Toeplitz determinants from compatibility conditions

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Abstract

In this paper we show, how a straightforward and natural application of a pair of fundamental identities valid for polynomials orthogonal over the unit circle, can be used to calculate the determinant of the finite Toeplitz matrix,

$$\Delta_n = \det(w_{j-k})_{j,k=0}^{n-1} := \det\left(\int_{|z|=1} \frac{w(z)}{z^{j-k+1}}\frac{dz}{2\pi i}\right)_{j,k=0}^{n-1},$$

with the Fisher-Hartwig symbol,

$$w(z) = C(1-z)^{\alpha+i\beta}(1-1/z)^{\alpha-i\beta}, \quad |z|=1,\ \ \alpha > -1/2,\ \beta\in\mathbb{R}\,.$$

Here C is the normalisation constant chosen so that $w_0 = \frac{1}{2\pi}$. We use the same approach to compute a difference equation for expressions related to the determinants of the symbol

$$w(z) = \mathrm{e}^{t(z+1/z)},$$

a symbol important in the study of random permutations. Finally, we study the analogous equations for the symbol

$$w(z) = \mathrm{e}^{tz}\prod_{\alpha=1}^{M}\left(\frac{z-a_\alpha}{z}\right)^{g_\alpha}.$$

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1 Introduction

The large n behavior of the determinant of $n \times n$ Toeplitz matrices have seen many diverse physical applications, from the Ising model [19], Random Matrix Theory [15], String Theory [10, 16] and Combinatorics [8, 14, 18]. For an arbitrary weight or symbol $w(z)$, the $n \times n$ Toeplitz matrix is defined by

$$T_n(w) = \left(\int_{|z|=1} \frac{w(z)}{z^{j-k+1}} \frac{dz}{2\pi i} \right)_{j,k=0}^{n-1}. \tag{1.1}$$

For smooth symbols w the asymptotic behavior is a consequence of the classis Szegö Limit Theorem. For symbols that have jumps or other kinds of singularities, the asymptotic behavior is harder to determine. The earliest general conjecture for asymptotics of determinants for such singular symbols dates back to the formulation of the the Fisher-Hartwig conjecture. One of the key ingredients in the proof of the conjecture was the exact computation of the determinants for symbols of the form

$$w_{\alpha,\beta}(z) = (1-z)^{\alpha+i\beta}(1-1/z)^{\alpha-i\beta}.$$

These symbols are called symbols with a pure Fisher-Hartwig singularity. The exact computation was then combined with a localization technique to find more general answers [2]. A good account of these results can be found in [3]. It is worth noting that in the case of only jumps ($\alpha = 0$) the matrix reduces to a Cauchy matrix and thus the determinants are straightforward to calculate. The more general case was done by factoring the Toeplitz matrix into a product of triangular and diagonal matrices [7].

Although, as mentioned above, asymptotic behavior in the case of smooth symbols is given by the Strong Szegö Limit Theorem, there are applications where more exact information is desired. In the analysis of problems involving random permutations certain difference equations for quotients of the determinants also arise. These are related to the symbol $w(z) = e^{t(z+1/z)}$. For more results about the connection to random permutations and random matrices see [8, 14, 18].

The purpose of this paper is to re-derive these results and obtain new ones using some very simple ideas from the theory of orthogonal polynomials. While the analysis of such singular symbols and smooth ones is generally very different the two classes of examples share the property that the derivative of their logarithms are rational functions. This is what allows us to treat these classes as two examples of a general theory.

In the course of his investigation into the inversion of finite Toeplitz matrices, Szegö introduced orthogonal polynomials supported on the unit circle and the corresponding Szegö kernel. To fix notations for the rest of the paper, let

$$\phi_n(z) = k_n z^n + l_n z^{n-1} + ... + \phi_n(0), \quad k_n > 0 \;\; n = 1, 2, ... \tag{1.2}$$

satisfy the orthogonality condition

$$\int_{|z|=1} \phi_m(z)\overline{\phi_n(z)} w(z) \frac{dz}{iz} = \delta_{m,n}, \quad m, n = 0, 1, 2, ... \tag{1.3}$$

