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Pre-Poisson submanifolds

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Abstract

We show that a pre-Poisson submanifold C of a Poisson manifold (i.e. a submanifold satisfying a certain constant rank condition) sits coisotropically inside some larger cosymplectic submanifold, which is naturally endowed with a Poisson structure. We present several examples, including cases when C is an affine subspace in the dual of a Lie algebra. We deduce the existence of deformation quantizations of the Poisson algebras of basic functions on a pre-Poisson submanifold.

1 Introduction

The following result in symplectic geometry is well known: a submanifold C of a symplectic manifold (M, Ω) is contained coisotropically in some symplectic submanifold of M iff the pullback of Ω to C has constant rank. In this work we extend this result to the setting of Poisson geometry.

Part of the motivation comes from applications of the Poisson sigma model to quantization problems. The Poisson sigma model is a topological field theory, whose fields are bundle maps from $T\Sigma$ (for Σ a surface) to the cotangent bundle T^*P of a Poisson manifold (P, Π). It was used by Felder and the first author [6] to derive and interpret Kontsevich's formality theorem and his star product on the Poisson manifold P. The Poisson sigma model with boundary conditions on a coisotropic[1] submanifold C, when suitable assumptions on C are satisfied and P is assumed to be an open subset of $\mathbb{R}^n$, provides [7] a deformation quantization of the Poisson algebra of basic (invariant) functions $C^\infty_{bas}(C)$ on C, i.e. of the functions on C whose differentials annihilate the distribution $\sharp N^*C$. This result was globalized using a supergeometric version of Kontsevich's formality theorem [8]: when the first and second cohomology of the Lie algebroid N^*C vanish, for C a coisotropic submanifold of any Poisson manifold P, the Poisson algebra $C^\infty_{bas}(C)$ admits a deformation quantization. Notice that the quotient of C by the distribution $\sharp N^*C$ is usually not a smooth manifold. Hence $C^\infty_{bas}(C)$ is usually not the algebra of functions on any Poisson manifold, and

[1] Recall that a submanifold C of P is coisotropic if $\sharp N^*C \subset TC$, where $N^*C = TC^\circ$ is the conormal bundle of C and $\sharp\colon T^*P \to TP$ is contraction with the bivector Π.

one can not apply directly Kontsevich's theorem [13] on deformation quantization of Poisson manifolds.

Calvo and Falceto considered the Poisson sigma model with boundary conditions on submanifolds of (P, Π) more general than the coisotropic ones, namely submanifolds C for which $TC + \sharp N^*C$ has constant rank. They call this condition "strong regularity", and we will refer to such submanifolds as *pre-Poisson*. Calvo and Falceto show [3] that when P is an open subset of $\mathbb{R}^n$ the problem of deformation quantizing the Poisson algebra of basic functions on C can be reduced to the results of [7], hence the algebra of basic functions can be deformation quantized when assumptions analog to those of [7] are satisfied. The computations in [3] are carried out choosing local coordinates on P adapted to C. The strong regularity condition allows to choose locally constraints for C such that the number of first class constraints (X^μ whose Poisson bracket with all other constrains vanish on C) and second class constraints (the remaining constraints X^A, which automatically satisfy $det\{X^A, X^B\} \neq 0$ on C) be constant along C. Setting the second class constraints X^A to zero locally gives a submanifold with an induced Poisson structure, and the fact that only first class constraints are left means that C lies in it as a coisotropic submanifold.

We consider the natural question of whether one can carry out the above procedure globally, i.e. without resorting to adapted coordinates. More generally we ask the following question:

- Given an arbitrary submanifold C of a Poisson manifold (P, Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that

  a) $\tilde{P}$ has a Poisson structure induced from Π

  b) C is a coisotropic submanifold of $\tilde{P}$?

  When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence, (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

We answer the question giving sufficient conditions and necessary conditions in section 5, where we also provide examples. In particular, we answer the above globalization question positively for pre-Poisson submanifolds in section 3: for any pre-Poisson submanifold C of a Poisson manifold P there is a submanifold $\tilde{P}$ which is cosymplectic (and hence has a canonically induced Poisson structure) such that C lies coisotropically in $\tilde{P}$. Further (section 4) this cosymplectic submanifold is unique up to neighborhood equivalence. In section 6 we discuss the case when P has a linear Poisson structure (i.e. when P is the dual of a Lie algebra $\mathfrak{g}$) and C is either the translate of the annihilator of a Lie subalgebra or the annihilator of some subspace of $\mathfrak{g}$. Then in section 7 we deduce statements about the algebra of basic functions on C and its deformation quantization: if C is a pre-Poisson submanifold so that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish, then the

Poisson algebra of basic functions on C admits a deformation quantization. Finally in section 8, assuming that the symplectic groupoid $\Gamma_s(P)$ of P exists, we describe two subgroupoids (an isotropic and a presymplectic one) naturally associated to a pre-Poisson submanifold C of P.

With the exception of Section 6 the content of this letter is a shorter version of a the work [9] by the authors, where the reader can find the proofs of the statements that appear without proof in this letter.

Conventions: We use the term "presymplectic manifold" to denote a manifold endowed with a closed 2-form of *constant rank*, i.e. such that its kernel have constant rank.

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2 Basic definitions

We will use some notions from Dirac linear algebra [10] [1]. A Dirac structure on a vector space P is a subspace $L \subset P \oplus P^*$ which is maximal isotropic w.r.t. the natural symmetric inner product on $P \oplus P^*$ (i.e. L is isotropic and has same dimension as P). A Dirac structure L specifies a subspace $\mathcal{O}$, defined as the image of L under the projection $P \oplus P^* \to P$, and a skew-symmetric bilinear form ω on $\mathcal{O}$, given by $\omega(X_1, X_2) = \langle \xi_1, X_2 \rangle$ where ξ_1 is any element of P^* such that $(X_1, \xi_1) \in L$. The kernel of ω (which in terms of L is given as $L \cap P$) is called *characteristic subspace*. Conversely, any choice of bilinear form defined on a subspace of P determines a Dirac structure on P. Given this equivalence, we will sometimes work with the bilinear form ω on $\mathcal{O}$ instead of working with L.

