}\left[\frac{(\sigma_2) + (\sigma_{12})\,v_1 + \cdots}{(\sigma_{11})\,v_1 + (\frac{1}{2}\sigma_{111})\,v_1^2 + (\sigma_{12})\,v_2 + \cdots}\right]\\
&= \lim_{v_1\to 0}\left[\left(\frac{\sigma_2}{\sigma_{11}}\right)\,v_1^{-1} + \left(\frac{\sigma_{12}}{\sigma_{11}} - \frac{1}{2}\frac{\sigma_2\,\sigma_{111}}{{\sigma_{11}}^2} - \frac{\sqrt{\mu_0}}{2}\frac{\sigma_2\sigma_{12}}{{\sigma_{11}}^2}\right) + \mathrm{O}(v_1)\right]\\
&= \lim_{v_1\to 0}\left[\left(-\frac{1}{\sqrt{\mu_0}}\right)\,v_1^{-1} + \left(\frac{1}{4}\frac{\mu_1}{\mu_0}\right) + \mathrm{O}(v_1)\right].
\end{aligned}$$

Since $\widetilde{C}$ is constant we set $k = (-1)^{\,g+1}\sqrt{\mu_0}$ and hence

$$\widetilde{C} = (-1)^{\,g}\sqrt{\mu_0}\,\left(\boldsymbol{A} + \widetilde{B}_2\,\boldsymbol{e}_1\right)^{\mathrm{T}} \boldsymbol{.}\, \boldsymbol{u}_0 + \frac{2}{\sqrt{\mu_0}}\frac{\sigma_{12}}{\sigma_2}(\boldsymbol{u}_0) + \frac{1}{2}\frac{\mu_1}{\mu_0}.$$

This gives the following result.

Theorem 2 *Let*

$$\begin{aligned}
\lambda(p) &= p + \int_\infty^p \frac{\prod_{i=1}^{g+1}(p' - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2g+2}(p' - p_i)}}\,\mathrm{d}p',\\
k &= (-1)^{\,g+1}\left(\frac{-4}{\prod_{i=1}^{2g+1}(p_{2g+2} - p_i)}\right)^{\frac{1}{2}},\\
\widetilde{B}_2 &= (-1)^{g+1}\left(\frac{1}{\sqrt{\mu_0}}\frac{\sigma_{22}}{\sigma_2}(\boldsymbol{u}_0) + \frac{2}{\mu_0}\frac{\sigma_{112}}{\sigma_2}(\boldsymbol{u}_0) - \frac{2}{\mu_0}\frac{{\sigma_{12}}^2}{{\sigma_2}^2}(\boldsymbol{u}_0)\right)
\end{aligned}$$

and $\boldsymbol{A}^{\mathrm{T}} = (\mathrm{A}_2, \mathrm{A}_3, \ldots, \mathrm{A}_{g+1})$ *where the* A_i *are defined as*

$$\sum_{i=0}^{g+1}\mathrm{A}_i\,t^i = \prod_{i=1}^{g+1}[(p_{2g+2} - \hat{p}_i)\,t - 1].$$

Then, if we set

$$p = p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\boldsymbol{u})$$

with $\boldsymbol{u}$, $\boldsymbol{u}_0 \in \Theta_1$ *and* $\sigma_1(\boldsymbol{u}_0) = 0$, *we have*

$$\lambda(p) = (-1)^{g+1}\sqrt{\mu_0}\left(\boldsymbol{A} + \widetilde{B}_2\,\boldsymbol{e}_1\right)^{\mathrm{T}} \cdot (\boldsymbol{u} - \boldsymbol{u}_0) - \frac{1}{\sqrt{\mu_0}}\frac{\sigma_{11}}{\sigma_1}(\boldsymbol{u}) + p_{2g+2} + \frac{2}{\sqrt{\mu_0}}\frac{\sigma_{12}}{\sigma_2}(\boldsymbol{u}_0) + \frac{1}{2}\frac{\mu_1}{\mu_0} \tag{20}$$

on sheet R_g^+ *of the Riemann surface*

$$R_g = \left\{(v,p) \in \mathbb{C}^g : v^2 = \prod_{i=1}^{2g+2}(p - p_i)\right\}$$

associated with the relation $p \to \infty_+ \Leftrightarrow \boldsymbol{u} \to +\boldsymbol{u}_0$.