Note that with the normalisation on the weight, $k_0 = \phi_0(0) = 1$. From the orthogonality condition, we find, [17]

$$\begin{aligned} k_n z\phi_n(z) &= k_{n+1}\phi_{n+1}(z) - \phi_{n+1}(0)\phi^*_{n+1}(z) && (1.4)\\ k_n\phi_{n+1}(z) &= k_{n+1}z\phi_n(z) + \phi_{n+1}(0)\phi^*_n(z), && (1.5)\end{aligned}$$

and

$$k_n\phi_n(0)\phi_{n+1}(z) + k_{n-1}\phi_{n+1}(0)\phi_{n-1}(z) = (k_n\phi_{n+1}(0) + k_{n+1}\phi_n(0)z)\,\phi_n(z) \qquad (1.6)$$

by eliminating $\phi^*_n(z)$ from (1.3) and (1.4). The * operation is defined as follows: If

$$\pi(z) := \sum_{j=0}^{n} a_j z^j$$

then

$$\pi^*(z) := \sum_{j=0}^{n} \overline{a_j} z^{n-j}.$$

The coefficient of z^{n-1}, l_n can be obtained from the first order difference equation,

$$\frac{l_{n+1}}{k_{n+1}} = \frac{l_n}{k_n} + \overline{\left(\frac{\phi_n(0)}{k_n}\right)}\frac{\phi_{n+1}(0)}{k_{n+1}}, \qquad (1.7)$$

if we are able to find k_n and $\phi_n(0)$ from the weight. A simple calculation shows that the Toeplitz determinant is

$$\Delta_n[w] = \prod_{j=0}^{n-1}\frac{1}{2\pi k_j^2}. \qquad (1.8)$$

There are, as one may expect, very few cases where explicit formulae were found for the Toeplitz determinants. However, it is clear that in order to find information about the Toeplitz determinants. a possible approach is to find explicitly the $k_j s$.

Our first step in that attempt is the derivation of two fundamental identities valid for all $z \in \mathbb{CP}^1$. We first introduce the functions $A_n(z)$ and $B_n(z)$ given by

$$\begin{aligned} A_n(z) &= n\frac{k_{n-1}}{k_n} + i\frac{k_{n-1}}{\phi_n(0)}z\int_{|\xi|=1}\frac{v'(z)-v'(\xi)}{z-\xi}\phi_n(\xi)\overline{\phi^*_n(\xi)}w(\xi)d\xi\\ B_n(z) &= \frac{k_n}{k_{n-1}}\frac{A_n(z)}{z} - \frac{n}{z} - i\int_{|\xi|=1}\frac{v'(z)-v'(\xi)}{z-\xi}\phi_n(\xi)\overline{\phi_n(\xi)}w(\xi)d\xi,\end{aligned}$$

where $w(z) =: \exp(-v(z))$. Then we claim

$$B_n(z) + B_{n-1}(z) = \frac{k_{n-1}}{k_{n-2}}\frac{A_{n-1}(z)}{z} + \frac{k_n}{k_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}A_{n-1}(z) - \frac{n-1}{z} - v'(z), \qquad (S_1)$$

$$
\begin{aligned}
&(B_{n+1}(z)-B_n(z))(zk_{n+1}\phi_n(0)+k_n\phi_{n+1}(0))\\
&\quad = k_n\phi_n(0)A_{n+1}(z)-\frac{k_{n-1}^2}{k_{n-2}}\frac{\phi_{n-1}(0)\phi_{n+1}(0)}{\phi_n(0)}A_{n-1}(z)-k_{n+1}\phi_n(0). \qquad (S_2)
\end{aligned}
$$

The equations (S_1) and (S_2) are the circular analogues obtained earlier [5, 6] in the case where polynomials are orthogonal with respect to weights supported on the interval $[a,b]$. The functions $A_n(z)$ and $B_n(z)$ appear in the differentiation formula

$$
\left(\frac{d}{dz}+B_n(z)\right)\phi_n(z)=A_{n-1}(z)\phi_{n-1}(z) \tag{1.9}
$$

$$
\begin{aligned}
\Big(\frac{d}{dz}+B_{n-1}(z)-\frac{k_{n-1}}{k_{n-2}}\frac{A_n(z)}{z} &- \frac{k_n}{k_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}A_{n-1}(z)\Big)\phi_{n-1}(z)\\
&= \frac{k_n}{k_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}\frac{A_{n-1}(z)}{z}\phi_n(z), \qquad (1.10)
\end{aligned}
$$

which can be thought of as generalised "creation" and "annihilation" operators. Equations (1.9) and (1.10) can be derived [12] in a manner completely analogous that of the interval case.

The equation (S_1) is simply found by the orthogonality condition and the circular analogue of the Christoffel-Darboux formula. Let

$$
\Psi_n(z):=\begin{pmatrix}\phi_n(z)\\ \phi_{n-1}(z)\end{pmatrix}.
$$

Regarding (S_2), this is found by re-writing (1.9) and (1.10), and the recurrence relations as

$$
\begin{aligned}
\Psi_n'(z) &= M_n(z)\Psi_n(z)\\
\Psi_{n+1}(z) &= U_n(z)\Psi_n(z),
\end{aligned}
$$

and demanding that above over-determined systems are compatible; entirely analogues to what was done in [6]. It is now clear that if w'/w is a rational function then, (S_1) and (S_2) will supply the basic equations for the determination of k_n and $\phi_n(0)$. Furthermore, by eliminating $\phi_{n-1}(z)$ from (1.9) and (1.10), we arrive at a second order differential equation satisfy by $\phi_n(z)$:

