Differential Geometry of Hydrodynamic Vlasov Equations

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Abstract

We consider hydrodynamic chains in $(1+1)$ dimensions which are Hamiltonian with respect to the Kupershmidt-Manin Poisson bracket. These systems can be derived from single $(2+1)$ equations, here called *hydrodynamic Vlasov equations*, under the map $A^n = \int_{-\infty}^{\infty} p^n f dp$. For these equations an analogue of the Dubrovin-Novikov Hamiltonian structure is constructed. The Vlasov formalism allows us to describe objects like the Haantjes tensor for such a chain in a much more compact and computable way. We prove that the necessary conditions found by Ferapontov and Marshall in [1] for the integrability of these hydrodynamic chains are also sufficient.

1 Systems of hydrodynamic type

Systems of hydrodynamic type are quasilinear first order PDE of the form

$$u_t^i = v_j^i(u) u_x^j, \qquad i, j = 1...N, \tag{1}$$

where (x, t) are the independent and $(u^1, ..., u^N)$ the dependent variables. Here and below sums over repeated indices are assumed. A Hamiltonian formalism for systems of this type was introduced in [2] by Dubrovin and Novikov, who defined a Poisson bracket of the form

$$\{I_\alpha, I_\beta\} = \int \frac{\delta I_\alpha}{\delta u^i(x)} \Pi^{ij} \frac{\delta I_\beta}{\delta u^j(x)} dx. \tag{2}$$

Here I_α, I_β are functionals of $u(x)$ and the first order differential operator A_{ij} is given by:

$$\Pi^{ij} = g^{ij}(u) \frac{\partial}{\partial x} + b_k^{ij}(u) u_x^k. \tag{3}$$

They showed that this is a Hamiltonian structure if g^{ij} is a nonsingular contravariant metric and $b_k^{ij} = -g^{is}\Gamma_{sk}^j$, where Γ_{jk}^i is a symmetric connection of zero curvature that is compatible with the metric g^{ij}. It is immediate to see that a Hamiltonian of the form

$$H = \int h(u) dx, \tag{4}$$

where $h(u)$ is independent of $u_x, u_{xx}, \ldots,$ together with the Hamiltonian structure (2), leads to an equation of hydrodynamic type, specifically

$$u_t^i = \{u^i, H\} = \left(g^{ij}\partial_x + b_k^{ij} u_x^k\right) \frac{\partial h}{\partial u^j}. \tag{5}$$

An obvious problem related with systems of hydrodynamic type (1) is to determine whether such a system is integrable, in the sense that it admits infinitely many conserved densities and commuting flows; in [3], Tsarev proved that this is true if the system is hyperbolic and can be written in diagonal form

$$R_t^i = \lambda^i(R) R_x^i,$$

where R^i are called the Riemann invariants and where the λ^i (called the characteristic velocities) satisfy the *semi-Hamiltonian* condition

$$\partial_k \left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \right) = \partial_j \left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} \right),$$

where $\partial_k = \partial / \partial R^k$. With these hypotheses, the system can then be integrated by the generalized hodograph transformation ([3]). We remark that the semi-Hamiltonian property is automatically satisfied for a Hamiltonian system with Dubrovin-Novikov Hamiltonian structure, and that the conditions for the system to be respectively diagonalizable, or semi-Hamiltonian, can be written invariantly; each corresponds to the vanishing of some tensor ([4],[1]). In particular, for the diagonalizability condition, if one defines the *Nijenhuis tensor* of the matrix v^i_j by:

$$N^i_{jk} = v^s_j \frac{\partial v^i_k}{\partial u^s} - v^s_k \frac{\partial v^i_j}{\partial u^s} - v^i_s \left(\frac{\partial v^s_k}{\partial u^j} - \frac{\partial v^s_j}{\partial u^k} \right),$$

and then the *Haantjes tensor* by:

$$H^i_{jk} = N^i_{\alpha\beta} v^\alpha_j v^\beta_k - N^\alpha_{j\beta} v^i_\alpha v^\beta_k - N^\alpha_{\beta k} v^i_\alpha v^\beta_j + N^\alpha_{jk} v^i_\beta v^\beta_\alpha$$

then we have the following:

Theorem 1.1. *[5] A hydrodynamic type system with mutually distinct characteristic speeds is diagonalizable if and only if the corresponding Haantjes tensor vanishes identically.*

1.1 Hydrodynamic chains

Hydrodynamic chains are defined as a natural generalization of systems of hydrodynamic type, letting the number of variables and equations go to infinity. More specifically, we consider, following Ferapontov and Marshall ([1]), systems of the type

$$A_t = V(A) A_x, \tag{6}$$

where $A = (A^0, A^1, ...)^t$ is an infinite column vector and $V(A)$ is an $\infty \times \infty$ matrix, with the following properties (see [1], [6]),

1) for every row only finitely many elements are nonzero
2) every element of the matrix depends only on a finite number of variables.

