Jacobi Polynomials from Compatibility Conditions

Yang Chen
Department of Mathematics
Imperial College
180 Queen’s Gate,
London SW7 2BZ, UK
e-mail: ychen@imperial.ac.uk

Mourad Ismail
Department of Mathematics
University of South Florida,
Tampa, Florida 33620
USA
e-mail: ismail@math.usf.edu

Abstract

We revisit the ladder operators for orthogonal polynomials and re-
interpret two supplementary conditions as compatibility conditions of
two linear over-determined systems; one involves the variation of the
polynomials with respect to the variable \( z \) (spectral parameter) and
the other a recurrence relation in \( n \) (the lattice variable). For the
Jacobi weight \( w(x) = (1-x)^\alpha(1+x)^\beta \), \( x \in [-1,1] \), we show how to
use the compatibility conditions to explicitly determine the recurrence
coefficients of the monic Jacobi polynomials.

Running title: Jacobi polynomials

1 Introduction and Preliminaries.

We begin with some notation. Let \( P_n(x) \) be monic polynomials of degree \( n \)
in \( x \) and orthogonal, with respect to a weight, \( w(x) \), \( x \in [a,b] \);

\[
\int_a^b P_m(x)P_n(x)w(x)dx = h_n\delta_{m,n}.
\]

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We further assume that \( v'(z) := -w'(z)/w(z) \) exists and that 
\( y^n [v'(x) - v'(y)] w(y)/(x - y) \) is integrable on \([a, b]\) for all \( n, n = 0, 1, \ldots \).
From the orthogonality condition there follows the recurrence relation,

\[
(1.2) \quad zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), \quad n = 0, 1, \ldots
\]

where \( \beta_0 P_{-1}(z) := 0, \alpha_n, \quad n = 0, 1, 2, \ldots \) is real and \( \beta_n > 0, \quad n = 1, 2, \ldots \)

In this paper we describe a formalism which derives properties of orthogonal polynomials, and their recurrence coefficients, from the knowledge of the weight function. We believe this is a new and interesting approach to orthogonal polynomials. In order to keep this work accessible we will only include the example of Jacobi polynomials. We defer in a future publication, the analysis in the case of the generalized Jacobi weights \([11],[8]\). In the Jacobi case we find the recurrence relations in §2. In §3 we show how our approach leads to the evaluation of monic Jacobi polynomials at \( x = \pm 1 \).

We also show that the evaluation of a Jacobi polynomial at \( x = 1 \) or \( x = -1 \) leads to explicit representations of the Jacobi polynomials. Closed form expressions for the normalization constants \( h_n \) are also found.

The actions of the ladder operators on \( P_n(z) \) and \( P_{n-1}(z) \) are,

\[
(1.3) \quad \left( \frac{d}{dz} + B_n(z) \right) P_n(z) = \beta_n A_n(z) P_{n-1}(z)
\]

\[
(1.4) \quad \left( \frac{d}{dz} - B_n(z) - v'(z) \right) P_{n-1}(z) = -A_{n-1}(z) P_n(z),
\]

with

\[
(1.5) \quad A_n(z) := \frac{w(y) P_n^2(y)}{h_n(y - z)} \Big|_{y=a}^{y=b} + \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(y)}{z - y} P_n^2(y) w(y) dy,
\]

\[
(1.6) \quad B_n(z) := \frac{w(y) P_n(y) P_{n-1}(y)}{h_n(y - z)} \Big|_{y=a}^{y=b}
\]

\[
+ \frac{1}{h_{n-1}} \int_a^b \frac{v'(z) - v'(y)}{z - y} P_{n-1}(y) P_n(y) w(y) dy,
\]

where we have used the supplementary condition,

\[
B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - v'(z), \quad (S_1)
\]
to arrive at (1.4). The equations (1.3)–(1.6) and the supplementary condition $(S_1)$, was derived by Bonan and Clark [4], Bauldry [3], and Mhaskar [10] for polynomial $v$, and the Authors [7] for general $v$. Ismail and Wimp [9] identified the additional supplementary condition, 

\[ B_{n+1}(z) - B_n(z) = \frac{\beta_{n+1}A_{n+1}(z) - \beta_nA_{n-1}(z) - 1}{z - \alpha_n}. \]

\((S_2)\)

Our thesis in this work is that the supplementary conditions, $(S_1)$ and $(S_2)$, being identities in $n, (n > 0)$ and $z \in \mathbb{C} \cup \mathbb{\infty}$ have the information needed to determine the recurrence coefficients and other auxiliary quantities. We illustrate this by systematically using $(S_1)$ and $(S_2)$ to determine most of the properties of the Jacobi polynomials. See [13], [2], and [12], for information concerning the Jacobi polynomials. In describing our results we shall follow the standard notation for shifted factorial and hypergeometric functions in [2], [12].