$$
Y''(z,n)+P(z,n)Y'(z,n)+Q(z,n)Y(z,n)=0, \tag{1.11}
$$

where,

$$
P(z,n) = -\frac{n-1}{z}-v'(z)-\frac{A_n'(z)}{A_n(z)} \tag{1.12}
$$

$$
\begin{aligned}
Q(z,n) &= B_n'(z)-B_n(z)\frac{A_n'(z)}{A_n(z)}+B_n(z)B_{n-1}(z)-\frac{k_{n-1}}{k_{n-2}}\frac{A_{n-1}(z)}{z}B_n(z)\\
&- \frac{k_n}{k_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}A_{n-1}(z)B_n(z)+\frac{k_{n-1}}{k_{n-2}}\frac{\phi_{n-1}(0)}{\phi_n(0)}\frac{A_{n-1}(z)A_n(z)}{z}. \qquad (1.13)
\end{aligned}
$$

We now give one more version of (S_1) and (S_2) which will also prove to be useful in what follows. It turns out to be profitable to use the alternative parametrisation,

$$\begin{aligned} r_n &= \frac{\phi_n(0)}{k_n} \\ m_n &= \frac{k_n}{k_{n+1}}, \\ s_n &= \frac{r_{n+1}}{r_n}, \end{aligned}$$

and the equations (S_1) and (S_2) become,

$$B_{n+1}(z) + B_n(z) = \frac{1}{m_{n-1}}\frac{A_n(z)}{z} + \frac{A_n(z)}{m_{n-1}s_n} - \frac{n}{z} - v'(z). \tag{T_1}$$

$$(B_{n+1}(z) - B_n(z))(z + s_n) = m_n A_{n+1}(z) - \frac{s_n}{s_{n-1}}\frac{m_{n-1}^2}{m_{n-2}}A_{n-1}(z) - 1. \tag{T_2}$$

2 The Pure Fisher-Hartwig Symbol

Let us return to the pure Fisher-Hartwig symbol,

$$w_{\alpha,\beta}(z) = (1-z)^{\alpha+i\beta}(1-1/z)^{\alpha-i\beta}$$

where the arguments are chosen so that the first factor is analytic in the interior of the disc and one at $z = 0$ and that the second factor is analytic outside the circle and one at infinity. The Toeplitz determinants for this symbol have been previously computed exactly. This was done using a hypergeometric approach and a factorization of the corresponding Toeplitz operator discovered by Roch. See Theorem 6.20 in [3] for the factorization. An alternative direct approach was also done by Widom many years ago, but only recently in printed form [4]. We now give a difference equation derivation.

For our purposes we normalize w so that $w_0 = 1/2\pi$. Note also that v' is rational and that

$$\begin{aligned} v'(z) &= -\frac{2\alpha}{z-1} + \frac{\alpha - i\beta}{z} \\ \frac{v'(z) - v'(\xi)}{z - \xi} &= \frac{2\alpha}{(z-1)(\xi-1)} + \frac{i\beta - \alpha}{z\xi}, \end{aligned}$$

and thus

$$A_n(z) = n\frac{k_{n-1}}{k_n} + \frac{k_{n-1}}{\phi_n(0)}\left(2\alpha J_n - (\alpha + i\beta)I_n\right) + \frac{2\alpha\frac{k_{n-1}}{\phi_n(0)}(J_n - I_n)}{z-1} \tag{2.1}$$

$$B_n(z) = \frac{i\beta - \alpha - n}{z} + \frac{2\alpha(1-L_n)}{z-1} + \frac{k_n}{k_{n-1}}\frac{A_n(z)}{z}, \tag{2.2}$$

where

$$\begin{aligned} I_n &:= \int_{|\xi|=1} \phi_n(\xi)\overline{\phi_n^*(\xi)}\frac{w(\xi)}{i\xi}d\xi \\ J_n &:= \int_{|\xi|=1} \frac{\phi_n(\xi)\overline{\phi_n^*(\xi)}}{1-\xi}\frac{w(\xi)}{i\xi}d\xi \\ L_n &:= \int_{|\xi|=1} \frac{\phi_n(\xi)\overline{\phi_n(\xi)}}{1-\xi}\frac{w(\xi)}{i\xi}d\xi. \end{aligned}$$

Now we have 5 unknowns; I_n, J_n, L_n, k_n and $\phi_n(0)$. Notice that $A_n(z)$ is of the form $c_1+c_2/(z-1)$ and $B_n(z)$ is of the form $d_1/z+d_2/(z-1)$. We now show how five quite simple steps yield the difference equation for the $k_n s$.

