Jacobi Polynomials from Compatibility Conditions *

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Abstract

We revisit the ladder operators for orthogonal polynomials and reinterpret two supplementary conditions as compatibility conditions of two linear over-determined systems; one involves the variation of the polynomials with respect to the variable z (spectral parameter) and the other a recurrence relation in n (the lattice variable). For the Jacobi weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta},\ x \in [-1,1]$, we show how to use the compatibility conditions to explictly determine the recurrence coefficients of the monic Jacobi polynomials.

Running title: Jacobi polynomials

1 Introduction and Preliminaries.

We begin with some notation. Let $P_n(x)$ be monic polynomials of degree n in x and orthogonal, with respect to a weight, $w(x),\ x \in [a,b]$;

$$\int_a^b P_m(x)P_n(x)w(x)dx = h_n\delta_{m,n}. \tag{1.1}$$

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We further assume that $\mathsf{v}'(z) := -w'(z)/w(z)$ exists and that $y^n[\mathsf{v}'(x) - \mathsf{v}'(y)]w(y)/(x-y)$ is integrable on $[a,b]$ for all n, $n = 0, 1.\cdots$. From the orthogonality condition there follows the recurrence relation,

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), n = 0, 1, ..., \tag{1.2}$$

where $\beta_0 P_{-1}(z) := 0$, $\alpha_n,\ n = 0, 1, 2, ...$ is real and $\beta_n > 0,\ n = 1, 2, ...$

In this paper we describe a formalism which derives properties of orthogonal polynomials, and their recurrence coefficients, from the knowledge of the weight function. We believe this is a new and interesting approach to orthogonal polynomials. In order to keep this work accessible we will only include the example of Jacobi polynomials. We defer in a future publication, the analysis in the case of the generalized Jacobi weights [11], [8]. In the Jacobi case we find the recurrence relations in §2. In §3 we show how our approach leads to the evaluation of monic Jacobi polynomials at $x = \pm 1$. We also show that the evaluation of a Jacobi polynomial at $x = 1$ or $x = -1$ leads to explicit representations of the Jacobi polynomials. Closed form expressions for the normalization constants h_n are also found.

The actions of the ladder operators on $P_n(z)$ and $P_{n-1}(z)$ are,

$$\left(\frac{d}{dz} + B_n(z)\right) P_n(z) = \beta_n A_n(z) P_{n-1}(z) \tag{1.3}$$

$$\left(\frac{d}{dz} - B_n(z) - \mathsf{v}'(z)\right) P_{n-1}(z) = -A_{n-1}(z) P_n(z), \tag{1.4}$$

with

$$A_n(z) := \left.\frac{w(y)\,P_n^2(y)}{h_n(y-z)}\right|_{y=a}^{y=b} + \frac{1}{h_n}\int_a^b \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} P_n^2(y) w(y) dy, \tag{1.5}$$

$$\begin{aligned} B_n(z) := &\left.\frac{w(y)\,P_n(y)P_{n-1}(y)}{h_n(y-z)}\right|_{y=a}^{y=b} \\ &+ \frac{1}{h_{n-1}}\int_a^b \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} P_{n-1}(y) P_n(y) w(y) dy, \end{aligned} \tag{1.6}$$

where we have used the supplementary condition,

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \mathsf{v}'(z), \tag{S_1}$$

to arrive at (1.4). The equations (1.3)–(1.6) and the supplementary condition (S_1), was derived by Bonan and Clark [4], Bauldry [3], and Mhaskar [10] for polynomial v, and the Authors [7] for general v. Ismail and Wimp [9] identified the additional supplementary condition,

$$B_{n+1}(z) - B_n(z) = \frac{\beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z) - 1}{z - \alpha_n}. \qquad (S_2)$$

Our thesis in this work is that the supplementary conditions, (S_1) and (S_2), being identities in $n, (n > 0)$ and $z \in \mathbf{C} \cup \infty$ have the information needed to determine the recurrence coefficients and other auxilliary quantities. We illustrate this by systematically using (S_1) and (S_2) to determine most of the properties of the Jacobi polynomials. See [14], [2], and [12], for information concerning the Jacobi polynomials. In describing our results we shall follow the standard notation for shifted factorial and hypergeometric functions in [2], [12].