Below, we reinterpret $(S_2)$. We set

\begin{align*}
(1.7) \quad \Phi_n(z) &:= \begin{pmatrix} P_n(z) \\ P_{n-1}(z) \end{pmatrix}, \\
(1.8) \quad M_n(z) &:= \begin{pmatrix} -B_n(z) & \beta_nA_n(z) \\ -A_{n-1}(z) & B_n(z) + v'(z) \end{pmatrix}, \\
(1.9) \quad U_n(z) &:= \begin{pmatrix} z - \alpha_n & -\beta_n \\ 1 & 0 \end{pmatrix}.
\end{align*}

Now equations (1.3) and (1.4) become,

\begin{align*}
(1.10) \quad \Phi'_n(z) &= M_n(z)\Phi_n(z),
\end{align*}

and the recurrence relations become,

\begin{align*}
(1.11) \quad \Phi_{n+1}(z) &= U_n(z)\Phi_n(z).
\end{align*}

We find, by requiring (1.10) and (1.11) be compatible;

\begin{align*}
\Phi'_{n+1}(z) &= M_{n+1}(z)\Phi_{n+1}(z) \\
&= M_{n+1}(z)U_n(z)\Phi_n(z).
\end{align*}
On the other hand

$$
\Phi'_{n+1}(z) = U'_n(z)\Phi_n(z) + U_n(z)\Phi'_n(z) \]
$$
$$
= U'_n(z)\Phi_n(z) + U_n(z)M_n(z)\Phi_n(z).
$$

We now write the above equations in matrix form as

$$
S_n(z)\Phi_n(z) = 0, \tag{1.12}
$$

where $S_n(z)$ is the matrix whose entries are

\[
\begin{align*}
S_n(z) & := U'_n(z) + U_n(z)M_n(z) - M_{n+1}(z)U_n(z) \\
S^{11}_n(z) & = 1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) + \beta_n A_{n-1}(z) - \beta_{n+1} A_{n+1}(z) \\
S^{12}_n(z) & = -\beta_n (B_{n+1}(z) + B_n(z) + \nu'(z) - (z - \alpha_n)A_n(z)) \\
S^{21}_n(z) & = S^{12}_n(z)/\beta_n \\
S^{22}_n(z) & = 0.
\end{align*}
\]

Here $n = 1, 2, \ldots$ and $z \in \mathbb{C} \cup \infty$. Observe that with $(S_1)$, $S^{12}_n(z) = S^{21}_n(z) = 0$. This leaves $S^{11}_n(z)P_n(z) = 0$. Since $P_n(z)$ does not vanish identically, we must have $S^{11}_n(z) = 0$, which is $(S_2)$. It is clear from (1.5) and (1.6) that, if $\nu'(z)$ is a rational function then $A_n(z)$ and $B_n(z)$ are also rational functions. This is particularly useful for our purpose, which is to determine the recurrence coefficients, $\alpha_n$ and $\beta_n$. In the next section, we illustrate the method by considering the Jacobi weight $w^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$ for $x \in [-1, 1]$.

Recall that the numerator polynomials $[13]$, $[14]$ are

$$
Q_n(z) := \int_{-\infty}^{\infty} \frac{P_n(z) - P_n(y)}{z - y} w(y) \, dy, \tag{1.14}
$$

and $\{P_n(z)\}$ and $\{Q_n(z)\}$ form a basis of solutions of the recurrence relation. We shall also use the notation

$$
F(z) = \int_{-\infty}^{\infty} \frac{w(y)}{z - y} \, dy, \tag{1.15}
$$

for the Stieltjes transform of the weight function.
2 Jacobi Weight

The Jacobi weight is \( w^{(\alpha,\beta)}(x) = (1 - x)^\alpha (1 + x)^\beta; x \in [-1,1] \), and for now we take \( \alpha \) and \( \beta \) to be strictly positive. It will become clear, using a real analyticity argument, the results that follows are also valid for \( \alpha, \beta > -1 \).