Step 1. Take the limit $z\to\infty$ in (T_1).

Since $B_n(z)$ has no constant term the constant term of A_n must be zero. Thus we immediately know that

$$A_n(z) = \frac{2\alpha m_{n-1}(J_n - I_n)}{r_n(z-1)}$$

Step 2. Compare the coefficients of $1/z$ in (T_1).

A simple computation yields

$$i\beta - n - \alpha = \frac{2\alpha(J_n - I_n)}{r_n}$$

and this implies

$$A_n(z) = \frac{m_{n-1}(-\alpha + i\beta - n)}{(z-1)}.$$

This of course also gives information about B_n and at this point we can conclude that

$$B_n(z) = \frac{2\alpha(1-L_n) - \alpha + i\beta - n}{z-1}.$$

Step 3. Compare the coefficients of $1/(z-1)$ in (T_1).

From this we have

$$4\alpha - 2\alpha(L_{n+1}+L_n) - \alpha + i\beta - n - 1 = \frac{r_n}{r_{n+1}}(-\alpha + i\beta - n) + 2\alpha. \tag{2.3}$$

We will return to this equation in a moment.

Step 4. Take the limit $z\to\infty$ in (T_2).

This limit shows

$$-1 - 2\alpha(L_{n+1} - L_n) = -1$$

or that $L_{n+1} = L_n$. A direct computation shows that $L_0 = \frac{\alpha+i\beta}{2\alpha}$. Thus $B_n(z) = \frac{-n}{z-1}$ and returning to step three we see that we have an equation for r_n, namely,

$$-\alpha - i\beta - n - 1 = \frac{r_n}{r_{n+1}}(-\alpha + i\beta - n).$$

Step 5. Compare the coefficients of $1/(z-1)$ **in** (T_2).

We have that $s_n = \frac{r_{n+1}}{r_n}$ so this residue produces the equation

$$1 + \frac{r_{n+1}}{r_n} = m_n^2(\alpha - i\beta + n + 1) - \frac{r_{n+1}}{r_n}\frac{r_{n-1}}{r_n}m_{n-1}^2(\alpha + i\beta + n - 1)$$

and since we have an expression for the quotients of the $r_n s$ this becomes

$$2\alpha + 2n + 1 \;=\; m_n^2(\alpha + i\beta + n + 1)(\alpha - i\beta + n + 1) - m_{n-1}^2(\alpha + i\beta + n)(\alpha - i\beta + n), \tag{2.4}$$

We easily verify that,

$$m_n^2 = \frac{(n+1)(2\alpha + n + 1)}{(\alpha + i\beta + n + 1)(\alpha - i\beta + n + 1)}, \tag{2.5}$$

solves (2.9) with the initial condistions $k_{-1} = 0$ and $k_0 = 1$. From this it follows

$$k_n^2 \;=\; \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha - i\beta + 1)\Gamma(\alpha + i\beta + 1)}\frac{\Gamma(\alpha + i\beta + n + 1)}{\Gamma(n+1)}\frac{\Gamma(\alpha - i\beta + n + 1)}{\Gamma(n + 2\alpha + 1)} \tag{2.6}$$

$$|\phi_n(0)|^2 \;=\; \frac{\Gamma(2\alpha+1)}{\Gamma(n+1)\Gamma(n+2\alpha+1)}\frac{|\Gamma(\alpha + i\beta + n)|^2}{|\Gamma(\alpha + i\beta)|^2} \tag{2.7}$$

$$l_n \;=\; \frac{(\alpha - i\beta)n}{n + \alpha + i\beta}k_n. \tag{2.8}$$

Incidentally I_n and J_n can also be easily determined. In Step 1, the vanishing of the constant term in $A_n(z)$ implies

$$n + \frac{2\alpha J_n - (\alpha + i\beta)I_n}{r_n} = 0,$$

and when combined with the first equation in Step 2 gives

$$I_n \;=\; r_n$$

$$J_n \;=\; \frac{(\alpha + i\beta - n)r_n}{2\alpha}.$$

3 Toeplitz determinant and discriminant.

The computation of the Toeplitz determinant Δ_n, is now immediate

$$\begin{aligned}\Delta_n &= \prod_{j=0}^{n-1} \frac{1}{2\pi k_j^2} \\ &= C^n \frac{G(n+1)G(n+2\alpha+1)G(\alpha+i\beta+1)G(\alpha-i\beta+1)}{G(2\alpha+1)G(n+\alpha+i\beta+1)G(n+\alpha-i\beta+1)}, \qquad (3.1)\end{aligned}$$

where $G(z)$ is Barnes $G-$function. Let

$$\pi_n(z) = \gamma \prod_{j=1}^{n} (z - z_j(n)),$$

where $z_j(n)$ are the n (simple) zeros, then the discriminant is

$$D[\pi_n] = \gamma^{2n-2} \prod_{1\le j<k\le n} (z_j(n) - z_k(n))^2. \qquad (3.2)$$

