COISOTROPIC EMBEDDINGS IN POISSON MANIFOLDS

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Abstract. We consider existence and uniqueness of two kinds of coisotropic embeddings and deduce the existence of deformation quantizations of certain Poisson algebras of basic functions. First we show that any submanifold of a Poisson manifold satisfying a certain constant rank condition, already considered by Calvo and Falceto [4], sits coisotropically inside some larger cosymplectic submanifold, which is naturally endowed with a Poisson structure. Then we give conditions under which a Dirac manifold can be embedded coisotropically in a Poisson manifold, extending a classical theorem of Gotay.

Contents

1. Introduction

The following two results in symplectic geometry are well known. First: a submanifold C of a symplectic manifold (M, Ω) is contained coisotropically in some symplectic submanifold of M iff the pullback of Ω to C has constant rank. Second: a manifold endowed with a closed 2-form ω can be embedded coisotropically into a symplectic manifold (M, Ω) so that $i^*\Omega = \omega$ (where i is the embedding) iff ω has constant rank [14]. In this work we extend these results to the setting of Poisson geometry.

Part of the motivation comes from applications of the Poisson sigma model to quantization problems. The Poisson sigma model is a topological field theory, whose fields are bundle maps from $T\Sigma$ (for Σ a surface) to the cotangent bundle T^*P of a Poisson manifold (P, Π). It was used by Felder and the first author [8] to derive and interpret Kontsevich's formality theorem and his star product on the Poisson manifold P. The Poisson sigma model with boundary conditions on a coisotropic[1] submanifold C, when suitable assumptions on C are satisfied and P is assumed to be an open subset of $\mathbb{R}^n$, provides [9] a deformation quantization of the Poisson algebra of basic (invariant) functions $C^\infty_{bas}(C)$ on C, i.e. of the functions on C whose differentials annihilate the distribution $\sharp N^*C$. This result was globalized using a supergeometric version of Kontsevich's formality theorem [10]: when the first and second cohomology of the Lie algebroid N^*C vanish, for C a coisotropic submanifold of any Poisson manifold P, the Poisson algebra $C^\infty_{bas}(C)$ admits a deformation quantization. Notice that the quotient of C by the distribution $\sharp N^*C$ is usually not a smooth manifold. Hence $C^\infty_{bas}(C)$ is usually not the algebra of functions on any Poisson manifold, and one can not apply directly Kontsevich's theorem [15] on deformation quantization of Poisson manifolds.

Calvo and Falceto considered the Poisson sigma model with boundary conditions on submanifolds of (P,Π) more general than the coisotropic ones, namely submanifolds C for which $TC + \sharp N^*C$ has constant rank. They call this condition "strong regularity", and we will refer to such submanifolds as *pre-Poisson*. Calvo and Falceto show [5] that when P is an open subset of $\mathbb{R}^n$ the problem of deformation quantizing the Poisson algebra of basic functions on C can be reduced to the results of [9], hence the algebra of basic functions can be deformation quantized when assumptions analog to those of [9] are satisfied. The computations in [5] are carried out choosing local coordinates on P adapted to C. The strong regularity condition allows to choose locally constraints for C such that the number of first class constraints (X^μ whose Poisson bracket with all other constrains vanish on C) and second class constraints (the remaining constraints X^A, which automatically satisfy $det\{X^A, X^B\} \neq 0$ on C) be constant along C. Setting the second class constraints X^A to zero locally gives a submanifold with an induced Poisson structure, and the fact that only first class constraints are left means that C lies in it as a coisotropic submanifold.

In the first part of this paper (sections 3- 6) we consider the natural question of whether one can carry out the above procedure globally, i.e. without resorting to adapted coordinates. More generally we ask the following question:

- Given an arbitrary submanifold C of a Poisson manifold (P,Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that
  a) $\tilde{P}$ has a Poisson structure induced from Π
  b) C is a coisotropic submanifold of $\tilde{P}$?

  When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence, (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

We answer the question giving sufficient conditions and necessary conditions in section 5, where we also provide examples. In particular, we answer the above globalization question positively for pre-Poisson submanifolds in section 3: for any pre-Poisson submanifold C of a Poisson manifold P there is a submanifold $\tilde{P}$ which is cosymplectic (and hence has a canonically induced Poisson structure) such that C lies coisotropically in $\tilde{P}$. Further (section 4) this cosymplectic submanifold is unique up to neighborhood equivalence. Then in section 6 we deduce statements about the algebra of basic functions on C and its deformation quantization: if C is a pre-Poisson submanifold so that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish, then the Poisson algebra of basic functions on C admits a deformation quantization. Finally in section 7, assuming that the symplectic

[1] Recall that a submanifold C of P is coisotropic if $\sharp N^*C \subset TC$, where $N^*C = TC^\circ$ is the conormal bundle of C and $\sharp\colon T^*P \to TP$ is contraction with the bivector Π.

groupoid $\Gamma_s(P)$ of P exists, we describe two subgroupoids (an isotropic and a presymplectic one) naturally associated to a pre-Poisson submanifold C of P.

The second part of this paper (sections 8 and 9) deals with a different embedding problem, where we start with an abstract manifold instead of a submanifold of some Poisson manifold. The question we ask is:

- Let (M, L) be a Dirac manifold. Is there an embedding $i\colon (M, L) \to (P, \Pi)$ into a Poisson manifold such that
  a) $i(M)$ is a coisotropic submanifold of P
  b) the Dirac structure L is induced by the Poisson structure Π?
  Is such embedding unique up to neighborhood equivalence?

In the symplectic setting it is a classical theorem of Gotay [14] that both existence and uniqueness hold. One motivation for this question is the deformation quantization of the Poisson algebra of so-called admissible functions on (M, L), for a coisotropic embedding as above allows to reduce the problem to [10], i.e. to the deformation quantization of the basic functions on a coisotropic submanifold of a Poisson manifold.

It turns out (section 8) that the above question admits a positive answer iff the distribution $L \cap TM$ on the Dirac manifold M is regular. In that case one expects the Poisson manifold $\tilde{P}$ to be unique (up to a Poisson diffeomorphism fixing M), provided $\tilde{P}$ has minimal dimension. We are not able to prove this global uniqueness; we can just show in section 9 that the Poisson vector bundle $T\tilde{P}|_M$ is unique (an infinitesimal statement along M) and that around each point of M a small neighborhood of $\tilde{P}$ is unique (a local statement). We remark that A. Wade [18] has been considering similar questions too. Our result about deformation quantization is the following (Thm. 8.5): let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. If the first and second foliated de Rham cohomologies of the foliation $\mathcal{F}$ vanish then the Poisson algebra of admissible functions on (M, L) has a deformation quantization. In Remark 8.6 we also notice that the foliated de Rham cohomology $\Omega^\bullet_{\mathcal{F}}(M)$ admits the structure of an L_∞-algebra, generalizing a result of Oh and Park in the presymplectic setting (Thm. 9.4 of [16]).

Conventions: We use the term "presymplectic manifold" to denote a manifold endowed with a closed 2-form of *constant rank*, i.e. such that its kernel have constant rank. However we stick to the denominations "presymplectic groupoid" coined in [2] and "presymplectic leaves" (of a Dirac manifold) despite the fact that the 2-forms on these objects do not have constant rank, for these denominations seem to be established in the literature.

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2. Basic definitions

We will use some notions from Dirac linear algebra [12] [3]. A Dirac structure on a vector space P is a subspace $L \subset P \oplus P^*$ which is maximal isotropic w.r.t. the natural symmetric inner product on $P \oplus P^*$ (i.e. L is isotropic and has same dimension as P). A Dirac structure L specifies a subspace $\mathcal{O}$, defined as the image of L under the projection $P \oplus P^* \to P$, and a skew-symmetric bilinear form ω on $\mathcal{O}$, given by $\omega(X_1, X_2) = \langle \xi_1, X_2 \rangle$ where ξ_1 is any element of P^* such that $(X_1, \xi_1) \in L$. The kernel of ω (which in terms of L is given as $L \cap P$) is called *characteristic subspace*. Conversely, any choice of bilinear form defined on a subspace of P determines a Dirac structure on P. Given this equivalence, we will sometimes work with the bilinear form ω on $\mathcal{O}$ instead of working with L.

We consider now Poisson vector spaces (P, Π) (i.e. $\Pi \in \wedge^2 P$; we denote by $\sharp \colon P^* \to P$ the map induced by contraction with Π). The Poisson structure on P is encoded by the Dirac structure $L_P = \{(\sharp\xi, \xi) \colon \xi \in P^*\}$. The image of L_P under the projection onto the first factor is $\mathcal{O} = \sharp P^*$, and the bilinear form ω is non-degenerate.

Remark 2.1. We recall that any subspace W of a Dirac vector space (P, L) has an induced Dirac structure L_W; the bilinear form characterizing L_W is just the pullback of ω (hence it is defined on $W \cap \mathcal{O}$). When (P, Π) is actually a Poisson vector space, one shows[2] that the symplectic orthogonal of $W \cap \mathcal{O}$ in $(\mathcal{O}, \omega)$ is $\sharp W^\circ$. Hence $\sharp W^\circ \cap W$ is the kernel of the restriction of ω to $W \cap \mathcal{O}$, i.e. it is the characteristic subspace of the Dirac structure L_W, and we will refer to it as the *characteristic subspace of* W. Notice that pulling back Dirac structure is functorial [3] (i.e. if W is contained in some other subspace W' of P, pulling back L first to W' and then to W gives the Dirac structure L_W), hence L_W, along with the corresponding bilinear form and characteristic subspace, is *intrinsic* to W.

Let W be a subspace of the Poisson vector space (P, Π). W is called *coisotropic* if $\sharp W^\circ \subset W$, which by the above means that $W \cap \mathcal{O}$ is coisotropic in $(\mathcal{O}, \omega)$.

W is called *Poisson-Dirac* subspace [13] when $\sharp W^\circ \cap W = \{0\}$; equivalent conditions are that $W \cap \mathcal{O}$ be a symplectic subspace of $(\mathcal{O}, \omega)$ or that the pullback Dirac structure L_P correspond to a Poisson bivector on W.

W is called *cosymplectic* subspace if $\sharp W^\circ \oplus W = P$, or equivalently if the pushforward of Π via the projection $P \to P/W$ is an invertible bivector. Notice that if W is cosymplectic then it has a canonical complement $\sharp W^\circ$ which is a symplectic subspace of $(\mathcal{O}, \omega)$. Clearly a cosymplectic subspace is automatically a Poisson-Dirac subspace, and the Poisson bivector on W can be expressed in a particularly simple way [13]: its sharp map $\sharp_W \colon W^* \to W$ is given by $\sharp_W \tilde{\xi} = \sharp \xi$, where $\xi \in P^*$ is the extension of $\tilde{\xi}$ which annihilates $\sharp W^\circ$.

Now we pass to the global definitions. A Dirac structure on P is a maximal isotropic subbundle $L \subset TP \oplus T^*P$ which is closed under the so-called Courant bracket (see [12]). The image of L under the projection onto the first factor is an integrable singular distribution, whose leaves (which are called presymplectic leaves) are endowed with closed 2-forms. A Poisson structure on P is a bivector Π such that $[\Pi, \Pi] = 0$. Coisotropic and cosymplectic submanifolds of a Poisson manifold are defined exactly as in the linear case; a Poisson-Dirac

[2] Indeed, writing vectors in $\mathcal{O}$ as $\sharp\xi$ for some $\xi \in P^*$, one sees that the symplectic orthogonal of $W \cap \mathcal{O}$ is $\sharp(W \cap \mathcal{O})^\circ$. This space coincides with $\sharp W^\circ$ because their respective annihilators $\sharp^{-1}(W \cap \mathcal{O})$ and $\sharp^{-1} W$ coincide.

submanifold additionally requires that the bivector induced on the submanifold by the pointwise condition be smooth [13]. Cosymplectic submanifolds are automatically Poisson-Dirac submanifolds[3]. The Poisson bracket on a cosymplectic submanifold $\tilde{P}$ of (P,Π) is computed as follows: $\{\tilde{f}_2,\tilde{f}_2\}_{\tilde{P}}$ is the restriction to $\tilde{P}$ of $\{f_1,f_2\}$, where the f_i are extensions of $\tilde{f}_i$ to P such that $df_i|_{\sharp N^*\tilde{P}}=0$ (for at least one of the two functions).

We will also need a definition which does not have a linear algebra counterpart.

Definition 2.2. A submanifold C of a Poisson manifold (P,Π) is called *pre-Poisson* if the rank of $TC+\sharp N^*C$ is constant along C.

Calvo and Falceto already considered [4][5] such submanifolds and called them "strongly regular submanifolds". We prefer to call them "pre-Poisson" because when P is a symplectic manifold they reduce to presymplectic submanifolds[4]. See Section 5 for several examples.

3. Existence of coisotropic embeddings for pre-Poisson submanifolds

In this section we consider the problem of embedding a submanifold of a Poisson manifold coisotropically in a Poisson-Dirac submanifold, and show that this can be always done for pre-Poisson submanifolds.

We start with some linear algebra. Given a subspace C of (P,Π), we want to determine the subspaces $\tilde{P}$ such that $\tilde{P}\subset(P,\Pi)$ is Poisson-Dirac and $C\subset(\tilde{P},L_{\tilde{P}})$ is coisotropic. We want to use the characterization given in section 2 of Poisson-Dirac and coisotropic subspaces in terms of the corresponding bilinear forms, hence we need a statement about symplectic vector spaces.

