A Riemann-Hilbert Approach to the Akhiezer Polynomials

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Abstract

In this paper, we study those polynomials, orthogonal with respect to a particular weight, over the unioin of disjoint intervals, first introduced by N. I. Akhiezer, via a reformulation as a matrix factorization or Riemann-Hilbert problem. This approach complements the method proposed in a previous paper, that involves the construction of a certain meromorphic function on a hyperelliptic Riemann surface. The method described here is based on the general Riemann-Hilbert scheme of the theory of integrable systems and will enable us to derive, in a very strightforward way, the relevant system of Fuchsian differential equations for the polynomials and the associated system of the Schlesinger deformation equations for certain quantaties involving the corresponding recurrence coefficients. Both of these equations were obtained earlier by A. Magnus. In our approach, however, we are able to go beyond Magnus's results by actually solving the equations in terms of the Riemann Θ-functions. We also show that the related Hankel determinant can be interpreted as the relevant $\tau-$ function.

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1 Introduction

The Chebyshev polynomials are those monic polynomials characterised by the property that $\max|\pi_n(x)|,\ \ x \in [-1,1],$ is as small as possible. Indeed, it is also known that π_n is orthogonal with respect to $\frac{1}{\pi\sqrt{1-x^2}}$ over $[-1,1]$. The polynomials π_n—the Chebyshev polynomials of the first kind—which satisfy a constant coefficients three term recurrence relations, can be thought of as the "Hydrogen Atom" model of those polynomials orthogonal over $[-1,1]$. These play a fundamental role in the large n asymptotics of the Bernstein-Szegö polynomials which are orthogonal with respect a "deformed" Chebyshev weight, $p(x)/\sqrt{1-x^2}$, over $[-1,1]$, where $p(x)$ is strictly positive, absolutely continuous and satisfies the Szegö condition [21]

$$\int_{-1}^{1}\frac{\ln p(x)}{\sqrt{1-x^2}}dx > -\infty.$$

Many years ago N. I. Akhiezer and also Yu. Ya. Tomchuk [1], [2], [3] considered a generalization of the Chebyshev polynomials, where the interval of orthogonality is a union of disjoint intervals henceforth denoted as

$$E := (\beta_0,\alpha_1)\cup(\beta_1,\alpha_2)\cup\cdots\cup(\beta_g,\beta_{g+1}). \tag{1.1}$$

For comparison with those of Akhiezer, we assume here $\beta_0 = -1$, and $\beta_{g+1} = 1$. For later convenience, when the end points become independent variables we shall adopt the convention,

$$(\alpha_1,\alpha_2,...,\alpha_g,\ \beta_0,\beta_1,...,\beta_{g+1}) \longrightarrow (\delta_1,\delta_2,...,\delta_{g+1},\delta_{g+2},...,\delta_{2g+2}). \tag{1.2}$$

Let

$$w(z) := \frac{i}{\pi}\sqrt{\frac{\Pi_{j=1}^{g}(z-\alpha_j)}{\Pi_{j=0}^{g+1}(z-\beta_j)}}, \tag{1.3}$$

be defined in the $\mathbb{CP}^1 \setminus E$. The multi-interval analog of the Chebyshev weight is

$$w_+(t) = \frac{1}{\pi}\sqrt{\frac{\Pi_{j=1}^{g}(t-\alpha_j)}{(\beta_{g+1}-t)(t-\beta_0)\Pi_{j=1}^{g}(t-\beta_j)}} > 0, \quad t \in E, \tag{1.4}$$

and is obtained from the continuation $w(z)$ to the top of the cut, E. The generalized Chebyshev or Akhiezer polynomials P_n are monic polynomials orthogonal with respect to w_+, i.e.,

$$\int_E P_m(x)P_n(x)w_+(x)dx = h_n\delta_{m,n}, \tag{1.5}$$

where h_n is the square of the L^2 norm.

In the construction of the Bernstein-Szegö asymptotics over E, for polynomials orthogonal with respect to the weight $p(t)w_+(t)$, where p is an absolutely continuous positive function, exact information on P_n would be required. This would entail the solution of the "Hydrogen Atom" problem in the multiple interval situation. In the case of two intervals, $[-1,\alpha]\cup[\beta,1]$, P_n was constructed by Akhiezer with an innovation which we would now recognise as the Baker-Akhiezer function, associated with the discrete Schrödinger equation, namely, the three term recurrence relations, where the degree of the polynomials n is the "coordinates", and z is spectral variable. Akhiezer, based his construction on the conformal mapping of a doubly connected domain, with the aid of the Jacobian elliptic functions, as a demonstration for his students, the applications of elliptic functions [4]. It is not at all clear how the conformal mapping could be adepted to handle the situation when there are more then two intervals. In the early 1960's, Akhiezer and also with Tomchuk published several very short and very deep papers regarding the Bernstein-Szegö asymptotics. Akhiezer and Tomchuk gave a description of P_n and Q_n (the second solution of the recurrence relations) with the aid of theory of Hyperelliptic integrals in terms of a cerian Abelian integral of the third kind. However, certain unknown points on Riemann surface appear in this respresentation, later circumvented in [5].

In a recent work of A. P. Magnus [6], a general class of semi-classical orthogonal polynomials, which includes the Akhiezer polynomials P_n, was introduced and shown that these polynomials satisfy a certain system of linear Fuchsian equations. It was also demonstrated there that the recurrence coefficients, as functions of the natural parameters of the semi-classical weights, obey the nonlinear Schlesinger equations, i.e. the differential equations describing the isomonodromy deformations of the Fuchsian systems.

In this paper we will study the Akhiezer polynomials P_n using the Riemann-Hilbert approach introduced in the theory of orthogonal polynomials in [7]. This will allow us to exploit the well-developed Riemann-Hilbert and algebro-geometric schemes of the Soliton theory [8], [9], [10] - with certain important technical modifications though, and not only re-derive the previous results of [5] and [6] but also unite them in a single approach and produce further facts concerning the Akhiezer polynomials. Specifically, in addition to the derivation of Magnus's equations, we will solve them in terms of the multidimensional Θ-functions, and we will identify the corresponding Hankel determinant with the relevant $\tau-$function, i.e. with one of the central objects associated with an integrable system, in our case - with the Magnus-Schlesinger equation. It should also be mentioned that part of our Θ - formulae, namely the ones describing the recurrence coefficients and the related Baker-Akhiezer function, reproduce the known expressions obtained in the late 70s (the works of

I. Krichever, D. Mumford, S. Novikov, and M. Salle) for the finite-gap discrete Schrödinger operators which were then intensively studied in connection with the periodic Toda lattice (see the pioneering paper of H. Flaschka and D. McLaughlin [11] and also [8] and [9] for more on the history of the subject).

We would like to think of our paper as a tribute to the pioneering works of N. I. Akhiezer which layed the foundation for the construction, in the 1970's of the algebro-geometric method in the theory of integrable systems, whose modern "Riemann-Hilbert" version we are using here.

2 Riemann-Hilbert problem

According to the classical theory of orthogonal polynomials the monic P_n, (with $P_0 = 1$ and $P_{-1} = 0$) and the polynomials of the second kind,

$$Q_n(z) := \int_E \frac{P_n(z) - P_n(t)}{z - t} w_+(t) dt, \tag{2.1}$$

of degree $n-1$, are linearly independent solutions of the second order difference equation,

$$z v_n(z) = v_{n+1}(z) + b_{n+1} v_n(z) + a_n v_{n-1}(z). \tag{2.2}$$

Following the general scheme of [7] (see also [12], [13]), let us introduced the 2×2 matrix $Y_n(z)$ be defined for $n = 1, 2, ..$ and $z \in \mathbb{C}$ as follows:

$$\begin{aligned} Y_n(z) &= \begin{pmatrix} P_n(z) & \int_E \frac{P_n(t) w_+(t)}{z-t} dt \\ \frac{P_{n-1}(z)}{h_{n-1}} & \frac{1}{h_{n-1}} \int_E \frac{P_{n-1}(t) w_+(t)}{z-t} dt \end{pmatrix} \\ &= \begin{pmatrix} P_n(z) & \psi(z) P_n(z) - Q_n(z) \\ \frac{P_{n-1}}{h_{n-1}} & \frac{\psi(z) P_{n-1}(z) - Q_{n-1}(z)}{h_{n-1}} \end{pmatrix} \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \psi(z) &:= \int_E \frac{w_+(t)}{z-t} dt = \sqrt{\frac{\Pi_{i=1}^g (z - \alpha_i)}{\Pi_{j=0}^{g+1} (z - \beta_i)}} \\ &= -i\pi w(z). \end{aligned} \tag{2.4}$$

Proposition 1. The function $Y_n(z)$ satisfies the following conditions,

RH1. $Y_n(z)$ is analytic in $\mathbb{C} \setminus E$

RH2. $Y_{n,-}(z) = Y_{n,+}(z) \begin{pmatrix} 1 & 2\pi i w_+(z) \\ 0 & 1 \end{pmatrix}, \quad z \in E \setminus \{\beta_j\}_{j=0}^{g+1}$.

RH3. $Y_n(z) z^{-n\sigma_3} \to I, \quad z \to \infty$.

RH4. $Y_n(z) = \hat{Y}_n^{(\beta_j)}(z) \begin{pmatrix} \sqrt{z - \beta_j} & 0 \\ \frac{1}{\mathsf{b}_j} & \frac{1}{\sqrt{z-\beta_j}} \end{pmatrix}, \quad z \in \mathcal{U}_{\beta_j}, \ 0 \le j \le g+1,$

where $\mathcal{U}_{z_0}$ denote a neighborhood of a point z_0. The matrix valued function $\hat{Y}_n^{(\beta_j)}(z)$ is holomorphic in $\sqrt{z-\beta_j}$ and b_j is defined by the equation,

$$w(z) = (z-\beta_j)^{-1/2}\mathsf{b}_j \frac{i}{\pi}(1 + O(z-\beta_j)). \tag{2.5}$$

We shall also assume that the branch of $\sqrt{z-\beta_j}$ is defined by the condition,

$$0 < \arg(z-\beta_j) < 2\pi, \quad \text{if} \quad j \le g, \quad \text{and} \quad -\pi < \arg(z-\beta_{g+1}) < \pi.$$

In addition, we assert that

$$\det \hat{Y}_n^{(\beta_j)}(\beta_j) = 1 \neq 0. \tag{2.6}$$

Proof.