Below, we reinterpret (S_2). We set

$$(1.7) \qquad \Phi_n(z) := \begin{pmatrix} P_n(z) \\ P_{n-1}(z) \end{pmatrix},$$

$$(1.8) \qquad \mathsf{M}_n(z) := \begin{pmatrix} -B_n(z) & \beta_n A_n(z) \\ -A_{n-1}(z) & B_n(z) + \mathsf{v}'(z) \end{pmatrix},$$

$$(1.9) \qquad \mathsf{U}_n(z) := \begin{pmatrix} z - \alpha_n & -\beta_n \\ 1 & 0 \end{pmatrix}.$$

Now equations (1.3) and (1.4) become,

$$(1.10) \qquad \Phi_n'(z) = \mathsf{M}_n(z)\Phi_n(z),$$

and the recurrence relations become,

$$(1.11) \qquad \Phi_{n+1}(z) = \mathsf{U}_n(z)\Phi_n(z).$$

We find, by requiring (1.10) and (1.11) be compatible;

$$\begin{aligned} \Phi_{n+1}'(z) &= \mathsf{M}_{n+1}(z)\Phi_{n+1}(z) \\ &= \mathsf{M}_{n+1}(z)\mathsf{U}_n(z)\Phi_n(z). \end{aligned}$$

On the other hand

$$\begin{aligned}\Phi'_{n+1}(z) &= \mathsf{U}'_n(z)\Phi_n(z) + \mathsf{U}_n(z)\Phi'_n(z) \\ &= \mathsf{U}'_n(z)\Phi_n(z) + \mathsf{U}_n(z)\mathsf{M}_n(z)\Phi_n(z).\end{aligned}$$

We now write the above equations in matrix form as

$$\mathsf{S}_n(z)\Phi_n(z) = 0, \tag{1.12}$$

where $\mathsf{S}_n(z)$ is the matrix whose entries are

$$\begin{aligned}\mathsf{S}_n(z) &:= \mathsf{U}'_n(z) + \mathsf{U}_n(z)\mathsf{M}_n(z) - \mathsf{M}_{n+1}(z)\mathsf{U}_n(z) \\ \mathsf{S}_n^{11}(z) &= 1 + (z-\alpha_n)(B_{n+1}(z) - B_n(z)) \\ &\qquad +\beta_n A_{n-1}(z) - \beta_{n+1}A_{n+1}(z) \\ \mathsf{S}_n^{12}(z) &= -\beta_n\left(B_{n+1}(z) + B_n(z) + \mathsf{v}'(z) - (z-\alpha_n)A_n(z)\right) \\ \mathsf{S}_n^{21}(z) &= S_n^{12}(z)/\beta_n \\ \mathsf{S}_n^{22}(z) &= 0.\end{aligned} \tag{1.13}$$

Here $n = 1, 2, ...$ and $z \in \mathbf{C} \cup \infty$. Observe that with (S_1), $S_n^{12}(z) = S_n^{21}(z) = 0$. This leaves $S_n^{11}(z)P_n(z) = 0$. Since $P_n(z)$ does not vanish identically, we must have $S_n^{11}(z) = 0$, which is (S_2). It is clear from (1.5) and (1.6) that, if $\mathsf{v}'(z)$ is a rational function then $A_n(z)$ and $B_n(z)$ are also rational functions. This is particularly useful for our purpose, which is to determine the recurrence coefficients, α_n and β_n. In the next section, we illustrate the method by considering the Jacobi weight $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ for $x \in [-1, 1]$.

Recall that the numerator polynomials [13], [1] are

$$Q_n(z) := \int_{-\infty}^{\infty} \frac{P_n(z) - P_n(y)}{z-y}\, w(y)\, dy, \tag{1.14}$$

and $\{P_n(z)\}$ and $\{Q_n(z)\}$ form a basis of solutions of the recurrence relation. We shall also use the notation

$$F(z) = \int_{-\infty}^{\infty} \frac{w(y)}{z-y}\, dy, \tag{1.15}$$

for the Stieltjes transform of the weight function.