Lemma 3.1. *Let $(\mathcal{O},\omega)$ be a symplectic vector space and D any subspace. Then the symplectic subspaces of $\mathcal{O}$ in which D sits coisotropically are exactly those of the form $R'\oplus D$, where R' is such that $R'\oplus(D+D^\omega)=\mathcal{O}$. Here D^ω denotes the symplectic orthogonal to D.*

Proof. First we show that a subspace $R'\oplus D$ as above is symplectic. Notice that $R\cap(D+D^\omega)=\{0\}$ implies that $(R'\oplus D)\cap(D+D^\omega)$ is contained in (hence equal to) $D\cap(D+D^\omega)=D$. Hence

$$(R'\oplus D)\cap(R'\oplus D)^\omega=((R'\oplus D)\cap D^\omega)\cap R'^\omega=(D\cap D^\omega)\cap R'^\omega,$$

and this is zero because its symplectic orthogonal is $D^\omega+D+R'$, which we assumed to be the whole of $\mathcal{O}$. Next we show that D is coisotropic in $R'\oplus D$: $D\cap D^\omega$ is surely contained in the symplectic orthogonal of D in $R'\oplus D$, and by dimension counting we see that it is the whole symplectic orthogonal.

Conversely let us consider a symplectic subspace of $\mathcal{O}$ in which D sits coisotropically; we write this subspace as $R'\oplus D$ for some R'. By the coisotropicity condition $(R'\oplus D)\cap D^\omega$,

[3]Indeed the bivector induced on a cosymplectic submanifold $\tilde{P}$ is always smooth: denote by L_P the Dirac structure corresponding to the Poisson structure on P and by $L_{\tilde{P}}$ its pullback to $\tilde{P}$. $L_{\tilde{P}}\cong L_P\cap(T\tilde{P}\oplus T^*P)/L_P\cap(\{0\}\oplus N^*\tilde{P})$ (see [12]), and both numerator and denominator have constant rank because $L_P\cap(\{0\}\oplus N^*\tilde{P})=ker\sharp|_{N^*\tilde{P}}=\{0\}$.

[4]Further reasons are the following: the subgroupoid associated to a pre-Poisson manifold, when it exists, is presymplectic (see Prop. 7.5). The Hamiltonian version of the Poisson Sigma Model with boundary conditions on P (at $t=0$) and on a submanifold C (at $t=1$) delivers a space of solutions which is presymplectic iff C is pre-Poisson.

the symplectic orthogonal of D in $R' \oplus D$, is contained in D. This has two consequences: first, using the fact that $R' \oplus D$ is a symplectic subspace,

$$\{0\} = ((R' \oplus D) \cap D^{\omega}) \cap R'^{\omega} = (D \cap D^{\omega}) \cap R'^{\omega},$$

which taking symplectic orthogonals gives $\mathcal{O} = R' + (D + D^{\omega})$. Second, this last sum is direct because $\dim R' = \dim(D \cap D^{\omega}) = codim(D + D^{\omega})$. □

Lemma 3.2. *Let (P, Π) be a Poisson vector space and C a subspace. The Poisson-Dirac subspaces of P in which C sits coisotropically are exactly those of the form $R \oplus C$, where R is such that*

$$R \oplus (C + \sharp C^{\circ}) \supset \mathcal{O}, \tag{1}$$

where $\mathcal{O} = \sharp P^*$. Among the Poisson-Dirac subspaces above the cosymplectic ones are exactly those of maximal dimension, i.e. those for which $R \oplus (C + \sharp C^{\circ}) = P$.

Proof. Notice that the symplectic subspaces determined in Lemma 3.1 can be described (without making a choice of complement to D) as those whose sum with $D + D^{\omega}$ is the whole of $\mathcal{O}$ and whose intersection with $D + D^{\omega}$ is D. Hence[5] the Poisson-Dirac subspaces $\tilde{P}$ of P that contain C as a coisotropic subspace are characterized by

$$(\tilde{P} \cap \mathcal{O}) + ((C \cap \mathcal{O}) + \sharp C^{\circ}) \equiv (\tilde{P} \cap \mathcal{O}) + \sharp C^{\circ} \stackrel{!}{=} \mathcal{O}. \tag{2}$$

$$(\tilde{P} \cap \mathcal{O}) \cap ((C \cap \mathcal{O}) + \sharp C^{\circ}) \equiv \tilde{P} \cap ((C \cap \mathcal{O}) + \sharp C^{\circ}) \stackrel{!}{=} C \cap \mathcal{O} \tag{3}$$

The equality (2) is really $(\tilde{P} \cap \mathcal{O}) + \sharp C^{\circ} \supset \mathcal{O}$ and is equivalent[6] to $\tilde{P} + \sharp C^{\circ} \supset \mathcal{O}$. For any choice of splitting $\tilde{P} = R \oplus C$ this just means $R + (C + \sharp C^{\circ}) \supset \mathcal{O}$.

The equality (3) is really $\tilde{P} \cap ((C \cap \mathcal{O}) + \sharp C^{\circ}) \subset C \cap \mathcal{O}$ and is equivalent[7] to $\tilde{P} \cap (C + \sharp C^{\circ}) \subset C$. For any choice of splitting $\tilde{P} = R \oplus C$ this inclusion means[8] to $R \cap (C + \sharp C^{\circ}) = \{0\}$. This proves the first part of the Lemma.

Now let $\tilde{P} = R \oplus C$ satisfy eq. (1); in particular $\tilde{P}$ is Poisson-Dirac. By dimension counting $\tilde{P}$ is cosymplectic iff the restriction of $\sharp$ to $\tilde{P}^{\circ}$ is injective, i.e. iff $\tilde{P}^{\circ} \cap \mathcal{O}^{\circ} = \{0\}$ or $\tilde{P} + \mathcal{O} = P$. This is equivalent to $\tilde{P} + \sharp C^{\circ} = P$: the direction "⇒" follows using eq. (1), the reverse direction simply because $\sharp C^{\circ} \subset \mathcal{O}$. □

Now we pass from linear algebra to global geometry. Given a submanifold C of a Poisson manifold P, one might try to construct a Poisson-Dirac submanifold in which C embeds coisotropically applying the corresponding symplectic contruction "leaf by leaf" in a smooth way. In view of Lemma 3.1 it would be then natural to require that the characteristic "distribution" $TC \cap \sharp N^*C$ of C have constant rank. However this approach generally does not work because even when it has constant rank $TC \cap \sharp N^*C$ might not be smooth (see example 5.4). Lemma 3.2 suggests instead to require that C be pre-Poisson and extend C *not* only "along the symplectic leaves of P".

[5]Here we use the characterization of subspaces of (P, Π) in terms of their intersections with $\mathcal{O}$, see section 2.

[6]The direction "⇒" is clear. The other direction follows because if $v \in \mathcal{O}$ is written as the sum of an element $v_{\tilde{P}}$ of $\tilde{P}$ and an element $v_{\sharp C^{\circ}}$ of $\sharp C^{\circ}$, then $v_{\tilde{P}} = v - v_{\sharp C^{\circ}} \in \mathcal{O}$.

[7]The "⇐" direction follows because the r.h.s. implies that $\tilde{P} \cap ((C \cap \mathcal{O}) + \sharp C^{\circ})$ is contained in C, and it is clearly contained in $\mathcal{O}$ too. For the direction "⇒", write an element v of $\tilde{P} \cap (C + \sharp C^{\circ})$ as the sum of an element v_C of C and an element $v_{\sharp C^{\circ}}$ of $\sharp C^{\circ}$; because of $C \subset \tilde{P}$ we have $v_{\sharp C^{\circ}} = v - v_C \in \tilde{P}$, so using the l.h.s. (i.e. eq. (3)) we get $v_{\sharp C^{\circ}} \in C \cap \mathcal{O}$. Hence v, as the sum of two elements of C, lies in C.

[8]The direction "⇒" is clear. The other direction follows by taking a vector $v \in (R \oplus C) \cap (C + \sharp C^{\circ})$ and writing it as $v_R \oplus v_C$. Then $v_R \in C + \sharp C^{\circ}$, so it follows by the r.h.s. that $v_R = 0$, hence $v = v_C \in C$.

Theorem 3.3. *Let C be a pre-Poisson submanifold of a Poisson manifold (P,Π). Then there exists a cosymplectic submanifold $\tilde{P}$ containing C such that C is coisotropic in $\tilde{P}$.*

Proof. Because of the rank condition on C we can choose a smooth subbundle R of $TP|_C$ which is a complement to $TC + \sharp N^*C$. Then by Lemma 3.2 at every point p of C we have that $T_pC \oplus R_p$ is a cosymplectic subspace of T_pP in which T_pC sits coisotropically. "Thicken" C to a smooth submanifold $\tilde{P}$ of P satisfying $T\tilde{P}|_C = TC \oplus R$. If we can show that, in a neighborhood of C, $\tilde{P}$ is a cosymplectic submanifold, then we are done.

First we show that at points p near C the restriction of $\sharp$ to $N_p^*\tilde{P}$ is injective. By the proof of Lemma 3.2 we know that this is equivalent to $T_p\mathcal{O}_p + T_p\tilde{P} = T_pP$ (where $\mathcal{O}_p$ the symplectic leaf of P through p) and that it is true if p belongs to C. The case $p \notin C$ is reduced to this using Weinstein's local structure theorem [19] which states that, near any $q \in C$, P is isomorphic (as a Poisson manifold) to the product of the symplectic leaf $\mathcal{O}_q$ and a Poisson manifold whose bivector vanishes at q. Under this isomorphism $T_q\mathcal{O}_q$ can be identified with a subspace of $T_p\mathcal{O}_p$, hence from $T\mathcal{O}_q + T\tilde{P} = TP$ at q we deduce $T\mathcal{O}_p + T\tilde{P} = TP$ at p. So we showed that the restriction of $\sharp$ to $N^*\tilde{P}$ is injective, hence $\sharp N^*\tilde{P}$ is a smooth constant rank subbundle of TP. The rank of $T\tilde{P} \cap \sharp N^*\tilde{P}$, which is the intersection of two smooth subbundles, can locally only decrease, and since it is zero along C it is zero also in a neighborhood of C. By dimension counting we deduce $T\tilde{P} \oplus \sharp N^*\tilde{P} = TP$, i.e. $\tilde{P}$ is cosymplectic. □

Remark 3.4. The above proposition says that if C is pre-Poisson then we can choose a subbundle R over C with fibers as in eq. (1) and "extend" C in direction of R to obtain a Poisson-Dirac submanifold of P containing C coisotropically. If C is not a pre-Poisson submanifold of (P,Π), we might still be able to find a smooth bundle R over C consisting of subspaces as in eq. (1). However "extending" C in direction of this subbundle will usually not give a submanifold with a smooth Poisson-Dirac structure, see Example 5.7 below.

Corollary 3.5. *Let $C, \tilde{P}$ be as in Thm. 3.3. The map $T^*\tilde{P} \to T^*P$ given by the splitting $T\tilde{P} \oplus \sharp N^*\tilde{P} = TP$ is a Lie algebroid map. Further $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$.*

Proof. Recall that a Lie-Dirac submanifold of a Poisson manifold P is one for which there exists a subbundle E containing $\sharp N^*M$ such that $E \oplus TM = TP$ and such that the induced map $T^*M \to T^*P$ be a Lie algebroid map. By Cor. 2.11 of [20] any cosymplectic submanifold $\tilde{P}$ is automatically Lie-Dirac with $E = \sharp N^*\tilde{P}$.

To prove $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$ we notice that the inclusion "$\supset$" is obvious because $C \subset \tilde{P}$. The other inclusion follows by dimension counting or by the following argument: write any $\xi \in N^*C$ uniquely as $\xi_1 + \xi_2$ where ξ_1 annihilates $\sharp N^*\tilde{P}$ and ξ_2 annihilates $T\tilde{P}$. Then $\sharp\xi_1 = \tilde{\sharp}(\xi_1|_{T\tilde{P}}) \in TC$, where $\tilde{\sharp}$ denotes the sharp map of $\tilde{P}$, since C is coisotropic in $\tilde{P}$. Hence $\sharp\xi = \sharp\xi_1 + \sharp\xi_2 \in TC \oplus \sharp N^*\tilde{P}$, and "$\subset$" follows. Finally, we have a direct sum in $TC \oplus \sharp N^*\tilde{P}$ because $\sharp N^*\tilde{P} \cap T\tilde{P} = \{0\}$ and $C \subset \tilde{P}$. □

Now we deduce consequences about Lie algebroids. See section 7 for the corresponding integrated statement.

Proposition 3.6. *Let C be a submanifold of a Poisson manifold (P,Π). Then $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P iff C is pre-Poisson. Further, for any cosymplectic submanifold $\tilde{P}$ in which C sits coisotropically, $N^*C \cap \sharp^{-1}TC$ is isomorphic as a Lie algebroid to the annihilator of C in $\tilde{P}$.*

Proof. At every point $N^*C \cap \sharp^{-1}TC$ is the annihilator of $TC + \sharp N^*C$, so it is a vector bundle iff C is pre-Poisson. So assume that C be pre-Poisson. We saw in Corollary 3.5 that for any cosymplectic submanifold $\tilde{P}$ constructed as in Thm. 3.3, the natural Lie algebroid embedding $T^*\tilde{P} \to T^*P$ is obtained by extending a covector in $T^*\tilde{P}$ so that it annihilates $\sharp N^*\tilde{P}$. By the same corollary $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$. Hence $N^*_{\tilde{P}}C$, the conormal bundle of C in $\tilde{P}$, is mapped *isomorphically* onto $(TC \oplus \sharp N^*\tilde{P})^\circ = (TC + \sharp N^*C)^\circ = N^*C \cap \sharp^{-1}TC$. Since $N^*_{\tilde{P}}C$ is a Lie subalgebroid of $T^*\tilde{P}$ [7], we are done. □

Remark 3.7. The fact that $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid if C is pre-Poisson can also be deduced as follows. The Lie algebra $(\mathcal{F} \cap I)/I^2$ forms a Lie-Rinehart algebra over the commutative algebra $C^\infty(P)/I$, where I is the vanishing ideal of C and $\mathcal{F}$ its Poisson-normalizer in $C^\infty(P)$. Lemma 1 of [4] states that C being pre-Poisson is equivalent to $N^*C \cap \sharp^{-1}TC$ being spanned by differentials of functions in $\mathcal{F} \cap I$. From this one deduces easily that $(\mathcal{F} \cap I)/I^2$ is identified with the sections of $N^*C \cap \sharp^{-1}TC$, and since $C^\infty(P)/I$ are just the smooth functions on C we deduce that $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid over C.