We consider now Poisson vector spaces (P, Π) (i.e. $\Pi \in \wedge^2 P$; we denote by $\sharp \colon P^* \to P$ the map induced by contraction with Π). The Poisson structure on P is encoded by the Dirac structure $L_P = \{(\sharp\xi, \xi) \colon \xi \in P^*\}$. The image of L_P under the projection onto the first factor is $\mathcal{O} = \sharp P^*$, and the bilinear form ω is non-degenerate.

Remark 2.1. We recall that any subspace W of a Dirac vector space (P, L) has an induced Dirac structure L_W; the bilinear form characterizing L_W is just the pullback of ω (hence it is defined on $W \cap \mathcal{O}$). When (P, Π) is actually a Poisson vector space, one shows[2] that the symplectic orthogonal of $W \cap \mathcal{O}$ in $(\mathcal{O}, \omega)$ is $\sharp W^\circ$. Hence $\sharp W^\circ \cap W$

[2]Indeed, writing vectors in $\mathcal{O}$ as $\sharp\xi$ for some $\xi \in P^*$, one sees that the symplectic orthogonal of $W \cap \mathcal{O}$ is $\sharp(W \cap \mathcal{O})^\circ$. This space coincides with $\sharp W^\circ$ because their respective annihilators $\sharp^{-1}(W \cap \mathcal{O})$ and $\sharp^{-1} W$ coincide.

is the kernel of the restriction of ω to $W \cap \mathcal{O}$, i.e. it is the characteristic subspace of the Dirac structure L_W, and we will refer to it as the *characteristic subspace of* W. Notice that pulling back Dirac structure is functorial [1] (i.e. if W is contained in some other subspace W' of P, pulling back L first to W' and then to W gives the Dirac structure L_W), hence L_W, along with the corresponding bilinear form and characteristic subspace, is *intrinsic* to W.

Let W be a subspace of the Poisson vector space (P, Π). W is called *coisotropic* if $\sharp W^\circ \subset W$, which by the above means that $W \cap \mathcal{O}$ is coisotropic in $(\mathcal{O}, \omega)$.

W is called *Poisson-Dirac* subspace [11] when $\sharp W^\circ \cap W = \{0\}$; equivalent conditions are that $W \cap \mathcal{O}$ be a symplectic subspace of $(\mathcal{O}, \omega)$ or that the pullback Dirac structure L_P correspond to a Poisson bivector on W.

W is called *cosymplectic* subspace if $\sharp W^\circ \oplus W = P$, or equivalently if the push-forward of Π via the projection $P \to P/W$ is an invertible bivector. Notice that if W is cosymplectic then it has a canonical complement $\sharp W^\circ$ which is a symplectic subspace of $(\mathcal{O}, \omega)$. Clearly a cosymplectic subspace is automatically a Poisson-Dirac subspace, and the Poisson bivector on W can be expressed in a particularly simple way [11]: its sharp map $\sharp_W \colon W^* \to W$ is given by $\sharp_W \tilde{\xi} = \sharp \xi$, where $\xi \in P^*$ is the extension of $\tilde{\xi}$ which annihilates $\sharp W^\circ$.

Now we pass to the global definitions. A Dirac structure on P is a maximal isotropic subbundle $L \subset TP \oplus T^*P$ which is closed under the so-called Courant bracket (see [10]). The image of L under the projection onto the first factor is an integrable singular distribution, whose leaves (which are called presymplectic leaves) are endowed with closed 2-forms. A Poisson structure on P is a bivector Π such that $[\Pi, \Pi] = 0$. Coisotropic and cosymplectic submanifolds of a Poisson manifold are defined exactly as in the linear case; a Poisson-Dirac submanifold additionally requires that the bivector induced on the submanifold by the point-wise condition be smooth [11]. Cosymplectic submanifolds are automatically Poisson-Dirac submanifolds[3]. The Poisson bracket on a cosymplectic submanifold $\tilde{P}$ of (P, Π) is computed as follows: $\{\tilde{f}_2, \tilde{f}_2\}_{\tilde{P}}$ is the restriction to $\tilde{P}$ of $\{f_1, f_2\}$, where the f_i are extensions of $\tilde{f}_i$ to P such that $df_i|_{\sharp N^*\tilde{P}} = 0$ (for at least one of the two functions).

We will also need a definition which does not have a linear algebra counterpart.

Definition 2.2. A submanifold C of a Poisson manifold (P, Π) is called *pre-Poisson* if the rank of $TC + \sharp N^*C$ is constant along C.

Calvo and Falceto already considered [2][3] such submanifolds and called them "strongly regular submanifolds". We prefer to call them "pre-Poisson" because when

[3]Indeed the bivector induced on a cosymplectic submanifold $\tilde{P}$ is always smooth: denote by L_P the Dirac structure corresponding to the Poisson structure on P and by $L_{\tilde{P}}$ its pullback to $\tilde{P}$. $L_{\tilde{P}} \cong L_P \cap (T\tilde{P} \oplus T^*P)/L_P \cap (\{0\} \oplus N^*\tilde{P})$ (see [10]), and both numerator and denominator have constant rank because $L_P \cap (\{0\} \oplus N^*\tilde{P}) = ker\sharp|_{N^*\tilde{P}} = \{0\}$.

P is a symplectic manifold they reduce to presymplectic submanifolds[4]. See Section 5 for several examples.

3 Existence of coisotropic embeddings for pre-Poisson submanifolds

In this section we consider the problem of embedding a submanifold of a Poisson manifold coisotropically in a Poisson-Dirac submanifold, and show that this can be always done for pre-Poisson submanifolds.