Indeed (see (5.15) of [12])

$$\begin{aligned}D[\phi_n] &= (-1)^{n(n-1)/2} \frac{(\phi_n(0))^{n-1}}{k_n k_{n-1}^n} \prod_{j=1}^{n-1} k_j^2 \prod_{l=1}^{n} A_n(z_l(n)) \\ &= (-1)^{n(n-1)/2} \left(\frac{\phi_n(0)}{k_n}\right)^{n-1} \frac{1}{k_n^2} \left(\frac{k_n}{k_{n-1}}\right)^n \prod_{j=1}^{n-1} k_j^2 \prod_{l=1}^{n} A_n(z_l(n)) \\ &= (-1)^{n(n-1)/2} \frac{r_n^{n-1}}{m_{n-1}^n k_n^2} \prod_{j=1}^{n-1} k_j^2 \prod_{l=1}^{n} A_n(z_l(n)). \qquad (3.3)\end{aligned}$$

The only missing information required for the computation of $D[\phi_n]$ is

$$\prod_{l=1}^{n} A_n(z_l).$$

The derivative of $\phi_n(z)$ is (see (1.10)),

$$\phi_n'(z) = A_n(z)\phi_{n-1}(z) - B_n(z)\phi_n(z), \qquad (3.4)$$

where in our example,

$$\begin{aligned}A_n(z) &= \frac{a_n}{z-1}, \;\; a_n = m_{n-1}(i\beta - \alpha - n) \\ B_n(z) &= \frac{b_n}{z-1}, \;\; b_n = -n.\end{aligned}$$

Now

$$\prod_{l=1}^{n} A_n(z_l) = a_n^n \prod_{l=1}^{n} \frac{1}{z_l - 1} = a_n^n (-1)^n \frac{k_n}{\phi_n(1)},$$

from the fact that $\phi_n(z) = k_n \prod_{l=1}^{n}(z - z_l)$. To determine $\phi_n(1)$, evaluate both sides of (3.4) at $z = 1$; keeping in mind that the l.h.s. is regular at every finite z. Therefore the residue of the r.h.s. at $z = 1$ must vanish:

$$\begin{aligned}
\phi_n(1) &= \frac{a_n}{b_n}\phi_{n-1}(1) \\
&= \prod_{j=1}^{n} \frac{a_j}{b_j}, \\
\prod_{l=1}^{n} A_n(z_l) &= (-1)^n a_n^n \frac{k_n}{\phi_n(1)}, \\
D[\phi_n] &= (-1)^{n(n+1)/2}(i\beta - \alpha - n)^n \frac{r_n^{n-1}}{k_n} \left(\prod_{j=1}^{n-1} k_j^2\right) \left(\prod_{j=1}^{n} \frac{b_j}{a_j}\right) \\
&= (-1)^{n(n+3)/2} \left(\frac{n + \alpha - i\beta}{2\pi}\right)^n \frac{r_n^{n-1}}{k_n}\, \Delta_n^{-1} \left(\prod_{j=1}^{n} \frac{b_j}{a_j}\right) \qquad (3.5)
\end{aligned}$$

Now

$$\begin{aligned}
\prod_{j=1}^{n} \frac{b_j}{a_j} &= \sqrt{\prod_{j=1}^{n} \frac{\alpha + i\beta + j}{\alpha - i\beta + j}\, \frac{j}{j + 2\alpha}} \\
&= \sqrt{\frac{\Gamma(n+1+\alpha+i\beta)}{\Gamma(n+1+\alpha-i\beta)} \frac{\Gamma(\alpha-i\beta+1)}{\Gamma(\alpha+i\beta+1)} \frac{\Gamma(2\alpha+1)\Gamma(n+1)}{\Gamma(n+2\alpha+1)}}. \qquad (3.6)
\end{aligned}$$

Therefore,

$$\Delta_n k_n |D[\phi_n]| = \left(\frac{|n + \alpha - i\beta|}{2\pi}\right)^n |r_n|^{n-1} \left|\prod_{j=1}^{n} \frac{b_j}{a_j}\right|. \qquad (3.7)$$

Note that according to general theory [17],

$$\lim_{n\to\infty} k_n = \kappa > 0,$$

and in particular

$$\lim_{n\to\infty} \phi_n(0) = 0.$$

Indeed this is the case for the Fisher-Hartwig symbol.