The variables $(A^0, A^1, ...)$ are usually called *moments*. The most famous example of a hydrodynamic chain is the Benney chain,

$$A^n_t + A^{n+1}_x + nA^{n-1}A^0_x = 0, \qquad n = 0, 1, \dots \tag{7}$$

which was derived in [7] from the study of long nonlinear waves on a shallow perfect fluid with a free surface. Kupershmidt and Manin ([8],[9]) found a Hamiltonian formulation,

$$A^n_t = \{A^n, H\}_{KM} = \Pi^{nm} \frac{\delta H}{\delta A^m},$$

given by the Poisson operator

$$\Pi^{nm} = (n+m)A^{n+m-1}\frac{d}{dx} + mA^{n+m-1}_x, \qquad n, m = 0, 1, \ldots \tag{8}$$

called the Kupershmidt-Manin bracket (KM bracket), together with the Hamiltonian

$$H = \int \left(\frac{1}{2}(A^0)^2 + \frac{1}{2}A^2 \right) dx. \tag{9}$$

The KM bracket (8) has been considered (see, for example, [10]) as an infinite dimensional example of the Dubrovin-Novikov structure. The general chain arising in this way, with a Hamiltonian density

$$h(A^0, \ldots, A^{N-1}), \tag{10}$$

takes the form

$$A^n_t = \sum_{m,l=0}^{N-1} (m+n)A^{m+n-1} h_{ml} A^l_x + \sum_{m=0}^{N-1} m h_m A^{m+n-1}_x, \quad n = 0, 1, \ldots \tag{11}$$

Here we have used the notation $h_i = \frac{\partial h}{\partial A^i}$, where $i = 0, 1, \ldots$. Compared with the finite dimensional case of Section 1, the theory of infinite dimensional Poisson brackets of hydrodynamic type is not so well developed; however, other examples of such Poisson brackets were given in [11] as a generalization of the Kupershmidt-Manin bracket (8), while the problem of the classification of such chains has been approached recently in [12]. The problem of finding integrable hydrodynamic chains was firstly approached in a systematic way by Kupershmidt ([11]), and, more recently and with different approaches, by Pavlov ([12, 13, 14, 15, 16]), Ferapontov and Marshall ([1]), and Ferapontov, Khusnutdinova, Marshall and Pavlov ([6]).

Particularly, in [1], the authors introduced an approached based on the Haantjes tensor, generalizing Tsarev's results [3] for finite dimensional systems. For hydrodynamic chains, calculation of any one component of this tensor only involves finite sums and hence is computable. Following this criterion, Ferapontov and Marshall considered Hamiltonian densities depending only on the first three moments

$$h = h(A^0, A^1, A^2),$$

together with the KM bracket (8), and they looked for the condition on the Hamiltonian for the system to have zero Haantjes tensor. They found that the conditions

$$H^0_{jk} = 0, \qquad j, k = 0, 1, 2, \dots$$

on the first upper component give a complete system of ten third order quasilinear partial differential equations, of which the simplest are

$$\begin{aligned}
&h_{222} = \frac{5h_{22}^2}{2h_2}, \qquad h_{022} = \frac{5h_{02}h_{22}}{2h_2}, \qquad h_{122} = \frac{5h_{12}h_{22}}{2h_2},\\
&h_{002} = \frac{3h_{02}^2 + 2h_{00}h_{22}}{2h_2},\\
&h_{012} = \frac{3h_{02}h_{12} + 2h_{01}h_{22}}{2h_2},\\
&h_{112} = \frac{3h_{12}^2 + 2h_{11}h_{22}}{2h_2}.
\end{aligned} \tag{12}$$

The last four equations are much more cumbersome.

Remark 1. *The list of equations above differs from the original paper [1], only in the names of the variables, as [1] uses* $u^n = A^{n-1}, n = 1, 2, \dots$

What Ferapontov and Marshall found were thus *necessary* conditions for the integrability of the chain. Remarkably, they were able to solve this system and all the solutions they found correspond to known integrable systems or systems since shown to be integrable. This result suggested a conjecture that the conditions above are not only necessary but also sufficient.

Remark 2. *In a subsequent paper ([6]), a similar problem was discussed, but with a Hamiltonian* $H(A^0, A^1)$*, and the* $(\alpha{-}\beta)$ *Hamiltonian structure,*

$$\Pi^{ij}_{(\alpha\beta)} = \big(\alpha(i+j) + \beta\big)A^{i+j-1}\frac{d}{dx} + \big(\alpha j + \beta\big)A^{i+j-1}_x,$$

which generalizes the KM bracket (8). It is interesting to remark that in this case too, the conditions for $H^0_{jk} = 0$ give a complete set of equations for the third derivatives of the Hamiltonian, and that for these systems it was shown that the conditions are indeed sufficient.

In the last section of this paper we will prove that the conditions (12) found by Ferapontov and Marshall are not only necessary for the vanishing of the Haantjes tensor, but also sufficient. In order to do this, though, we need do develop a somewhat different formalism.

2 Vlasov formalism for Hamiltonian hydrodynamic chains

In this section we define the Vlasov equations, and we recall ([17]) how a special class of these equations can be related with hydrodynamic chains which are Hamiltonian with respect to the Kupershmidt-Manin bracket. Moreover, we show how to construct all the differential geometric objects related with such chains in the Vlasov picture.