Using the basic properties of the Cauchy integrals and the Plemelj formulae we directly verify that $Y_n(z)$ satisfies **RH1 - RH2**. To check property **RH3** it is enough to note that *because of the orthogonality condition (1.5)*, we have (cf. [7, 5])

$$\begin{aligned}\int_E \frac{P_n(t)w_+(t)}{z-t}dt &= \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int_E P_n(t)w_+(t)t^k dt \\ &= \frac{h_n}{z^{n+1}} + \mathrm{O}\left(\frac{1}{z^{n+2}}\right), \quad z \to \infty.\end{aligned}$$

To prove **RH4** we observe that the matrix product,

$$Y_n(z)\begin{pmatrix} \frac{1}{\sqrt{z-\beta_j}} & 0 \\ -\frac{1}{\mathsf{b}_j} & \sqrt{z-\beta_j}\end{pmatrix},$$

is bounded near β_j (the singular terms in the first column cancel out), and hence the function $\hat{Y}_n^{(\beta_j)}(z)$ defined by equation **RH4** is indeed holomorphic in $\sqrt{z-\beta_j}$. To complete the prove of the proposition we only need to establish equation (2.6). To this end, we notice that we have already established **RH1 - RH4** but short of equation (2.6). One can see, however, that **RH1 - RH4** already yield even stronger statement. Namely, we claim that

$$\det Y_n(z) \equiv 1. \tag{2.7}$$

Indeed, the (scalar) function $\det Y_n(z)$ is holomorphic in $\mathbb{CP}^1 \setminus E$, has no jumps across E and has removable singularities at the end points of E; moreover, it approaches 1 as $z \longrightarrow \infty$. By the Liouville theorem, equation (2.7) follows. Equation (2.6) is a direct consequence of equation (2.7). The proposition is proven.

Remark 2.1 Equation (2.7) can be also derived by using the first line of (2.3) and the Christoeffel-Darbooux formula,

$$\det Y_n(z) = \frac{1}{h_{n-1}}\int_E \frac{P_n(z)P_{n-1}(t) - P_{n-1}(z)P_n(t)}{z-t}w_+(t)dt$$

$$= \int_E \sum_{k=0}^{n-1} \frac{1}{h_k} P_k(z) P_k(t) w_+(t) dt$$

$$= \int_E K_n(z,t) w_+(t) dt = P_0(z) h_0 = 1, \tag{2.8}$$

or from the recurrence relations,

$$\det Y_n(z) = \frac{1}{h_{n-1}} (Q_n(z) P_{n-1}(z) - P_n(z) Q_{n-1}(z)) = 1. \tag{2.9}$$

Remark 2.2 $Y_n(z)$ also depend on $\{\delta_j : 1 \le j \le 2g+2\}$.

Proposition 2. Conditions **RH1** - **RH4** defines the function $Y_n(z)$ uniquely.
Proof. If $\tilde{Y}_n(z)$ is another function that satisfies **RH1** - **RH4** then $X_n(z) := \tilde{Y}_n(z) Y_n^{-1}(z)$ is holomorphic for $z \in \mathbb{CP}^1 \setminus \{\beta_j : 0 \le j \le g+1\}$. Furthermore, for $z \in \mathcal{U}_{\beta_j}$,

$$Y_n^{-1}(z) = \begin{pmatrix} \frac{1}{\sqrt{z-\beta_j}} & 0 \\ -\frac{1}{\mathsf{b}_j} & \sqrt{z-\beta_j} \end{pmatrix} \hat{Y}_n^{(\beta_j)-1}(z), \tag{2.10}$$

where $\hat{Y}_n^{(\beta_j)-1}(z)$ is holomorphic (see equation (2.6) !) in $\sqrt{z-\beta_j}$. This implies,

$$X_n(z) = O(1), \quad z \sim \beta_j, \tag{2.11}$$

which in turn implies $X_n(z)$ is holomorphic for $z \in \mathbb{CP}^1$, and $X_n(z) = I$, for all $z \in \mathbb{CP}^1$.

The conditions **RH1** - **RH4** constitute the Riemann-Hilbert problem whose unique solution is given by equation (2.3), due to Proposition 1.

The Riemann-Hilbert problem **RH1** - **RH4** together with the equation

$$P_n(z) = (Y_n(z))_{11} \tag{2.12}$$

will be used as an *alternative definition* of the Akhiezer polynomials. Notice also that the asymptotic condition **RH3** can be extended to the full Laurent series,

$$Y_n(z) = \left(I + \sum_{k=1}^{\infty} \frac{m_k(n)}{z^k} \right) z^{n\sigma_3}, \quad |z| > 1 \tag{2.13}$$

and from (2.3) we have,

$$m_1(n) = \begin{pmatrix} p_1(n) & h_n \\ 1/h_{n-1} & -p_1(n) \end{pmatrix} \tag{2.14}$$

where $p_1(n)$ is the coefficient of z^{n-1} of $P_n(z)$. Taking into account the recurrence relations (2.2), we have,

$$a_n = \frac{h_n}{h_{n-1}}$$

and

$$b_{n+1} = p_1(n) - p_1(n+1),$$

and the following relations supplementing (2.12)

$$\begin{aligned} h_n &= (m_1(n))_{12} && (2.15) \\ a_n &= (m_1(n))_{12}\,(m_1(n))_{21} && (2.16) \\ b_{n+1} &= (m_1(n))_{11} - (m_1(n+1))_{11}\,. && (2.17) \end{aligned}$$

Therefore, all the basic ingredients of the theory of polynomials $P_n(z)$ (including the polynomials themselves) can be obtained directly from the solution $Y_n(z)$ of the Riemann-Hilbert problem.

Remark 2.3 In the *a prior* setting of the Riemann-Hilbert problem **RH1** - **RH4**, the condition **RH3** can be replaced by the following weaker one

$$\textbf{RH4}.\quad Y_n(z)\begin{pmatrix} \frac{1}{\sqrt{z-\beta_j}} & 0 \\ -\frac{1}{\mathsf{b}_j} & \sqrt{z-\beta_j} \end{pmatrix} = O(1),\ z \sim \beta_j,\ 0 \le j \le g+1. \qquad (2.18)$$

3 Differential Equations

Having obtained equations (2.12) - (2.17) which represent orthogonal polynomials $P_n(z)$ and the corresponding norm and recurrence coefficients in terms of the solution $Y_n(z)$ of the Riemann-Hilbert problem **RH1** - **RH4**, we can now use the powerfull techniques of the Soliton theory. Specifically, in this and the two following sections we will apply a certain modification of the standard Zakharov-Shabat dressing method (see e.g. [8]) to obtain the relevant differential and difference equations for the Akhiezer polynomials. The modification needed is caused by the presence of the condition **RH4**. This condition indicates the relation of the problem under consideration to the theory of Fuchsian systems. Indeed, our derivations will be close to the Zakharov - Shabat scheme and to the constructions of the Jimbo-Miwa-Ueno monodromy theory [14] (see also [15] were both methods are unified in a single general Riemann-Hilbert formalism).

To describe the change of $Y_n(z)$ with respect to z for a fixed n, it is advantageous to transform the Riemann-Hilbert problem satisfied by $Y_n(z)$ in to a form where jump matrix has constant entries. To this end, put

$$\Phi_n(z) = Y_n(z)\begin{pmatrix} 1 & 0 \\ 0 & w^{-1}(z) \end{pmatrix}\begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & 1/\sqrt{2\pi i} \end{pmatrix}. \qquad (3.1)$$

A direct computation shows that

$$\Phi_{n,-}(z) = \Phi_{n,+}(z)\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}. \tag{3.2}$$

To specify the behavior of the new function near the end points of the set E let us observe that the new (constant !) jump matrix admits the following spectral representation,

$$\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathbf{P},$$

where

$$\mathbf{P} = \begin{pmatrix}0 & 1\\ 2 & -1\end{pmatrix}.$$

This implies that the function

$$\Phi^{(\beta_j)}(z) := \begin{pmatrix}\sqrt{z-\beta_j} & 0\\ 0 & 1\end{pmatrix}\mathbf{P}$$

satisfies the jump condition (3.2) in the neighborhood of β_j. Indeed, assuming $z \in \mathcal{U}_{\beta_j} \cap E$, we find,

$$\begin{aligned}[\Phi_+^{(\beta_j)}(z)]^{-1}\Phi_-^{(\beta_j)}(z) &= \mathbf{P}^{-1}\begin{pmatrix}\frac{1}{(\sqrt{z-\beta_j})_+} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}(\sqrt{z-\beta_j})_- & 0\\ 0 & 1\end{pmatrix}\mathbf{P}\\ &= \mathbf{P}^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathbf{P} = \begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}.\end{aligned}$$

Hence the matrix valued fucntion

$$\Phi_n(z)[\Phi^{(\beta_j)}(z)]^{-1}$$

has no jump accross E and therefore is holomorphic in the punctured neighborhood $\mathcal{U}_{\beta_j}\setminus\{\beta_j\}$. Observe in addition that in the product,

$$\begin{pmatrix}\sqrt{z-\beta_j} & 0\\ \frac{2\pi i}{\mathsf{b}_j} & \frac{1}{\sqrt{z-\beta_j}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & w^{-1}(z)\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{z-\beta_j}} & 0\\ 0 & 1\end{pmatrix}, \tag{3.3}$$

the negative powers of $\sqrt{z-\beta_j}$ cancel out.