To determine $|D[\phi_n]|$, as $n \to \infty$, we give here some preliminary results:

$$\begin{aligned}
\lim_{n\to\infty} k_n &= \frac{\sqrt{\Gamma(2\alpha+1)}}{|\Gamma(\alpha+i\beta+1)|} =: \kappa, \\
\phi_n(0) &\sim \kappa \frac{\Gamma(\alpha+i\beta+1)}{\Gamma(\alpha-i\beta)} n^{-1-2i\beta}, \\
|r_n|^{n-1} &\sim \left(\frac{|\alpha+i\beta|}{n}\right)^{n-1}, \\
\left|\prod_{j=1}^{n} \frac{b_j}{a_j}\right| &\sim \frac{\sqrt{\Gamma(2\alpha+1)}}{n^\alpha}, \\
\Delta_n &\sim \frac{|G(\alpha+i\beta+1)|^2}{G(2\alpha+1)} C^n n^{\alpha^2+\beta^2}.
\end{aligned}$$

Therefore,

$$|D[\phi_n]| \sim \frac{|\Gamma(\alpha+i\beta+1)|}{|\alpha+i\beta|} \frac{G(2\alpha+1)}{|G(\alpha+i\beta+1)|^2} \left(\frac{\Gamma(2\alpha+1)}{C|\alpha+i\beta||\Gamma(\alpha+i\beta)|^2}\right)^n n^{1-\alpha-\alpha^2-\beta^2}. \quad (3.8)$$

4 Differential Equation.

From the general theory $\phi_n(z)$ satisfies

$$Y''(z) + P(z,n)Y'(z) + Q(z,n)Y(z) = 0,$$

where

$$\begin{aligned}
P(z,n) &= \frac{1-n-\alpha+i\beta}{z} + \frac{2\alpha+1}{z-1}, \\
Q(z,n) &= -\frac{n(\alpha+i\beta+1)}{z(z-1)}.
\end{aligned} \quad (4.1)$$

The general solution of the differential equation is

$$\begin{aligned}
& A \quad {}_2F_1(-n, \alpha+i\beta+1; 1-n-\alpha+i\beta; z) \\
+ & B \quad z^{n+\alpha-i\beta}\, {}_2F_1(n+2\alpha+1, \alpha-i\beta; n+\alpha-i\beta+1; z),
\end{aligned} \quad (4.2)$$

and since we know that ϕ_n is a polynomial we have that

$$\phi_n(z) = A\, {}_2F_1(-n, \alpha+i\beta+1; 1-n-\alpha+i\beta; z)$$

where

$$A = \sqrt{\frac{\Gamma(2\alpha+1)\Gamma(\alpha+i\beta+n+1)\Gamma(\alpha-i\beta+n+1)}{\Gamma(\alpha-i\beta+a)\Gamma(\alpha+i\beta+1)\Gamma(n+1)\Gamma(n+2\alpha+1)}} \frac{\Gamma(n+\alpha-i\beta)\Gamma(\alpha+i\beta+1)}{\Gamma(n+\alpha+i\beta+1)\Gamma(\alpha-i\beta)}.$$

This value comes from computing the coefficient for the nth term of the Hypergeometric function and also using the value of k_n given in (2.6). Indeed this particular Hypergeometric function was found by Askey to be orthogonal with respect to the pure Fisher-Hartwig symbol in a commentary on Szegö's collected papers [1].

5 The weight $C\exp[t(z+1/z)/2]$.

For this example

$$\begin{aligned} v'(z) &= -\frac{t}{2} + \frac{t}{2z^2} \\ \frac{v'(z)-v'(\xi)}{z-\xi} &= -\frac{1}{z}\frac{t}{2\xi^2} - \frac{1}{z^2}\frac{t}{2\xi}. \end{aligned}$$

We find,

$$\begin{aligned} A_n(z) &= (n+b_n)m_{n-1} + \frac{m_{n-1}a_n}{z}. \\ B_n(z) &= \frac{L_n+b_n}{z} + \frac{a_n - t/2}{z^2} \\ a_n &= \frac{t}{2r_n}\int_{|\xi|=1} \frac{\phi_n(\xi)\overline{\phi_n^*(\xi)}}{i\xi} w(\xi)d\xi \\ b_n &= \frac{t}{2r_n}\int_{|\xi|=1} \frac{\phi_n(\xi)\overline{\phi_n^*(\xi)}}{i\xi^2} w(\xi)d\xi \\ L_n &= -\frac{t}{2}\int_{|\xi|=1} \frac{\phi_n(\xi)\overline{\phi_n(\xi)}}{i\xi^2} w(\xi)d\xi. \end{aligned}$$

There are 5 unknowns: r_n, m_n, a_n, b_n, and L_n. Again we use the same basic steps as was done in the previous example.