Let $f(x,p,t)$ be a distribution function in the in the (1+1)-dimensional phase space, and consider the Lie-Poisson bracket

$$\{J,H\}_{LP} := \iint f \left\{ \frac{\delta J}{\delta f}, \frac{\delta H}{\delta f} \right\}_{x,p} dpdx, \tag{13}$$

where H, K are functionals of f and where the bracket $\{\ ,\ \}_{x,p}$ is the canonical 'single-particle' Poisson bracket. Hamilton's equations related with such brackets,

$$f_t = \{f,H\}_{LP},$$

or, equivalently,

$$f_t + \left\{ f, \frac{\delta H}{\delta f} \right\}_{x,p} = 0,$$

are called *Vlasov equations*, and they arise in the theories of plasma physics and vortex dynamics. The relation between these equations and the hydrodynamic chains of the previous section is obtained by defining

$$\mu \ : f(p,x,t) \longmapsto \{A^n(x,t)\}_{n=0}^{\infty} \qquad (14)$$
$$A^n = \int p^n f dp,$$

where the integral above converges, for example, if f is bounded and $|f| \to 0$ faster than $|p|^{-n}$, $\forall n \geqslant 1$.

As was shown in [17] by one of the present authors, if we restrict the bracket (13) to functionals depending on the moments alone:

$$H = H(A^0, \ldots, A^{N-1})$$

then the Lie-Poisson bracket restricts to the Kupershmidt-Manin Poisson bracket (8):

$$\{J, H\}_{LP} \equiv \{J, H\}_{KM}.$$

In order to prove this, is sufficient to use the chain rule for the map μ:

$$\frac{\delta H(A^0, \ldots A^{N-1})}{\delta f} = \sum_{n=0}^{N-1} \frac{\delta H}{\delta A^n}\frac{\delta A^n}{\delta f} = \sum_{n=0}^{N-1} \frac{\delta H}{\delta A^n} p^n, \qquad (15)$$

then the KM bracket arises as the push forward of the L-P bracket under this map. If we look at the evolution equations described by this bracket, for Hamiltonian functionals of type

$$H = \int h(A^0, \ldots A^{N-1}) dx, \qquad (16)$$

we obtain a relation between a class of Vlasov equations

$$f_t = \{f, H\}_{LP}, \qquad (17)$$

and the Kupershmidt-Manin hydrodynamic chains (11)

$$A^n_t = \{A^n, H\}_{KM}, \quad n = 0, 1, \ldots.$$

We call equations (17) *hydrodynamic Vlasov equations*; more explicitly, recalling that for functionals of type (16) we have

$$\frac{\delta H(A^0, \ldots A^{N-1})}{\delta f} = \sum_{n=0}^{N-1} h_n p^n, \qquad (18)$$

these equations take the form

$$f_t = \left(\sum_{n,m=0}^{N-1} p^n h_{nm} A_x^m \right) f_p - \left(\sum_{n=0}^{N-1} n p^{n-1} h_n \right) f_x. \tag{19}$$

Ferapontov and Marshall started to study the differential-geometric properties of such hydrodynamic chains using the countably infinite set of discrete coordinates A^n, but instead, it is possible to study these properties by looking at the corresponding hydrodynamic Vlasov equations directly.

In order to do so, we want to consider equation (19) as a kind of $(1+1)$-dimensional hydrodynamic type system (1). Indeed, we notice that equations (19) are linear with respect to the derivatives f_t and f_x; thus, we consider the function

$$f(p, x, t),$$

as a vector; the independent variable p is treated as a continuous parameter, analogous to the discrete index in the components of a finite dimensional vector. We will, for brevity, suppress the dependence on (x, t). In this way, the hydrodynamic Vlasov equations (19) can be viewed as $(1+1)$ hydrodynamic type systems of continuously infinitely many equations and variables; indeed they can be written in the form

$$f_t(p) = \int V\binom{p}{q} f_x(q) dq, \tag{20}$$

where the kernel $V\binom{p}{q}$ is given by

$$V\binom{p}{q} = \left(\sum_{n,m=0}^{N-1} p^n q^m h_{nm} \right) f_p - \left(\sum_{n=0}^{N-1} n p^{n-1} h_n \right) \delta(p-q). \tag{21}$$

It is important here that we do not consider (19) as a $(2+1)$-dimensional hydrodynamic type system. Rather, we consider f_p as a functional of f, namely

$$K[f] := \int f(r) \delta'(p-r) dr = f_p,$$

where $\delta'(p-r)$ is the derivative of the Dirac delta function. Thus, the kernel (21) may be considered as depending on f, analogously to discrete non-linear hydrodynamic type systems.

To complete our construction, we need to substitute, in a formal way, discrete objects with continuously indexed ones; namely:

$n=0,1,2,\dots$	$p\in\mathbb{R}$
$A^n(x,t)$	$f(p,x,t)$
$\frac{\partial h(A^0,\dots,A^{N-1})}{\partial A^n}$	$\frac{\delta h[f]}{\delta f(p)}$
Sums on repeated discrete indices	Integrals on repeated continuous indices

Using these coordinates, we can construct any tensor object related with a hydrodynamic chain (11), the relation being an analogue of the classical change of coordinates of a tensor under the map μ. The advantage of this formulation is that, instead of studying infinite-component tensors, we can consider integral operators, which are much more compact and computable.