Therefore we conclude that the product $\Phi_n(z)[\Phi^{(\beta_j)}(z)]^{-1}$ is in fact holomorphic in the whole neighborhood $\mathcal{U}_{\beta_j}$. Similar is also true for the matrix product

$$\Phi_n(z)[\Phi^{(\alpha_j)}(z)]^{-1} \equiv \Phi_n(z)\left[\begin{pmatrix}\frac{1}{\sqrt{z-\alpha_j}} & 0\\ 0 & 1\end{pmatrix}\mathbf{P}\right]^{-1}$$

in the neighborhood $\mathcal{U}_{\alpha_j}$ of the endpoint α_j. Here we shall assume that the branch of $\sqrt{z-\alpha_j}$ is defined by the condition,

$$-\pi < \arg(z-\alpha_j) < \pi.$$

In summary, $\Phi_n(z)$ solves the following Riemann-Hilbert problem:

$$\mathbf{\Phi 1}. \quad \Phi_n(z) \text{ is holomorphic for } z \in \mathbb{C} \setminus E$$

$$\mathbf{\Phi 2}. \quad \Phi_{n,-}(z) = \Phi_{n,+}(z) \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \ z \in E$$

$$\mathbf{\Phi 3}. \quad \Phi_n(z) = \left(I + \mathrm{O}\left(\frac{1}{z}\right)\right) z^{\begin{pmatrix} n & 0 \\ 0 & -n+1 \end{pmatrix}} \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & -\sqrt{\frac{\pi i}{2}} \end{pmatrix}, \ z \longrightarrow \infty,$$

$$\mathbf{\Phi 4}. \quad \Phi_n(z) = \hat{\Phi}_n^{(\beta_j)}(z) \begin{pmatrix} \sqrt{z-\beta_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \qquad z \in \mathcal{U}_{\beta_j}$$

$$\mathbf{\Phi 5}. \quad \Phi_n(z) = \hat{\Phi}_n^{(\alpha_j)}(z) \begin{pmatrix} \frac{1}{\sqrt{z-\alpha_j}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \qquad z \in \mathcal{U}_{\alpha_j},$$

where $\hat{\Phi}_n^{(\beta_j)}(z)$ and $\hat{\Phi}_n^{(\alpha_j)}(z)$ are holomorphic in the neighborhoods of the points β_j and α_j, respectively. Moreover, the matrices $\hat{\Phi}_n^{(\beta_j)}(\beta_j)$ and $\hat{\Phi}_n^{(\alpha_j)}(\alpha_j)$ are invertible. In fact,

$$\hat{\Phi}_n^{(\beta_j)}(\beta_j) = \hat{Y}_n^{(\beta_j)}(\beta_j) \begin{pmatrix} \sqrt{\frac{\pi i}{2}} & 0 \\ 0 & \frac{1}{\mathsf{b}_j}\sqrt{\frac{\pi i}{2}} \end{pmatrix}$$

and

$$\hat{\Phi}_n^{(\alpha_j)}(\alpha_j) = Y_n(\alpha_j) \begin{pmatrix} 0 & \sqrt{\frac{\pi i}{2}} \\ -\frac{1}{\mathsf{a}_j}\sqrt{\frac{\pi i}{2}} & 0 \end{pmatrix}$$

where a_j is defined by the equation (cf. 2.5)

$$w(z) = (z-\alpha_j)^{1/2} \mathsf{a}_j \frac{i}{\pi}(1 + O(z-\alpha_j)). \tag{3.4}$$

We want to emphasize, that unlike the case of the Y - Riemann-Hilbert problem, in the case of the Φ - Riemann-Hilbert problem the left multipliers $\hat{\Phi}_n^{(\beta_j)}(z)$ and $\hat{\Phi}_n^{(\alpha_j)}(z)$ are holomorphic *with respect to* z.

Remark 3.1 From $\mathbf{\Phi 1} - \mathbf{\Phi 5}$ it follows (independent of (3.1)) that

$$\det \Phi_n(z) = \frac{1}{w(z)}. \tag{3.5}$$

Consider now, the logarithmic derivative of $\Phi_n(z)$,

$$A(z,n) \ := \ \frac{d\Phi_n(z)}{dz}\Phi_n^{-1}(z). \tag{3.6}$$

Since all the right matrix multipliers in the r.h.s. of $\mathbf{\Phi 2} - \mathbf{\Phi 5}$ are constant matrices, $A(z,n)$ enjoys the following properties:

$A1.$ $A(z,n)$ is holomorphic for $z \in \mathbb{CP}^1 \setminus \{\alpha_j, \beta_j\}$,

$$A2. \quad A(z,n) = \frac{\begin{pmatrix} n & 0 \\ 0 & -n+1 \end{pmatrix}}{z} + \mathrm{O}\left(\frac{I}{z^2}\right), \quad z \longrightarrow \infty,$$

$$A3. \quad A(z,n) = \frac{1}{2}\hat{\Phi}_n^{(\beta_j)}(\beta_j)\frac{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{z-\beta_j}\hat{\Phi}_n^{(\beta_j)-1}(\beta_j) + \mathrm{O}(1), \quad z \sim \beta_j,$$

$$A4. \quad A(z,n) = -\frac{1}{2}\hat{\Phi}_n^{(\alpha_j)}(\alpha_j)\frac{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{z-\alpha_j}\hat{\Phi}_n^{(\alpha_j)-1}(\alpha_j) + \mathrm{O}(1), \quad z \sim \alpha_j.$$

By virtue of the Liouville theorem, it follows that,

$$A(z,n) = \sum_{j=0}^{g+1} \frac{B_j(n)}{z-\beta_j} + \sum_{j=1}^{g} \frac{A_j(n)}{z-\alpha_j} \tag{3.7}$$

where

$$B_j(n) := \frac{1}{2}\hat{\Phi}_n^{(\beta_j)}(\beta_j)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\hat{\Phi}_n^{(\beta_j)-1}(\beta_j) = \frac{1}{2}\hat{Y}_n^{(\beta_j)}(\beta_j)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\hat{Y}_n^{(\beta_j)-1}(\beta_j) \tag{3.8}$$

$$A_j(n) := -\frac{1}{2}\hat{\Phi}_n^{(\alpha_j)}(\alpha_j)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\hat{\Phi}_n^{(\alpha_j)-1}(\alpha_j) = -\frac{1}{2}Y_n(\alpha_j)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}Y_n^{-1}(\alpha_j). \tag{3.9}$$

Note also,

$$\sum_{j=0}^{g+1} B_j(n) + \sum_{j=1}^{g} A_j(n) = \begin{pmatrix} n & 0 \\ 0 & -n+1 \end{pmatrix}.$$

Using (2.3) and **RH4** give

$$\hat{Y}_n^{(\beta_j)}(\beta_j) = \begin{pmatrix} Q_n(\beta_j)/\mathsf{b}_j & \mathsf{b}_j P_n(\beta_j) \\ \frac{Q_{n-1}(\beta_j)}{\mathsf{b}_j h_{n-1}} & \mathsf{b}_j P_{n-1}(\beta_j)/h_{n-1} \end{pmatrix}. \tag{3.10}$$

We conclude this section by recording the linear matrix differential equation with Fuchsian singularities at $\{\alpha_j, \beta_j\}$, mentioned in the Abstract,

$$\frac{d\Phi_n(z)}{dz} = A(z,n)\Phi_n(z), \tag{3.11}$$

with $A(z,n)$ defined by (3.7), (3.8) and (3.9). Furthermore, using the second line of (2.3), the matrix valued residues are expressed in terms of the evalutions of the polynomials at the branch points:

$$B_j(n) = \frac{1}{2}\begin{pmatrix} Q_n(\beta_j)P_{n-1}(\beta_j)/h_{n-1} & -Q_n(\beta_j)P_n(\beta_j) \\ Q_{n-1}(\beta_j)P_{n-1}(\beta_j)/h_{n-1}^2 & -Q_{n-1}(\beta_j)P_n(\beta_j)/h_{n-1} \end{pmatrix} \tag{3.12}$$

$$A_j(n) = \frac{1}{2}\begin{pmatrix} P_n(\alpha_j)Q_{n-1}(\alpha_j)/h_{n-1} & -Q_n(\alpha_j)P_n(\alpha_j) \\ Q_{n-1}(\alpha_j)P_{n-1}(\alpha_j)/h_{n-1}^2 & -P_{n-1}(\alpha_j)Q_n(\alpha_j)/h_{n-1} \end{pmatrix}. \tag{3.13}$$

Note that from (3.8) and (3.9) it follows that

$$\begin{aligned} \mathrm{tr}B_j(n) &\equiv \frac{1}{2h_{n-1}}(Q_{n-1}(\beta_j)P_n(\beta_j) - Q_n(\beta_j)P_{n-1}(\beta_j)) = 1/2 \\ \det B_j(n) &= 0, \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}A_j(n) &\equiv -\frac{1}{2h_{n-1}}(Q_n(\alpha_j)P_{n-1}(\alpha_j) - P_n(\alpha_j)Q_{n-1}(\alpha_j)) = -1/2. \\ \det A_j(n) &= 0. \end{aligned}$$

We note that this leads to a discrete analogue of the "Wronskian" relation,

$$P_{n-1}(z)Q_n(z) - P_n(z)Q_{n-1}(z) = h_{n-1},$$

which, of course, can be independently derived from the recurrence relations.

As it has already been mentioned in Introduction, equation (3.11), even for more general weights of the type $\prod_j (t-\delta_j)^{\kappa_j}$, was first obtained in [6]. In [6] the Riemann-Hilbert problem is not used explicitely; rather, the author analyses directly the monodromy properties of the function $Y_n(z)$, i.e. the approach of [6] is based more on the ideas of [14] than of [8]. It is also worth mentioning that our approach can be extended to the general semi-classical weights without any serious modifications.

4 Derivatives with respect to the branch points.

In this section we determine differentiation formulas for $\Phi_n(z)$ with respect to $\{\alpha_j, \beta_j\}$. First let us consider the logarithmic derivative of $\Phi_n(z)$ with respect to a particular β_j;

$$V_j(z) := \frac{\partial \Phi_n(z)}{\partial \beta_j}\Phi_n^{-1}(z), \tag{4.1}$$

and note that $V_j(z)$ has the following properties

$V1.$ $V_j(z)$ is holomorphic for $z \in \mathbf{C} \setminus \{\beta_j\}$.

$$V2. \quad V_j(z) = \mathrm{O}(I/z), \quad z \to \infty.$$

$$V3. \quad V_j(z) \sim -\frac{1}{2}\hat{\Phi}_n^{(\beta_j)}(\beta_j)\frac{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}{z-\beta_j}\hat{\Phi}_n^{(\beta_j)-1}(\beta_j) + \mathrm{O}(1). \tag{4.2}$$

By comparing with (3.8) and again invoking the Liouville theorem, we conclude that

$$V_j(z) = -\frac{B_j(n)}{z-\beta_j}, \tag{4.3}$$

which implies

$$\partial_{\beta_j}\Phi_n(z) \quad = \quad -\frac{B_j(n)}{z-\beta_j}\Phi_n(z). \tag{4.4}$$

A similar analysis gives

$$\partial_{\alpha_j}\Phi_n(z) \quad = \quad -\frac{A_j(n)}{z-\alpha_j}\Phi_n(z) \tag{4.5}$$

5 Difference Equation.

Consider the "difference logarithmic derivative"

$$U_n(z) := \Phi_{n+1}(z)\Phi_n^{-1}(z) \equiv Y_{n+1}(z)Y_n^{-1}(z).$$

Taking into account that all the right matrix multipliers in the r.h.s of **RH1 - RH4** are constant with respect to n we conclude that $U_n(z)$ is an entire function. Moreover, from the asymptotics (2.13) we have that

$$\begin{aligned} U_n(z) &= \left(I + \frac{m_1(n+1)}{z}\right) z^{\sigma_3} \left(I - \frac{m_1(n)}{z}\right) + O\left(\frac{1}{z}\right) \\ &= \left(I + \frac{m_1(n+1)}{z}\right) \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \left(I - \frac{m_1(n)}{z}\right) + O\left(\frac{1}{z}\right) \\ &= z\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + m_1(n+1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} m_1(n) + O\left(\frac{1}{z}\right), \quad z \to \infty. \end{aligned}$$

Appealing once again to the Liouville theorem, we conclude that $U_n(z)$ is linear function in z defined by the equations

$$\begin{aligned} U_n(z) &= \begin{pmatrix} z + (m_1(n+1))_{11} - (m_1(n))_{11} & -(m_1(n))_{12} \\ (m_1(n+1))_{21} & 0 \end{pmatrix} \\ &= \begin{pmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix}, \end{aligned}$$

where in the last equation we have taken into account (2.15)- (2.17). To summarize, the difference equation for the function $\Phi_n(z)$ reads

$$\Phi_{n+1}(z) = \begin{pmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix} \Phi_n(z). \tag{5.1}$$

Of course, equation (5.1) is just the matrix form of the basic recurrence equation (2.2). Nevertheless, we gave its "Riemann-Hilbert" derivation to emphasize the "master" role of the Riemann-Hilbert problem **RH1** - **RH4** in our analysis.