Lemma 3.1. *Let* (P, Π) *be a Poisson vector space and* C *a subspace. The Poisson-Dirac subspaces of* P *in which* C *sits coisotropically are exactly those of the form* $R \oplus C$*, where* R *is such that*

$$R \oplus (C + \sharp C^\circ) \supset \mathcal{O}, \tag{3.1}$$

where $\mathcal{O} = \sharp P^*$*. Among the Poisson-Dirac subspaces above the cosymplectic ones are exactly those of maximal dimension, i.e. those for which* $R \oplus (C + \sharp C^\circ) = P$*.*

Now we pass from linear algebra to global geometry. Given a submanifold C of a Poisson manifold P, one might try to construct a Poisson-Dirac submanifold in which C embeds coisotropically applying the corresponding symplectic contruction "leaf by leaf" in a smooth way. It would be then natural to require that the characteristic "distribution" $TC \cap \sharp N^*C$ of C have constant rank. However this approach generally does not work because even when it has constant rank $TC \cap \sharp N^*C$ might not be smooth (see example 5.4). Lemma 3.1 suggests instead to require that C be pre-Poisson and extend C *not* only "along the symplectic leaves of P".

Theorem 3.2. *Let* C *be a pre-Poisson submanifold of a Poisson manifold* (P, Π)*. Then there exists a cosymplectic submanifold* $\tilde{P}$ *containing* C *such that* C *is coisotropic in* $\tilde{P}$*.*

Proof. Because of the rank condition on C we can choose a smooth subbundle R of $TP|_C$ which is a complement to $TC + \sharp N^*C$. Then by Lemma 3.1 at every point p of C we have that $T_pC \oplus R_p$ is a cosymplectic subspace of T_pP in which T_pC sits coisotropically. "Thicken" C to a smooth submanifold $\tilde{P}$ of P satisfying $T\tilde{P}|_C = TC \oplus R$. If we can show that, in a neighborhood of C, $\tilde{P}$ is a cosymplectic submanifold, then we are done.

First we show that at points p near C the restriction of $\sharp$ to $N_p^*\tilde{P}$ is injective. By the proof of Lemma 3.1 we know that this is equivalent to $T_p\mathcal{O}_p + T_p\tilde{P} = T_pP$

[4]Further reasons are the following: the subgroupoid associated to a pre-Poisson manifold, when it exists, is presymplectic (see Prop. 8.5). The Hamiltonian version of the Poisson Sigma Model with boundary conditions on P (at $t = 0$) and on a submanifold C (at $t = 1$) delivers a space of solutions which is presymplectic iff C is pre-Poisson.

(where $\mathcal{O}_p$ the symplectic leaf of P through p) and that it is true if p belongs to C. The case $p \notin C$ is reduced to this using Weinstein's local structure theorem [15] which states that, near any $q \in C$, P is isomorphic (as a Poisson manifold) to the product of the symplectic leaf $\mathcal{O}_q$ and a Poisson manifold whose bivector vanishes at q. Under this isomorphism $T_q\mathcal{O}_q$ can be identified with a subspace of $T_p\mathcal{O}_p$, hence from $T\mathcal{O}_q + T\tilde{P} = TP$ at q we deduce $T\mathcal{O}_p + T\tilde{P} = TP$ at p. So we showed that the restriction of $\sharp$ to $N^*\tilde{P}$ is injective, hence $\sharp N^*\tilde{P}$ is a smooth constant rank subbundle of TP. The rank of $T\tilde{P} \cap \sharp N^*\tilde{P}$, which is the intersection of two smooth subbundles, can locally only decrease, and since it is zero along C it is zero also in a neighborhood of C. By dimension counting we deduce $T\tilde{P} \oplus \sharp N^*\tilde{P} = TP$, i.e. $\tilde{P}$ is cosymplectic. □

Remark 3.3. The above proposition says that if C is pre-Poisson then we can choose a subbundle R over C with fibers as in eq. (3.1) and "extend" C in direction of R to obtain a Poisson-Dirac submanifold of P containing C coisotropically. If C is not a pre-Poisson submanifold of (P, Π), we might still be able to find a smooth bundle R over C consisting of subspaces as in eq. (3.1). However "extending" C in direction of this subbundle will usually not give a submanifold with a smooth Poisson-Dirac structure, see Example 5.7 below.

Now we deduce consequences about Lie algebroids. See section 8 for the corresponding integrated statement.

Proposition 3.4. *Let C be a submanifold of a Poisson manifold (P, Π). Then $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P iff C is pre-Poisson. Further, for any cosymplectic submanifold $\tilde{P}$ in which C sits coisotropically, $N^*C \cap \sharp^{-1}TC$ is isomorphic as a Lie algebroid to the annihilator of C in $\tilde{P}$.*

4 Uniqueness of coisotropic embeddings for pre-Poisson submanifolds

Given a submanifold C of a Poisson manifold (P, Π) in this section we investigate the uniqueness (up Poisson diffeomorphisms fixing C) of *cosymplectic* submanifolds in which C is embedded coisotropically.

This lemma tells us that we need consider only the case that C be pre-Poisson and the construction of Thm. 3.2:

Lemma 4.1. *A submanifold C of a Poisson manifold (P, Π) can be embedded coisotropically in a cosymplectic submanifold $\tilde{P}$ iff it is pre-Poisson. In this case all such $\tilde{P}$ are constructed (in a neighborhood of C) as in Thm. 3.2.*

Proof. In Thm. 3.2 we saw that given any pre-Poisson submanifold C, choosing a smooth subbundle R with $R \oplus (TC + \sharp N^*C) = TP|_C$ and "thickening" C in direction of R gives a submanifold $\tilde{P}$ with the required properties.