2 Jacobi Weight

The Jacobi weight is $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta; x \in [-1,1]$, and for now we take α and β to be strictly positive. It will become clear, using a real analyticity argument, the results that follows are also valid for $\alpha, \beta > -1$. Let $\{\mathcal{P}_n^{(\alpha,\beta)}(x)\}$ and $\{\mathcal{Q}_n^{(\alpha,\beta)}(x)\}$ denote the monic Jacobi polynomials, and their numerators, respectively, see (1.14). Moreover in the present example, the Stieltjes transform of $w^{\alpha,\beta}$ will be denoted by $F^{(\alpha,\beta)}(z)$.

From (1.5)–(1.6) we find

$$\begin{aligned} h_n A_n(z) &= \frac{\alpha}{1-z}\int_{-1}^{1}[\mathcal{P}_n^{(\alpha.\beta)}(y)]^2(1-y)^{\alpha-1}(1+y)^\beta\, dy \\ &\quad +\frac{\beta}{1+z}\int_{-1}^{1}[\mathcal{P}_n^{(\alpha.\beta)}(y)]^2(1-y)^{\alpha}(1+y)^{\beta-1}\, dy. \end{aligned}$$

Through integration by parts, it readily follows that,

$$A_n(z) = -\frac{R_n}{z-1} + \frac{R_n}{z+1}, \tag{2.1}$$

for some constant R_n. Similarly we find

$$B_n(z) = -\frac{n+r_n}{z-1} + \frac{r_n}{z+1}. \tag{2.2}$$

Here R_n and r_n are given by

$$R_n = R_n(\alpha,\beta) := \frac{\beta}{h_n}\int_{-1}^{1}\frac{[\mathcal{P}_n^{(\alpha,\beta)}(y)]^2}{1+y}\, w^{(\alpha,\beta)}(y)\, dy, \tag{2.3}$$

$$r_n = r_n(\alpha,\beta) := \frac{\beta}{h_{n-1}}\int_{-1}^{1}\frac{\mathcal{P}_n^{(\alpha,\beta)}(y)\mathcal{P}_{n-1}^{(\alpha,\beta)}(y)}{1+y}\, w^{(\alpha,\beta)}(y)\, dy. \tag{2.4}$$

It is easy to see that

$$\begin{aligned} R_n(\alpha,\beta) &= \frac{\beta}{h_n}\mathcal{P}_n^{(\alpha,\beta)}(-1)\left[\mathcal{Q}_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1)P_n^{(\alpha,\beta)}(-1)\right], \\ r_n(\alpha,\beta) &= \frac{\beta}{h_{n-1}}\mathcal{P}_{n-1}^{(\alpha,\beta)}(-1)\left[\mathcal{Q}_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1)\mathcal{P}_n^{(\alpha,\beta)}(-1)\right]. \end{aligned}$$

The reader may ask, "What is the point of this formalism? Since in the attempt to find α_n and β_n, two new unknown quantities, R_n and r_n, have been introduced." However when v' is a rational function both sides of (S_1) and (S_2), are rational functions and by equating coefficients and residues of both sides of (S_1) and (S_2), we shall arrive at four equations which should be sufficient for the determination of R_n and r_n as well as α_n and β_n. Equating residues at $z = -1$ and $z = +1$, of (S_1), gives

$$-2n - 1 - r_n - r_{n+1} = \alpha - R_n(1 - \alpha_n) \tag{2.5}$$

$$r_n + r_{n+1} = \beta - R_n(1 + \alpha_n). \tag{2.6}$$

Similarly, from (S_2), we obtain

$$(r_n - r_{n+1} - 1)(1 - \alpha_n) = \beta_n R_{n-1} - \beta_{n+1} R_{n+1} \tag{2.7}$$

$$(r_n - r_{n+1})(1 + \alpha_n) = \beta_{n+1} R_{n+1} - \beta_n R_{n-1}. \tag{2.8}$$

Observe that R_n can be obtained immediately by adding (2.5) and (2.6);

$$R_n = \frac{1}{2}(\alpha + \beta + 2n + 1). \tag{2.9}$$

The sum (2.7) of (2.8) gives,

$$1 - \alpha_n = 2(r_n - r_{n+1}), \tag{2.10}$$

while (2.8) minus (2.7) and with (2.9) gives,

$$\beta - \alpha - 2n - 1 - (\alpha + \beta + 2n + 1)\alpha_n = 2(r_n + r_{n+1}). \tag{2.11}$$