4. Uniqueness of coisotropic embeddings for pre-Poisson submanifolds

Given a submanifold C of a Poisson manifold (P, Π) in this section we investigate the uniqueness (up Poisson diffeomorphisms fixing C) of *cosymplectic* submanifolds in which C is embedded coisotropically.

This lemma tells us that we need consider only the case that C be pre-Poisson and the construction of Thm. 3.3:

Lemma 4.1. *A submanifold C of a Poisson manifold (P, Π) can be embedded coisotropically in a cosymplectic submanifold $\tilde{P}$ iff it is pre-Poisson. In this case all such $\tilde{P}$ are constructed (in a neighborhood of C) as in Thm. 3.3.*

Proof. In Thm. 3.3 we saw that given any pre-Poisson submanifold C, choosing a smooth subbundle R with $R \oplus (TC + \sharp N^*C) = TP|_C$ and "thickening" C in direction of R gives a submanifolds $\tilde{P}$ with the required properties.

Now let C be any submanifold embedded coisotropically in a cosymplectic submanifold $\tilde{P}$. By Lemma 3.2, for any complement R of TC in $T\tilde{P}|_C$ we have $R \oplus (TC + \sharp N^*C) = TP|_C$. This has two consequences: first the rank of $TC + \sharp N^*C$ must be constant, concluding the proof of the "iff" statement of the lemma. Second, it proves the final statement of the lemma. □

When C is a point $\{x\}$ then $\tilde{P}$ as above is a slice transverse to the symplectic leaf through x (see Ex. 5.1) and $\tilde{P}$ is unique up Poisson diffeomorphism by Weinstein's splitting theorem (Lemma 2.2 in [19]; see also Thm. 2.16 in [17]). A generalization of its proof gives

Proposition 4.2. *Let $\tilde{P}_0$ be a cosymplectic submanifold of a Poisson manifold P and $\pi\colon U \to \tilde{P}_0$ a projection of some tubular neighborhood of $\tilde{P}_0$ onto $\tilde{P}_0$. Let $\tilde{P}_t$, $t \in [0, 1]$, be a smooth family of cosymplectic submanifolds such that all $\tilde{P}_t$ are images of sections of π. Then, for t close enough to zero, there are Poisson diffeomorphisms ϕ_t mapping open sets of $\tilde{P}_0$ to open sets of $\tilde{P}_t$.*

Remark 4.3. Since each $\tilde{P}_t$ is cosymplectic it has a canonical transverse direction given by $\sharp N^*\tilde{P}_t$. The family of diffeomorphisms ϕ_t can be constructed[9] so that the curve $t \mapsto \phi_t(y)$ (for $y \in \tilde{P}_0$) is tangent to $\sharp N^*\tilde{P}_t$ at time t.

[9]To achieve this just choose H_t in the proof so that it vanishes on $\tilde{P}_t$.

Proof. We will use the following fact, whose straightforward proof we omit: let $\tilde{P}_t$, $t \in [0,1]$, be a smooth family of submanifold of a manifold U, and Y_t a time-dependent vector field on U. Then $Y + \frac{\partial}{\partial t}$ (considered as a vector field on $U \times [0,1]$) is tangent to the submanifold $\bigcup_{t\in[0,1]}(\tilde{P}_t, t)$ iff for each $\bar{t}$ and each integral curve γ of Y_t in U with $\gamma(\bar{t}) \in \tilde{P}_{\bar{t}}$ we have $\gamma(t) \in \tilde{P}_t$ (at all times where γ is defined).

Denote by s_t the section of π whose image is $\tilde{P}_t$. We will be interested in time-dependent vector fields Y_t on U such that for all $\bar{t}$ and $y \in \tilde{P}_{\bar{t}}$

$$Y_{\bar{t}}(y) = s_{\bar{t}*}(\pi_* Y_y) + \frac{d}{dt}|_{\bar{t}} s_t(\pi(y)). \tag{4}$$

We claim that, for such a vector field, $(Y + \frac{\partial}{\partial t})$ will be tangent to $\bigcup_{t\in[0,1]}(\tilde{P}_t, t)$. Indeed

$$(Y + \frac{\partial}{\partial t})(y, \bar{t}) = Y_{\bar{t}}(y) + \frac{\partial}{\partial t} \tag{5}$$

$$= s_{\bar{t}*}(\pi_* Y_y) + \frac{d}{dt}|_{\bar{t}} s_t(\pi(y)) + \frac{\partial}{\partial t}. \tag{6}$$

Since $s_{\bar{t}*}(\pi_* Y_y)$ is tangent to $(\tilde{P}_{\bar{t}}, \bar{t})$, and $\frac{d}{dt}|_{\bar{t}} s_t(\pi(y)) + \frac{\partial}{\partial t}$ is the velocity at time $\bar{t}$ of the curve $(s_t(\pi(y)), t)$, the claimed tangency follows. Hence by the fact recalled in the first paragraph we deduce that the flow ϕ_t of Y_t takes points y of $\tilde{P}_0$ to $\tilde{P}_{\bar{t}}$ (if $\phi_t(y)$ is defined until time $\bar{t}$).

So we are done if we realize such Y_t as the hamiltonian vector fields of a smooth family of functions H_t on U. For each fixed $\bar{t}$, eq. (4) for $Y_{\bar{t}}$ is just a condition on the vertical[10] component of $Y_{\bar{t}}$ at points of $\tilde{P}_{\bar{t}}$, and the latter is determined exactly by the effect of $Y_{\bar{t}}$ on functions f vanishing on $\tilde{P}_{\bar{t}}$. We have

$$Y_{\bar{t}}(f) = X_{H_{\bar{t}}}(f) = -dH_{\bar{t}}(\sharp df),$$

and the restriction of $\sharp$ to $N^*\tilde{P}_{\bar{t}}$ is injective because $\tilde{P}_{\bar{t}}$ is cosymplectic. Together we obtain that specifying the vertical component of $X_{H_{\bar{t}}}$ at points of $\tilde{P}_{\bar{t}}$ is equivalent to specifying the derivative of $H_{\bar{t}}$ in direction of $\sharp N^*\tilde{P}_{\bar{t}}$, which is transverse to $\tilde{P}_{\bar{t}}$. We can clearly find a function $H_{\bar{t}}$ satisfying the required conditions on its derivative at $\tilde{P}_{\bar{t}}$. Choosing H_t smoothly for every t we conclude that the vector field X_{H_t} will satisfy eq. (4), hence its flow ϕ_t, which obviously consists of Poisson diffeomorphisms, will take $\tilde{P}_0$ (or rather any subset of it on which the flow is defined up to time $\bar{t}$) to $\tilde{P}_{\bar{t}}$. □

Now we are ready to prove the uniqueness of $\tilde{P}$:

Theorem 4.4. *Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}_0$, $\tilde{P}_1$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then, shrinking $\tilde{P}_0$ and $\tilde{P}_1$ to a smaller tubular neighborhood of C if necessary, there is a Poisson diffeomorphism Φ from $\tilde{P}_0$ to $\tilde{P}_1$ which is the identity on C.*

Proof. In a neighborhood U of $\tilde{P}_0$ take a projection $\pi\colon U \to \tilde{P}_0$; choose it so that at points of $C \subset \tilde{P}_0$ the fibers of π are tangent to $\sharp N^*\tilde{P}_0|_C$. For $i = 0, 1$ make some choices of maximal dimensional subbundles R_i satisfying eq. (1) to write $T\tilde{P}_i|_C = TC \oplus R_i$, and join R_0 to R_1 by a smooth curve of subbundles R_t satisfying eq. (1) (there is no topological obstruction to this because R_0 and R_1 are both complements to the same subbundle $TC + \sharp N^*C$). By Thm. 3.3 we obtain a curve of cosymplectic submanifolds $\tilde{P}_t$, which moreover by Cor. 3.5 at points of C are all transverse to $\sharp N^*\tilde{P}_0|_C$, i.e. to the fibers of π.

[10]Vertical w.r.t. the splitting $T_yP = T_y\tilde{P}_{\bar{t}} \oplus ker_y \pi_*$.

Hence we are in the situation of Prop.4.2, which allows us to construct a Poisson diffeomorphism from $\tilde{P}_0$ to $\tilde{P}_t$ for small t. Since $M \subset \tilde{P}_t$ for all t, in the proof of Prop.4.2 we have that the sections s_t are trivial on M, hence by eq. (4) the vertical part of X_{H_t} at points of $M \subset \tilde{P}_t$ is zero. Choosing H_t to vanish on $\tilde{P}_t$ we obtain, $X_{H_t} = 0$ at points of $M \subset \tilde{P}_t$. From this we deduce two things: in a tubular neighborhood of M the flow ϕ_t of X_{H_t} is defined for all $t \in [0,1]$, and each ϕ_t keeps points of M fixed. Now just let $\Phi := \phi_1$. □

The derivative at points of C of the Poisson diffeomorphism Φ constructed in Thm. 4.4 gives an isomorphism of Poisson vector bundles $T\tilde{P}_0|_C \to T\tilde{P}_1|_C$ which is the identity on TC. The construction of Φ involves many choices; we wish now to give a *canonical* construction for such a vector bundle isomorphism.

Proposition 4.5. *Let C be a pre-Poisson submanifold (P,Π), and $\tilde{P}$, $\hat{P}$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then there is a canonical isomorphism of Poisson vector bundles $\varphi\colon T\tilde{P}|_C \to T\hat{P}|_C$ which is the identity on TC.*

Proof. We construct φ in two steps, and to simplify notation we will omit the restriction to C in expressions like $T\tilde{P}|_C$.

First we consider the vector bundle map

$$A\colon T\tilde{P} \to \sharp N^*\tilde{P}$$

determined by the requirement that $T\hat{P} = \{v + Av : v \in T\tilde{P}\}$. A is well-defined since $\sharp N^*\tilde{P}$ is a complement in TP both to $T\tilde{P}$ (because $\tilde{P}$ is cosymplectic) and to $T\hat{P}$ (because $T\hat{P} \cap (TC + \sharp N^*C) = TC$ by Lemma 3.2 and $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}$ by Lemma 3.5). Notice that, since C lies in both $\tilde{P}$ and $\hat{P}$, the restriction of A to TC is zero. The map $A + Id\colon T\tilde{P} \to T\hat{P}$ is an isomorphism of vector bundles. Further at each x it maps $T_x\tilde{P} \cap T_x\mathcal{O}$ isomorphically onto $T_x\hat{P} \cap T_x\mathcal{O}$ (which has the same dimension since both vector spaces contain $T_xC \cap T_x\mathcal{O}$ as a coisotropic subspace) because $\sharp N^*\tilde{P} \subset T\mathcal{O}$, however it does not match the symplectic forms there. We deform $A + Id$ by adding the following vector bundle map[11]($x \in C$):

$$B\colon T_x\tilde{P} \to T_xC, v \mapsto \frac{1}{2}\tilde{\sharp}(\Omega_x(Av, A\bullet)).$$

Here $\tilde{\sharp}$ is the sharp map of the cosymplectic submanifold $\tilde{P}$, Ω_x denotes the symplectic form at x of the symplectic leaf $\mathcal{O}$ through x, and $\Omega_x(Av, A\bullet)$ is an element of $T^*_x\tilde{P}$. To show that B is a smooth vector bundle map, it is enough to show that if X is a smooth section of (the restriction to C of) $\sharp N^*\tilde{P}$, then $\Omega(X,\bullet)|_{\sharp N^*\tilde{P}} : \sharp N^*\tilde{P} \to \mathbb{R}$ is smooth. But this follows from the fact that $\tilde{P}$ is cosymplectic: since $\sharp\colon N^*\tilde{P} \to \sharp N^*\tilde{P}$ is bijective, there is a smooth section ξ of $N^*\tilde{P}$ with $\sharp\xi = X$, and $\Omega(X,\bullet)|_{\sharp N^*\tilde{P}} = \xi|_{\sharp N^*\tilde{P}}$. Next we show that B is well-defined and that it actually maps into $TC \cap \tilde{\sharp}N^*C$: this is true because the section $\Omega(Av, A\bullet)$ of $T^*\tilde{P}$ annihilates TC (recall that $A|_{TC} = 0$) and because C is coisotropic in $\tilde{P}$. Further it is clear that the restriction of B to TC is zero.

At this point we are ready to define

$$\varphi\colon T\tilde{P} \to T\hat{P}, v \mapsto v + Av + Bv.$$

This is a well-defined (since $TC \subset T\hat{P}$), smooth map of vector bundles, and it is an isomorphism: if $v + Bv + Av = 0$ then $v + Bv = 0$ and $Av = 0$ (because $T\tilde{P}$ is transversal to

[11] Here we mimic a construction in symplectic linear algebra where one deforms canonically a complement of a coisotropic subspace C to obtain an isotropic complement of C; see [6] for the case when C is Lagrangian.