Now let C be any submanifold embedded coisotropically in a cosymplectic submanifold $\tilde{P}$. By Lemma 3.1, for any complement R of TC in $T\tilde{P}|_C$ we have $R \oplus (TC + \sharp N^*C) = TP|_C$. This has two consequences: first the rank of $TC + \sharp N^*C$ must be constant, concluding the proof of the "iff" statement of the lemma. Second, it proves the final statement of the lemma. □

When C is a point $\{x\}$ then $\tilde{P}$ as above is a slice transverse to the symplectic leaf through x (see Ex. 5.1) and $\tilde{P}$ is unique up Poisson diffeomorphism by Weinstein's splitting theorem (Lemma 2.2 in [15]; see also Thm. 2.16 in [14]). A generalization of its proof gives

Theorem 4.2. *Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}_0$, $\tilde{P}_1$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then, shrinking $\tilde{P}_0$ and $\tilde{P}_1$ to a smaller tubular neighborhood of C if necessary, there is a Poisson diffeomorphism Φ from $\tilde{P}_0$ to $\tilde{P}_1$ which is the identity on C.*

5 Conditions and examples

Let C be as usual a submanifold of the Poisson manifold (P, Π); in Section 3 we considered the question of existence of a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically. In Thm. 3.2 we showed that a *sufficient* condition is that C be pre-Poisson, which by Prop. 3.4 is equivalent to saying that $N^*C \cap \sharp^{-1}TC$ be a Lie algebroid.

A *necessary* condition is that the (intrinsically defined) characteristic distribution $TC \cap \sharp N^*C$ of C be the distribution associated to a Lie algebroid over C; in particular its rank locally can only increase. This is a necessary condition since the concept of characteristic distribution is an intrinsic one (see Remark 2.1), and the characteristic distribution of a coisotropic submanifold of a Poisson manifold is the image of the anchor of its conormal bundle, which is a Lie algebroid.

The following are examples of pre-Poisson submanifolds; see section 6 for more examples.

Example 5.1. An obvious example is when C is a coisotropic submanifold of P, and in this case the construction of Thm. 3.2 delivers $\tilde{P} = P$ (or more precisely, a tubular neighborhood of C in P).

Another obvious example is when C is just a point x: then the construction of Thm. 3.2 delivers as $\tilde{P}$ any slice through x transversal to the symplectic leaf $\mathcal{O}_x$.

Now if $C_1 \subset P_1$ and $C_2 \subset P_2$ are pre-Poisson submanifolds of Poisson manifolds, the cartesian product $C_1 \times C_2 \subset P_1 \times P_2$ also is, and if the construction of Thm. 3.2 gives cosymplectic submanifolds $\tilde{P}_1 \subset P_1$ and $\tilde{P}_2 \subset P_2$, the same construction applied to $C_1 \times C_2$ (upon suitable choices of complementary subbundles) delivers the cosymplectic submanifold $\tilde{P}_1 \times \tilde{P}_2$ of $P_1 \times P_2$. In particular, if C_1 is coisotropic and C_2 just a point x, then $C_1 \times \{x\}$ is pre-Poisson.

The following are two examples of submanifolds C which surely can not be imbedded coisotropically in any Poisson-Dirac submanifold:

Example 5.2. The submanifold $C = \{(x_1, x_2, x_2^2, x_1^2)\}$ of the symplectic manifold $(P, \omega) = (\mathbb{R}^4, dx_1 \wedge dx_3 + dx_2 \wedge dx_4)$ has characteristic distribution of rank 2 on the points with $x_1 = x_2$ and rank zero on the rest of C. The rank of the characteristic distribution locally decreases, hence C does not satisfies the necessary condition above.

Remark 5.3. If C is a submanifold of a symplectic manifold (P, ω), then the necessary and the sufficient conditions coincide, both being equivalent to saying that the characteristic distribution of C (which can be described as $\ker(i_C^*\omega)$ for i_C the inclusion) have constant rank, i.e. that C be presymplectic.

Example 5.4. Consider the Poisson[5] manifold $(\mathbb{R}^6, x_1\partial_{x_2} \wedge \partial_{x_4} + (\partial_{x_3} + x_1\partial_{x_5}) \wedge \partial_{x_6})$. Let C be the three-dimensional subspace given by setting $x_4 = x_5 = x_6 = 0$. The characteristic subspaces are all one-dimensional, spanned by ∂_{x_3} at points of C where $x_1 = 0$ and by ∂_{x_2} on the rest of C. Hence the characteristic subspaces don't form a smooth distribution, and can not be the image of the anchor map of any Lie algebroid over C. Hence C does not satisfies the necessary condition above.

The *sufficient* condition above is not necessary (i.e. the first inclusion in the diagram above is strict), as either of the following simple examples shows.

Example 5.5. Take C to be the vertical line $\{x = y = 0\}$ in the Poisson manifold $(P, \Pi) = (\mathbb{R}^3, f(z)\partial_x \wedge \partial_y)$, where f is any function with at least one zero. Then C is a Poisson-Dirac submanifold (with zero induced Poisson structure), hence taking $\tilde{P} := C$ we obtain a Poisson-Dirac submanifold in which C embeds coisotropically. The sufficient conditions here is not satisfied, for the rank of $TC + \sharp N^*C$ at $(0, 0, z)$ is 3 at points where f does not vanishes and 1 at points where f vanishes.

Example 5.6. Consider the Poisson manifold $(P, \Pi) = (\mathbb{R}^4, x^2\partial_x \wedge \partial_y + z\partial_z \wedge \partial_w)$ as in Example 6 of [11] and the submanifold $C = \{(z^2, 0, z, 0) : z \in \mathbb{R}\}$. The rank of $TC + \sharp N^*C$ is 3 away from the origin (because there C is an isotropic submanifold in an open symplectic leaf of P) and 1 at the origin (since Π vanishes there). The submanifold $\tilde{P} = \{(z^2, 0, z, w) : z, w \in \mathbb{R}\}$ is Poisson-Dirac and it clearly contains C as a coisotropic submanifold.

The *necessary* condition above is not a sufficient (i.e. the second inclusion in the diagram above is strict):

[5]This is really a Poisson structure because the bracket of any two coordinates is a Casimir functions (indeed either a constant or x_1), so that the Jacobiator of any three coordinate functions vanishes.