As an example, we write down explicitly the Vlasov formalism for the Kupershmidt-Manin structure (8). This is given by the metric

$$G^{mn}[A^0,A^1,\dots]=(m+n)A^{m+n-1},$$

and to this metric corresponds, in the Vlasov coordinate, an operator $g[f]$, depending on two real parameters

$$g^{(p,q)}[f],\qquad p,q\in\mathbb{R} \tag{22}$$

and symmetric with respect to p,q. The relation with the metric in the Vlasov coordinates is the identity

$$\iint g^{(p,q)}[f]\frac{\delta A^m}{\delta f(p)}\frac{\delta A^n}{\delta f(q)}dpdq=G^{mn}[A^0,A^1,\dots] \tag{23}$$

which is an analogue of the classical change of variables of a $(2,0)-$tensor under the map (14). Of course, when a continuous index is repeated, we integrate with respect to the repeated index. If we take

$$g^{(p,q)}[f] = -f_p(p)\delta(p-q), \tag{24}$$

then, substituting in (23), we indeed have

$$\iint g^{(p,q)}[f]p^m q^n dpdq = -\int f_p(p)p^{m+n}dp = (m+n)A^{m+n-1}.$$

We notice that, in the new coordinates, the Dirac delta function plays the role of the Kronecker delta δ^i_j, indeed, we have

$$\sum_j \frac{\delta A^j}{\delta f(p)}\delta^i_j = \int \frac{\delta A^i}{\delta f(q)}\delta(p-q)dq.$$

As a consequence of this, we notice that the metric (24) in the Vlasov coordinates has diagonal form. In addition, we will say that a metric $g^{(p,q)}$ is *non degenerate* if there exists an inverse metric $g_{(p,q)}$ such that

$$\int g_{(p,\alpha)}g^{(\alpha,q)}d\alpha = \delta(p-q). \tag{25}$$

Continuing the analogy, we can now pursue a direct computation of the differential geometric object we need, directly in the Vlasov coordinates. So, we define the *Christoffel symbols*, which are given, in components, by the following formula:

$$b\binom{p}{q,r} := \frac{1}{2}\int g^{(p,\alpha)}\left(\frac{\delta g_{(\alpha,q)}}{\delta f(r)} + \frac{\delta g_{(\alpha,r)}}{\delta f(q)} - \frac{\delta g_{(q,r)}}{\delta f(\alpha)}\right)d\alpha. \tag{26}$$

For the metric (24), the Christoffel symbols are

$$b\binom{p}{q,r} = -\delta'(p-q)\delta(r-q),$$

and then the curvature, defined in analogy with the classical case as

$$R\begin{pmatrix} s \\ p,q,r \end{pmatrix} := \frac{\delta b\binom{s}{p,r}}{\delta f(q)} - \frac{\delta b\binom{s}{p,q}}{\delta f(r)} + \int b\begin{pmatrix} s \\ q,\alpha \end{pmatrix} b\begin{pmatrix} \alpha \\ p,r \end{pmatrix} d\alpha - \int b\begin{pmatrix} s \\ r,\alpha \end{pmatrix} b\begin{pmatrix} \alpha \\ p,q \end{pmatrix} d\alpha \tag{27}$$

is found to be identically zero.

3 The Haantjes tensor for hydrodynamic Vlasov equations

We introduce now the Nijenhuis and Haantjes tensors for a Vlasov equation of hydrodynamic type (17,19). Particularly, in the second part of the section we will consider the special case when the Hamiltonian density depends only on the first three moments, $h(A^0, A^1, A^2)$, so that

$$\frac{\delta h}{\delta f} = h_0 + ph_1 + p^2 h_2,$$

and we calculate the conditions for a system with such a Hamiltonian to have vanishing Haantjes tensor. As in the previous examples, this differential geometric result for a Vlasov hydrodynamic equation can be lifted to the corresponding hydrodynamic chain. Consider first the general case of a Hamiltonian function of type (10). In order to simplify our notation, we write the kernel (21) as

$$V\begin{pmatrix} p \\ q \end{pmatrix} = B(p,q) f_p - A(p)\delta(p-q), \tag{28}$$

where

$$A(p) := \sum_{n=0}^{N-1} n h_n p^{n-1} \quad , \quad B(p,q) := \sum_{n,m=0}^{N-1} h_{nm} p^n q^m$$

are polynomials in p and p, q respectively, whose coefficients are the derivatives of the Hamiltonian. As with the discrete case, we define the Nijenhuis tensor for a hydrodynamic Vlasov equation as

$$N\binom{p}{q,r} := \int V\binom{\alpha}{q}\frac{\delta V\binom{p}{r}}{\delta f(\alpha)} - V\binom{\alpha}{r}\frac{\delta V\binom{p}{q}}{\delta f(\alpha)} - V\binom{p}{\alpha}\left(\frac{\delta V\binom{\alpha}{r}}{\delta f(q)} - \frac{\delta V\binom{\alpha}{q}}{\delta f(r)}\right) d\alpha, \quad (29)$$

while the Haantjes tensor is then given by

$$\begin{aligned} H\binom{p}{q,r} := \iint \Bigg(& N\binom{p}{\alpha,\beta} V\binom{\beta}{q} V\binom{\alpha}{r} - N\binom{\alpha}{\beta,r} V\binom{p}{\alpha} V\binom{\beta}{q} + \\ & - N\binom{\beta}{q,\alpha} V\binom{p}{\beta} V\binom{\alpha}{r} + N\binom{\beta}{q,r} V\binom{p}{\alpha} V\binom{\alpha}{\beta} \Bigg) d\alpha d\beta. \end{aligned} \quad (30)$$