Now, (2.10) plus (2.11) implies,

$$4r_n = \beta - \alpha - 2n - (\alpha + \beta + 2n + 2)\alpha_n \tag{2.12}$$

and (2.11) minus (2.10) implies,

$$4r_{n+1} = \beta - \alpha - 2n - 2 - (\alpha + \beta + 2n)\alpha_n. \tag{2.13}$$

When (2.12) and (2.13) are made compatible, we obtain a first order difference equation satisfied by α_n :

$$\alpha_{n+1}(\alpha+\beta+2n+4)-\alpha_n(\alpha+\beta+2n)=0, \tag{2.14}$$

which has a very simple "integrating factor," $\alpha+\beta+2n+2$. Using this, we find,

$$\alpha_n = \frac{C_1}{(2R_n-1)(2R_n+1)}.$$

where C_1 is a "integration" constant, determined by the initial condition,

$$\alpha_0 = \frac{\mu_1}{\mu_0} = \frac{\beta-\alpha}{\alpha+\beta+2}, \qquad C_1 = \beta^2-\alpha^2.$$

Here $\mu_j := \int_{-1}^{1} t^j w(t)dt,\ j=0,1,\cdots$ are the moments. Therefore we have established

$$\alpha_n = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}. \tag{2.15}$$

Going back to (2.8) and using (2.10), we see that β_n satisfies the linear difference equation:

$$\beta_{n+1}R_{n+1}-\beta_n R_{n-1} = \frac{1-\alpha_n^2}{2}, \tag{2.16}$$

which has the "integrating factor"R_n. Therefore,

$$\begin{aligned}\beta_n R_n R_{n-1} &= C_2 + \frac{1}{2}\sum_{j=0}^{n-1}\left(1-\alpha_j^2\right)R_j \\ &= C_2 + \frac{1}{2}\sum_{j=0}^{n-1}\left(1-\frac{C_1^2}{(4R_j^2-1)^2}\right)R_j, \qquad (2.17)\end{aligned}$$

where C_2 is another integration constant to be determined by the initial condition

$$\beta_1 = \frac{h_1}{h_0} = \frac{h_1}{\mu_0} = \frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2 = \frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}.$$

After some computations,

$$C_2 = \beta_1 R_0 R_1 - \frac{1}{2}(1-\alpha_0^2)R_0 = 0.$$

Now the sum (2.17), may look complicated, however, with a partial fraction expansion, the sum can be taken and leads to

$$\beta_n = \frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2 R_n R_{n-1}}.$$

Therefore, after some simplifications we establish

$$\beta_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}. \tag{2.18}$$

3 Explicit Formulas

We first determine $\mathcal{P}_n^{(\alpha,\beta)}(\pm 1)$. Write (1.3) as,

$$\begin{aligned}\frac{d}{dz}\mathcal{P}_n^{(\alpha,\beta)}(z) = & \frac{(n+r_n)\mathcal{P}_n^{(\alpha,\beta)}(z) - \beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(z)}{z-1} \\ & + \frac{\beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(z) - r_n \mathcal{P}_n^{(\alpha,\beta)}(z)}{z+1},\end{aligned}$$

and since $\frac{d}{dz}\mathcal{P}_n^{(\alpha,\beta)}(z)$ is regular at $z=\pm 1$, we arrive at

$$(n+r_n)\mathcal{P}_n^{(\alpha,\beta)}(1) - \beta_n \mathcal{R}_n P_{n-1}^{(\alpha,\beta)}(1) = 0,$$

$$\beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(-1) - r_n \mathcal{P}_n^{(\alpha,\beta)}(-1) = 0.$$

Thus we find

$$\mathcal{P}_n^{(\alpha,\beta)}(1) = \mathcal{P}_0^{(\alpha,\beta)}(1)\prod_{j=1}^{n}\frac{\beta_j R_j}{r_j + j}, \tag{3.1}$$

and

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \mathcal{P}_0^{(\alpha,\beta)}(-1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j}.$$