Example 5.7. In Example 3 in Section 8.2 of [11] the authors consider the manifold $P = \mathbb{C}^3$ with complex coordinates x, y, z and specify a Poisson structure on it by declaring the symplectic leaves to be the complex lines given by $dy = 0, dz - ydx = 0$, the symplectic forms being the restrictions of the canonical symplectic form on $\mathbb{C}^3$. They consider submanifold C the complex plane $\{z = 0\}$ and show that C is point-wise Poisson-Dirac (i.e. $TC \cap \sharp N^*C = \{0\}$ at every point), but that the induced bivector field is not smooth. Being point-wise Poisson-Dirac, C satisfies the necessary condition above. However there exists no Poisson-Dirac submanifold $\tilde{P}$ of P in which C embeds coisotropically. Indeed at points p of C where $y \neq 0$ we have $T_pC \oplus T_p\mathcal{O} = TP$ (where as usual $\mathcal{O}$ is a symplectic leaf of P through p), from which follows that $\sharp|_{N_p^*C}$ is injective and $T_pC \oplus \sharp N_p^*C = TP$. From Lemma 3.1 (notice that the subspace R there must have trivial intersection with $T_pC \oplus \sharp N_p^*C$, so R must be the zero subbundle over C) it follows that the only candidate for $\tilde{P}$ is C itself. However, as we have seen, the Poisson bivector induced on C is not smooth. (More generally, examples are provided by any submanifold C of a Poisson manifold P which is point-wise Poisson-Dirac but not Poisson-Dirac and for which there exists a point p at which $T_pC \oplus T_p\mathcal{O} = TP$.)
Notice that this provides an example for the claim made in Remark 3.3, because the zero subbundle R over C satisfies the condition of Lemma 3.1 at every point of C and is obviously a smooth subbundle.

6 Duals of Lie algebras

In this subsection $\mathfrak{g}$ will always denote a finite dimensional Lie algebra. It is well known that its dual $\mathfrak{g}^*$ is a Poisson manifold, whose Poisson bracket on linear functions (which can be identified with elements of $\mathfrak{g}$) is given by $\{g_1, g_2\} := [g_1, g_2]$. It is known that if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then its annihilator $\mathfrak{h}^\circ$ is a coisotropic submanifold of $\mathfrak{g}^*$ (also see Prop. 6.2 below). We want to look at two generalizations: the first considers affine subspaces obtained translating $\mathfrak{h}^\circ$; the second one is obtained weakening the condition that $\mathfrak{h}$ be a subalgebra.

We start with a lemma that holds not only for linear Poisson structures:

Lemma 6.1. *Let P_1, P_2 be Poisson manifolds and $f : P_1 \to P_2$ be a submersive Poisson morphism. If $C \subset P_2$ is a pre-Poisson submanifold then $f^{-1}(C)$ is a pre-Poisson submanifold of P_1. Further, if $\tilde{P}_2$ is a cosymplectic submanifold containing C as a coisotropic submanifold, then $f^{-1}(\tilde{P}_2)$ is a cosymplectic submanifold containing $f^{-1}(C)$ as a coisotropic submanifold.*

Proof. Let $y \in C$ and $x \in f^{-1}(y)$. Since

$$f_*(\sharp N_x^*(f^{-1}(C))) = f_*(\sharp f^*(N_y^*C)) = \sharp N_y^*C$$

it follows that the restriction of f_* to $T_x(f^{-1}(C)) + \sharp N_x^*(f^{-1}(C))$ has image $T_yC + \sharp N_y^*C$, whose rank is independent of $y \in C$ by assumption. Since the kernel of this restricion, being $T_x(f^{-1}(y))$, also has constant rank, it follows that $f^{-1}(C)$ is pre-Poisson.

Further it is clear that f_* maps a complement R_x of $T_x(f^{-1}(C)) + \sharp N_x^*(f^{-1}(C))$ in T_xP_1 isomorphically onto a complement R_y of $T_yC + \sharp N_y^*C$ in T_yP_2, so that $R_x + T_x(f^{-1}(C))$ is the pre-image of $R_y + T_yC$ under f_*. Using Lemma 4.1 this proves the second assertion. □

Hence we have

Proposition 6.2. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and fix $\lambda \in \mathfrak{g}^*$. Then the affine subspace $C := \mathfrak{h}^\circ + \lambda$ is always pre-Poisson, and it is coisotropic iff λ is a character of $\mathfrak{h}$ (i.e. by definition $\lambda \in [\mathfrak{h}, \mathfrak{h}]^\circ$).*

Proof. The restriction $f : \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map because $\mathfrak{h}$ is a Lie subalgebra. Every point ν of $\mathfrak{h}^*$ is a pre-Poisson submanifold (see Ex. 5.1), hence by Lemma 6.1 its pre-image $f^{-1}(\nu)$ (which will be a translate of $\mathfrak{h}^\circ$) is pre-Poisson. Notice that by Lemma 6.1 we also know that, for any slice $S \subset \mathfrak{h}^*$ transverse to the H-coadjoint orbit through ν, $f^{-1}(S)$ is a cosymplectic submanifold containing coisotropically $f^{-1}(\nu)$. Further from the proof of Lemma 6.1 it is clear that $f^{-1}(\nu)$ is coisotropic in $\mathfrak{g}^*$ iff $\{\nu\}$ is coisotropic in $\mathfrak{h}^*$, i.e. if ν is a fixed-point of the H-coadjoint action or equivalently $\nu|_{[\mathfrak{h},\mathfrak{h}]} = 0$. □

Remark 6.3. An alternative proof of Prop. 6.2 can be given using Lemma 6.5 below. Indeed any $x \in C$ can be written uniquely as $y + \lambda$ where $y \in \mathfrak{h}^\circ$. Notice that $ad_h^*(y) \in \mathfrak{h}^\circ$ for all $h \in \mathfrak{h}$, because $\langle ad_h^*(y), \mathfrak{h}\rangle = \langle y, [h, \mathfrak{h}]\rangle$ vanishes since $\mathfrak{h}$ is a subalgebra. Hence

$$T_xC + \sharp N_x^*C = \mathfrak{h}^\circ + \{ad_h^*(y) + ad_h^*(\lambda) : h \in \mathfrak{h}\} = \mathfrak{h}^\circ + ad_{\mathfrak{h}}^*(\lambda),$$

which is independent on the point x. From the first computation above (applied to λ instead of y) it is clear that $ad_{\mathfrak{h}}^*(\lambda) \in \mathfrak{h}^\circ$ iff $\lambda \in [\mathfrak{h}, \mathfrak{h}]^\circ$.