$\sharp N^*P$); from $Av = 0$ we deduce $Bv = 0$ hence $v = 0$. At each $x \in C$ the map φ restricts to an isomorphism from $T_x\tilde{P} \cap T_x\mathcal{O}$ to $T_x\hat{P} \cap T_x\mathcal{O}$ (because the images if A and B lie in $T_x\mathcal{O}$); we show that this restriction is a linear symplectomorphism. If $v_1, v_1 \in T_x\tilde{P} \cap T_x\mathcal{O}$ we have $\Omega(\varphi v_1, \varphi v_2) = \Omega(v_1 + Bv_1, v_2 + Bv_2) + \Omega(Av_1, Av_2)$, for the cross terms vanish since A takes values in $\sharp N^*\tilde{P}$. Now $\Omega(Bv_1, \bullet)|_{T_x\tilde{P}\cap T_x\mathcal{O}} = -\frac{1}{2}\Omega(Av_1, A\bullet)|_{T_x\tilde{P}\cap T_x\mathcal{O}}$ using the fact that $\Omega(\sharp\xi, \bullet) = -\xi|_{T\mathcal{O}}$ for any covector ξ of P. Further $\Omega(Bv_1, Bv_2)$ vanishes because B takes values in $T_xC \cap \sharp N_x^*C$. So altogether we obtain $\Omega(\varphi v_1, \varphi v_2) = \Omega(v_1, v_2)$ as desired. □

Remark 4.6. The isomorphism φ constructed in Prop. 4.5 can be extended to a Poisson vector bundle automorphism of $TP|_C$ as follows: define

$$(\varphi, pr) \colon T\tilde{P} \oplus \sharp N^*\tilde{P} \to T\hat{P} \oplus \sharp N^*\hat{P}$$

where pr denotes the projection of $N^*\tilde{P}$ onto $N^*\hat{P}$ along TC (recall from Cor. 3.5 that $TC \oplus N^*\tilde{P} = TC \oplus N^*\hat{P}$). (φ, pr) restricts to a linear automorphism of $T\mathcal{O} = (T\tilde{P} \cap T\mathcal{O}) \oplus \sharp N^*\tilde{P}$ which preserves the symplectic form: the only non-trivial check is $\Omega(pr(v_1), pr(v_2)) = \Omega(v_1, v_2)$ for $v_i \in \sharp N^*\tilde{P}$, which follows because $pr(v_1) - v_1 \in TC \cap \sharp N^*C$.

5. Conditions and examples

Let C be as usual a submanifold of the Poisson manifold (P, Π); in Section 3 we considered the question of existence of a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically. In Thm. 3.3 we showed that a *sufficient* condition is that C be pre-Poisson, which by Prop. 3.6 is equivalent to saying that $N^*C \cap \sharp^{-1}TC$ be a Lie algebroid.

A *necessary* condition is that the (intrinsically defined) characteristic distribution $TC \cap \sharp N^*C$ of C be the distribution associated to a Lie algebroid over C; in particular its rank locally can only increase. This is a necessary condition since the concept of characteristic distribution is an intrinsic one (see Remark 2.1), and the characteristic distribution of a coisotropic submanifold of a Poisson manifold is the image of the anchor of its conormal bundle, which is a Lie algebroid.

The submanifolds C which are not covered by the above conditions are those for which $N^*C \cap \sharp^{-1}TC$ is not a Lie algebroid but its image $TC \cap \sharp N^*C$ under $\sharp$ is the image of the anchor of some Lie algebroid over C. Diagrammatically:

$\{C$ s.t. $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid, i.e. C is pre-Poisson $\} \subset$

$\{C$ sitting coisotropically in some Poisson-Dirac submanifold $\tilde{P}$ of $P\} \subset$

$\{C$ s.t. $TC \cap \sharp N^*C$ is the distribution of some Lie algebroid over $C\}$

The following are examples of pre-Poisson submanifolds.

Example 5.1. An obvious example is when C is a coisotropic submanifold of P, and in this case the construction of Thm. 3.3 delivers $\tilde{P} = P$ (or more precisely, a tubular neighborhood of C in P).

Another obvious example is when C is just a point x: then the construction of Thm. 3.3 delivers as $\tilde{P}$ any slice through x transversal to the symplectic leaf $\mathcal{O}_x$.

Now if $C_1 \subset P_1$ and $C_2 \subset P_2$ are pre-Poisson submanifolds of Poisson manifolds, the cartesian product $C_1 \times C_2 \subset P_1 \times P_2$ also is, and if the construction of Thm. 3.3 gives cosymplectic submanifolds $\tilde{P}_1 \subset P_1$ and $\tilde{P}_2 \subset P_2$, the same construction applied to $C_1 \times C_2$

(upon suitable choices of complementary subbundles) delivers the cosymplectic submanifold $\tilde{P}_1 \times \tilde{P}_2$ of $P_1 \times P_2$. In particular, if C_1 is coisotropic and C_2 just a point x, then $C_1 \times \{x\}$ is pre-Poisson.

The following are two examples of submanifolds C which surely can not be imbedded coisotropically in any Poisson-Dirac submanifold:

Example 5.2. The submanifold $C = \{(x_1, x_2, x_2^2, x_1^2)\}$ of the symplectic manifold $(P, \omega) = (\mathbb{R}^4, dx_1 \wedge dx_3 + dx_2 \wedge dx_4)$ has characteristic distribution of rank 2 on the points with $x_1 = x_2$ and rank zero on the rest of C. The rank of the characteristic distribution locally decreases, hence C does not satisfies the necessary condition above.

Remark 5.3. If C is a submanifold of a symplectic manifold (P, ω), then the necessary and the sufficient conditions coincide, both being equivalent to saying that the characteristic distribution of C (which can be described as $\ker(i_C^*\omega)$ for i_C the inclusion) have constant rank, i.e. that C be presymplectic.

Example 5.4. Consider the Poisson[12] manifold $(\mathbb{R}^6, x_1\partial_{x_2} \wedge \partial_{x_4} + (\partial_{x_3} + x_1\partial_{x_5}) \wedge \partial_{x_6})$. Let C be the three-dimensional subspace given by setting $x_4 = x_5 = x_6 = 0$. The characteristic subspaces are all one-dimensional, spanned by ∂_{x_3} at points of C where $x_1 = 0$ and by ∂_{x_2} on the rest of C. Hence the characteristic subspaces don't form a smooth distribution, and can not be the image of the anchor map of any Lie algebroid over C. Hence C does not satisfies the necessary condition above.

The *sufficient* condition above is not necessary (i.e. the first inclusion in the diagram above is strict), as either of the following simple examples shows.

Example 5.5. Take C to be the vertical line $\{x = y = 0\}$ in the Poisson manifold $(P, \Pi) = (\mathbb{R}^3, f(z)\partial_x \wedge \partial_y)$, where f is any function with at least one zero. Then C is a Poisson-Dirac submanifold (with zero induced Poisson structure), hence taking $\tilde{P} := C$ we obtain a Poisson-Dirac submanifold in which C embeds coisotropically. The sufficient conditions here is not satisfied, for the rank of $TC + \sharp N^*C$ at $(0, 0, z)$ is 3 at points where f does not vanishes and 1 at points where f vanishes.

Example 5.6. Consider the Poisson manifold $(P, \Pi) = (\mathbb{R}^4, x^2\partial_x \wedge \partial_y + z\partial_z \wedge \partial_w)$ as in Example 6 of [13] and the submanifold $C = \{(z^2, 0, z, 0) : z \in \mathbb{R}\}$. The rank of $TC + \sharp N^*C$ is 3 away from the origin (because there C is an isotropic submanifold in an open symplectic leaf of P) and 1 at the origin (since Π vanishes there). The submanifold $\tilde{P} = \{(z^2, 0, z, w) : z, w \in \mathbb{R}\}$ is Poisson-Dirac and it clearly contains C as a coisotropic submanifold.

The *necessary* condition above is not a sufficient (i.e. the second inclusion in the diagram above is strict):

Example 5.7. In Example 3 in Section 8.2 of [13] the authors consider the manifold $P = \mathbb{C}^3$ with complex coordinates x, y, z and specify a Poisson structure on it by declaring the symplectic leaves to be the complex lines given by $dy = 0, dz - ydx = 0$, the symplectic forms being the restrictions of the canonical symplectic form on $\mathbb{C}^3$. They consider submanifold C the complex plane $\{z = 0\}$ and show that C is point-wise Poisson-Dirac (i.e. $TC \cap \sharp N^*C = \{0\}$ at every point), but that the induced bivector field is not smooth. Being point-wise

[12]This is really a Poisson structure because the bracket of any two coordinates is a Casimir functions (indeed either a constant or x_1), so that the Jacobiator of any three coordinate functions vanishes.

Poisson-Dirac, C satisfies the necessary condition above. However there exists no Poisson-Dirac submanifold $\tilde{P}$ of P in which C embeds coisotropically. Indeed at points p of C where $y \neq 0$ we have $T_pC \oplus T_p\mathcal{O} = TP$ (where as usual $\mathcal{O}$ is a symplectic leaf of P through p), from which follows that $\sharp|_{N_p^*C}$ is injective and $T_pC \oplus \sharp N_p^*C = TP$. From Lemma 3.2 (notice that the subspace R there must have trivial intersection with $T_pC \oplus \sharp N_p^*C$, so R must be the zero subbundle over C) it follows that the only candidate for $\tilde{P}$ is C itself. However, as we have seen, the Poisson bivector induced on C is not smooth. (More generally, examples are provided by any submanifold C of a Poisson manifold P which is point-wise Poisson-Dirac but not Poisson-Dirac and for which there exists a point p at which $T_pC \oplus T_p\mathcal{O} = TP$.) Notice that this provides an example for the claim made in Remark 3.4, because the zero subbundle R over C satisfies the condition of Lemma 3.2 at every point of C and is obviously a smooth subbundle.

We refer the reader to Section 6 of [11] for more examples in which the Poisson manifold P is the dual of a Lie algebra and C an affine subspace.

6. Reduction of submanifolds and deformation quantization of pre-Poisson submanifolds

In this section we consider the set of basic functions on a submanifold of a Poisson manifold, and show that in certain cases it is a Poisson algebra and that it can be deformation quantized.

Given any submanifold C of a Poisson manifold (P, Π), it is natural to consider the characteristic "distribution" $\sharp N^*C \cap TC$, which by Remark 2.1 consists of the kernels of the restriction to C of the symplectic forms on the symplectic leaves of P. We used quotation marks because $\sharp N^*C \cap TC$ usually does not have constant rank. We will consider the set of basic functions on C, i.e.

$$C^\infty_{bas}(C) = \{f \in C^\infty(C) : df|_{\sharp N^*C \cap TC} = 0\}.$$

When the characteristic distribution is regular and smooth and the quotient $\underline{C}$ is a smooth manifold, then these are exactly the pullbacks of functions on $\underline{C}$.

If we endow C with the (possibly non-smooth[13]) point-wise Dirac structure i^*L_P, where $i \colon C \to P$ is the inclusion and L_P is the Dirac structure corresponding to Π, then $C^\infty_{bas}(C)$ is exactly the set of basic functions in the sense of Dirac geometry, i.e. the set of functions whose differentials annihilate at each point the characteristic subspaces $i^*L_P \cap TC$. Given basic functions f, g the expression

$$\{f, g\}_C(p) := Y(g),$$

[13]A sufficient condition for the induced Dirac structure to be smooth and integrable is that the rank of $\sharp N^*C$ be constant, because $L_P \cap (\{0\} \oplus N^*C) = ker(\sharp|_{N^*C})$.

where Y is any element of T_pC such that $(Y, df_p) \in i^*L_P$, is well-defined. However it does not usually vary smoothly[14] with p, so we can *not* conclude that $C^\infty_{bas}(C)$ with this bracket is a Poisson algebra.

As pointed out in [4] $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ inherits a Poisson bracket from the Poisson manifold P, where $\mathcal{I}$ denotes the set of functions on P that vanish on C and $\mathcal{F} := \{\hat{f} \in C^\infty(P) : \{\hat{f}, \mathcal{I}\} \subset \mathcal{I}\}$ (the so-called first class functions) its normalizer. $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ is exactly the subset of functions f on C which admits an extension to some function $\hat{f}$ on P whose differential annihilates $\sharp N^*C$ (or equivalently $X_{\hat{f}}|_C \subset TC$). The bracket is computed as follows:

$$\{f, g\} = \{\hat{f}, \hat{g}\}_P|_C = X_{\hat{f}}(g)|_C$$

for extensions as above. Notice that $\mathcal{F}/(\mathcal{F} \cap \mathcal{I}) \subset C^\infty_{bas}(C)$, and that the Poisson bracket $\{\bullet, \bullet\}$ on $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ coincides with $\{\bullet, \bullet\}_C$ (if f, g belong to $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ we can compute $\{f, g\}_C$ by choosing $Y = X_{\hat{f}}$ for some extension $\hat{f} \in \mathcal{F}$).