Step 1. Take the limit as $z \to \infty$ in (T_1).

$$b_n = -n - \frac{t}{2}s_n. \tag{5.1}$$

Step 2. Compare the coefficients of $1/z$ in (T_1).

$$L_{n+1} + L_n - \frac{t}{2}\left(s_{n+1} + \frac{2a_n/t}{s_n}\right) = n+1. \tag{5.2}$$

Step 3. Compare the coefficients of $1/z^2$ in (T_1).

$$\begin{aligned} a_{n+1} + a_n - t &= a_n - t/2 \\ a_n &= \frac{t}{2}. \end{aligned} \tag{5.3}$$

Step 4. Take the limit as $z \to \infty$ in (T_2). This limit yields after simplifying

$$L_{n+1} - \frac{t}{2}s_{n+1}(1-m_n^2) - \left(L_n - \frac{t}{2}s_n(1-m_{n-1}^2)\right) = 0$$

which implies,

$$L_n = \alpha + \frac{t}{2}s_n(1-m_{n-1}^2),$$

where α is an "integration" constant. To determine α put $n=0$ and note that $m_{-1}=0$ and $s_0 = r_1/r_0 = r_1 = -I_1(t)/I_0(t)$, where $I_j(t)$ is the $I-$Bessel function of order j. So,

$$\alpha = L_0 - \frac{t}{2}r_1.$$

But

$$\begin{aligned} L_0 &= -\frac{tC}{2}\int_{|z|=1}\frac{\exp(t(z+1/z)/2)}{iz^2}dz \\ &= Ct\int_0^{\pi}\mathrm{e}^{-t\cos\psi}\cos\psi d\psi \\ &= -\pi C t I_1(t) = -\frac{\pi t I_1(t)}{2\pi I_0(t)} = -\frac{t}{2}\frac{I_1(t)}{I_0(t)}. \end{aligned}$$

So $\alpha = 0$, and

$$L_n = \frac{t}{2}s_n(1-m_{n-1}^2). \tag{5.4}$$

Step 5. Compare the coefficients of $1/z$ in (T_2). Again after simplifying

$$L_{n+1} - \frac{t}{2}\left(s_{n+1} + \frac{m_n^2}{s_n}\right) - \left(L_n - \frac{t}{2}\left(s_n + \frac{m_{n-1}^2}{s_{n-1}}\right)\right) = 1,$$

which implies,

$$L_n - \frac{t}{2}\left(s_n + \frac{m_{n-1}^2}{s_{n-1}}\right) = \beta + n.$$

It turns out that the "integration" constant β is also 0. Therefore,

$$L_n = n + \frac{t}{2}\left(s_n + \frac{m_{n-1}^2}{s_{n-1}}\right). \tag{5.5}$$

Step 6. Compare coefficients of $1/z^2$ in (T_2) This gives $0=0$.

Eliminating L_n from (5.4) and (5.5) gives,

$$-\frac{2n}{t} = m_{n-1}^2\left(s_n + \frac{1}{s_{n-1}}\right). \tag{5.6}$$

Now substitute (5.5), (5.4) into (5.2) and note that $a_n = t/2$, we get after some simplification,

$$s_n^2 = \frac{1-m_n^2}{1-m_{n-1}^2}. \tag{5.7}$$

Now since $s_n = r_{n+1}/r_n$, we see that

$$\frac{r_{n+1}^2}{1-m_n^2} = \frac{r_n^2}{1-m_{n-1}^2} = \gamma = \text{constant}.$$

which implies,

$$m_{n-1}^2 = 1 - r_n^2, \tag{5.8}$$

where an easy computation shows that $\gamma = 1$. This last equation hold in general [9].

Using this on (5.6) gives the discrete Painleve II [11, 16]

$$r_{n+1} + r_{n-1} = -\frac{2n}{t}\,\frac{r_n}{1-r_n^2}. \tag{5.9}$$

6 Another non-linear difference equation for r_n.

In [13] the authors considered a symbol of the form

$$w(z) = \mathrm{e}^{tz}\prod_{\alpha=1}^{M}\left(\frac{z-z_\alpha}{z}\right)^{g_\alpha}, \quad \sum_{\alpha=1}^{M} g_\alpha = g > 0, \quad -1 < z_\alpha < 0, \quad \sum_{\alpha=1}^{M} g_\alpha z_\alpha = -1. \tag{6.1}$$