Example 6.4. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. In a suitable basis the Lie algebra structure is given by $[e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$. The symplectic leaves of $\mathfrak{g}^*$ are given essentially[6] by the connected components of level sets of the Casimir function $\nu_1^2 + \nu_2^2 - \nu_3^2$ (where ν_i is just e_i viewed as a linear function on $\mathfrak{g}^*$), and they consist of a familiy of two-sheeted hyperboloids, the cone $\nu_1^2 + \nu_2^2 - \nu_3^2 = 0$ and a family of two-sheeted hyperboloids [4]. $C := \{(0, t, t) : t \in \mathbb{R}\} \subset \mathfrak{g}^*$ is contained in the cone and is clearly a coisotropic submanifold; indeed its the annihilator of the Lie subalgebra $\mathfrak{h} := span\{e_1, e_2 - e_3\}$ of $\mathfrak{g}$. If we translate C by an element in the annihilator of $[\mathfrak{h}, \mathfrak{h}] = \mathbb{R}(e_2 - e_3)$ we obtain an affine line contained in one of the hyperboloids, which hence is lagrangian in there and therefore coisotropic in $\mathfrak{g}^*$. If we translate C by any other $\lambda \in \mathfrak{g}^*$ we obtain a line that intersects transversely the hyperboloids, so at every point of such a line C' we have $TC' + \sharp N^*C' = T\mathfrak{g}^*$, showing that C' is pre-Poisson.

[6]The cone is the union of 3 leaves, one being the origin.

Before considering the case when $\mathfrak{h}$ is *not* a subalgebra of $\mathfrak{g}$ we need a

Lemma 6.5. *Let $C \subset \mathfrak{g}^*$ be an affine subspace obtained by translating the annihilator of some linear subspace $\mathfrak{h} \subset \mathfrak{g}$. Then $\sharp N^*_x C = ad^*_{\mathfrak{h}}(x) := \{ad^*_h(x) : h \in \mathfrak{h}\}$ for all $x \in C$.*

Proof. $N^*_x C$ is given by the differentials at x of the functions $h \in \mathfrak{h} \subset C^\infty(\mathfrak{g}^*)$. Now for any $g \in \mathfrak{g}$ we have

$$\langle \sharp d_x h, g\rangle = d_x g(\sharp d_x h) = \{h, g\}(x) = \langle [h, g], x\rangle = \langle ad^*_h(x), g\rangle,$$

i.e. $\sharp d_x h = ad^*_h(x)$. □

Now we consider the case when $\mathfrak{h}$ is just a linear subspace of $\mathfrak{g}$ and $\mathfrak{h}^\circ \subset \mathfrak{g}^*$ its dual. Since the Poisson tensor of $\mathfrak{g}^*$ vanishes at the origin we have $T(\mathfrak{h}^\circ)+\sharp N^*(\mathfrak{h}^\circ) = T(\mathfrak{h}^\circ)$ at the origin, so $\mathfrak{h}^\circ$ is pre-Poisson iff it is coisotropic (i.e. if $\mathfrak{h}$ is a Lie subalgebra). However the open subset C of $\mathfrak{h}$ on which $T(\mathfrak{h}^\circ) + \sharp N^*(\mathfrak{h}^\circ)$ has maximal rank will be pre-Poisson. Further, shrinking C if necessary, we can find a subspace $R \subset \mathfrak{g}^*$ (independent of $x \in C$) with $R \oplus (T_x C + \sharp N^*_x C) = \mathfrak{g}^*$ for all $x \in C$. For example we can construct such an R at one point $\bar{x}$ of C, and since transversality is an open condition R will be transverse to $TC + \sharp N^*C$ in a neighborhood of $\bar{x}$ in C. By Thm. 3.2 an open subset $\tilde{P}$ (containing C) of the subspace $\mathfrak{p}^\circ := R \oplus C$ is cosymplectic. If we *assume* that $\sharp N^*_y \tilde{P}$ is independent of the footpoint $y \in \tilde{P}$ then we are in the situation of the following proposition.

Proposition 6.6. *Let $\mathfrak{p}$ be a linear subspace of $\mathfrak{g}$ such that an open subset $\tilde{P} \subset \mathfrak{p}^\circ$ is cosymplectic and $\mathfrak{k}^\circ := \sharp N^*_y \tilde{P}$ is independent of $y \in \tilde{P}$. Then $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Hence, whenever $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $(\mathfrak{k}, \mathfrak{p})$ forms a symmetric pair [12].*

Proof. The fact that $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ follows from $\mathfrak{k}^\circ \oplus \mathfrak{p}^\circ = \mathfrak{g}^*$, which holds because $\tilde{P}$ is cosymplectic. Recall that given functions f_1, f_2 on $\tilde{P}$, the bracket $\{f_1, f_2\}_{\tilde{P}}$ is obtained by extending the functions in a constant way along $\mathfrak{k}^\circ$ to obtain functions $\hat{f}_1, \hat{f}_2$ on $\mathfrak{g}^*$, taking their Poisson bracket and restricting to $\tilde{P}$. Further (see Cor. 2.11 of [16]) the differential of $\{\hat{f}_1, \hat{f}_2\}$ at any point of $\tilde{P}$ kills $\mathfrak{k}^\circ$. So if the f_i are restrictions of linear functions on $\mathfrak{p}^\circ$ then the $\hat{f}_i$ will be linear functions on $\mathfrak{g}^*$ corresponding to elements of $\mathfrak{k}$, and $\{\hat{f}_1, \hat{f}_2\}$, which is a linear function on $\mathfrak{g}^*$, will also correspond to an element of $\mathfrak{k}$. We deduce that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ (and that the Poisson structure on $\tilde{P}$ induced from $\mathfrak{g}^*$ is the restriction of a linear Poisson structure on $\mathfrak{p}^\circ$).