Let \( \{P_n^{(\alpha,\beta)}(x)\} \) and \( \{Q_n^{(\alpha,\beta)}(x)\} \) denote the monic Jacobi polynomials, and their numerators, respectively, see (1.14). Moreover in the present example, the Stieltjes transform of \( w^{\alpha,\beta} \) will be denoted by \( F^{(\alpha,\beta)}(z) \).

From (1.5)–(1.6) we find

\[
\begin{align*}
A_n(z) &= \frac{\alpha}{1 - z} \int_{-1}^{1} \left[ P_n^{(\alpha,\beta)}(y) \right]^2 (1 - y)^{\alpha-1} (1 + y)^{\beta} \, dy \\
&+ \frac{\beta}{1 + z} \int_{-1}^{1} \left[ P_n^{(\alpha,\beta)}(y) \right]^2 (1 - y)^{\alpha} (1 + y)^{\beta-1} \, dy.
\end{align*}
\]

Through integration by parts, it readily follows that,

\[ A_n(z) = \frac{R_n}{z - 1} + \frac{R_n}{z + 1}, \tag{2.1} \]

for some constant \( R_n \). Similarly we find

\[ B_n(z) = -\frac{n + r_n}{z - 1} + \frac{r_n}{z + 1}. \tag{2.2} \]

Here \( R_n \) and \( r_n \) are given by

\[ R_n = R_n(\alpha, \beta) := \frac{\beta}{h_n} \int_{-1}^{1} \frac{\left[ P_n^{(\alpha,\beta)}(y) \right]^2}{1 + y} w^{(\alpha,\beta)}(y) \, dy, \tag{2.3} \]

\[ r_n = r_n(\alpha, \beta) := \frac{\beta}{h_{n-1}} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y) P_{n-1}^{(\alpha,\beta)}(y)}{1 + y} w^{(\alpha,\beta)}(y) \, dy. \tag{2.4} \]

It is easy to see that

\[
\begin{align*}
R_n(\alpha, \beta) &= \frac{\beta}{h_n} \left[ P_n^{(\alpha,\beta)}(-1) \right] Q_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1) P_n^{(\alpha,\beta)}(-1) \\
r_n(\alpha, \beta) &= \frac{\beta}{h_{n-1}} \left[ P_n^{(\alpha,\beta)}(-1) \right] Q_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1) P_{n-1}^{(\alpha,\beta)}(-1).
\end{align*}
\]
The reader may ask, “What is the point of this formalism? Since in the attempt to find $\alpha_n$ and $\beta_n$, two new unknown quantities, $R_n$ and $r_n$, have been introduced.” However when $v'$ is a rational function both sides of $(S_1)$ and $(S_2)$, are rational functions and by equating coefficients and residues of both sides of $(S_1)$ and $(S_2)$, we shall arrive at four equations which should be sufficient for the determination of $R_n$ and $r_n$ as well as $\alpha_n$ and $\beta_n$. Equating residues at $z = -1$ and $z = +1$, of $(S_1)$, gives

\begin{align*}
(2.5) \quad -2n - 1 - r_n - r_{n+1} &= \alpha - R_n(1 - \alpha_n) \\
(2.6) \quad r_n + r_{n+1} &= \beta - R_n(1 + \alpha_n).
\end{align*}

Similarly, from $(S_2)$, we obtain

\begin{align*}
(2.7) \quad (r_n - r_{n+1} - 1)(1 - \alpha_n) &= \beta_n R_{n-1} - \beta_{n+1} R_{n+1} \\
(2.8) \quad (r_n - r_{n+1})(1 + \alpha_n) &= \beta_{n+1} R_{n+1} - \beta_n R_{n-1}.
\end{align*}

Observe that $R_n$ can be obtained immediately by adding (2.5) and (2.6);

\begin{equation}
(2.9) \quad R_n = \frac{1}{2}(\alpha + \beta + 2n + 1).
\end{equation}