Let us calculate the Nijenhuis tensor for a general kernel (28); first of all, we have to compute the variational derivative of V with respect to f. A direct calculation shows that

$$\begin{aligned} \frac{\delta V\binom{p}{q}}{\delta f(r)} = & \left(\sum_{n,m,l=0}^{N-1} p^n q^m r^l h_{nml}\right) f_p + \left(\sum_{n,m=0}^{N-1} p^n q^m h_{nm}\right) \delta'(p-r) + \\ & - \left(\sum_{n,m=0}^{N-1} n p^{n-1} q^m h_{nm}\right) \delta(p-q). \end{aligned} \quad (31)$$

If we denote

$$C(p,q,r) := \sum_{n,m,l=0}^{N-1} p^n q^m r^l h_{nml},$$

then the identity (31) may be written

$$\frac{\delta V\binom{p}{q}}{\delta f(r)} = C(p,q,r) f_p + B(p,q)\delta'(p-r) - \frac{\partial B(p,q)}{\partial p}\delta(p-q). \quad (32)$$

Substituting equations (28) and (32) into the definition of the Nijenhuis tensor (29), and using properties of the delta function, we obtain

$$N\binom{p}{q,r} = E(p,q,r) f_p + F(p,r)\delta(p-q) - F(p,q)\delta(p-r),$$

where E and F are polynomials given by

$$F(p,q) = (A(q) - A(p))\frac{\partial B(p,q)}{\partial p} + \frac{\partial A(p)}{\partial p} B(p,q) - \int B(\alpha,q)\frac{\partial B(p,\alpha)}{\partial p} f_\alpha d\alpha$$

and

$$E(p,q,r) = (A(r)-A(q))C(p,q,r) + B(p,q)\frac{\partial B(p,r)}{\partial p} - B(p,r)\frac{\partial B(p,q)}{\partial p} +$$
$$+ B(q,r)\left(\frac{\partial B(p,q)}{\partial q} - \frac{\partial B(p,r)}{\partial r}\right).$$

Remark 3. *It is easy to verify that E is a polynomial in p, q, r whose coefficients are quadratic expressions in the derivatives of h, for it is defined as a product of polynomials which are linear in the derivatives of h. For F, though, this fact is less clear, because of the integral in the last term. However, it is possible to write the integrand as a polynomial in α, since we have*

$$\begin{aligned}\int B(\alpha,q)\frac{\partial B(p,\alpha)}{\partial p} f_\alpha d\alpha &= \int \sum_{n=0}^{2N-2} P_n(p,q)\alpha^n f_\alpha d\alpha = \\ &= -\sum_{n=0}^{2N-2} P_n(p,q) \int n\alpha^{n-1} f(\alpha) d\alpha = \\ &= -\sum_{n=0}^{2N-2} nP_n(p,q)A^{n-1}.\end{aligned}$$

Here the P_n are suitable polynomials with coefficients quadratic in the derivatives of the Hamiltonian. We observe that the number of moments appearing in these expressions will generally be bigger than N. Similar dependence on the A^n will appear in the calculation of the Haantjes tensor as well.

The calculation of the Haantjes tensor is similar. It follows from the above, with a long but essentially straightforward calculation, that

$$H\begin{pmatrix} p \\ q, r \end{pmatrix} = Q(p,q,r) f_p. \tag{33}$$

We will call the polynomial $Q(p,q,r)$, above, the *Haantjes polynomial* for the related hydrodynamic Vlasov equation. Remarkably, in the above expression there do not appear any coefficients in the δ-function or its derivative. It would be interesting to find a deeper explanation for this. In addition, given the Haantjes tensor H^i_{jk} for the corresponding hydrodynamic chain, we have the following relation:

$$\sum_j \sum_k H^i_{jk} \frac{\delta A^j}{\delta f(q)} \frac{\delta A^k}{\delta f(r)} = \int H\begin{pmatrix} p \\ q, r \end{pmatrix} \frac{\delta A^i}{\delta f(p)} dp,$$

which is the change of variables under the map (14), introduced in Section (2), for a tensor of type $(1, 2)$. Explicitly:

$$\sum_{j,k} q^j r^k H^i_{jk} = \int p^i Q(p, q, r) f_p dp. \tag{34}$$

So, to study the properties of the Haantjes tensor of a chain it is sufficient to study the properties of the corresponding Haantjes polynomial. It is possible to show that, for $N > 2$, this polynomial has the form

$$Q(p, q, r) = \sum_{l=0}^{4(N-2)} \sum_{m=0}^{3N-5} \sum_{n=0}^{3N-5} Q_{lmn} p^l q^m r^n, \tag{35}$$

where the coefficients Q_{lmn} are linear or quadratic expressions of type

$$Q_{lnm} = Q_{lnm}\left(h_i, h_{ij}, h_{ijk}, A^0, \ldots, A^{4N-7}\right) \qquad i, j, k = 0, \ldots, N-1,$$

involving the first, second and third derivatives of the Hamiltonian

$$h(A^0, \ldots A^{N-1}),$$

as well as explicit dependence on the moments

$$A^0, \ldots A^{N-1},$$

and on 'extra' moments not appearing in the Hamiltonian,

$$A^N, \ldots A^{4N-7}. \tag{36}$$

These appear, as explained in Remark 3, when integrals of the form $\int \alpha^n f_\alpha d\alpha$ are evaluated. The Haantjes polynomial is antisymmetric with respect to q