To show $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ pick any $k \in \mathfrak{k}, p \in \mathfrak{p}$ and $y \in \tilde{P}$. Then $\langle [k, p], y\rangle = -\langle k, ad^*_p(y)\rangle = \langle k, \sharp d_y p\rangle = 0$, using Lemma 6.5 in the second equality, because $\sharp d_y p \subset \sharp N^*_y \tilde{P} = \mathfrak{k}^\circ$. This shows that $[k, p]$ annihilates $\tilde{P}$, hence it must annihilate its span $\mathfrak{p}^\circ$. □

Remark 6.7. If $\mathfrak{g}$ admits a non-degenerate Ad-invariant bilinear form B, then the B-orthogonal $\mathfrak{p}$ of any subalgebra $\mathfrak{k}$ satisfies $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, because for any $k, k' \in \mathfrak{k}$ and $p \in \mathfrak{p}$ we have $B([k,p],k') = -B(p,[k,k']) = 0$. If B is positive-definite (for example if the simply connected Lie group integrating $\mathfrak{g}$ is compact), then we clearly also have $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, so for such Lie algebras the construction of Prop. 6.6 is uninteresting.

A converse statement to Prop. 6.6 is given by

Proposition 6.8. *Assume that $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and there exists a point $y \in \mathfrak{p}^\circ$ at which none of the fundamental vector fields $\frac{d}{dt}|_0 Ad^*_{exp(tp)}(y)$ vanish, where p ranges over $\mathfrak{p}/\{0\}$. Then there is an open subset $\tilde{P} \subset \mathfrak{p}^\circ$ which is cosymplectic and $\mathfrak{k}^\circ := \sharp N^*_x \tilde{P}$ is independent of $x \in \tilde{P}$. (Hence applying Prop. 6.6 it follows that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$).*

Proof. For all $x \in \mathfrak{p}^\circ$ we have $\sharp N^*_x(\mathfrak{p}^\circ) = ad^*_{\mathfrak{p}}(x) \subset \mathfrak{k}^\circ$, as can be seen using $\langle ad^*_p(x), \mathfrak{k}\rangle = \langle x, [p,\mathfrak{k}]\rangle = 0$ for all $p \in \mathfrak{p}$ (which holds because of $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$). The assumption on the coadjoint action at y means that the map $\mathfrak{p} \to \mathfrak{g}^*, p \mapsto ad^*_p(y)$ is injective; by continuity it is injective also on an open subset $\tilde{P} \subset \mathfrak{p}^\circ$, and by dimension counting we get $\sharp N^*_x(\mathfrak{p}^\circ) = \mathfrak{k}^\circ$ on $\tilde{P}$. □

Now we display an example for Prop. 6.6

Example 6.9. Let $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the non-degenerate (indefinite) inner product $(A,B) = Tr(A \cdot B)$. Since it is Ad-invariant, the action of ad_X and ad^*_X on $\mathfrak{g}$ and $\mathfrak{g}^*$ are intertwined (up to sign).

Now take $\mathfrak{h} = \{\left(\begin{smallmatrix} 0 & b \\ c & d\end{smallmatrix}\right) : b,c,d \in \mathbb{R}\}$, which is not a subalgebra. Its annihilator is identified with the line C spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right)$. Since C is one-dimensional and the Poisson structure on $\mathfrak{g}^*$ linear it is clear that $\sharp N^*_x C$ is independent of $x \in C/\{0\}$ and $C/\{0\}$ is pre-Poisson. Using Lemma 6.5 we compute $\sharp N^*_x C = \{\left(\begin{smallmatrix} 0 & b \\ c & 0\end{smallmatrix}\right) : b,c \in \mathbb{R}\}$, so as complement R to $T_xC + \sharp N^*_x C$ we can take the line spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then $\mathfrak{p}^\circ := R \oplus C$ is given by the diagonal matrices, and $\mathfrak{p} \subset \mathfrak{g}$ is given by matrices with only zeros on the diagonal. For any $\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right) \in \mathfrak{p}^\circ$ we compute $\sharp N^*_{\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right)} \mathfrak{p}^\circ$ using Lemma 6.5 and obtain the set of matrices with only zeros on the diagonal if $a \neq d$ and $\{0\}$ otherwise. So the open set $\tilde{P}$ on which $\mathfrak{p}^\circ$ is cosymplectic is a plane with a line removed, and $\mathfrak{k}^\circ := \sharp N^*_{\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right)} \tilde{P}$ is independent of the footpoint $\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right) \in \tilde{P}$. $\mathfrak{k} \subset \mathfrak{g}$ coincides hence with the set of diagonal matrices. As predicted by Lemma 6.6 $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$; one can check easily that $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ too.