In some cases $(C^\infty_{bas}(C), \{\bullet, \bullet\}_C)$ actually is a Poisson algebra:

Proposition 6.1. *Let C be any submanifold of a Poisson manifold (P, Π). If there exists a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically, then the set of basic functions on C has an intrinsic Poisson algebra structure, and $(\mathcal{F}/(\mathcal{F} \cap \mathcal{I}), \{\bullet, \bullet\})$ is a Poisson subalgebra.*

Proof. We add a tilde in the notation introduced above when we view C as a submanifold of the Poisson manifold $\tilde{P}$ instead of P. So is $\tilde{\mathcal{I}}$ the vanishing ideal of C in $\tilde{P}$ and by $\tilde{\mathcal{F}}$ its normalizer. Since $\sharp N^*C \subset TC$ it follows that $\tilde{\mathcal{F}}/\tilde{\mathcal{I}}$ fills up the space of basic functions of $(C, \tilde{i}^*L_{\tilde{P}})$. By the above the bracket on $\tilde{\mathcal{F}}/\tilde{\mathcal{I}}$ and the bracket coming from the point-wise Dirac structure $\tilde{i}^*L_{\tilde{P}}$ agree; in particular the latter endows $C^\infty_{bas}(C)$ with a Poisson algebra structure. It is intrinsic to C in the following sense: if $\bar{P}$ is any other submanifold of (P, Π) containing C, $L_{\bar{P}}$ the point-wise Dirac structure on $\bar{P}$ induced by P and $\bar{i} : C \to \bar{P}$ the inclusion, then the Poisson bracket on $C^\infty_{bas}(C)$ induced by $\bar{P}$ via $\bar{i}^*L_{\bar{P}}$ agrees with the above, because $\bar{i}^*L_{\bar{P}} = \tilde{i}^*L_{\tilde{P}}$ by the functoriality of pullback. Hence it makes sense to denote this bracket by $\{\bullet, \bullet\}_C$. This allows us to show that $(C^\infty_{bas}(C), \{\bullet, \bullet\}_C)$ contains $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ as a Poisson subalgebra, because as we saw above the bracket on $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ coincides with the bracket of basic functions on (C, i^*L_P). ☐

By Thm. 3.3 pre-Poisson submanifolds C satisfy the assumption of Prop. 6.1, hence they admit a Poisson algebra structure on their space of basic functions. This fact was already established in Theorem 3 of [4], where furthermore it is shown that $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ is the whole space of basic functions. We need our Prop. 6.1 because it tells us that the Poisson algebra $(C^\infty_{bas}(C), \{\bullet, \bullet\}_C)$ is intrinsic to C; this allows us to finally state our result about deformation quantization.

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[14]In the case of smooth Dirac structure the set of so-called admissible functions, endowed with this bracket, is a Poisson algebra [12]. In our case it is tempting to define the set of admissible functions as functions f on C for which there is a smooth vector field X such that $(X, df) \subset i^*L_P$, however these does not seem to be closed under $\{\bullet, \bullet\}_C$.

We can instead consider a larger set of functions. Denote by C_{reg} the open, dense subset of C where the rank of $\sharp N^*C$ is locally constant; on this set the point-wise Dirac structure i^*L_P is actually smooth and integrable [12]. The set of functions f for which there is a smooth vector field X such that $(X, df)|_{C_{reg}} \subset i^*L_P|_{C_{reg}}$ is a Poisson algebra. The reason is essentially that the set of smooth sections of $TC \oplus T^*C$ whose restriction to C_{reg} lie in $i^*L_P|_{C_{reg}}$ are closed under the Courant bracket. However the latter set of functions is usually not contained in $C^\infty_{bas}(C)$.

Theorem 6.2. *Let C be a pre-Poisson submanifold, and assume that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish. Then $(C^{\infty}_{bas}(C), \{\bullet,\bullet\}_C)$, the Poisson algebra of basic functions on C, admits a deformation quantization.*

Proof. By Thm. 3.3 we can embed C coisotropically in some cosymplectic submanifold $\tilde{P}$. Further by Prop. 6.1 the Poisson bracket $\{\bullet,\bullet\}_C$ on $C^{\infty}_{bas}(C)$ is induced by the embedding of C in $\tilde{P}$. Now we invoke Corollary 3.3 of [10]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold admits a deformation quantization. The conditions in Corollary 3.3 of [10] translate into the conditions stated in the proposition because the conormal bundle of C in $\tilde{P}$ is isomorphic to $N^*C \cap \sharp^{-1}TC$ as a Lie algebroid, see Prop. 3.6. □

7. Subgroupoids associated to pre-Poisson submanifolds

Let C be a pre-Poisson submanifold of a Poisson manifold (P,Π). In Prop. 3.6 we showed that $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P. When $\sharp N^*C$ has constant rank there is another Lie subalgebroid associated[15] to C, namely $\sharp^{-1}TC = (\sharp N^*C)^\circ$. Now we assume that T^*P is an integrable Lie algebroid, i.e. that the source simply connected (s.s.c.) symplectic groupoid $(\Gamma_s(P),\Omega)$ of (P,Π) exists. In this section we study the (in general only immersed) subgroupoids of $\Gamma_s(P)$ integrating $N^*C\cap\sharp^{-1}TC$ and $\sharp^{-1}TC$. Here, for any Lie subalgebroid A of T^*P integrating to a s.s.c. Lie groupoid G, we take "subgroupoid" to mean the (usually just immersed) image of the (usually not injective) morphism $G \to \Gamma_s(P)$ induced from the inclusion $A \to T^*P$.

By Thm. 3.3 we can find a cosymplectic submanifold $\tilde{P}$ in which C lies coisotropically. We first make few remarks on the subgroupoid corresponding to $\tilde{P}$.

Lemma 7.1. *The subgroupoid of $\Gamma_s(P)$ integrating $\sharp^{-1}T\tilde{P}$ is $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$ and is a symplectic subgroupoid. Its source (target) map is a Poisson (anti-Poisson) map onto $\tilde{P}$, where the latter is endowed with the Poisson structure induced by (P,Π).*

Proof. According to Thm. 3.7 of [20] the subgroupoid[16] of $\Gamma_s(P)$ corresponding to $\tilde{P}$, i.e. the one integrating $(\sharp N^*\tilde{P})^\circ$, is a symplectic subgroupoid of $\Gamma_s(P)$. It is given by $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$, because[17] $(\sharp N^*\tilde{P})^\circ = \sharp^{-1}T\tilde{P}$.

To show that the maps $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P}) \to \tilde{P}$ given by the source and target maps of $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$ are Poisson (anti-Poisson) maps proceed as follows. Take a function $\tilde{f}$ on $\tilde{P}$, and extend it to a functions f on P so that X_f is tangent to $\tilde{P}$ along $\tilde{P}$ (i.e. exactly as was done in section 2 to compute the Poisson bracket on $\tilde{P}$ in terms of the one on P). Since $s:\Gamma_s(P)\to P$ is a Poisson map and s-fibers are symplectic orthogonal to t-fiber we know that the vector field X_{s^*f} on $\Gamma_s(P)$ is tangent to $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$. Hence, denoting by $\tilde{s}$ the source map of $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$, we have

$$\tilde{s}^*\{\tilde{f}_1,\tilde{f}_2\} = \tilde{s}^*(\{f_1,f_2\}|_{\tilde{P}}) = \{s^*f_1, s^*f_2\}|_{\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})} = \{\tilde{s}^*f_1,\tilde{s}^*f_1\},$$

[15] More generally for any Lie algebroid $A\to M$ with anchor ρ, if N is a submanifold of M such that $\rho^{-1}TN$ has constant rank then $\rho^{-1}TN\to N$ is a Lie subalgebroid of $A\to M$.

[16] In [20] this is claimed only when the subgroupoid integrating $(\sharp N^*\tilde{P})^\circ$ is an embedded subgroupoid, however the proof there is valid for immersed subgroupoids too.

[17] More generally we claim the following: if $\Gamma \rightrightarrows M$ is any Lie groupoid integrating the Lie algebroid $A\to M$ and $N\subset M$ a submanifold such that $\rho^{-1}TN\to N$ has constant rank, then the Lie subalgebroid $\rho^{-1}TN$ is integrated by the source-connected part of the subgroupoid $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$, and this intersection is clean.

i.e. $\tilde{s}$ is a Poisson map. A similar reasoning holds for $\tilde{t}$. □

Now we describe the subgroupoid integrating $N^*C \cap \sharp^{-1}TC$:

Proposition 7.2. *Let C be a pre-Poisson submanifold of (P, Π). Then the subgroupoid of $\Gamma_s(P)$ integrating $N^*C \cap \sharp^{-1}TC$ is an isotropic subgroupoid of $\Gamma_s(P)$.*

Proof. The canonical vector bundle isomorphism $i : T^*\tilde{P} \cong (\sharp N^*\tilde{P})^\circ$ is a Lie algebroid isomorphism, where $T^*\tilde{P}$ is endowed with the cotangent algebroid structure coming from the Poisson structure on $\tilde{P}$. Indeed both the anchor and the brackets of exact (hence by the Leibniz rule of all) 1-forms on $\tilde{P}$ match, as follows from section 2. Integrating this algebroid isomorphism we obtain a Lie groupoid morphism from $\Gamma_s(\tilde{P})$, the s.s.c. Lie groupoid integrating $T^*\tilde{P}$, to $\Gamma_s(P)$, and the image of this morphism is $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. Since by Lemma 7.1 the symplectic form on $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ is multiplicative, symplectic and the source map is a Poisson map, pulling back the symplectic form on $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ endows $\Gamma_s(\tilde{P})$ with the structure of the s.s.c. symplectic groupoid of $\tilde{P}$. The subgroupoid of $\Gamma_s(\tilde{P})$ integrating $N^*_{\tilde{P}}C$, the annihilator of C in $\tilde{P}$, is Lagrangian ([7], Prop. 5.5). Hence $i(N^*_{\tilde{P}}C)$, which by Prop. 3.6 is equal to $N^*C \cap \sharp^{-1}TC$, integrates to a Lagrangian subgroupoid of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, which therefore is an isotropic subgroupoid of $\Gamma_s(P)$. □

Now we consider $\sharp^{-1}TC$. For any submanifold N, $\sharp^{-1}TN$ has constant rank iff it is a Lie subalgebroid of T^*P, integrating to the subgroupoid $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ of $\Gamma_s(P)$. So the constant rank condition on $\sharp^{-1}TN$ corresponds to a smoothness condition on $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$.

Remark 7.3. 1) If $\sharp^{-1}TN$ has constant rank it follows that the Poisson structure on P pulls back to a smooth Dirac structure on N, and that $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ is an over-pre-symplectic[18] groupoid inducing the same Dirac structure on N (Ex. 6.7 of [2]). $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ has dimension equal to $2dimN + rk(N^*N \cap N^*\mathcal{O})$, where $\mathcal{O}$ the symplectic leaves of P intersecting C.
2) For a pre-Poisson submanifold C, the condition that $\sharp^{-1}TC$ have constant rank is equivalent to the characteristic distribution $TC \cap \sharp N^*C$ having constant rank[19].

Proposition 7.4. *Let C be a pre-Poisson submanifold with constant-rank characteristic distribution. Then for any cosymplectic submanifold $\tilde{P}$ in which C embeds coisotropically, $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a coisotropic subgroupoid of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$.*

Proof. By the comments above we know that $\sharp^{-1}TC$ is a Lie subalgebroid, hence $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (smooth) subgroupoid of $\Gamma_s(P)$. We saw in Lemma 7.1 that $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ is endowed with a symplectic multiplicative 2-form for which its source and target maps are (anti-)Poisson maps onto $\tilde{P}$. Further its source and target fibers symplectic orthogonals of each other. Since $C \subset \tilde{P}$ is coisotropic, the above (together with the fact that the preimage of coisotropic submanifolds under Poisson maps are again coisotropic) implies that $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is coisotropic in $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. □

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[18]Recall from Def. 4.6 of [2] that an over-pre-symplectic groupoid is a Lie groupoid G over M equipped with a closed multiplicative 2-form ω such that $ker\omega_x \cap ker(ds)_x \cap ker(dt)_x$ has rank $dimG - 2dimM$ at all $x \in M$.

[19]Indeed more generally we have the following for any submanifold C of P: if any two of $\sharp^{-1}TC$, $\sharp N^*C + TC$ or $TC \cap \sharp N^*C$ have constant rank, then the remaining one also has constant rank. This follows trivially from $rk(\sharp N^*C + TC) = rk(\sharp N^*C) + \dim C - rk(TC \cap \sharp N^*C)$.

We now describe the subgroupoids corresponding to pre-Poisson manifolds.

Proposition 7.5. *Let C be any submanifold of P. Then $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (immersed) presymplectic submanifold iff C is pre-Poisson and its characteristic distribution has constant rank. In this case the characteristic distribution of $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ has rank $2rk(\sharp N^*C \cap TC) + rk(N^*C \cap N^*\mathcal{O})$, where $\mathcal{O}$ denotes the symplectic leaves of P intersecting C.*

Proof. Assume that $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (immersed) presymplectic submanifold. We apply the same proof as in Prop. 8 of [13]: there is an isomorphism of vector bundles $T\Gamma_s(P)|_P \cong TP \oplus T^*P$, under which the non-degenerate bilinear form $\Omega|_P$ corresponds to $(X_1 \oplus \xi_1, X_2 \oplus \xi_2) := \langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle + \Pi(\xi_1, \xi_2)$. Under the above isomorphism $T(\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C))$ corresponds to $TC \oplus \sharp^{-1}TC$, and a short computation shows that the restriction of $(\bullet, \bullet)$ to $TC \oplus \sharp^{-1}TC$ has kernel $(TC \cap \sharp N^*C) \oplus (\sharp^{-1}TC \cap N^*C)$. From the smoothness of $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ it follows that $(\sharp N^*C)^\circ = \sharp^{-1}TC$ has constant rank, so this kernel is a direct sum of two intersections of smooth subbundles. We deduce that $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is "presymplectic at points of C" (i.e. the pullback of Ω to $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ has constant rank along C) iff $\sharp^{-1}TC \cap N^*C$ has constant rank, i.e. (taking annihilators) iff C is pre-Poisson. By the comments before Prop. 7.4 we also know that C has characteristic distribution of constant rank.