and r. Writing the Haantjes polynomial Q as in (35), the equation (34) becomes

$$\sum_{j,k=0}^{\infty} q^j r^k H^i_{jk} = \int p^i \left(\sum_{j,k=0}^{3N-5} \sum_{l=0}^{4(N-2)} Q_{ljk} p^l q^j r^k \right) f_p dp =$$
$$= \sum_{j,k=0}^{3N-5} \sum_{l=0}^{4(N-2)} \left(Q_{ljk} \int p^{i+l} f_p dp \right) q^j r^k,$$

so that

$$H^i_{jk} = - \sum_{j,k=0}^{3N-5} \sum_{l=0}^{4(N-2)} (i+l) Q_{ljk} A^{i+l-1}.$$

As a consequence of the equation above, we have, for every fixed i, that

$$H^i_{jk} = 0 \qquad \forall j, k > 3N-5.$$

This fact, noticed in [1] by Ferapontov and Marhall, in this setting turns out to be a straightforward consequence of the dependence of the Hamiltonian on finitely many moment variables. In order for the Haantjes tensor to vanish identically, we note that the remaining H^i_{jk} must vanish provided that all the Q_{ljk} do so. Hence, the problem of the vanishing of a tensor with infinitely many components has been reduced to the vanishing of the coefficients of a polynomial,

$$Q_{ljk} = 0,$$

$$\forall \quad l = 0, \ldots, 4(N-2), \quad j, k = 0, \ldots, 3N-5.$$

We look at these conditions as a system on the derivatives of the Hamiltonian h. Using the antisymmetry of the Haantjes polynomial in q and r, we can reduce the number of conditions, since Q is divisible by $(q-r)$. In the case $N=3$, the Haantjes polynomial reduces to

$$Q(p,q,r) = \sum_{l=0}^{4} \sum_{m=0}^{4} \sum_{n=0}^{4} Q_{lmn} p^l q^m r^n.$$

We write

$$Q(p,q,r) = (q-r)\sum_{l=0}^{4} M_l(q,r)p^l;$$

then, successively requiring the coefficients of M_4 and then M_3 to vanish leads to 10 partial differential equations of the form

$$h_{ijk} = F_{ijk}\left(h_n, h_{nm}, A^l\right), \qquad i,j,k,n,m,l = 0,\ldots,N-1.$$

If these conditions hold, it is easy to verify directly that the Haantjes polynomial $Q(p,q,r)$ is identically zero. We also recalculated the conditions on the zeroth upper component

$$H^0_{ij} = 0 \tag{37}$$

which Ferapontov and Marshall used as *necessary* conditions for the Haantjes tensor to vanish (see Section 1.1). It is then straightforward to verify that if these conditions (37) hold, then Q vanishes identically. It thus follows that the necessary conditions are also sufficient, as Ferapontov and Marshall had conjectured.

4 Dubrovin-Novikov Hamiltonian formalism for hydrodynamic Vlasov equations

Hydrodynamic Vlasov equations can be viewed as a generalization of systems of hydrodynamic type. In particular, the Lie-Poisson bracket (13) can be seen as a Dubrovin-Novikov Poisson bracket (2). In this section we formalize an analogue of the DN Poisson bracket for these equations; and we provide two explicit examples, for a class of diagonal metrics and for the second Hamiltonian structure of the Benney chain.

The main objects for the construction of a bracket of this form has already been defined: given a metric

$$g^{(p,q)}[f],$$

we can define the Christoffel symbols and the curvature

$$b\binom{p}{q,r}, \qquad R\binom{s}{p,q,r},$$

given explicitly by (26), (27). Moreover, in view of the definition of the new Poisson bracket, we notice that the elements b^{ij}_k are here replaced by

$$\begin{aligned} b\binom{p,q}{r} &= \int g^{(p,\alpha)} b\binom{q}{\alpha, r} d\alpha = \\ &= \frac{1}{2}\iint g^{(p,\alpha)} g^{(q,\beta)} \left(\frac{\delta g_{(\alpha,\beta)}}{\delta f(r)} + \frac{\delta g_{(\beta,r)}}{\delta f(\alpha)} - \frac{\delta g_{(\alpha,r)}}{\delta f(\beta)} \right) d\alpha d\beta \end{aligned}$$

Given these objects, we can define an *infinite dimensional Poisson bracket of hydrodynamic type* as

$$\{K,H\}_g := \iiiint \frac{\delta K}{\delta f(p,x)} \left(g^{(p,q)} \frac{\partial}{\partial x} + b\binom{p,q}{r} \frac{\partial f(r)}{\partial x} \right) \frac{\delta H}{\delta f(q,y)} dp dq dr dx, \tag{38}$$

where K, H are functionals of the type (16). Given a Poisson bracket of type (38), the related Hamiltonian evolution equations are then

$$\begin{aligned} f_t(p) &= \{f, H\}_V = \\ &= \int g^{(p,q)} \frac{\partial}{\partial x} \frac{\delta H}{\delta f(q)} dq + \iint b\binom{p,q}{r} \frac{\partial f(r)}{\partial x} \frac{\delta H}{\delta f(q)} dq dr. \end{aligned}$$