Since $\mathfrak{k}$ is abelian, the linear Poisson structure induced on $\tilde{P}$ is the zero Poisson structure. This can be seen also looking at the explicit Poisson structure on $\mathfrak{g}^*$, which with respect to the coordinates given by the basis $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$, $c = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0\end{smallmatrix}\right)$ and $d = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right)$ of $\mathfrak{g}^*$ is

$$-b\partial_a \wedge \partial_b + c\partial_a \wedge \partial_c + (d-a)\partial_b \wedge \partial_c - b\partial_b \wedge \partial_d + c\partial_c \wedge \partial_d.$$

Indeed at a point $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ of $\mathfrak{p}^\circ$ the bivector reduces to $(d-a)\partial_b \wedge \partial_c$. Finally remark that if we had chosen R to be spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ instead we would have obtained as $\sharp N^*_{\left(\begin{smallmatrix} a & b \\ 0 & b \end{smallmatrix}\right)}\mathfrak{p}^\circ$ the span of $\left(\begin{smallmatrix} -b & b \\ a-b & b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & b-a \\ 0 & 0 \end{smallmatrix}\right)$, which obviously is not constant on any open subset of $\mathfrak{p}^\circ$

7 Reduction of submanifolds and deformation quantization of pre-Poisson submanifolds

Given any submanifold C of a Poisson manifold (P, Π), it is natural to consider the characteristic "distribution" $\sharp N^*C \cap TC$, which by Remark 2.1 consists of the kernels of the restriction to C of the symplectic forms on the symplectic leaves of P. We used quotation marks because $\sharp N^*C \cap TC$ usually does not have constant rank. We will consider the set of basic functions on C, i.e.

$$C^\infty_{bas}(C) = \{f \in C^\infty(C) : df|_{\sharp N^*C \cap TC} = 0\}.$$

When C is pre-Poisson the Poisson structure on P induces a Poisson bracket $\{\bullet, \bullet\}_C$ on $C^\infty_{bas}(C)$.

Theorem 7.1. *Let C be a pre-Poisson submanifold, and assume that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish. Then $(C^\infty_{bas}(C), \{\bullet, \bullet\}_C)$, the Poisson algebra of basic functions on C, admits a deformation quantization.*

Proof. By Thm. 3.2 we can embed C coisotropically in some cosymplectic submanifold $\tilde{P}$. One can show that the Poisson bracket $\{\bullet, \bullet\}_C$ on $C^\infty_{bas}(C)$ concides with the one induced by the embedding of C in $\tilde{P}$. Now we invoke Corollary 3.3 of [8]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold admits a deformation quantization. The conditions in Corollary 3.3 of [8] translate into the conditions stated in the proposition because the conormal bundle of C in $\tilde{P}$ is isomorphic to $N^*C \cap \sharp^{-1}TC$ as a Lie algebroid, see Prop. 3.4. □

8 Subgroupoids associated to pre-Poisson submanifolds

Let C be a pre-Poisson submanifold of a Poisson manifold (P, Π). In Prop. 3.4 we showed that $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P. When $\sharp N^*C$ has constant rank there is another Lie subalgebroid associated[7] to C, namely $\sharp^{-1}TC = (\sharp N^*C)^\circ$. Now we assume that T^*P is an integrable Lie algebroid, i.e. that the source simply

[7]More generally for any Lie algebroid $A \to M$ with anchor ρ, if N is a submanifold of M such that $\rho^{-1}TN$ has constant rank then $\rho^{-1}TN \to N$ is a Lie subalgebroid of $A \to M$.

connected (s.s.c.) symplectic groupoid $(\Gamma_s(P),\Omega)$ of (P,Π) exists. In this section we study the (in general only immersed) subgroupoids of $\Gamma_s(P)$ integrating $N^*C\cap\sharp^{-1}TC$ and $\sharp^{-1}TC$. Here, for any Lie subalgebroid A of T^*P integrating to a s.s.c. Lie groupoid G, we take "subgroupoid" to mean the (usually just immersed) image of the (usually not injective) morphism $G\to\Gamma_s(P)$ induced from the inclusion $A\to T^*P$.

By Thm. 3.2 we can find a cosymplectic submanifold $\tilde{P}$ in which C lies coisotropically. We first make few remarks on the subgroupoid corresponding to $\tilde{P}$.

Lemma 8.1. *The subgroupoid of $\Gamma_s(P)$ integrating $\sharp^{-1}T\tilde{P}$ is $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$ and is a symplectic subgroupoid. Its source (target) map is a Poisson (anti-Poisson) map onto $\tilde{P}$, where the latter is endowed with the Poisson structure induced by (P,Π).*

Using lemma 8.1 we can reduce the problem of describing the subgroupoid integrating $N^*C\cap\sharp^{-1}TC$ to [5].

Proposition 8.2. *Let C be a pre-Poisson submanifold of (P,Π). Then the subgroupoid of $\Gamma_s(P)$ integrating $N^*C\cap\sharp^{-1}TC$ is an isotropic subgroupoid of $\Gamma_s(P)$.*

Now we consider $\sharp^{-1}TC$. For any submanifold N, $\sharp^{-1}TN$ has constant rank iff it is a Lie subalgebroid of T^*P, integrating to the subgroupoid $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$ of $\Gamma_s(P)$. So the constant rank condition on $\sharp^{-1}TN$ corresponds to a smoothness condition on $\mathbf{s}^{-1}(N)\cap\mathbf{t}^{-1}(N)$.

Remark 8.3. For a pre-Poisson submanifold C, the condition that $\sharp^{-1}TC$ have constant rank is equivalent to the characteristic distribution $TC\cap\sharp N^*C$ having constant rank[8].

Proposition 8.4. *Let C be a pre-Poisson submanifold with constant-rank characteristic distribution. Then for any cosymplectic submanifold $\tilde{P}$ in which C embeds coisotropically, $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ is a coisotropic subgroupoid of $\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})$.*

The above proposition is used to prove one implication in

Proposition 8.5. *Let C be any submanifold of P. Then $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ is a (immersed) presymplectic submanifold iff C is pre-Poisson and its characteristic distribution has constant rank. In this case the characteristic distribution of $\mathbf{s}^{-1}(C)\cap\mathbf{t}^{-1}(C)$ has rank $2rk(\sharp N^*C\cap TC)+rk(N^*C\cap N^*\mathcal{O})$, where $\mathcal{O}$ denotes the symplectic leaves of P intersecting C.*

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[8]Indeed more generally we have the following for any submanifold C of P: if any two of $\sharp^{-1}TC$, $\sharp N^*C+TC$ or $TC\cap\sharp N^*C$ have constant rank, then the remaining one also has constant rank. This follows trivially from $rk(\sharp N^*C+TC)=rk(\sharp N^*C)+\dim C-rk(TC\cap\sharp N^*C)$.

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