6 Schlesinger Equations and the Hankel Determinant.

With the unified notation mentioned in the Introduction, we write,

$$A(z,n) = \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j}, \tag{6.1}$$

and the correspondence,

$$(A_1(n), ..., A_g(n), B_0(n), ..., B_{g+1}(n)) \longrightarrow (C_1(n), ..., C_g(n), C_{g+1}(n), ..., C_{2g+2}(n)). \tag{6.2}$$

Note that, $C_j(n)$, depend on δ_j's. We of course have,

$$\begin{aligned} \partial_z \Phi_n(z) &= \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j} \Phi_n(z), &(6.3)\\ \partial_{\delta_k} \Phi_n(z) &= -\frac{C_k(n)}{z - \delta_k} \Phi_n(z). &(6.4) \end{aligned}$$

Applying ∂_z on (6.4) gives

$$\partial_z \partial_{\delta_k} \Phi_n(z) = \frac{C_k(n)}{(z - \delta_k)^2} \Phi_n - \frac{C_k(n)}{z - \delta_k} \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j} \Phi_n, \tag{6.5}$$

and ∂_{δ_k} on (6.3) gives,

$$\partial_{\delta_k} \partial_z \Phi_n(z) = \frac{C_k(n)}{(z - \delta_k)^2} \Phi_n + \sum_{j=1}^{2g+2} \frac{\partial_{\delta_k} C_j(n)}{z - \delta_j} \Phi_n - \left(\sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j} \right) \frac{C_k(n)}{z - \delta_k} \Phi_n. \tag{6.6}$$

Since $\partial_z \partial_{\delta_k} \Phi_n = \partial_{\delta_k} \partial_z \Phi_n$ and $\det \Phi_n \neq 0$, we get,

$$\begin{aligned} \sum_{j=1}^{2g+2} \frac{\partial_{\delta_k} C_j(n)}{z - \delta_k} &= \sum_{j=1}^{2g+2} \frac{[C_j(n), C_k(n)]}{(z - \delta_j)(z - \delta_k)} \\ &= \sum_{j=1}^{2g+2} \frac{[C_j(n), C_k(n)]}{\delta_j - \delta_k} \left(\frac{1}{z - \delta_j} - \frac{1}{z - \delta_k} \right). &(6.7) \end{aligned}$$

We now send z to a particular δ_j in (6.7), with $j \neq k$, and find by equating residues,

$$\partial_{\delta_k} C_j(n) = \frac{[C_j(n), C_k(n)]}{\delta_j - \delta_k}, \quad j \neq k. \tag{6.8}$$

If $j = k$, then a similar calculation gives,

$$\partial_{\delta_k} C_k(n) = -\sum_{l(\neq k)} \frac{[C_l(n), C_k(n)]}{\delta_l - \delta_k}. \tag{6.9}$$

The equations (6.8) and (6.9) are the Schlesinger Equations satisfied by $C_j(n)$. This is the equation first derived for the general semi-classical orthogonal polynomials in [6]. We are now going to move beyond the results of [6] and show that the corresponding τ - function can be identified with the Hankel determinant associated with the weight $w_+(t)$. To this end we first recall Jimbo-Miwa-Ueno definition of the τ -function.

Let $\Omega^{(1)}$ be the one-form,

$$\begin{aligned}\Omega^{(1)}(\delta_1, ..., \delta_{2g+2}) &:= \sum_{1 \leq j < k \leq 2g+2} \mathrm{tr}\,(C_j(n) C_k(n)) \frac{d\delta_j - d\delta_k}{\delta_j - \delta_k} \\ &= \sum_{1 \leq j < k \leq 2g+2} \mathrm{tr}\,(C_j(n) C_k(n))\, d \ln |\delta_j - \delta_k|, \end{aligned} \tag{6.10}$$

then it can be verified [14] using the Schlesinger Equations that,

$$d\Omega^{(1)} = 0, \tag{6.11}$$

which implies that, localy, $\Omega^{(1)}$ is an exact form. The $\tau-$ function of the completely integrable system of partial differential equations (6.8) and (6.9) is then defined by the relation,

$$\Omega^{(1)} = d \ln \tau_n(\delta_1, ..., \delta_{2g+2}). \tag{6.12}$$

In the theory orthogonal polynomials, the Hankel determinant,

$$D_n[w_+] := \det\left(\int_E t^{j+k} w_+(t) dt\right)_{j,k=0}^{n-1}, \tag{6.13}$$

has two other equivalent expressions,

$$\begin{aligned} D_n[w_+] &= \frac{1}{n!} \int_E ... \int_E \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{l=1}^{n} w_+(x_l) dx_l, \\ &= \prod_{j=0}^{n-1} h_j. \end{aligned} \tag{6.14}$$

It is to be expected from the structure of the Riemann-Hilbert formulation that, D_n, considered as a function of $\{\delta_j\}_{j=1}^{2g+2}$, is the $\tau-$ function for this problem. To understand this, we require the derivatives of h_n w.r.t. to δ_k. To begin with, we use that,

$$\partial_{\delta_k}\Phi_n(z) = -\frac{C_k(n)}{z-\delta_k}\Phi_n(z), \tag{6.15}$$

must be compatible with

$$\Phi_{n+1}(z) = \begin{pmatrix} z-b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix}\Phi_n(z). \tag{6.16}$$

This results is

$$\begin{aligned}\begin{pmatrix} z-b_{n+1} & -h_n \\ h_n & 0 \end{pmatrix}\frac{C_k(n)}{z-\delta_k} \quad &- \quad \frac{C_k(n+1)}{z-\delta_k}\begin{pmatrix} z-b_{n+1} & -h_n \\ h_n & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\partial_{\delta_k}b_{n+1} & -\partial_{\delta_k}h_n \\ -(1/h_n)\partial_{\delta_k}\ln h_n & 0 \end{pmatrix},\end{aligned} \tag{6.17}$$

which holds for all $z \in \mathbb{CP}^1 \setminus \{\delta_1, ...\delta_{2g+2}\}$. Putting $z=\infty$ in (6.17), gives,

$$\begin{pmatrix} C_k^{11}(n) - C_k^{11}(n+1) & C_k^{12}(n) \\ -C_k^{21}(n+1) & 0 \end{pmatrix} = \begin{pmatrix} -\partial_{\delta_k}b_{n+1} & -\partial_{\delta_k}h_n \\ -(1/h_n)\partial_{\delta_k}\ln h_n & 0 \end{pmatrix},$$

which implies, amongst others,

$$\partial_{\delta_k}h_n = -C_k^{12}(n). \tag{6.18}$$

Lemma 1. Let the asymptotic expansion of $A(z,n)$ about $z=\infty$ be

$$A\,(z,n) = \sum_{k=0}^{\infty}\mathcal{A}_k(n)z^{-k-1} \tag{6.19}$$

where

$$\mathcal{A}_k(n) := \sum_{j=1}^{2g+2}C_j(n)\delta_j^k(n). \tag{6.20}$$

Then the first two $\mathcal{A}_k(n)$ are

$$\mathcal{A}_0(n) = \begin{pmatrix} n & 0 \\ 0 & 1-n \end{pmatrix}, \tag{6.21}$$

$$\mathcal{A}_1(n) = \begin{pmatrix} 0 & 0 \\ 0 & c_1 \end{pmatrix} + m_1(n)\begin{pmatrix} n-1 & 0 \\ 0 & -n \end{pmatrix} - \begin{pmatrix} n & 0 \\ 0 & 1-n \end{pmatrix}m_1(n), \tag{6.22}$$

where $c_1 = \sum_{j=1}^{g} (\beta_j - \alpha_j)$.

Proof. Putting (3.1) into (3.11) we find

$$\frac{d}{dz} Y_n(z) + Y_n(z) \begin{pmatrix} 0 & 0 \\ 0 & -\frac{d}{dz} \ln w(z) \end{pmatrix} = A(z, n) Y_n(z). \tag{6.23}$$

Expansion of (6.23) in z^{-1} gives the desired results.

Theorem 2. The Hankel determinant is the τ function of the Magnus - Schlesinger Equations.