The other direction follows from Prop. 7.4. □

Remark 7.6. One can wonder whether any subgroupoid of a symplectic groupoid $(\Gamma_s(P), \Omega)$ which is a presymplectic submanifold (i.e. Ω pulls back to a constant rank 2-form) is contained coisotropically in some symplectic subgroupoid of $\Gamma_s(P)$. This would be exactly the "groupoid" version of Thm. 3.3. The above Prop. 7.4 and Prop. 7.5 together tell us that this is the case when the subgroupoid has the form $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$, where $C \subset P$ is its base. In general the answer to the above question is negative, as the following counterexample shows.

Let (P, ω) be some simply connected symplectic manifold, so that $\Gamma_s(P) = (P \times P, \omega_1 - \omega_2)$ and the units are embedded diagonally. Take C to be any 1-dimensional closed submanifold of P. $C \rightrightarrows C$ is clearly a subgroupoid and a presymplectic submanifold; since $\omega_1 - \omega_2$ there pulls back to zero, any subgroupoid G of $P \times P$ in which $C \rightrightarrows C$ embeds coisotropically must have dimension 2. If the base of G has dimension 2 then G is contained in the identity section of $P \times P$, which is Lagrangian. So let us assume that the base of G is C. Then G must be contained in $C \times C$, on which $\omega_1 - \omega_2$ vanishes because $C \subset P$ is isotropic. So we conclude that there is no symplectic subgroupoid of $P \times P$ containing $C \rightrightarrows C$ as a coisotropic submanifold.

8. Existence of coisotropic embeddings of Dirac manifolds in Poisson manifolds

Let (M, L) be a (smooth) Dirac manifold. We ask when (M, L) can be embedded coisotropically in some Poisson manifold (P, Π), i.e. when there exists an embedding i such that $i^*L_P = L$ and $i(M)$ is a coisotropic submanifold of P [20].

[20] Notice that when P is symplectic any coisotropic submanifold has an induced smooth presymplectic form, however when we take P to be Poisson the induced structure is generally not even continuous: for example the x-axis in $(\mathbb{R}^2, x\partial_x \wedge \partial_y)$ is coisotropic, but its pullback Dirac structure is not continuous at the origin.

When M consists of exactly one leaf, i.e. when M is a manifold endowed with a closed 2-form ω, the existence and uniqueness of coisotropic embeddings in symplectic manifolds was considered by Gotay in the short paper [14]: the coisotropic embedding iff $\ker \omega$ has constant rank, and in that case one has uniqueness up to neighborhood equivalence. Our strategy will be to check if we can apply Gotay's arguments "leaf by leaf" *smoothly* over M. Recall that $L \cap TM$ is the kernel of the 2-forms on the presymplectic leaves of (M, L).

Theorem 8.1. *(M, L) can be embedded coisotropically in a Poisson manifold iff $L \cap TM$ has constant rank.*

Proof. Suppose that an embedding $i \colon M \to P$ as above exists. Then $L \cap TM$ is equal $\sharp N^*C$ (where N^*C is the normalizer of C in P), the image of a vector bundle under a smooth bundle map, hence its rank can locally only increase. On the other hand the rank of $L \cap TM$, which is the intersection of two smooth bundles, can locally only decrease. Hence the rank of $L \cap TM$ must be constant on M.

Conversely, assume that the rank of $E := L \cap TM$ be constant and define P to be the total space of the vector bundle $\pi \colon E^* \to M$. We define the Poisson structure on P as follows. First take the pullback Dirac structure $\pi^* L$ (which is smooth and integrable since π is a submersion). Then choose a smooth distribution V such that $E \oplus V = TM$. This choice gives an embedding $i_M \colon E^* \to T^*M$, which we can use to pull back the canonical symplectic form ω_{T^*M}. Our Poisson structure is $L_{E^*} := \tau_{i_M^* \omega_{T^*M}} \pi^* L$, i.e. it is obtained applying to $\pi^* L$ the gauge transformation by the closed 2-form $i_M^* \omega_{T^*M}$. It is clear that L_{E^*} is a smooth Dirac structure; we still have to show that it is actually Poisson, and that the zero section is coisotropic. In more concrete terms (E^*, L_{E^*}) can be described as follows: the leaves are all of the form $\pi^{-1}(F_\alpha)$ for $(F_\alpha, \omega_\alpha)$ a presymplectic leaf of M. The 2-form on the leaf is given by adding to $(\pi|_{\pi^{-1}(F_\alpha)})^* \omega_\alpha$ the 2-form $i_\alpha^* \omega_{T^* F_\alpha}$. The latter is defined considering the transverse distribution $V \cap TF_\alpha$ to $E|_{F_\alpha}$ in TF_α, the induced embedding $i_\alpha \colon \pi^{-1}(F_\alpha) = E^*|_{F_\alpha} \to T^* F_\alpha$, and pulling back the canonical symplectic form. One can check that $i_\alpha^* \omega_{T^* F_\alpha}$ is the pullback of $i_M^* \omega_{T^*M}$ via the inclusion of the leaf in E^*. But this is exactly Gotay's recipe to endow $\pi^{-1}(F_\alpha)$ with a symplectic form so that F_α is embedded as a coisotropic submanifold. Hence we conclude that a neighborhood of the zero section of E^*, with the above Dirac structure, is actually a Poisson manifold and that M is embedded as a coisotropic submanifold. □

We comment on how choices affect the construction of Thm. 8.1. We need the following version of Moser's theorem for Poisson structures (see Section 3.3. of [1]) : suppose we are given Poisson structures Π_t on some manifold P, $t \in [0, 1]$. Assume that each Π_t be related to Π_0 via the gauge transformation by some closed 2-form B_t, i.e. $\Pi_t = \tau_{B_t} \Pi_0$. This means that the symplectic foliations agree and on each symplectic leaf $\mathcal{O}$ we have $\Omega_t = \Omega_0 + i_{\mathcal{O}}^* B_t$, where Ω_0,Ω_t are the symplectic forms on the leaf $\mathcal{O}$ and $i_{\mathcal{O}}$ the inclusion. Assume further that each $\frac{d}{dt} B_t$ be exact, and let α_t be a smooth family of primitives vanishing on some submanifold M. Then the time-1 flow of the Moser vector field[21] $\sharp_t \alpha_t$ is defined in a tubular neighborhood of M, it fixes M and maps Π_0 to Π_1.

Proposition 8.2. *Different choices of splitting V in the construction of Thm. 8.1 yield (canonically) isomorphic Poisson structures on E^*. Hence, given a Dirac manifold (M, L) for which $L \cap TM$ has constant rank, there is a canonical (up to neighborhood equivalence) Poisson manifold in which M embeds coisotropically.*

[21] Here $\sharp_t$ denotes the map $T^*P \to TP$ induced by Π_t.

Proof. Let V_0, V_1 be two different splittings as in Thm. 8.1, i.e. $E \oplus V_i = TM$ for $i = 0, 1$. We can interpolate between them by defining the graphs $V_t := \{v + tAv : v \in V_0\}$ for $t \in [0, 1]$, where $A\colon V_0 \to E$ is determined by requiring that its graph be V_1. Obviously each V_t also gives a splitting $E \oplus V_t = TM$; denote by $i_t\colon E^* \to T^*M$ the corresponding embedding. We obtain Dirac structures $\tau_{i_t^*\omega_{T^*M}}\pi^*L$ on the total space of $\pi\colon E^* \to M$; by Thm. 8.1 they correspond to Poisson bivectors, which we denote by Π_t. These Poisson structures are related by a gauge transformation: $\Pi_t = \tau_{B_t}\Pi_0$ for $B_t := i_t^*\omega_{T^*M} - i_0^*\omega_{T^*M}$. A primitive of $\frac{d}{dt}B_t$ is given by $\frac{d}{dt}i_t^*\alpha_{T^*M}$; notice that this primitive vanishes at points of M, because the canonical 1-form α_{T^*M} on T^*M vanishes along the zero section. Hence the time-1 flow of $\sharp_t(\frac{d}{dt}i_t^*\alpha_{T^*M})$ fixes M and maps Π_0 to Π_1.

□

Assuming that (M, L) is integrable we describe the symplectic groupoid of (E^*, L_{E^*}), the Poisson manifold constructed in Thm. 8.1 with a choice of distribution V. It is $\pi^*(\Gamma_s(M))$, the pullback via $\pi\colon E^* \to M$ of the presymplectic groupoid of M, endowed with the following symplectic form: the pullback via $\pi^*(\Gamma_s(M)) \to \Gamma_s(M)$ of the presymplectic form on the groupoid $\Gamma_s(M)$, plus $\mathbf{s}^*(i_M^*\omega_{T^*M}) - \mathbf{t}^*(i_M^*\omega_{T^*M})$, where $i_M\colon E^* \to T^*M$ is the inclusion given by the choice of distribution V, ω_{T^*M} is the canonical symplectic form, and $\mathbf{s}, \mathbf{t}$ are the source and target maps of $\pi^*(\Gamma_s(M))$. This follows easily from Examples 6.3 and 6.6 in [2]. Notice that this groupoid is source simply connected when $\pi^*(\Gamma_s(M))$ is.

Now we can give an affirmative answer to the possibility raised in [13] (Remark (e) in Section 8.2), although we prove it "working backwards"; this is the "groupoid" version of Gotay's embedding theorem.

Proposition 8.3. *Any presymplectic groupoid in the sense*[22] *of* [2] *with constant rank characteristic distribution can be embedded coisotropically as a Lie subgroupoid in a symplectic groupoid.*

Proof. By Cor. 4.8 iv),v) of [2], a presymplectic groupoid $\Gamma_s(M)$ has characteristic distribution (the kernel of the multiplicative 2-form) of constant rank iff the Dirac structure L induced on its base M does. We can embed (M, L) coisotropically in the Poisson manifold (E^*, L_{E^*}) constructed in Thm. 8.1; we just showed that $\pi^*(\Gamma_s(M))$ is a symplectic groupoid for E^*. $\Gamma_s(M)$ embeds in $\pi^*(\Gamma_s(M))$ as $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$, and this embedding preserves both the groupoid structures and the 2-forms. $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a coisotropic subgroupoid of $\pi^*(\Gamma_s(M))$ because M lies coisotropically in E^* and $\mathbf{s}, \mathbf{t}$ are (anti)Poisson maps. □

Remark 8.4. A partial converse to this proposition is given as follows: if $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a coisotropic subgroupoid of a symplectic groupoid $\Gamma_s(P)$, then M is a coisotropic submanifold of the Poisson manifold P, it has an smooth Dirac structure (induced from P) with characteristic distribution of constant rank, and $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a *over*-pre-symplectic groupoid over M inducing the same Dirac structure. This follows from our arguments in section 7.

Now we draw the conclusions about deformation quantization. Recall that for any Dirac manifold (M, L) the set of admissible functions

(7) $C^\infty_{adm}(M) = \{f \in C^\infty(M) :$ there exists a smooth vector field X_f s.t. $(X_f, df) \subset L\}$

is naturally a Poisson algebra [12], with bracket $\{f, g\} = X_f(g)$.

[22]Recall from Def. 2.1 of [2] that a presymplectic groupoid is a Lie groupoid G over M equipped with a closed multiplicative 2-form ω such that $ker\omega_x \cap ker(ds)_x \cap ker(dt)_x = 0$ at all $x \in M$.

Theorem 8.5. *Let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. If the first and second foliated de Rham cohomologies of the foliation $\mathcal{F}$ vanish then the Poisson algebra of admissible functions on (M, L) admits a deformation quantization.*

Proof. By Thm. 8.1 we can embed (M, L) coisotropically in a Poisson manifold P; hence we can apply again Corollary 3.3 of [10]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold admits a deformation quantization. Since $L \cap TM$ has constant rank the inclusion $C^\infty_{adm}(M) \subset C^\infty_{bas}(M)$ is an equality[23]. Further the Poisson algebra structure on $C^\infty_{bas}(M)$ coming from (M, L) coincides with the one induced by M as a coisotropic submanifold of P, as follows from Prop. 6.1 and $i^* L_P = L$. So when the assumptions are satisfied we really deformation quantize $C^\infty_{adm}(M)$.

Notice that in Thm. 8.1 we constructed a Poisson manifold P of minimal dimension, i.e. of dimension $dim M + rk(L \cap TM)$. The anchor map $\sharp$ of the Lie algebroid N^*C is injective (see also Rem. 9.5 in the next section), hence the Lie algebroids N^*C and $L \cap TM$ are isomorphic. This allows us to state the assumptions of Corollary 3.3 of [10] in terms of the foliation $\mathcal{F}$ on M. □

Remark 8.6. Let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. Then the foliated de Rham complex $\Omega^\bullet_{\mathcal{F}}(M)$ admits the structure of an L_∞-algebra[24] $\{\lambda_n\}_{n \geq 1}$, the differential λ_1 being the foliated de Rham differential and the bracket λ_2 inducing on $H^0_{\lambda_1} = C^\infty_{bas}(M)$ the natural bracket (7).