We note that in the $g = 2$ case the analogous solution to (20) could be rewritten using the relation

$$\frac{\sigma_{11}}{\sigma_1}(\boldsymbol{u}) = \frac{\sigma_1}{\sigma}(\boldsymbol{u} + \boldsymbol{u}_0) + \frac{\sigma_1}{\sigma}(\boldsymbol{u} - \boldsymbol{u}_0) = \zeta_1(\boldsymbol{u} + \boldsymbol{u}_0) + \zeta_1(\boldsymbol{u} - \boldsymbol{u}_0)$$

for $\boldsymbol{u} \in \Theta_1$. In the case of higher genus reductions this is not possible since $(\boldsymbol{u}\pm\boldsymbol{u}_0) \in \Theta_2$ and ζ_1 is singular everywhere on Θ_2.

The formula (20) seems a little more complicated than the analogous results in genus 1 and 2; the reason for this is the difficulty of expanding the terms involving $\boldsymbol{u}_0$ in the general case. However, we consider it remarkable that essentially the same formula is valid for any genus.

Acknowledgments

We would like to thank V Z Enolski for bringing [11] to our attention and for the result given in Appendix A.

Appendix A. Reduction of the Inversion theorem to Θ_1.

Following Enolski and Previato [6], we begin by rewriting the main result of [11] in terms of first derivatives of the $\sigma-$function.

Theorem 3 *Let* $\boldsymbol{K}_P$ *be the vector of Riemann constants associated with the point* P, $\{P_1, P_2, \ldots, P_{g-1}\}$ *be a set of points on* R_g *and let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_g)^{\mathrm{T}}$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_g)^{\mathrm{T}} \in \mathbb{C}^g$ *be any nonzero vectors. Then the following identity holds*

$$\frac{\sum_{j=1}^{g}\sigma_j\,(\boldsymbol{u})\,a_j}{\sum_{j=1}^{g}\sigma_j\,(\boldsymbol{u})\,b_j} = \frac{\det[\boldsymbol{a}|\,\mathrm{d}\boldsymbol{u}(P_1)|\cdots|\,\mathrm{d}\boldsymbol{u}(P_{g-1})]}{\det[\boldsymbol{b}|\,\mathrm{d}\boldsymbol{u}(P_1)|\cdots|\,\mathrm{d}\boldsymbol{u}(P_{g-1})]}$$

where the point $\boldsymbol{u}$ *is given by*

$$\boldsymbol{u} = \sum_{k=1}^{g-1}\int_P^{P_k}\mathrm{d}\boldsymbol{u} + 2\,\omega\boldsymbol{K}_P.$$

Here, we take the $\mathrm{d}u_i$ *to be the holomorphic differentials defined above:*

$$\mathrm{d}u_i = \frac{t^{i-1}}{s}\,\mathrm{d}t \quad (i = 1, \ldots, g).$$

Corollary 3.1 *Let the points* $P_1, P_2, \ldots, P_{g-1}$ *coalesce to a point* P*. Then we obtain by L'Hôpital's rule*

$$\frac{\sum_{j=1}^{g} \sigma_j\,(2\omega\,\boldsymbol{K}_P)\,a_j}{\sum_{j=1}^{g} \sigma_j\,(2\omega\,\boldsymbol{K}_P)\,b_j} = \frac{\det\left[\boldsymbol{a}|\,\mathrm{d}\boldsymbol{u}(P)|\,\mathrm{d}\boldsymbol{u}(P)^{(1)}|\cdots|\,\mathrm{d}\boldsymbol{u}(P)^{(g-2)}\right]}{\det\left[\boldsymbol{b}|\,\mathrm{d}\boldsymbol{u}(P)|\,\mathrm{d}\boldsymbol{u}(P)^{(1)}|\cdots|\,\mathrm{d}\boldsymbol{u}(P)^{(g-2)}\right]} \tag{A.1}$$

where $\mathrm{d}\boldsymbol{u}(P)^{(k)}$ *denotes the column of* k^{th} *derivatives of the holomorphic differentials* $\mathrm{d}\boldsymbol{u}(P)$.

Expanding the RHS of (A.1) we find that the numerator is the determinant of the matrix

$$C\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & t & 0 & 0 & \cdots & 0 & 0 \\ a_3 & t^2 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{g-1} & t^{g-2} & 0 & 1 & \cdots & 0 & 0 \\ a_g & t^{g-1} & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

for some constant C. The matrix in the denominator of the RHS is of the same form, but with b_i instead of a_i $(i = 1, \ldots, g)$. It follows that (A.1) can be written as

$$\frac{\sum_{j=1}^{g} \sigma_j\,(2\omega\,\boldsymbol{K}_P)\,a_j}{\sum_{j=1}^{g} \sigma_j\,(2\omega\,\boldsymbol{K}_P)\,b_j} = \frac{a_1 t - a_2}{b_1 t - b_2}. \tag{A.2}$$