Proof: We start by equating the residues of (6.17) at $z = \delta_j$. This gives,

$$U_n(\delta_j) C_j(n) = C_j(n+1) U_n(\delta_j),$$

or

$$C_j(n+1) = U_n(\delta_j) C_j(n) U_n^{-1}(\delta_j), \tag{6.24}$$

where

$$\begin{aligned} U_n(z) &:= \begin{pmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix} \\ U_n^{-1}(z) &= \begin{pmatrix} 0 & h_n \\ -1/h_n & z - b_{n+1} \end{pmatrix}. \end{aligned} \tag{6.25}$$

A simple calculation shows that

$$U_n^{-1}(z) U_n(z') = \begin{pmatrix} 1 & 0 \\ \frac{z-z'}{h_n} & 1 \end{pmatrix} = I + \frac{z - z'}{h_n} \sigma_-, \tag{6.26}$$

where $\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Now,

$$d \ln \tau_n = \sum_k \partial_{\delta_k} \ln \tau_n \;\; d\delta_k, \tag{6.27}$$

where (from (6.10)),

$$\partial_{\delta_j} \ln \tau_n = \sum_{k(\neq j)} \frac{\mathrm{tr} C_j(n) C_k(n)}{\delta_j - \delta_k}, \tag{6.28}$$

which leads to

$$\partial_{\delta_j} \ln \frac{\tau_{n+1}}{\tau_n} = \sum_{k(\neq j)} \frac{\mathrm{tr}(C_j(n+1) C_k(n+1) - C_j(n) C_k(n))}{\delta_j - \delta_k}$$

$$
\begin{aligned}
&= \sum_{k(\neq j)} \frac{\mathrm{tr}(U_n(\delta_j)C_j(n)U_n^{-1}(\delta_j)U_n(\delta_k)C_k(n)U_n^{-1}(\delta_k) - C_j(n)C_k(n))}{\delta_j - \delta_k} \\
&= \sum_{k(\neq j)} \frac{\mathrm{tr}(U_n^{-1}(\delta_k)U_n(\delta_j)C_j(n)U_n^{-1}(\delta_j)U_n(\delta_k)C_k(n) - C_j(n)C_k(n))}{\delta_j - \delta_k} \\
&= \sum_{k(\neq j)} \frac{\mathrm{tr}[(I - \frac{\delta_j-\delta_k}{h_n}\sigma_-)C_j(n)(I + \frac{\delta_j-\delta_k}{h_n}\sigma_-)C_k(n) - C_j(n)C_k(n)]}{\delta_j - \delta_k}. \qquad (6.29)
\end{aligned}
$$

A calculation shows that the term [...] in (6.29) is

$$
\frac{\delta_j - \delta_k}{h_n}\left(C_j(n)\sigma_- C_k(n) - \sigma_- C_j(n)C_k(n)\right) - \left(\frac{\delta_j - \delta_k}{h_n}\right)^2 \sigma_- C_j(n)\sigma_- C_k(n).
$$

We also note here some useful identities;

$$
\begin{aligned}
\mathrm{tr}(C_j(n)\sigma_- C_k(n) &- \sigma_- C_j(n)C_k(n)) \\
&= C_j^{12}(n)(C_k^{11}(n) - C_k^{22}(n)) - C_k^{12}(n)(C_j^{11}(n) - C_j^{22}(n)),
\end{aligned}
$$

and

$$
\mathrm{tr}(\sigma_- C_j(n)\sigma_- C_k(n)) = C_j^{12}(n)C_k^{12}(n).
$$

Therefore

$$
\begin{aligned}
\partial_{\delta_j} \ln \frac{\tau_{n+1}}{\tau_n} &= \frac{1}{h_n}\sum_{k(\neq j)} \left(C_j^{12}(n)(C_k^{11}(n) - C_k^{22}(n)) - C_k^{12}(n)(C_j^{11}(n) - C_j^{22}(n))\right) \\
&- \frac{1}{h_n^2}\sum_{k(\neq j)} (\delta_j - \delta_k)C_j^{12}(n)C_k^{12}(n). \qquad (6.30)
\end{aligned}
$$

To simplify the r.h.s. of (6.30) we note, from (6.20), (6.21) and (6.22)

$$
\begin{aligned}
\sum_j C_j^{12}(n) &= 0, \\
\sum_j (C_j^{11}(n) - C_j^{22}(n)) &= 2n - 1, \\
\sum_j \delta_j C_j^{12}(n) &= -2nh_n.
\end{aligned}
$$

Using these, and $\sum_{k(\neq j)} f_k = -f_j + \sum_k f_k$, the r.h.s. of (6.30), becomes,

$$
\frac{C_j^{12}(n)}{h_n}\sum_k (C_k^{11}(n) - C_k^{22}(n)) + \frac{C_j^{12}(n)}{h_n^2}\sum_k \delta_k C_k^{12}(n) = -\frac{C_j^{12}(n)}{h_n}.
$$

Finally, using (6.18),

$$\partial_{\delta_j} \ln \frac{\tau_{n+1}}{\tau_n} = -\frac{C_j^{12}(n)}{h_n} = \partial_{\delta_j} \ln h_n. \tag{6.31}$$

Summing over n from 0 to $N-1$, we conclude that τ_N is a constant multiple of D_N, where the constant is independent of $\{\delta_j\}_{j=1}^{2g+2}$. Since the τ - function is defined up to such a constant, we can assume that the constant is unity;

$$\tau_N(\delta_1, ..., \delta_{2g+2}) = D_N[w_+]. \tag{6.32}$$

Remark 6.1. It is worth mentioning that equations (6.24) follow also (by putting $z = \delta_j$) from the equation

$$A(z, n+1)U_n(z) - U_n(z)A(z, n) = \frac{\partial U_n(z)}{\partial z}, \tag{6.33}$$

which, in turn, is the compatibility condition of the basic Fuchsian equation (3.11) and the difference equation (5.1). This is the matrix form of the so-called Freud equation which in principal can be written for any semi-classical polynomials - see [16] and [6] (and also [7]). In the physical language this is the "discrete string equation" corresponding to the weight $w_+(t)$. More precisely, equation (6.33) is the (discrete) Lax representation of the Freud equation which manifests its integrability from the algebraic point of view: linear equations (3.11) and (5.1) form a Lax pair for the Freud equation (cf. [7], [17]).

7 Non-linear difference equations.

As explained in Remark 6.1, the matrix equation (6.33) should lead to the nonlinear difference equations for the recurrence coefficients, following the *genre* of the Freud equations for the Akhiezer polynomials. To this end, we rewrite (6.24) elementwise, by first specializing δ_j to α_j and second to β_j. This will produce six difference equations, relating polynomial evaluations at the branch points and the recurrence coefficients. For later convenience we introduce four quantities

$$\begin{aligned}
r_n^{(\alpha)} &:= \frac{1}{2h_{n-1}} P_n(\alpha_j) Q_{n-1}(\alpha_j), \\
r_n^{(\beta)} &:= \frac{1}{2h_{n-1}} P_n(\beta_j) Q_{n-1}(\beta_j), \\
R_n^{(\alpha)} &:= \frac{1}{2h_n} P_n(\alpha_j) Q_n(\alpha_j), \\
R_n^{(\beta)} &:= \frac{1}{2h_n} P_n(\beta_j) Q_n(\beta_j).
\end{aligned}$$

Thus by specializing to α_j, $C_j(n)$ becomes,

$$\begin{pmatrix} r_n^{(\alpha)} & -h_n R_n^{(\alpha)} \\ R_{n-1}^{(\alpha)}/h_{n-1} & -r_n^{(\alpha)} - 1/2 \end{pmatrix},$$

where we have taken onto account that the trace of the above is $-1/2$. In component form (6.24) is equivalent to,

$$r_{n+1}^{(\alpha)} + r_n^{(\alpha)} + \frac{1}{2} = R_n^{(\alpha)}(\alpha_j - b_{n+1}) \tag{7.1}$$

$$a_{n+1}R_{n+1}^{(\alpha)} - a_n R_{n-1}^{(\alpha)} = (b_{n+1} - \alpha_j)\left(R_n^{(\alpha)}(b_{n+1} - \alpha_j) + 2r_n^{(\alpha)} + \frac{1}{2}\right). \tag{7.2}$$

Note that out of the four possible equations, the 21 element is a tautology and the 11 and 22 elements are equivalent. Similarly, specializing to β_j, $C_j(n)$ becomes

$$\begin{pmatrix} r_n^{(\beta)} + \frac{1}{2} & -h_n R_n^{(\beta)} \\ R_{n-1}^{(\beta)}/h_{n-1} & -r_n^{(\beta)} \end{pmatrix},$$

where the trace of the above is $1/2$. In component form, (6.24) becomes,

$$r_{n+1}^{(\beta)} + r_n^{(\beta)} + \frac{1}{2} = R_n^{(\beta)}(\beta_j - b_{n+1}) \tag{7.3}$$

$$a_{n+1}R_{n+1}^{(\beta)} - a_n R_{n-1}^{(\beta)} = (\beta_j - b_{n+1})\left(R_n^{(\beta)}(\beta_j - b_{n+1}) - 2r_n^{(\beta)} - \frac{1}{2}\right). \tag{7.4}$$

In addition to these we have

$$a_n R_n^{(\alpha)} R_{n-1}^{(\alpha)} = r_n^{(\alpha)}\left(r_n^{(\alpha)} + \frac{1}{2}\right) \tag{7.5}$$

$$a_n R_n^{(\beta)} R_{n-1}^{(\beta)} = r_n^{(\beta)}\left(\frac{1}{2} + r_n^{(\beta)}\right), \tag{7.6}$$

since $\det C_j(n) = 0$. The equations (7.1) - (7.6) are the difference equations mentioned above. We should be able to eliminate, $r_n^{(\alpha)}$, $r_n^{(\beta)}$, $R_n^{(\alpha)}$ and $R_n^{(\beta)}$ from these to obtain non-linear difference equations involving only a_n and b_n. These equations, are also discussed in [6].

8 The σ_1 Riemann-Hilbert Problem.

In this section we shall solve the Riemann-Hilbert problem **RH1** - **RH4** for the Akhiezer polynomials in terms of the Θ - functions. To this end we will need a further transformation of the Riemann-Hilbert problem satisfied by $\Phi_n(z)$ to the so-called σ_1 problem, first appeared in the theory of algebrogeometric solutions of integrable PDEs (see [18], [10]).

We notice that since the matrices $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and σ_1 have the same simple spectrum, they must be similar. Indeed, we have

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1.$$

Therefore, if we define

$$\Psi_n(z) := \begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} \Phi_n(z) \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} Y_n(z) \begin{pmatrix} 1 & 0 \\ 0 & 1/w(z) \end{pmatrix} \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 1/\sqrt{2\pi i} & -1/\sqrt{2\pi i} \end{pmatrix}, \qquad (8.1)$$

then the jump matrix of the new function becomes σ_1. The left diagonal constant matrix multiplier is introduced to normalize the asymptotic behavior of the function $\Psi_n(z)$ at $z = \infty$:

$$\begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix} \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & -\sqrt{\frac{\pi i}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$= \left(I + \mathrm{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix}.$$

Taking also into account that

$$\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

we can reformulate the Riemann-Hilbert problem in terms of $\Psi_n(z)$, as follows.

$\Psi 1.$ $\quad \Psi_n(z)$ is holomorphic for $z \in \mathbb{C} \setminus E$.