The sum (2.7) of (2.8) gives,

\begin{equation}
(2.10) \quad 1 - \alpha_n = 2(r_n - r_{n+1}),
\end{equation}

while (2.8) minus (2.7) and with (2.9) gives,

\begin{equation}
(2.11) \quad \beta - \alpha - 2n - 1 - (\alpha + \beta + 2n + 1)\alpha_n = 2(r_n + r_{n+1}).
\end{equation}

Now, (2.10) plus (2.11) implies,

\begin{equation}
(2.12) \quad 4r_n = \beta - \alpha - 2n - (\alpha + \beta + 2n + 2)\alpha_n
\end{equation}

and (2.11) minus (2.10) implies,

\begin{equation}
(2.13) \quad 4r_{n+1} = \beta - \alpha - 2n - 2 - (\alpha + \beta + 2n)\alpha_n.
\end{equation}
When (2.12) and (2.13) are made compatible, we obtain a first order difference equation satisfied by $\alpha_n$:

\begin{align}
\alpha_{n+1}(\alpha + \beta + 2n + 4) - \alpha_n(\alpha + \beta + 2n) &= 0, \\
(2.14)
\end{align}

which has a very simple “integrating factor,” $\alpha + \beta + 2n$. Using this, we find,

$$
\alpha_n = \frac{C_1}{(2R_n - 1)(2R_n + 1)},
$$

where $C_1$ is a “integration” constant, determined by the initial condition,

$$
\alpha_0 = \frac{\mu_1}{\mu_0} = \frac{\beta - \alpha}{\alpha + \beta + 2}, \quad C_1 = \beta^2 - \alpha^2.
$$

Here $\mu_j := \int_{t_1}^{t_2} t^j w(t) dt$, $j = 0, 1, \ldots$ are the moments. Therefore we have established

\begin{align}
\alpha_n &= \frac{\beta^2 - \alpha^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)}, \\
(2.15)
\end{align}

Going back to (2.8) and using (2.10), we see that $\beta_n$ satisfies the linear difference equation:

\begin{align}
\beta_{n+1}R_{n+1} - \beta_nR_{n-1} &= \frac{1 - \alpha_n^2}{2}, \\
(2.16)
\end{align}

which has the “integrating factor” $R_n$. Therefore,

$$
\beta_nR_nR_{n-1} = C_2 + \frac{1}{2} \sum_{j=0}^{n-1} (1 - \alpha_j^2) R_j \\
(2.17) = C_2 + \frac{1}{2} \sum_{j=0}^{n-1} \left( 1 - \frac{C_1^2}{(4R_j^2 - 1)^2} \right) R_j,
$$

where $C_2$ is another integration constant to be determined by the initial condition

$$
\beta_1 = \frac{h_1}{h_0} = \frac{h_1}{\mu_0} = \frac{\mu_2}{\mu_0} \left( \frac{\mu_1}{\mu_0} \right)^2 = \frac{4(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}.
$$
After some computations,

\[ C_2 = \beta_1 R_0 R_1 - \frac{1}{2} (1 - \alpha_0^2) R_0 = 0. \]

Now the sum (2.17), may look complicated, however, with a partial fraction expansion, the sum can be taken and leads to

\[ \beta_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2 R_n R_{n-1}}. \]

Therefore, after some simplifications we establish

\[ \beta_n = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2 (2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}. \]

3 Explicit Formulas

We first determine \( P_n^{(\alpha, \beta)}(\pm 1) \). Write (1.3) as,

\[
\frac{d}{dz} P_n^{(\alpha, \beta)}(z) = \frac{\alpha + \beta}{z - 1} \left( (n + r_n) P_n^{(\alpha, \beta)}(z) - \beta_n R_n P_n^{(\alpha, \beta)}(z) \right) \\
+ \beta_n R_n P_{n-1}^{(\alpha, \beta)}(z) - r_n P_n^{(\alpha, \beta)}(z),
\]

and since \( \frac{d}{dz} P_n^{(\alpha, \beta)}(z) \) is regular at \( z = \pm 1 \), we arrive at

\[ (n + r_n) P_n^{(\alpha, \beta)}(1) - \beta_n R_n P_n^{(\alpha, \beta)}(1) = 0, \]

\[ \beta_n R_n P_{n-1}^{(\alpha, \beta)}(-1) - r_n P_n^{(\alpha, \beta)}(-1) = 0. \]