Indeed by the proof of Thm. 8.1 we know that M can be embedded coisotropically in a Poisson manifold P so that the Lie algebroids N^*M and $L \cap TM$ are isomorphic. After choosing an embedding of $NM := TP|_M / TM$ in a tubular neighborhood of M in P, Thm. 2.2 of [10] gives the desired L_∞-structure. The L_∞-structure depends on the choice of embedding, but it is natural to expect that different embeddings give the same structure up to L_∞-isomorphism. This issue will be addressed in a forthcoming work.

9. Uniqueness of coisotropic embeddings of Dirac manifolds

The coisotropic embedding of Gotay [14] is unique up to neighborhood equivalence, i.e. any two coisotropic embeddings of a fixed presymplectic manifold in symplectic manifolds are intertwined by a symplectomorphism which is the identity on the coisotropic submanifold. It is natural to ask whether, given a Dirac manifold (M, L) such that $L \cap TM$ have constant rank, the coisotropic embedding constructed in Thm. 8.1 is the only one up to neighborhood equivalence. In general the answer will be negative: for example the origin is a coisotropic submanifold in $\mathbb{R}^2$ endowed either with the zero Poisson structure or with the Poisson structure $(x^2 + y^2)\partial_x \wedge \partial_y$, and the two Poisson structures are clearly not equivalent. As Aissa Wade pointed out to us, it is necessary to require that the Poisson manifold in which we embed be of minimal dimension, i.e. of dimension $\dim M + rk(L \cap TM)$.

[23] Use that since $L \cap TM$ is the kernel of the projection $L \to T^*M$, the image of this projection has constant rank.

[24] The λ_n are derivations w.r.t. the wedge product, so one actually obtains what in [10] is called a P_∞ algebra.

9.1. **Infinitesimal uniqueness and global issues.** We try to apply the construction of Gotay's uniqueness proof [14] on each presymplectic leaf of the Dirac manifold M; then we will show that under certain assumptions the resulting diffeomorphism varies smoothly from leaf to leaf.

We start establishing infinitesimal uniqueness.

Proposition 9.1. *Suppose we are given a Dirac manifold (M, L) for which $L \cap TM$ has constant rank k, and let (P_1, Π_1) and (P_2, Π_2) be Poisson manifolds of dimension $\dim M + k$ in which (M, L) embeds coisotropically. Then there is an isomorphism of Poisson vector bundles $\Phi\colon TP_1|_M \to TP_2|_M$ which is the identity on TM.*

Proof. Let P denote either of P_1 or P_2, by $(\mathcal{O}, \Omega)$ the symplectic leaf of P passing through some $x \in M$ and by F the presymplectic leaf of M passing through x. Since $T_xM \cap T_x\mathcal{O}$ is coisotropic in the symplectic vector space $T_x\mathcal{O}$, a simple dimension count shows that the assumption on the dimension of P is equivalent to $T_xM + T_x\mathcal{O} = T_xP$ [25]. Choose a distribution V such that $E \oplus V = TM$, where $E := L \cap TM$. We claim that $V \oplus \sharp V^\circ = TP|_M$: indeed $V_x \cap T_x\mathcal{O} = V_x \cap T_xF$ is a symplectic subspace of $(T_x\mathcal{O}, \Omega_x)$, being transverse to $E_x = ker(\Omega|_{T_xF})$. Hence $(V_x \cap T_x\mathcal{O})^{\Omega_x}$, which by section 2 is equal to $\sharp V_x^\circ$, is a complement to $V_x \cap T_x\mathcal{O}$ in $T_x\mathcal{O}$, so $V_x \oplus \sharp V_x^\circ = V_x + T_x\mathcal{O} = T_xP$ as we claimed. Now we repeat the construction of Gotay's uniqueness proof [14]: since E_x is Lagrangian in the symplectic subspace $\sharp V_x^\circ$, we can find a linear symplectomorphism $(\sharp V_x^\circ, \Omega|_{\sharp V_x^\circ}) \cong E_x \oplus E_x^*$, where the latter is equipped with the canonical antisymmetric pairing ω_E. This goes as follows: choose a complement to E_x in $\sharp V_x^\circ$, deform it canonically to a Lagrangian complement R_x (see [6]), and define the isomorphism $\sharp V_x^\circ = E_x \oplus R_x \to E_x \oplus E_x^*$ to be $(v_E, v_R) \mapsto (v_E, \Omega(v_R, \cdot)|_{E_x})$. Since V_x and the above linear symplectomorphism can be chosen to depend smoothly on $x \in M$ we obtain a smooth vector bundle isomorphism $TP|_M = V_x \oplus \sharp V_x^\circ \to V \oplus E \oplus E^*$. We equip the fibers of the latter vector bundle with bivectors as in Thm. 8.1, i.e. bivectors (depending only on the Dirac structure on M and V) so that the induced symplectic subspaces are $((V_x \cap T_xF) \oplus (E_x \oplus E_x^*), \Omega|_{V_x \cap T_xF} \oplus \omega_E)$. This isomorphism preserves the bivectors on the fibers because at each point it restricts to an isomorphism $T_x\mathcal{O} \to (V_x \cap T_xF) \oplus (E_x \oplus E_x^*)$ which matches the symplectic forms Ω_x and $\Omega|_{V_x \cap T_xF} \oplus \omega_E$. This shows that $TP_1|_M$ and $TP_2|_M$ are both isomorphic to the same Poisson vector bundle. □

Making a regularity assumption we can extend the infinitesimal uniqueness of Prop. 9.1 to a global statement.

Proposition 9.2. *Let M,P_1 and P_2 be as in Proposition 9.1, and assume additionally that the presymplectic leaves of (M, L) have constant dimension. Then P_1 and P_2 are neighborhood equivalent.*

Proof. Since the presymplectic leaves of (M, L) have constant dimension, by the proof of Proposition 9.1 the symplectic leaves of each P_i also have constant dimension in a tubolar neighborhood of P_i. We can find[26] identifications ϕ_i between normal bundles $N_i \subset TP_i|_M$

[25]In particular M intersects cleanly the symplectic leaves of P and the intersections are the presymplectic leaves of M.

[26]Let P denote either of P_1 or P_2. Let V be a distribution on M such that $V \oplus (L \cap TM) = TM$. We saw in the proof of Proposition 9.1 that $V \oplus \sharp V^\circ = TP|_M$ and $L \cap TM \subset \sharp V^\circ$. Define N as a smooth complement to $L \cap TM$ in $\sharp V^\circ$; then $TP|_M = TM \oplus N$ and N is tangent to the symplectic leaves of P at points of M. Choose a Riemannian metric on P and define $\phi(v_x)$ to be $exp^{\mathcal{O}} v$, where $exp^{\mathcal{O}}$ is the exponential map of the symplectic leaf $\mathcal{O}$ passing through x (with the induced metric). The resulting map $\phi : N \to P$ is well-defined since v is tangent to $\mathcal{O}$, it maps $N|_F$ onto an open neighborhood in $\mathcal{O}$, and it is smooth because the symplectic leaves of P form a regular foliation.

and tubular neighborhoods of M in P_i which identify $N|_F$ and $\mathcal{O}$ in an neighborhood of M (for each presymplectic leaf F of M and corresponding symplectic leaf $\mathcal{O}$ of P_i, $i = 1, 2$).

Using the Poisson vector bundle isomorphism $\Phi\colon TP_1|_M \to TP_2|_M$ of Proposition 9.1 we obtain an identification $\phi_2 \circ \Phi \circ \phi_1^{-1}$ between tubular neighborhoods of M in P_1 and P_2. Using this identification can view Π_2 as a Poisson structure on $P := P_1$ with two properties: it induces exactly the same foliation as Π_1, and it coincides with Π_1 on $TP|_M$. We want to show that there is a diffeomorphism near M, fixing M, which maps Π_1 to Π_2.

To this aim we want to apply Moser's theorem on each symplectic leaf of P (Thm. 7.1 of [6]). Denote by Ω_i the symplectic form given by Π_i on a leaf $\mathcal{O}$. Since the convex linear combination $(1-t)\Omega_1 + t\Omega_2$ is symplectic (because Ω_1 and Ω_2 coincide at points of M) and lies in the same cohomology class, by Moser's theorem there is a diffeomorphism of ψ of $\mathcal{O}$ such that $\psi^*\Omega_2 = \Omega_1$. Concretely this goes as follows (see Chapter 6 of [6]). We identify a neighborhood of P with N (via ϕ_1) and consider $\rho_t\colon N \to N, v \mapsto tv$, where $t \in [0, 1]$. Denote by v_t the vector tangent to the curve $\rho_s(v)$ at time $s = t$. Now just consider $N|_F$, where F is the presymplectic leaf $\mathcal{O} \cap M$. The operator

$$Q : \Omega^{\bullet}(\mathcal{O}) \to \Omega^{\bullet-1}(\mathcal{O}); \quad Q\omega = \int_0^1 \rho_t^*(i_{v_t}\omega)dt$$

has the property of providing primitives for closed differential forms whose pullback to M vanishes. So $\mu := Q(\Omega_2 - \Omega_1)$ is a primitive for $\Omega_2 - \Omega_1$. Consider the Moser vector field, obtained inverting via $(1-t)\Omega_1 + t\Omega_2$ the 1-form μ. Following from time 1 to time 0 the flow of the Moser vector field gives the desired diffeomorphism ψ (which keeps M fixed since μ vanishes at points of F).

This constructions varies smoothly from leaf to leaf: ρ_t and v_t are clearly smooth, and the foliated 2-forms $\Omega_2 - \Omega_1$ and $(1-t)\Omega_1 + t\Omega_2$ also are, as can be seen using coordinates adapted to the foliation. Hence we obtain a diffeomorphism ψ of a tubular neighborhood of M, fixing M, which maps Π_1 to Π_2. □

Since local uniqueness holds (see subsection 9.2) and since by Proposition 9.1 there is no topological obstruction, it seems that a global uniqueness statement should hold in the general case, i.e. when the presymplectic foliation of (M, L) is not necessarily regular. We conclude with some possible approaches to prove global uniqueness.

The argument from [1] just before Prop. 8.2 shows that the uniqueness of (minimal dimensional) coisotropic embeddings of a given Dirac manifold (M, L) is equivalent to the following: whenever (P_1, Π_1) and (P_2, Π_2) are minimal Poisson manifolds in which (M, L) embeds coisotropically there exists a diffeomorphism $\phi\colon P_1 \to P_2$ near M so that Π_2 and $\phi_*\Pi_1$ differ by the gauge transformation by a closed 2-form B vanishing on M. One could hope that if $\phi\colon P_1 \to P_2$ is chosen to match symplectic leaves and to match $\Pi_1|_M$ and $\Pi_2|_M$ then a 2-form B as above automatically exist. This is not the case, as the following example shows.

Example 9.3. Take $M = \mathbb{R}^3$ with Dirac structure

$$L = span\{(-x_1^2\partial_{x_2}, dx_1), (x_1^2\partial_{x_1}, dx_2), (\partial_{x_3}, 0)\}.$$

There are two open presymplectic leaves $(\mathbb{R}_{\pm} \times \mathbb{R}^2, \frac{1}{x_1^2}dx_1 \wedge dx_2)$ and 1-dimensional presymplectic leaves $\{0\} \times \{c\} \times \mathbb{R}$ with zero presymplectic form (for every real number c); hence our Dirac structure is a product of the Poisson structure $x_1^2\partial_{x_1} \wedge \partial_{x_2}$ and of the zero presymplectic form on the x_3-axis. The characteristic distribution $L \cap TM$ is always $span\partial_{x_3}$. Clearly the construction of Thm. 8.1 gives

$$P_1 := (\mathbb{R}^4, x_1^2 \partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{y_3})$$

where y_3 is the coordinate on the fibers of $P_1 \to M$.

Another Poisson structure on $\mathbb{R}^4$ with the same foliation as Π_1 and which coincides with Π_1 along M is the following:

$$\Pi_2 := x_1^2 \partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{y_3} + x_1 y_3 \partial_{x_2} \wedge \partial_{x_3}.$$

On each of the two open symplectic leaves $\mathbb{R}_\pm \times \mathbb{R}^3$ the symplectic form corresponding to Π_1 is $\Omega_1 = \frac{1}{x_1^2} dx_1 \wedge dx_2 + dx_3 \wedge dy_3$, whereas the one corresponding to Π_2 is $\Omega_2 = \Omega_1 + \frac{y_3}{x_1} dx_1 \wedge dy_3$. Clearly the difference $\Omega_1 - \Omega_2$ does not extend to smooth a 2-form on the whole of $\mathbb{R}^4$. Hence there is no smooth 2-form on $\mathbb{R}^4$ relating Π_1 and Π_2.

Nevertheless Π_1 and Π_2 are Poisson diffeomorphic: Prop. 9.7 in subsection 9.2 will tell us that they are in neighborhoods of the origin, and the construction of Prop. 9.7 will provide a global coordinate change that maps Π_2 into Π_1, namely the coordinate change that transforms x_2 into $x_2 + \frac{y_3^2}{2} x_1$ and leaves the other coordinates untouched.

One could try to obtain a $\phi\colon P_1 \to P_2$ as above by integrating the isomorphisms Φ constructed in Prop. 9.1. Alternatively one could show the existence, for any minimal Poisson manifold (P, Π) in which (M, L) embeds coisotropically, of a projection $\pi : P \to M$ such that Π and the pullback Dirac structure $\pi^* L$ be related by a (suitable) B-transformation (as happens for the Poisson manifold E^* of Thm. 8.1): by choosing a diffeomorphism $P_1 \to P_2$ intertwining the projections of P_1 and P_2 one would conclude that Π_1 and Π_2 are gauge equivalent. Another approach to prove global uniqueness is to construct a projection $\pi : P \to M$ with the weaker property that the Lie algebroids corresponding to Π and to $\pi^* L$ be isomorphic, for then the symplectic groupoids of any P_1,P_2 as above will be isomorphic as Lie groupoid; then one would try to relate the corresponding symplectic forms by the flow of a multiplicative Moser vector field.