$\Psi 2.$ $\quad \Psi_{n-}(z) = \Psi_{n+}(z)\sigma_1, \quad z \in E.$

$\Psi 3.$ $$\Psi_n(z) = \left(I + \mathrm{O}\left(\frac{1}{z}\right)\right) z^{\begin{pmatrix} n & 0 \\ 0 & -n+1 \end{pmatrix}}, \quad z \to \infty.$$

$\Psi 4.$ $$\Psi_n(z) = \hat{\Psi}_n^{(\beta_j)}(z) \begin{pmatrix} \sqrt{z-\beta_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \hat{\Psi}_n^{(\beta_j)}(z)(z-\beta_j)^{\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad (8.2)$$

$\Psi 5.$ $$\Psi_n(z) = \hat{\Psi}_n^{(\alpha_j)}(z) \begin{pmatrix} 1/\sqrt{z-\alpha_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \hat{\Psi}_n^{(\alpha_j)}(z)(z-\alpha_j)^{\begin{pmatrix} -1/2 & 0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

where $\hat{\Psi}_n^{(\alpha_j)}(z)$ is holomorphic in the neighbourhood of $z = \alpha_j$ and $\det \hat{\Psi}_n^{(\alpha_j)}(\alpha_j) \neq 0$,i.e.,

$$\hat{\Psi}_n^{(\alpha_j)}(z) = \sum_{k=0}^{\infty} \Psi_{nk}^{(\alpha_j)}(z-\alpha_j)^k, \quad \det \Psi_{n0}^{(\alpha_j)} \neq 0.$$

Similarly, $\hat{\Psi}_n^{(\beta_j)}(z)$ is holomorphic in the neighbourhood of $z = \beta_j$ and $\det \hat{\Psi}_n^{(\beta_j)}(\beta_j) \neq 0$,i.e.,

$$\hat{\Psi}_n^{(\beta_j)}(z) = \sum_{k=0}^{\infty} \Psi_{nk}^{(\beta_j)}(z - \beta_j)^k, \quad \det \Psi_{n0}^{(\beta_j)} \neq 0.$$

It is also worth noticing that the matrix products

$$\begin{pmatrix} \sqrt{z-\beta_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1/\sqrt{z-\alpha_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

have an exact σ_1- jump matrix in the respective neighborhoods.

Remark 8.1. From $\Psi 1 - \Psi 5$ it follows (independent of (8.1)) that

$$\det \Psi_n(z) = \frac{i}{\pi w(z)}, \tag{8.3}$$

Remark 8.2. The function $\Psi_n(z)$, in terms of $P_n(z)$ and $Q_n(z)$, is given as:

$$\Psi_n(z) = \frac{1}{2\pi i} \begin{pmatrix} \frac{i\pi w(z) P_n(z) - Q_n(z)}{w(z)} & \frac{i\pi w(z) P_n(z) + Q_n(z)}{w(z)} \\ 2\frac{i\pi w(z) P_{n-1}(z) - Q_{n-1}(z)}{h_{n-1} w(z)} & 2\frac{i\pi w(z) P_{n-1}(z) + Q_{n-1}(z)}{h_{n-1} w(z)} \end{pmatrix}, \tag{8.4}$$

and all the properties listed in $\Psi 1 - \Psi 5$ can be deduced from this representation. It is worth emphasizing here that our approach does not require this formula. Our logic is: The initial Riemann-Hilbert Problem for $Y_n(z)$, quite generally posed, is transformed via (8.1) to the σ_1 problem which in turn leads to the equations (8.2) and (8.3) by the completely general principals of the Riemann-Hilbert problem.

Let us now solve the σ_1 problem defined by $\Psi 1 - \Psi 5$, however, without any reference to (8.4). The philosophy we adopt here is similar to that in the asymptotic analysis of orthogonal polynomials via the Riemann-Hilbert problem (cf. [12], [13]): We simply "forget" the explicit formulas involving polynomials.

Introduce the genus g Riemann surface $\mathfrak{R}$ defined by

$$y^2 = (z - \beta_0)(z - \beta_{g+1}) \prod_{j=1}^{g} (z - \alpha_j)(z - \beta_j),$$

and let $\vec{\Psi}_n(P)$, where $P = (z, y) \in \mathfrak{R}$ be the vector Baker-Akhiezer function determined by the conditions:

BA1. $\vec{\Psi}_n(P)$ is meromorphic on $\mathfrak{R} \setminus \infty^{\pm}$ with the pole divisor,

$$(\vec{\Psi}_n(P)) = -\sum_{j=1}^{g} \alpha_j$$

BA2. The behaviour of $\vec{\Psi}_n(P)$ at $\infty^\pm$ is specified by the equations,

$$\vec{\Psi}_n(P) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\frac{1}{z}\right) \right) z^n, \quad P \to \infty^+,$$

$$\vec{\Psi}_n(P) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\frac{1}{z}\right) \right) z^{-n+1}, \quad P \to \infty^-,$$

in other words, ∞^+ is a pole of order n and ∞^- is a zero of order $n-1$. Here as usual, $\infty^\pm$ means

$$P \to \infty^\pm \iff z \to \infty, \;\; y \to \pm z^{g+1}.$$

Let $\pi : \mathfrak{R} \to \mathbb{CP}^1$ be the projection,

$$\pi(P) = z, \quad P = (z, y),$$

and $* : \mathfrak{R} \to \mathfrak{R}^*$ be the involution,

$$P \to P^* = (z, -y) \;\text{ if }\; P = (z, y).$$

The main observation (cf. [18], [10]) is that the matrix function,

$$\Psi_n(z) := \left(\vec{\Psi}_n(P), \vec{\Psi}_n(P^*) \right), \tag{8.5}$$

where $\pi(P) = z$, and $P \to \infty^+$ as $z \to \infty$, solves the RH problem $\Psi1 - \Psi5$.

1. Indeed $\Psi1$ is satisfied by construction since (8.5) defines $\Psi_n(z)$ uniquely as an analytic function on $\mathbb{CP}^1 \setminus E$.

2. If $z \to E$ from the "+"-side (or from above the cut), then

$$\begin{aligned} P &\to (z, y_+(z)) = P_+ \\ P^* &\to (z, -y_+(z)) = (z, y_-(z)) = P_-. \end{aligned}$$

If $z \to E$ from the "-" side, then

$$\begin{aligned} P &\to (z, y_-(z)) = P_- \\ P^* &\to (z, -y_-(z)) = (z, y_+(z)) = P_+. \end{aligned}$$

Hence,

$$\begin{aligned} \Psi_{n-}(z) &= \left(\vec{\Psi}_n(P_-), \; \vec{\Psi}_n(P_+) \right) \\ \Psi_{n+}(z) &= \left(\vec{\Psi}_n(P_+), \; \vec{\Psi}_n(P_-) \right) \end{aligned}$$

and it follows that,

$$\Psi_{n-}(z) = \Psi_{n+}(z)\sigma_1, \quad z \in E,$$

and therefore $\Psi 2$ is satisfied.

3. We have by construction, $z \to \infty$ implies $P \to \infty^+$ and $P^* \to \infty^-$.

Therefore from **BA2**,

$$\Psi_n(z) = \left(I + \mathrm{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix}, \tag{8.6}$$

which shows that $\Psi 3$ is satisfied.

4. The function $\Psi(P)$ is analytic in the neighborhood of $P = \beta_j$ as a point of the Riemann surface $\mathfrak{R}$. The local parameter at the point β_j is the square root of $z - \beta_j$. Therefore, in the neighborhood of $P = \beta_j$ we have,

$$\vec{\Psi}_n(P) = \sum_{k=0}^{\infty} \vec{\psi}_{jk}(z-\beta_j)^{k/2}, \tag{8.7}$$

$$\vec{\Psi}_n(P^*) = \sum_{k=0}^{\infty}(-1)^k \vec{\psi}_{jk}(z-\beta_j)^{k/2}, \tag{8.8}$$

so that

$$\Psi_n(z) = \left(\sum_{k=0}^{\infty} \vec{\psi}_{jk}(z-\beta_j)^{k/2}, \sum_{k=0}^{\infty}(-1)^k \vec{\psi}_{jk}(z-\beta_j)^{k/2}\right).$$

This in turn implies that the function $\hat{\Psi}_n^{(\beta_j)}(z)$ defined by the equation $\Psi 4$ is *a holomorphic function of* z. Indeed we have

$$\hat{\Psi}_n^{(\beta_j)}(z) \equiv \Psi_n(z)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{\sqrt{z-\beta_j}} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty}\left[\vec{\psi}_{jk} - (-1)^k\vec{\psi}_{jk}\right](z-\beta_j)^{k-1/2}, \sum_{k=0}^{\infty}\left[\vec{\psi}_{jk} + (-1)^k\vec{\psi}_{jk}\right](z-\beta_j)^{k/2}\right)$$

$$= \left(\sum_{l=0}^{\infty}\vec{\psi}_{j2l+1}(z-\beta_j)^l, \sum_{l=0}^{\infty}\vec{\psi}_{j2l}(z-\beta_j)^l\right).$$

5. Since $P = \alpha_j$ is a simple pole of $\Psi(P)$, the Taylor series (8.7) and (8.8) shoud be replaced by the Laurent series,

$$\vec{\Psi}_n(P) = \sum_{k=-1}^{\infty} \vec{\phi}_{jk}(z-\alpha_j)^{k/2},$$

$$\vec{\Psi}_n(P^*) = \sum_{k=-1}^{\infty} (-1)^k \vec{\phi}_{jk}(z-\alpha_j)^{k/2}.$$

The rest of the arguments is literaly the same as in the β -case, and we have that the function $\hat{\Psi}_n^{(\alpha_j)}(z)$ defined by the equation $\Psi 5$ is holomorphic at $z = \alpha_j$.