We have,

$$\begin{aligned}
v(z) &= -tz - \sum_\alpha g_\alpha\left(\ln(z-z_\alpha) - \ln z\right)\\
v'(z) &= -t - \sum_\alpha \frac{g_\alpha}{z-z_\alpha} + \frac{g}{z}\\
\frac{v'(z)-v'(\xi)}{z-\xi} &= -\frac{g}{z\xi} + \sum_\alpha \frac{g_\alpha}{z-z_\alpha}\,\frac{1}{\xi-z_\alpha}.
\end{aligned}$$

$$A_n(z) = (n+a_n)m_{n-1} + m_{n-1}\sum_\alpha \frac{b_n(\alpha)}{z-z_\alpha}$$

$$
\begin{aligned}
a_n &= \frac{1}{r_n}\int_{|\xi|=1}\left(\frac{g}{\xi}-\sum_\alpha \frac{g_\alpha}{\xi - z_\alpha}\right)\phi_n(\xi)\overline{\phi_n^*(\xi)}\frac{w(\xi)}{i}d\xi \\
b_n(\alpha) &= -\frac{g_\alpha z_\alpha}{r_n}\int_{|\xi|=1}\frac{\phi_n(\xi)\overline{\phi_n^*(\xi)}}{\xi - z_\alpha}\frac{w(\xi)}{i}d\xi. \\
B_n(z) &= \frac{A_n(z)}{m_{n-1}z} - \frac{n+g}{z} + \sum_\alpha \frac{g_\alpha + c_n(\alpha)}{z - z_\alpha} \\
c_n(\alpha) &= g_\alpha z_\alpha \int_{|\xi|=1}\frac{\phi_n(\xi)\overline{\phi_n(\xi)}}{\xi - z_\alpha}\frac{w(\xi)}{i\xi}d\xi.
\end{aligned}
$$

Step 1. $z \to \infty$ in (T_1) :

$$
a_n + n = -\frac{r_{n+1}}{r_n}t = -s_n t. \tag{6.2}
$$

Step 2. Residues at $z = 0$ of (T_1) :

$$
a_n = g + \sum_\alpha \frac{b_n(\alpha)}{z_\alpha}. \tag{6.3}
$$

Step 3. Residues at $z = z_\alpha$ of (T_1) :

$$
g_\alpha + c_{n+1}(\alpha) + c_n(\alpha) + \frac{b_{n+1}(\alpha)}{z_\alpha} = \frac{b_n(\alpha)}{s_n}. \tag{6.4}
$$

Step 4. $z \to \infty$ in (T_2) :

$$
1 + a_{n+1} - a_n + \sum_\alpha (c_{n+1}(\alpha) - c_n(\alpha)) = m_n^2(n+1+a_{n+1}) - \frac{s_n}{s_{n-1}}m_{n-1}^2(n-1+a_{n-1})
$$

which is simplifies to:

$$
tr_n^2 s_n - tr_{n+1}^2 + \sum_\alpha (c_{n+1}(\alpha) - c_n(\alpha)) = 0
$$

using (6.2) and $m_{n-1}^2 = 1 - r_n^2$. "Integrating" the above gives:

$$
tr_n r_{n+1} - \sum_\alpha c_n(\alpha) = \lambda(\alpha), \tag{6.5}
$$

where $\lambda(\alpha)$ is the $n-$ independent integration constant and depends on t and the set of $z_\alpha' s$.
Step 5. Residues at $z = 0$ of (T_2) :

$$
a_{n+1} - a_n = \sum_\alpha \frac{b_{n+1}(\alpha) - b_n(\alpha)}{z_\alpha},
$$

and upon "integration" gives,

$$a_n - \sum_\alpha \frac{b_n(\alpha)}{z_\alpha} = d(\alpha) = g, \tag{6.6}$$

the same as (6.3).

Step 6. Residues at $z = z_\alpha$ of (T_2) :

$$(z_\alpha + s_n)\left(c_{n+1}(\alpha) - c_n(\alpha) + \frac{b_{n+1}(\alpha) - b_n(\alpha)}{z_\alpha}\right) = m_n^2 b_{n+1}(\alpha) - \frac{s_n}{s_{n-1}} m_{n-1}^2 b_{n-1}(\alpha). \tag{6.7}$$

Let's see what all this means when there is just one z_α.

If we use Step 3, along with (6.5) and (6.6) we have that

$$g + tr_{n+2}r_{n+1} + \lambda + tr_n r_{n+1} + \lambda - g - s_{n+1}t - n - 1 = \frac{z_\alpha}{s_n}(-g - s_n t - n),$$

or, after simplifying,

$$tr_{n+1}r_n + +tr_n r_{n+1} + 2\lambda - s_{n+1}t - n - 1 = 1/s_n - tz_\alpha - nz_\alpha/s_n,$$

A little more algebra yields

$$r_{n+2} + r_n = -\frac{(nr_n - t)z_\alpha - 2\lambda + (n+1)}{t(1 - r_{n+1}^2)}$$

an equation very much like the one found for the previous example.

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$$\int_{-\infty}^{\infty} {}_2F_1\left(-n, x; x + y - 1; \frac{a+b}{a+it}\right) {}_2F_1\left(-m, y, x + y - 1; \frac{a+b}{b-it}\right) d\mu(t) = 0,$$

where

$$d\mu(t) := \frac{dt}{(a+it)^x(b-it)^y},$$

for $m \neq n$, $x + y > 1$ and $a > 0,\ b > 0$.

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