9.2. **Local uniqueness.** While we are not able to prove a global uniqueness statement in the general case, we prove in this subsection that a local uniqueness statements holds. We start with a normal form statement.

Proposition 9.4. *Suppose we are given a Dirac manifold (M^m, L) for which $L \cap TM$ have constant rank k, and let (P, Π) a Poisson manifold of dimension $m + k$ in which (M, L) embeds coisotropically. Then about any $x \in M$ there is a neighborhood $U \subset P$ and coordinates $\{q_1, \dots, q_k, p_1, \dots, p_k, y_1, \dots, y_{m-k}\}$ defined on U such that locally M is given by the constraints $p_1 = 0, \dots, p_k = 0$ and*

$$\Pi = \sum_{i=1}^{k} \partial_{q_i} \wedge \partial_{p_i} + \sum_{i,j=1}^{m-k} \varphi_{ij}(y) \partial_{y_i} \wedge \partial_{y_j} \tag{8}$$

for functions $\varphi_{ij} \colon \mathbb{R}^{m-k} \to \mathbb{R}$.

Remark 9.5. The existence of coordinates in which Π has the above split form is guaranteed by Weinstein's Splitting Theorem [19]; the point in the above proposition is that one can choose the coordinates (q, p, y) so that M is given by the constrains $p = 0$.

The assumption on the dimension of P is equivalent to $T_x M + T_x \mathcal{O} = T_x P$ at every $x \in M$, where $\mathcal{O}$ is the symplectic leaf through x, which in turn is equivalent to $\sharp|_{N^*M}$ being injective.

Proof. We adapt the proof of Weinstein's Splitting Theorem [19] to our setting. To simplify the notation we will often write P in place of U and M in place of $M \cap U$. Choose a function q_1 on P near x such that dq_1 does not annihilate $L \cap TM$. Then $X_{q_1}|_M$ doesn't vanish and is transverse to M, because there is a $\xi \in N^*M$ with $0 \neq \langle \sharp\xi, dq_1 \rangle = -\langle \xi, X_{q_1} \rangle$. Choose a hypersurface in P containing M and transverse to $X_{q_1}|_M$, and determine the function p_1 by requiring that it vanishes on the hypersurface and $dp_1(X_{q_1}) = -1$. Since $[X_{q_1}, X_{p_1}] = X_1 = 0$ the span of X_{p_1} and X_{q_1} is an integrable distribution giving rise to a foliation of P by surfaces. This foliation is transverse to P_1, the codimension two submanifold where p_1 and q_1 vanish. $M_1 := P_1 \cap M$ is clean intersection and is a codimension one submanifold of M. To proceed inductively we need

Lemma 9.6. *P_1 has an induced Poisson structure, $M_1 \subset P_1$ is coisotropic, and the sharp-map $\sharp_1$ of P_1 is injective on the conormal bundle to M_1.*

Proof. P_1 is cosymplectic because it is given by constraints whose matrix of Poisson brackets is non-degenerate: $\{q_1, p_1\} = 1$. Hence it has an induced Poisson structure, whose sharp map we denote by $\sharp_1$. Recall from section 2 that if $\xi_1 \in T^*_x P_1$ then $\sharp_1 \xi_1 \in TP_1$ is given as follows: extend ξ_1 to a covector ξ of P by asking that it annihilates $\sharp N^*_x P_1$ and apply $\sharp$ to it. Now in particular let $x \in M_1$ and ξ_1 be an element of the conormal bundle of M_1 in P_1. We have $T_x M = T_x M_1 \oplus \mathbb{R} X_{p_1}(x) \subset T_x M_1 + \sharp N^*_x P_1$, so $\xi \in N^*_x M$, and since M is coisotropic in P we have $\sharp\xi \in T_x M$. Hence $\sharp_1 \xi_1 \in T_x P_1 \cap T_x M = T_x M_1$, which shows the claimed coisotropicity. The injectivity of $\sharp_1$ on the conormal bundle follows by the above together with the injectivity of $\sharp|_{N^*M}$. □

Thanks to Lemma 9.6 we are allowed to apply the above procedure to the codimension $k-1$ coisotropic submanifold M_1 of P_1. We obtain functions q_2, p_2 on P_1 such that[27] $\{q_2, p_2\}_1 = 1$, a codimension two submanifold P_2 of P_1 given by the points where q_2 and p_2 vanish, and a codimension one submanifold $M_2 := P_2 \cap M$ of P_2. After repeating this other $k-2$ times we get to M_k, a codimension zero submanifold of P_k which hence coincides with P_k.

Now we start working backwards: choose arbitrary functions $y_1, , y_{m-k}$ on P_k, extend them to P_{k-1} constantly along the surfaces integrating $span\{X_{q_k}, X_{p_k}\}$. The Poisson bracket on P_{k-1} satisfies $\{y_i, q_k\}_{k-1} = 0$ and $\{y_i, p_k\}_{k-1} = 0$, and using the Jacobi identity one sees that any $\{y_i, y_j\}_{k-1}$ Poisson commutes with q_k and p_k, and hence the $\{y_i, y_j\}_{k-1}$ are functions of the y's only. Now continue extending the y's and q_k, p_k to P_{k-2}. After k steps we obtain functions on P for which the non-trivial brackets are $\{q_i, p_i\} = 1$ and $\{y_i, y_j\} =: \varphi_{ij}(y)$. Hence formula (8) for the Poisson bivector Π follows.

To show that M is given by the constraints $p_1 = 0, \dots, p_k = 0$ we notice the following. We chose p_1 to vanish on M. We chose p_2 on P_1 to vanish on M_1, and we extended it to P asking that it be constant along the foliation tangent to the span of X_{p_1} and X_{q_1}. Since $X_{p_1}|_M$ is tangent to M and $TM|_{M_1} = TM_1 \oplus X_{p_1}|_{M_1}$, it follows that p_2 vanishes on the whole of M. Inductively one shows that all the p_i vanish on M, and by dimension counting one obtains that the p_i define exactly M. This concludes the proof of Prop. 9.5. □

Using the normal forms derived above we can prove local uniqueness:

[27] Here $\{q_2, p_2\}_1$ denotes the Poisson bracket on P_1, which coincides with (the restriction to P_1 of) the Poisson bracket on P of the functions obtained extending q_2, p_2 to P constantly along the surfaces tangent to $span\{X_{q_1}, X_{p_1}\}$.

Proposition 9.7. *Let (P,Π) and $(\bar{P},\bar{\Pi})$ be Poisson manifolds as in Prop. 9.4 in which (M,L) embeds coisotropically. Then about each $x \in M$ there are neighborhoods $U \subset P$, $\bar{U} \subset \bar{P}$ and a Poisson diffeomorphism $(U,\Pi) \cong (\bar{U},\bar{\Pi})$ which is the identity on M.*

Proof. By integrating the vector bundle isomorphism Φ of Prop. 9.1 we may assume that $\bar{\Pi}$ is a Poisson bivector on P and that it coincides with Π at points of M. We will show below that we can make choices of coordinates $\{q_i,p_i,y_j\}$ on U and $\{\bar{q}_i,\bar{p}_i,\bar{y}_j\}$ on $\bar{U}$ which bring Π and $\bar{\Pi}$ respectively in the canonical form (8) and which are *compatible*, in the sense that these coordinate sets coincide once restricted to M. Then the diffeomorphism of P induced by the obvious coordinate change

$$q_1 \mapsto \bar{q}_i, p_i \mapsto \bar{p}_i, y_j \mapsto \bar{y}_j$$

is the identity on M. Further it is a Poisson diffeomorphism: one just has to check that the functions φ_{ij} appearing in (8), which are just $\{y_i,y_j\}_P$, coincide with $\bar{\varphi}_{ij} = \{\bar{y}_i,\bar{y}_j\}_{\bar{P}}$ when we consider them as functions of the $m-k$ variables y_i or $\bar{y}_i$. To this aim notice that $y_i|_M$ annihilates the characteristic distribution $L \cap TM$ of M, for $L \cap TM$ is spanned by $X_{p_1}|_M, \cdots, X_{p_k}|_M$. Hence $y_i|_M$ is an admissible function (7) for the Dirac manifold (M,L), and similarly $y_j|_M$, so we can apply to them the bracket $\{\bullet,\bullet\}_M$ of admissible functions on (M,L) which is of course determined only by the Dirac structure L on M. Since X_{y_i} is tangent to M it follows that $\{y_i|_M,y_j|_M\}_M$ is just the restriction to M of $\{y_i,y_j\}_P$. Similarly $\{\bar{y}_i|_M,\bar{y}_j|_M\}_M$ is the restriction to M of $\{\bar{y}_i,\bar{y}_j\}_{\bar{P}}$. Since as we saw $y_i|_M = \bar{y}_i|_M$, we deduce that φ_{ij} and $\bar{\varphi}_{ij}$ coincide on M, so $\varphi_{ij} = \bar{\varphi}_{ij}$ as functions $\mathbb{R}^{m-k} \to \mathbb{R}$.

In the rest of the proof we show that it is possible to perform the construction of the proof of Prop. 9.4 (which depended on several choices) to obtain *compatible* coordinates $\{q_i,p_i,y_j\}$ and $\{\bar{q}_i,\bar{p}_i,\bar{y}_j\}$. We refer to the proof of Prop. 9.4 for the notation and decorate with a bar the objects arising from Π. Choose functions $q_1,\bar{q}_1$ on P around x so that the functions and their differentials agree at points of M (of course here we could just take $q_1 = \bar{q}_1$). Then $M_1 = \{q_1 = 0\} \cap M$ coincides with $\bar{M}_1$. The conditions on the differentials, together with $\Pi|_M = \bar{\Pi}|_M$, imply $X_{q_1}|_M = \bar{X}_{\bar{q}_1}|_M$ (where the second hamiltonian vector field is taken w.r.t $\bar{\Pi}$). Choose two hypersurfaces of P containing M such that their tangent spaces at points of M coincide (of course we could take the hypersurfaces to be equal). This determines the functions $p_1,\bar{p}_1$ on P. Notice that $dp_1|_M$ (a section of the vector bundle $T^*P|_M \to M$) and $d\bar{p}_1|_M$ coincide, because they have the same kernel and both evaluate to -1 on $X_{q_1}|_M = X_{\bar{q}_1}|_M$. This has two consequences: first $X_{p_1}|_M = \bar{X}_{\bar{p}_1}|_M$. Second, even though $P_1 := \{\text{points of } P \text{ where } p_1 = 0, q_1 = 0\}$ and $\bar{P}_1$ do not coincide, they are tangent to each other along $M_1 = \bar{M}_1$, since the differentials of q_1 and p_1 coincide with their barred counterparts on M and in particular on $M_1 = \bar{M}_1$. Further the Poisson structures induced by Π on P_1 and $\bar{\Pi}$ on $\bar{P}_1$ coincide at points of $M_1 = \bar{M}_1$, because Π and $\bar{\Pi}$ there. To summarize we showed

$$(9) \qquad M_1 = \bar{M}_1, \qquad TP_1|_{M_1} = T\bar{P}_1|_{\bar{M}_1} \text{ as Poisson vector bundles}, \qquad X_{p_1}|_M = \bar{X}_{\bar{p}_1}|_M.$$

Now we would like to apply the above procedure to $M_1 = \bar{M}_1$, which is coisotropic in the two Poisson manifolds P_1 and $\bar{P}_1$. The only difference to the above situation is that now we have two Poisson manifolds which do not agree as spaces. However since their tangent spaces along $M_1 = \bar{M}_1$ agree we can still proceed as above: we choose q_2 on P_1 and $\bar{q}_2$ on $\bar{P}_1$ so that they agree on $M_1 = \bar{M}_1$ together with their first derivates; we choose hypersurfaces in P_1 and $\bar{P}_1$ so that their tangent spaces along $M_1 = \bar{M}_1$ coincide, and these in turn determine p_2 and $\bar{p}_2$. Proceeding inductively we have

$$M_i = \bar{M}_i, \quad TP_i|_{M_i} = T\bar{P}_i|_{\bar{M}_i} \text{ as Poisson vector bundles}, \quad X_{p_i}|_{M_{i-1}} = \bar{X}_{\bar{p}_i}|_{\bar{M}_{i-1}} \text{ for } i \leq k.$$

Now we start working backwards. We choose arbitrary coordinates $\{y_1, \cdots, y_{m-k}\}$ on $P_k = \bar{P}_k$, and extend them to P_{k-1} constantly along the surfaces spanned by X_{q_k}, X_{p_k}, as well as to $\bar{P}_{k-1}$ constantly along the surfaces spanned by $\bar{X}_{\bar{q}_k}, \bar{X}_{\bar{p}_k}$. Since X_{p_k} and $\bar{X}_{\bar{p}_k}$ coincide on $M_{k-1} = \bar{M}_{k-1}$, we see that the resulting y and $\bar{y}$ coincide on $M_{k-1} = \bar{M}_{k-1}$. The coordinates q_k and $\bar{q}_k$ coincide there too by definition (p_k and $\bar{p}_k$ trivially too, because they vanish there). After k steps we see that the coordinates $q_1, \cdots, q_k, y_1, \cdots, y_{m-k}$ on P, once restricted to M, coincide with their barred counterparts. □

We refer to Example 9.3 for an example of the construction of Prop. 9.7

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