Thus we find

\[ P_n^{(\alpha, \beta)}(1) = P_0^{(\alpha, \beta)}(1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j + j}, \]

(3.1)
and

\[ P_n^{(\alpha,\beta)}(-1) = P_0^{(\alpha,\beta)}(-1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j}. \]

Substituting for \( \beta_n, r_n \) and \( R_n \) from (2.9), (2.12), and (2.18), and applying (2.15) we prove that

\[ P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n}{2^n} \prod_{j=1}^{n} \frac{(j + \beta)(j + \alpha + \beta)}{[j + (\alpha + \beta/2)][j + (\alpha + \beta - 1/2)]}. \]

Using the facts \((\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda), (2\lambda)_{2n} = 4^2(\lambda)_n(\lambda + 1/2)_n\) we rewrite the above equation as

\[ (3.2) \quad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n 2^n (\beta + 1)_n}{(\alpha + \beta + n + 1)_n}. \]

Similarly

\[ (3.3) \quad P_n^{(\alpha,\beta)}(1) = \frac{2^n (\alpha + 1)_n}{(\alpha + \beta + n + 1)_n}. \]

We next evaluate \( h_n \), the squares of the \( L^2 \) norms. In general (1.1) and (1.2) yield, \[12\]

\[ (3.4) \quad h_n = h_0 \beta_1 \beta_2 \cdots \beta_n. \]

The beta integral evaluation gives

\[ (3.5) \quad h_0 = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)}. \]

Thus

\[ (3.6) \quad \int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) (1 - x)^\alpha (1 + x)^\beta dx = h_n \delta_{m,n}, \]
with
\[
(3.7) \quad h_n = \frac{2^{\alpha+\beta+n+1} \Gamma(\alpha + n + 1)\Gamma(\beta + 1) n!}{(\alpha + \beta + n + 1)n \Gamma(\alpha + \beta + n + 2n + 2)}.
\]

We now prove that
\[
(3.8) \quad \frac{d}{dz} P_n^{(\alpha,\beta)}(z) = nP_n^{(\alpha+1,\beta+1)}(z)
\]
For \(\alpha > -1, \beta > -1,\) and \(m < n - 1\) integration by parts gives
\[
\int_{-1}^{1} x^m \left( \frac{d}{dx} P_n^{(\alpha,\beta)}(x) \right) (1-x)^{\alpha+1}(1+x)^{\beta+1} \, dx = -\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) f(x)(1-x)^{\alpha}(1+x)^{\beta} \, dx
\]
where \(f(x) = x^{m-1}[m+x(\beta-\alpha)-x^2(\alpha+\beta+m+2)].\) Since \(f\) has degree at most \(n-1,\) the above integral must vanish and we conclude that \(\frac{d}{dz} P_n^{(\alpha,\beta)}(x)\)

is orthogonal to all polynomials of degree less than \(n - 1\) with respect to \(w^{(\alpha+1,\beta+1)}(x)\). The uniqueness of the orthogonal polynomials and the fact

that \(P_n^{(\alpha,\beta)}(x), n \geq 0\) are monic, establish (3.8). Clearly (3.8) and (3.2) give
\[
(3.9) \quad \left. \frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) \right|_{x=-1} = \frac{n!}{(n-k)!} P_n^{(\alpha+\beta+k)}(-1)
\]

\[
= \frac{(-2)^{n-k}(\beta+k)_{n-k}}{(\alpha+\beta+n+k+1)_{n-k}}
\]

The Taylor series about \(x = -1\) now gives the representation
\[
(3.10) \quad P_n^{(\alpha,\beta)}(x) = \frac{(-2)^n(\beta+1)_n}{(\alpha+\beta+1)_n}
\]

\[
\times \, _2F_1(-n,n+\alpha+\beta+1;\beta+1;(1+x)/2),
\]

which we recognized to be the monic Jacobi polynomials. Similarly (3.3) and (3.8) give the alternate representation
\[
(3.11) \quad P_n^{(\alpha,\beta)}(x) = \frac{(2)^n(\alpha+1)_n}{(\alpha+\beta+1)_n}
\]

\[
\times \, _2F_1(-n,n+\alpha+\beta+1;\alpha+1;(1-x)/2),
\]
References