To evaluate t in terms of the σ_j we can therefore set $\boldsymbol{a} = (1, 0, \ldots, 0)^{\mathrm{T}}$ and $\boldsymbol{b} = (0, 1, 0, \ldots, 0)^{\mathrm{T}}$. This gives

$$\frac{\sigma_1}{\sigma_2}(\boldsymbol{u}) = -t$$

for $\boldsymbol{u} \in \Theta_1$. Further, since only a_1, a_2 and b_1, b_2 appear in the RHS of (A.2), we obtain the following definition for Θ_1:

$$\Theta_1 = \{\boldsymbol{u} : \sigma(\boldsymbol{u}) = 0,\, \sigma_k(\boldsymbol{u}) = 0 \quad (k = 3, \ldots, g)\}\,.$$

Appendix B. Differential relations holding at $\boldsymbol{u} = \boldsymbol{u}_0$.

For any $\boldsymbol{u}$ in Θ_1 we have $\sigma(\boldsymbol{u}) = 0$. Expanding this identity near $\boldsymbol{u}_0$ we obtain a Taylor series in $v_k = \boldsymbol{e}_k{}^{\mathrm{T}} \cdot (\boldsymbol{u} - \boldsymbol{u}_0)$ equal to zero:

$$\begin{aligned} 0 &= \sigma(\boldsymbol{u}_0 - (\boldsymbol{u}_0 - \boldsymbol{u})) \\ &= \left[\frac{1}{2}\sigma_{11}(\boldsymbol{u}_0)\right]\, v_1^2 + [\sigma_2(\boldsymbol{u}_0)]\, v_2 + [\sigma_{12}(\boldsymbol{u}_0)]\, v_1\, v_2 + \left[\frac{1}{6}\,\sigma_{111}(\boldsymbol{u}_0)\right]\, v_1^3 + \cdots \end{aligned}$$

(since $\sigma(\boldsymbol{u}_0) = \sigma_1(\boldsymbol{u}_0) = \sigma_3(\boldsymbol{u}_0) = 0$). If we now substitute relations (13)

$$v_k = \left(\frac{1}{k}{\mu_0}^{(k-1)/2}\right) {v_1}^k + \mathrm{O}\left({v_1}^{k+1}\right) \qquad (k = 2, 3, \ldots, g)$$

into this expansion, then for $g \geq 3$ we have

$$0 = \left[\frac{1}{2}\,\sigma_{11}(\boldsymbol{u}_0) + \frac{1}{2}\sqrt{\mu_0}\,\sigma_2(\boldsymbol{u}_0)\right]\,v_1^2 + \left[\frac{1}{6}\sigma_{111}(\boldsymbol{u}_0) + \frac{1}{12}\,\mu_1\,\sigma_2(\boldsymbol{u}_0) + \frac{1}{2}\sqrt{\mu_0}\,\sigma_{12}(\boldsymbol{u}_0)\right]\,v_1^3 + \mathrm{O}(v_1^4).$$

Setting each coefficient to zero, we find

$$\sigma_{11}(\boldsymbol{u}_0) \;=-\sqrt{\mu_0}\,\sigma_2(\boldsymbol{u}_0) \qquad \text{(B.1)}$$

and

$$\sigma_{111}(\boldsymbol{u}_0) = -\frac{1}{2}\,\mu_1\,\sigma_2(\boldsymbol{u}_0) - 3\,\sqrt{\mu_0}\,\sigma_{12}(\boldsymbol{u}_0) \qquad \text{(B.2)}$$

for $\boldsymbol{u}_0 \in \Theta_1$ with $\sigma_1(\boldsymbol{u}_0) = 0$ and for $\forall g \geq 3$.

If we repeat the above procedure for the identity $\sigma_3(\boldsymbol{u}) = 0$ $(\forall\,\boldsymbol{u} \in \Theta_1)$, then we obtain the following expansion

$$\begin{aligned} 0 &= \sigma_3(\boldsymbol{u}_0 - (\boldsymbol{u}_0 - \boldsymbol{u})) \\ &= [\sigma_{13}(\boldsymbol{u}_0)]\;v_1 + [\sigma_{23}(\boldsymbol{u}_0)]\;v_2 + \left[\frac{1}{2}\sigma_{113}(\boldsymbol{u}_0)\right]\;{v_1}^2 + \cdots \\ &= [\sigma_{13}(\boldsymbol{u}_0)]\;v_1 + \mathrm{O}({v_1}^2). \end{aligned}$$

This gives the identity

$$\sigma_{13}(\boldsymbol{u}_0) = 0 \qquad \text{for } g \geq 3. \qquad \text{(B.3)}$$

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