4.1 Diagonal metrics

In order to find explicit expressions for the Christoffel symbols and of the curvature, we now restrict ourselves to the case when the metric g is diagonal with components $g_{(p,q)}$ of the form

$$g_{(p,q)} = \frac{1}{k[f]} \delta(p-q). \tag{39}$$

The function $k[f]$ can depend on f and finitely many of its derivatives with respect to p. The first advantage of a diagonal metric is that the inverse metric has components given by

$$g^{(p,q)} = k[f] \delta(p-q), \tag{40}$$

and so Hamilton's equations take the simpler form

$$f_t(p,x) = k[f]\frac{\partial}{\partial x}\frac{\delta H}{\delta f(p)} + \iint b\binom{p,q}{r}\frac{\partial f(r)}{\partial x}\frac{\delta H}{\delta f(q)}dqdr.$$

For general $k[f]$, the calculation of the Christoffel symbols and of the curvature presents many difficulties, due to the presence of higher p-derivatives of f. On the other hand, the simplest case, when $k[f]$ depends only on f and not on its derivatives, turns out to be of little interest, as Hamilton's equations became the direct sum over $\mathbb{R}$ of one-dimensional ($N = 1$) Poisson brackets of type (2). In this section, we develop the first non-trivial case, when the function k depends only on f_p. We have the following

Proposition 1. *If for all p, q, the function k depends only on on the first $p-$derivative of f, i.e.*

$$g_{(p,q)} = \frac{1}{k[f_p]}\delta(p-q), \tag{41}$$

then the Christoffel symbols have the form

$$b\binom{p}{q,r} = \frac{k'[f_q]}{k[f_q]}\delta'(p-q)\delta(r-q) - \frac{1}{2}\frac{k''[f_q]}{k[f_q]}f_{qq}\delta(p-q)\delta(r-q), \tag{42}$$

where k', k'' are the first and second derivatives of k respectively. The metric g is flat if and only if $k[f_p]$ is linear in f_p.

Proof. The proof is a direct computation. In the calculation of the Christoffel symbols (26), we note that for the metric (41) we have

$$\frac{\delta g_{(p,q)}}{\delta f(r)} = \frac{k'[f_p]}{k[f_p]^2}\delta'(p-r)\delta(p-q),$$

substituting in the definition (26) of the Christoffel symbols, this leads to

$$\begin{aligned}
b\binom{p}{q,r} =& \frac{1}{2}\int k[f_p]\delta(p-\alpha)\left(\frac{k'[f_\alpha]}{k[f_\alpha]^2}\delta'(\alpha-r)\delta(q-\alpha) + \frac{k'[f_\alpha]}{k[f_\alpha]^2}\delta'(\alpha-q)\delta(r-\alpha)+\right.\\
&\left.-\frac{k'[f_q]}{k[f_q]^2}\delta'(q-\alpha)\delta(r-q)\right)d\alpha = \\
=&\frac{1}{2}\frac{k'[f_p]}{k[f_p]}\delta'(p-r)\delta(p-q) + \frac{1}{2}\frac{k'[f_p]}{k[f_p]}\delta'(p-q)\delta(p-r)+\\
&-\frac{1}{2}\frac{k[f_p]k'[f_q]}{k[f_q]^2}\delta'(q-p)\delta(r-q).
\end{aligned}$$

Rearranging, we obtain equation (42). For the calculation of the curvature (27) the technique is the same, and we obtain

$$R\binom{s}{p,q,r} = \frac{k''[f_p]}{k[f_p]}\Big(\delta'(p-r)\delta'(p-s)\delta(p-q) - \delta'(p-q)\delta'(p-s)\delta(p-r)\Big) +$$
$$+\frac{1}{2}\frac{k''[f_p]}{k[f_p]}\Big(\delta''(p-q)\delta(p-s)\delta(p-r) - \delta''(p-r)\delta(p-s)\delta(p-q)\Big) +$$
$$+\frac{1}{2}\frac{k'''[f_p]}{k[f_p]}f_{pp}\Big(\delta'(p-q)\delta(p-s)\delta(p-r) - \delta'(p-r)\delta(p-s)\delta(p-q)\Big).$$

It is elementary to see that the condition $k''[f_p] = 0$ leads to the vanishing of the curvature tensor. On the other hand, evaluating the result above with suitable test functions, it is possible to prove that the condition is also sufficient.

□

So, a metric of type (39) is flat if and only if has the form

$$g_{(p,q)} = \frac{1}{af_p + b}\delta(p-q),$$

with a, b not depending on f. The related evolution equations are then

$$f_t(p,x) = (af_p + b)\frac{\partial}{\partial x}\frac{\delta H}{\delta f(p)} - a\frac{\partial f(p)}{\partial x}\frac{\partial}{\partial p}\frac{\delta H}{\delta f(p)},$$

In the special case $a = 1$, $b = 0$, we obtain the canonical Lie-Poisson bracket (17), with Poisson operator of the form

$$\pi^{(p,q)} = f_p\frac{\partial}{\partial x} + \int \delta'(p-q)\delta(p-r)f_x(r)dr. \tag{43}$$