Substituting for β_n r_n and R_n from (2.9), (2.12), and (2.18), and applying (2.15) we prove that

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n}{2^n} \prod_{j=1}^{n} \frac{(j+\beta)(j+\alpha+\beta)}{[j+(\alpha+\beta/2)]\,[j+(\alpha+\beta-1/2)]}.$$

Using the facts $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, $(2\lambda)_{2n} = 4^2(\lambda)_n(\lambda+1/2)_n$ we rewrite the above equation as

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\, 2^n\, (\beta+1)_n}{(\alpha+\beta+n+1)_n}. \tag{3.2}$$

Similarly

$$\mathcal{P}_n^{(\alpha,\beta)}(1) = \frac{2^n\, (\alpha+1)_n}{(\alpha+\beta+n+1)_n}. \tag{3.3}$$

We next evaluate h_n, the squares of the L^2 norms. In general (1.1) and (1.2) yield, [12]

$$h_n = h_0\beta_1\beta_2\cdots\beta_n. \tag{3.4}$$

The beta integral evaluation gives

$$h_0 = 2^{\alpha+\beta+1}\, \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}. \tag{3.5}$$

Thus

$$\int_{-1}^{1} \mathcal{P}_m^{(\alpha,\beta)}(x)\mathcal{P}_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = h_n\delta_{m,n}, \tag{3.6}$$

with

$$h_n = \frac{2^{\alpha+\beta+n+1}\,\Gamma(\alpha+n+1)\Gamma(\beta+1)\,n!}{(\alpha+\beta+n+1)_n\,\Gamma(\alpha+\beta+n+2n+2)}. \tag{3.7}$$

We now prove that

$$\frac{d}{dz}\mathcal{P}_n^{(\alpha,\beta)}(z) = n\mathcal{P}_n^{(\alpha+1,\beta+1)}(z) \tag{3.8}$$

For $\alpha > -1$, $\beta > -1$, and $m < n-1$ integration by parts gives

$$\begin{aligned}&\int_{-1}^{1} x^m \left(\frac{d}{dx}\mathcal{P}_n^{(\alpha,\beta)}(x)\right)(1-x)^{\alpha+1}(1+x)^{\beta+1}\,dx \\ &\quad = -\int_{-1}^{1} \mathcal{P}_n^{(\alpha,\beta)}(x)\; f(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx\end{aligned}$$

where $f(x) = x^{m-1}[m + x(\beta-\alpha) - x^2(\alpha+\beta+m+2)]$. Since f has degree at most $n-1$, the above integral must vanish and we conclude that $\frac{d}{dx}\mathcal{P}_n^{(\alpha,\beta)}(x)$ is orthogonal to all polynomials of degree less than $n-1$ with respect to $w^{(\alpha+1,\beta+1)}(x)$. The uniqueness of the orthogonal polynomials and the fact that $\mathcal{P}_n^{(\alpha,\beta)}(x), n \geq 0$ are monic, establish (3.8). Clearly (3.8) and (3.2) give

$$\begin{aligned}\frac{d^k}{dx^k}\mathcal{P}_n^{(\alpha,\beta)}(x)\Big|_{x=-1} &= \frac{n!}{(n-k)}\mathcal{P}_{n-k}^{(k+\alpha,k+\beta)}(-1) \\ &= \frac{(-2)^{n-k}(\beta+k)_{n-k}}{(\alpha+\beta+n+k+1)_{n-k}}\end{aligned} \tag{3.9}$$

The Taylor series about $x = -1$ now gives the representation

$$\begin{aligned}\mathcal{P}_n^{(\alpha,\beta)}(x) &= \frac{(-2)^n(\beta+1)_n}{(\alpha+\beta+1)_n} \\ &\times {}_2F_1(-n, n+\alpha+\beta+1; \beta+1; (1+x)/2),\end{aligned} \tag{3.10}$$

which we recognized to be the monic Jacobi polynomials. Similarly (3.3) and (3.8) give the alternate representation

$$\begin{aligned}\mathcal{P}_n^{(\alpha,\beta)}(x) &= \frac{(2)^n(\alpha+1)_n}{(\alpha+\beta+1)_n} \\ &\times {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2),\end{aligned} \tag{3.11}$$

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