Our final observation is that already established properties imply (8.3) (cf. our "Riemann-Hilbert" proof of (2.7) above) and hence the inequalities,

$$\det \hat{\Psi}_n^{(\alpha_j)}(\alpha_j) \neq 0, \quad \det \hat{\Psi}_n^{(\beta_j)}(\beta_j) \neq 0.$$

We now come to the $\Theta-$ formula for $\vec{\Psi}_n(P)$. First we assemble here for this purpose some facts about the Riemann surface $\mathfrak{R}$ realized as a two-sheet covering of the z plane in the usual way and with the first homology basis depicted in the figure below. Let

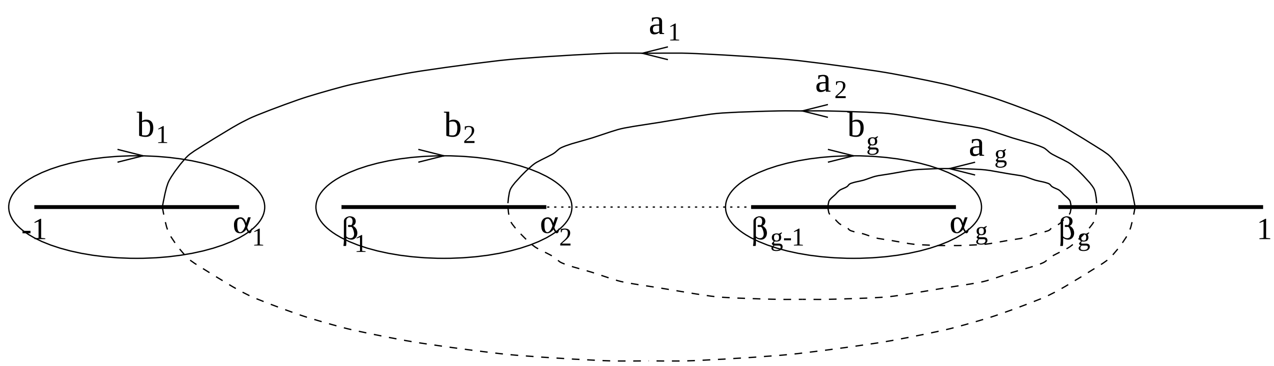

Figure 1: The dash curves represent the parts of the cannonical loops lying on the lower sheet. The lower (upper) sheet is fixed by the condition that it contains the point ∞^+ (∞^-).

$$\{d\omega_j\}_{j=1}^g, \quad \int_{a_j} d\omega_k = \delta_{jk},$$

be a set of normalised Abelian differentials of the first kind. As it is usual for a hyperelliptic curve, we shall chose the differentials $d\omega_j$ according to the equations,

$$\begin{aligned} d\omega_j &= \sum_{k=1}^{g} (A^{-1})_{jk} \frac{z^{g-k}}{y} dz, \\ A_{jk} &= \int_{a_k} \frac{z^{g-j}}{y} dz. \end{aligned}$$

The invertability of the matrix A is a (relatively simple) classical result. We refer the reader to the monograph [19] for the basic general facts concerning the theory of functions on the Riemann surfaces (see also chapter 1 of [10]). Let us also introduce the normalized Abelian differential of the third kind, having its only poles at $\infty^{\pm}$,

$$d\Omega(P) = \frac{z^g + \lambda_{g-1} z^{g-1} + ... + \lambda_0}{y} dz,$$

with vanishing $a-$period;

$$\int_{a_j} d\Omega = 0, \quad j = 1, ..., g.$$

The above g conditions uniquely determine [19] the coefficients, $\{\lambda_j\}_{j=0}^{g-1}$. Put

$$\Omega(P) = \int_{\beta_{g+1}}^{P} d\Omega.$$

One easily deduces,

$$\Omega(P) = \pm\left(\ln z - \ln C(E) + \mathrm{O}\left(\frac{1}{z}\right)\right), \quad P \to \infty^{\pm}, \tag{8.9}$$

where

$$C(E) = \exp\left(-\int_{\beta_{g+1}}^{\infty^+}\left(\frac{z^g + \sum_{j=0}^{g-1}\lambda_j z^j}{y(z)} - \frac{1}{z}\right)dz\right). \tag{8.10}$$

(We recall that $\beta_{j+1} = 1$.) Finally, the Riemann $\Theta-$function of $g-$ complex variables $\vec{s} \in \mathbb{C}^g$, is defined with the aid of the period matrix

$$B_{jk} := \int_{b_k} d\omega_j,$$

as follows:

$$\Theta(\vec{s}) \equiv \Theta(\vec{s}; B) := \sum_{\vec{t}\in\mathbb{Z}^g} \exp\left(i\pi(\vec{t}, B\vec{t}\,) + 2\pi i(\vec{t}, \vec{s}\,)\right).$$

Here are the fundamental periodic property of the $\Theta-$ function:

$$\Theta(\vec{s} + \vec{n} + B\vec{m}) = e^{-\pi i(B\vec{m},\vec{m}) - 2\pi i(\vec{s},\vec{m})}\Theta(\vec{s}), \tag{8.11}$$

and the obvious symmetry relation:

$$\Theta(-\vec{s}) = \Theta(\vec{s}).$$

Observe now that **BA1** $-$ **BA2** imply the following properties on the components of $\vec{\Psi}_n(P)$.

$$\begin{aligned}
&\Psi_{n1}(P) && \text{is meromorphic on } \mathfrak{R}\setminus\{\infty^+, \infty^-\} \\
&(\Psi_{n1}(P)) && = -\sum_{j=1}^{g}\alpha_j \\
&\Psi_{n1}(P) && = z^n + \mathrm{O}(z^{n-1}), \quad P \to \infty^+ \\
&\Psi_{n1}(P) && = \mathrm{O}(z^{-n}), \quad P \to \infty^-.
\end{aligned} \tag{8.12}$$

Similary for $\Psi_{n2}(P)$,

$$\begin{array}{rcl}\Psi_{n2}(P) &=& z^{-n+1} + \mathrm{O}(z^{-n}), \quad O \to \infty^-, \\ \Psi_{n2}(P) &=& \mathrm{O}(z^{n-1}), \quad P \to \infty^+. \end{array} \tag{8.13}$$

By standard technique of the algebrogeometric method (see e.g. [10]), we get,

$$\begin{array}{rcl}\Psi_{n1}(P) &=& \mathrm{e}^{n\Omega(P)} \dfrac{\Theta\left(\int_{\beta_{g+1}}^P d\vec{\omega} + n\vec{L} - \vec{D}\right)}{\Theta\left(\int_{\beta_{g+1}}^P d\vec{\omega} - \vec{D}\right)} \dfrac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} - \vec{D}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} + n\vec{L} - \vec{D}\right)} C^n(E), \\ \Psi_{n2}(P) &=& \mathrm{e}^{(n-1)\Omega(P)} \dfrac{\Theta\left(\int_{\beta_{g+1}}^P d\vec{\omega} + (n-1)\vec{L} - \vec{D}\right)}{\Theta\left(\int_{\beta_{g+1}}^P d\vec{\omega} - \vec{D}\right)} \dfrac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} + \vec{D}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} - (n-1)\vec{L} + \vec{D}\right)} C^{(1-n)}(E), \end{array}$$

where

$$\begin{array}{rcl} L_j &=& \dfrac{1}{2\pi i} \displaystyle\int_{bj} d\Omega \\ D_j &=& \displaystyle\sum_{k=1}^{g} \int_{\beta_{g+1}}^{\alpha_k} d\omega_j + C_j \\ &=& 2\displaystyle\sum_{k=1}^{g} \int_{\beta_{g+1}}^{\alpha_k} d\omega_j, \end{array}$$

and C_j form the vector of the Riemann constants (see again [19] and [10]). Indeed, by the Riemann theorem (see e.g. [19]), the first $\Theta-$functions in the denominators has zeros exactly at the points α_j; the front exponential factors provide the needed asymptotic behavior at $\infty^{\pm}$; the first $\Theta-$ functions in the numerators, by virture of the periodicity property (8.11), ensure the single-valuedness; and, finally, the P-independent Θ-factors together with the back exponential factors provide the needed normalizations ad $\infty^{\pm}$ (cf. (8.12) and (8.13)). We also assume that we choose the same path between β_{g+1} and P for all the integrals involved[1].

The formulae above can be simplified. To this end we observe that

$$\int_{\beta_{g+1}}^{\alpha_k} d\omega_j = \frac{1}{2}\delta_{jk} + \frac{1}{2}\sum_{l=1}^{k} B_{jl}, \tag{8.14}$$

where the path of integration from β_{g+1} to α_k lies on the upper plane of the upper sheet. Therefore, moduli the lattice periods,

$$D_j = 1 + \sum_{k=1}^{g} B_{jk}(g-k+1).$$

[1]Alternatively, one can choose for each integral its own path. In this case though the paths must not intersect the basic cycles.

In other words, the vector $\vec{D}$ belongs to the latice $\mathbb{Z}^g + B\mathbb{Z}^g$ and hence (property (8.11) again) can be droped from the above formulae for $\vec{\Psi}_n(P)$. This yields the following simplified $\Theta-$ representation for $\vec{\Psi}_n(P)$.

$$\Psi_{n1}(P) = \mathrm{e}^{n\Omega(P)} \frac{\Theta\left(\int_{\beta_{g+1}}^{P} d\vec{\omega} + n\vec{L}\right)}{\Theta\left(\int_{\beta_{g+1}}^{P} d\vec{\omega}\right)} \frac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} + n\vec{L}\right)} C^n(E), \tag{8.15}$$

$$\Psi_{n2}(P) = \mathrm{e}^{(n-1)\Omega(P)} \frac{\Theta\left(\int_{\beta_{g+1}}^{P} d\vec{\omega} + (n-1)\vec{L}\right)}{\Theta\left(\int_{\beta_{g+1}}^{P} d\vec{\omega}\right)} \frac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} - (n-1)\vec{L}\right)} C^{(1-n)}(E),$$

We conclude the $\Theta-$ function solution of the Akhiezer Riemann-Hilbert problem by noticing the following equation for the vector $\vec{L}$ of the b - periods of the integral $\Omega(P)$.

$$\begin{aligned}\vec{L} &= \mathrm{res}_{P=\infty^+}(\vec{\omega} d\Omega(P)) \; + \; \mathrm{res}_{P=\infty^-}(\vec{\omega} d\Omega(P)) \\ &= -\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} + \int_{\beta_{g+1}}^{\infty^-} d\vec{\omega} \; = \; -2\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega},\end{aligned} \tag{8.16}$$

The equation is just the classical Riemann bilinear identity (see e.g. [19] or [10]) applied to the pair of the Abelian integrals $\vec{\omega}(P)$ and $\Omega(P)$.

Remark 8.3 Using equation (8.14), one can check directly, with the help of the periodic condition (8.11), that the theta function,

$$\Theta\left(\int_{\beta_{g+1}}^{P} d\vec{\omega}\right)$$

has the points α_j as its zeros.