4.2 The second Hamiltonian structure for the Benney chain

The second Hamiltonian structure for the Benney equation (7) is defined by the local Poisson operator

$$(\Pi_2)^{kn} = (G_2)^{kn}\frac{\partial}{\partial x} + \sum_m (B_2)^{kn}_m A^m_x, \tag{44}$$

where the metric G_2 has components,

$$
\begin{aligned}
(G_2)^{kn} = {} & k\,nA^{k-1}A^{n-1} + (k+n+2)A^{k+n} + \sum_{i=0}^{n-1}(k+i)A^{k+i-1}A^{n-i-1} + \\
& - \sum_{i=0}^{n-2}(n-i-1)A^{k+i}A^{n-i-2},
\end{aligned} \tag{45}
$$

and the Christoffel symbols are given by

$$
\begin{aligned}
(B_2)^{kn}_m = {} & knA^{k-1}\delta^{n-1}_m + (n+1)\delta^{k+n}_m - \sum_{i=0}^{n-2}\left(nA^{k+i}\delta^{n-i-2}_m\right) + \\
& + \sum_{i=0}^{n-1}\left(iA^{n-i-1}\delta^{k+i-1}_m + (k+i)A^{k+i-1}\delta^{n-i-1}_m\right).
\end{aligned} \tag{46}
$$

The Hamiltonian density is $\frac{1}{2}A^1$. This structure appeared for the first time in [18], where Kupershmidt derived it as a dispersionless limit of the second Poisson structure of the KP hierarchy. Recently, Błaszak and Szablikowski rediscovered it ([19],[20]) using the semiclassical R-matrix approach. Using the techniques developed in the previous sections, we obtain that the metric (45) becomes, in the Vlasov picture,

$$
\begin{aligned}
g_2^{(p,q)} = {} & f_p f_q - p f_p \delta(p-q) + f(p)\delta(p-q) + \frac{f(p)f_q - f(q)f_p}{q-p} + \\
& \delta(p-q)f_p \int \frac{f(r)}{r-p}dr - \delta(p-q)f(p)\int \frac{f(r)}{(r-p)^2}dr,
\end{aligned}
$$

and the Christoffel symbols (46) are then found to be

$$
\begin{aligned}
b_2\binom{p,q}{r} = {} & f_p\delta'(q-r) - q\delta'(q-r)\delta(p-r) + \frac{f(q)}{(q-p)^2}\delta(p-r) + \\
& - \delta(p-q)\delta(p-s)\int \frac{f(s)}{(s-p)^2}ds + \\
& + \delta'(q-p)\delta(p-r)\int \frac{f(s)}{s-p}ds + \\
& + \frac{f_p}{r-p}\left(\delta(p-q) - \delta(r-q)\right) + \frac{f(p)}{r-p}\left(\delta'(q-r) - \delta(q-p)\right).
\end{aligned}
$$

In analogy with the Kupershmidt-Manin structure, if we consider a Hamiltonian density depending on a finite number of moments $h = h(A^0, \dots, A^{N-1})$, we obtain Vlasov equations of the form (20), with kernel $V\binom{p}{q}$ given by

$$
\begin{aligned}
V\begin{pmatrix} p \\ q \end{pmatrix} = \sum_{n,m=0}^{N-1} h_{nm} q^m \Bigg(& - nA^{n-1} f_p - p^{n+1} f_p + p^n f(p) + \\
& - \sum_{i=0}^{n-1} p^i A^{n-i-1} f_p + \sum_{i=0}^{n-2} (n-i-1) p^i A^{n-i-2} f(p) \Bigg) + \\
& + \sum_{n=0}^{N-1} h_n \left(-n f_p q^{n-1} + (n+1) p^n \delta(p-q) + \delta(p-q) \sum_{i=0}^{n-1} i p^i A^{n-i-1} + \right. \\
& \left. - f_p \sum_{i=0}^{n-1} p^i q^{n-i-1} - n f(p) \sum_{i=0}^{n-2} p^{i-1} q^{n-i-1} \right).
\end{aligned}
$$

Consider now the Galilean transformation

$$
p \longmapsto p + \alpha,
$$

where α is a constant. It is easy to verify that, under this change of coordinates, we have

$$
{\pi_2}^{(p+\alpha, q+\alpha)} = {\pi_2}^{(p,q)} + \alpha {\pi_1}^{(p,q)},
$$

where $\pi_1 = \pi$ is the Poisson operator (43). Thus the brackets (43) and (44) form a Poisson pencil.

5 Conclusions and open questions

We have considered the problem of the integrability of Hydrodynamic chains which are Hamiltonian with respect to the Kupershmidt-Manin Poisson bracket. It turns out that this problem can be reduced to the study of the corresponding hydrodynamic Vlasov equation, for which the differential geometric objects related with a chain become integral operators. Using this formulation, we calculated the Haantjes tensor explicitly and found the conditions for it to vanish, showing that the conditions found bt Ferapontov and Marshall are in fact sufficient. In addition, we have constructed a suitable Dubrovin-Novikov Hamiltonian formalism for hydrodynamic Vlasov equations, getting explicit conditions for a class of diagonal metrics to be flat. Finally, we have found the formulation, in Vlasov variables, of the second Hamitonian structure for the Benney hierarchy. It would be interesting to study the analogous conditions on the Hamiltonian for the vanishing of the Haantjes tensor for systems with this Hamiltonian structure.

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