Remark 8.4 The reader should not be misled by the formal possibility to diagonalize simultaneously the jump matrices of the Riemann-Hilbert problem $\Psi 1 - \Psi 5$ (which all are equal to σ_1) and by apparently following from this conclusion that the problem can be reduced to the scalar one on the complex plane and hence solved without any use of the $\Theta-$ functions. The obstractions come from the end points α_j, β_j and from the point at infinity, where the function $\Psi_n(z)$ must have the singularities specified by equations $\Psi 5$, $\Psi 4$ and $\Psi 3$, respectively. These singularities can be alternatively discribed as the addition *jump conditions* posed on the small circles around the end points and on the big circle around the infinity. The relevant jump matrices are

$$\begin{pmatrix} 1/\sqrt{z-\alpha_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{z-\beta_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$z^{\begin{pmatrix} n & 0 \\ 0 & -n+1 \end{pmatrix}},$$

respectively. Posed in this form, the σ_1 Riemann-Hilbert problem becomes the regular one - no singularities different from the jumps are prescribed. At the same, the additional jump matrices depend on z and the whole new set of jump matrices can not be simultaneously diagonalized. The only way to circumvent this obstacles, and not to use the $\Theta-$ functions, is the equation (8.4) which indeed gives an explicit representation of the solution of the σ_1 Riemann-Hilbert problem in terms of the elementary functions and their contour integrals. The $\Theta-$ function representation (8.15) for the solution $\Psi_n(z)$ obtained in this chapter has an important advantage comparing to (8.4). It reveals the nature of the dependence of $\Psi_n(z)$, and hence of the Akhiezer polynomials themselves (see (9.1) below), on the number n, as n varies over the whole range $1 \leq n \leq \infty$ (see [5] for more on the use of the Θ - representations in the analysis of the Akhiezer polynomials). Simultaneously, the comparison of equations (8.4) and (8.15) might, perhaps, be used to derive some new nontrivial identities for the hyperelliptic $\Theta-$ functions.

Remark 8.5 Up to a trivial diagonal gauge transformation, the matrix function $\Psi_n(z)$ satisfies the same Fuchsian equation (3.11) that is satisfied by the function $\Phi_n(z)$. Note that the corresponding monodromy group is very simple; indeed, it has just one generator - the matrix σ_1. Once again, the reader might be wondering about the appearance of the highly nontrivial theta-functional formulae in the describtion of the function $\Psi_n(z)$ which gives the solution of the corresponding inverse monodromy problem. Similar to the previous remark, the explanation comes from the fact that the solution $\Psi_n(z)$, in addition to the given monodromy group, must exhibit the local behavior at the singular points indicated by the conditions $\Psi 3 - \Psi 5$. This situation is typical in the theory of the finite-gap solutions of integrable PDEs [2] (see e.g. [14] and [10]).

9 A list of the Θ - formulae.

In this section, we give formulae expressing the polynomial $P_n(z)$, recurrence coefficients a_n, b_n, the square of the weighted L^2 norm h_n and the Hankel determinant in terms of the $\Theta-$ functions. The expressions will be derived as simple corollaries of the equations (8.5) and (8.15) representing the solution $\Psi_n(z)$ of the Riemann-Hilbert problem $\Psi 1 - \Psi 5$ in terms of the Θ - functions.

[2] Another example of an apparently simple but nontrivialy solved invesre monodromy problem can be also found in the theory of integrable PDEs. It is provided by the multi-soliton Baker-Akhiezer function whose monodromy group is just trivial. Of course, the formulae in this case are simplier than the finite-gap ones - they do not contain the $\Theta-$ functions. At the same time, the answer is still rather complicated; in fact, it involves degenerated $\Theta-$ functions corresponding to the singular curves of genus zero.

From (8.1) it follows that (see also (8.4))

$$P_n(z) = (Y_n(z))_{11} = (\Psi_n(z))_{11} + (\Psi_n(z))_{12}.$$

This together with (8.5) and (8.15) leads to the following Θ - representation of the Akhiezer polynomials,

$$P_n(z) = \frac{\Theta\left(n\vec{L} + \int_{\beta_{g+1}}^{z} d\vec{\omega}\right) \mathrm{e}^{n\Omega(z)} + \Theta\left(n\vec{L} - \int_{\beta_{g+1}}^{z} d\vec{\omega}\right) \mathrm{e}^{-n\Omega(z)}}{\Theta\left(\int_{\beta_{g+1}}^{z} d\vec{\omega}\right)}$$

$$\times \frac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} + n\vec{L}\right)} C^n(E), \tag{9.1}$$

where all the hyperelliptic integrals are taken in the upper sheet of the curve $\mathfrak{R}$ (and along the same path).

Remark 9.1 It is a simple but an instructive exercise to check directly, using equation (8.14), the similar equation for the integral $\Omega(P)$, i.e.

$$\Omega(\alpha_k) = \pi i + \pi i \sum_{j=1}^{k} L_j,$$

and, once again, the periodicity property of the Θ-function, that the right side of (9.1) is indeed a *polynomial*.

To evaluate the quantities a_n, b_n, and h_n we shall use the relation

$$\psi_1 = \begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} m_1 \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & \sqrt{\frac{\pi i}{2}} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix}, \quad n > 1, \tag{9.2}$$

between the first matrix coefficients, ψ_1 and m_1, of the Laurent series

$$\Psi_n(z) = \left(I + \sum_{k=1}^{\infty} \frac{\psi_k(n)}{z^k}\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix}, \quad |z| > 1,$$

and

$$Y_n(z) = \left(I + \sum_{k=1}^{\infty} \frac{m_k(n)}{z^k}\right) z^{n\sigma_3}, \quad |z| > 1,$$

respectively. In (9.2), the parameter κ is defined via the expansion,

$$w(z) = \frac{i}{\pi z}\left(1 + \frac{\kappa}{z} + ...\right),$$

and the matrix,

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

should be added to the r.h.s if $n = 1$. Combaining equation (9.2) with the formula (2.15) we obtain that

$$h_n = 2\left(\psi_1(n)\right)_{12}.$$

On the other hand, let us introduce the coefficient matrix c_{jk}, $j,k = 1,2$ by the relations (cf. (8.12) and (8.13)),

$$\Psi_{n1}(P) = z^n + c_{11}z^{n-1} + \mathrm{O}(z^{n-2}), \quad P \to \infty^+ \tag{9.3}$$

$$\Psi_{n1}(P) = c_{12}z^{-n} + \mathrm{O}(z^{-n-1}), \quad P \to \infty^-, \tag{9.4}$$

and

$$\Psi_{n2}(P) = z^{-n+1} + c_{22}z^{-n} + \mathrm{O}(z^{-n-1}), \quad O \to \infty^-, \tag{9.5}$$

$$\Psi_{n2}(P) = c_{21}z^{n-1} + \mathrm{O}(z^{n-2}), \quad P \to \infty^+. \tag{9.6}$$

Then, it is obvious that

$$\left(\psi_1(n)\right)_{jk} = c_{jk}, \tag{9.7}$$

and, in particular, we arrive to the equation

$$h_n = 2c_{12}. \tag{9.8}$$

The coefficient c_{12}, in its turn, can be immediately evaluated from the $\Theta-$ formula (8.15) by letting $P \to \infty^-$. In fact, we have

$$c_{12} = C^{2n}(E)\frac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} - n\vec{L}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega} + n\vec{L}\right)}. \tag{9.9}$$

Taking into account the Riemann bilinear relation (8.16) we can present the formula for h_n in the following final form,

$$h_n = 2C^{2n}(E)\frac{\Theta\left(\left(n+\frac{1}{2}\right)\vec{L}\right)}{\Theta\left(\left(n-\frac{1}{2}\right)\vec{L}\right)}, \quad n = 1,2,..., \tag{9.10}$$

$$h_0 := 1.$$

An important direct consequence of this equation is the explicit $\Theta-$ functional representation for determinant of the $(n+1)\times(n+1)$ Hankel matrix:

$$D_{n+1}[w_+] = \prod_{j=0}^{n} h_j = 2^n (C(E))^{n(n+1)}\frac{\Theta\left(\left(n+\frac{1}{2}\right)\vec{L}\right)}{\Theta\left(\frac{1}{2}\vec{L}\right)}$$

$$= 2^n (C(E))^{n(n+1)} \frac{\Theta\left((2n+1)\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega}\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\vec{\omega}\right)}. \tag{9.11}$$

A similar use of the remaining equations in (9.2), (9.7) and the formulae (2.16), (2.17) leads at once to the $\Theta-$ representations of the recurrence coefficients a_n and b_n:

$$a_n = \begin{cases} 2C^2(E)\frac{\Theta\left(\frac{3}{2}\vec{L}\right)}{\Theta\left(\frac{1}{2}\vec{L}\right)} & \text{if } n = 1 \\ \\ C^2(E)\frac{\Theta\left((n+\frac{1}{2})\vec{L}\right)\Theta\left((n-\frac{3}{2})\vec{L}\right)}{\Theta^2\left((n-\frac{1}{2})\vec{L}\right)} & \text{if } n > 1 \end{cases}, \tag{9.12}$$

and

$$\begin{aligned} b_n &= \frac{1}{2}\sum_{j=1}^{g}(\beta_j - \alpha_j) \\ &+ \sum_{j=1}^{g}(A^{-1})_{j1}\left[\frac{\Theta_j'\left((n-\frac{1}{2})\vec{L}\right)}{\Theta\left((n-\frac{1}{2})\vec{L}\right)} - \frac{\Theta_j'\left((n-\frac{3}{2})\vec{L}\right)}{\Theta\left((n-\frac{3}{2})\vec{L}\right)} - 2\frac{\Theta_j'\left(\frac{1}{2}\vec{L}\right)}{\Theta\left(\frac{1}{2}\vec{L}\right)}\right]. \end{aligned} \tag{9.13}$$

Here,

$$\Theta_j'(\vec{s}) := \frac{\partial\Theta(\vec{s})}{\partial s_j}.$$

Equations (9.1), (9.10), (9.11), (9.12) and (9.13) were previously obtained in [5] by a direct analysis of Akhiezer's function defined as the sum $\frac{i\pi w(z)P_n(z)-Q_n(z)}{w(z)}$ (cf. (8.4)). In [5] it was also shown that the above formulae allow to identify the quantity $C(E)$ as the transfinite diameter of the set E. We remind that in our approach, $C(E)$ appears as a first nontrivial coefficient in the asymptotic expansion of the Abelian integral $\Omega(P)$, see (8.9) and (8.10). Finally, we should note that equations (8.15), (9.12) and (9.13), as the equations describing the eigenfunctions and the coefficients of a finite-gap discrete Schrd̈inger operator, have already been known (see e.g. [20]) in the theory of the periodic Toda lattice.

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