Universality of a double scaling limit near singular edge points in random matrix models

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Abstract

We consider unitary random matrix ensembles $Z_{n,s,t}^{-1}e^{-n\,\mathrm{tr}\,V_{s,t}(M)}dM$ on the space of Hermitian $n\times n$ matrices M, where the confining potential $V_{s,t}$ is such that the limiting mean density of eigenvalues (as $n\to\infty$ and $s,t\to 0$) vanishes like a power $5/2$ at a (singular) endpoint of its support. The main purpose of this paper is to prove universality of the eigenvalue correlation kernel in a double scaling limit. The limiting kernel is built out of functions associated with a special solution of the P_I^2 equation, which is a fourth order analogue of the Painlevé I equation. In order to prove our result, we use the well-known connection between the eigenvalue correlation kernel and the Riemann-Hilbert (RH) problem for orthogonal polynomials, together with the Deift/Zhou steepest descent method to analyze the RH problem asymptotically. The key step in the asymptotic analysis will be the construction of a parametrix near the singular endpoint, for which we use the model RH problem for the special solution of the P_I^2 equation.

In addition, the RH method allows us to determine the asymptotics (in a double scaling limit) of the recurrence coefficients of the orthogonal polynomials with respect to the varying weights $e^{-nV_{s,t}}$ on $\mathbb{R}$. The special solution of the P_I^2 equation pops up in the $n^{-2/7}$-term of the asymptotics.

1 Introduction and statement of results

1.1 Unitary random matrix ensembles

On the space $\mathcal{H}_n$ of Hermitian $n\times n$ matrices M, we consider for $n\in\mathbb{N}$ and $s,t\in\mathbb{R}$ the unitary random matrix ensemble,

$$\frac{1}{Z_{n,s,t}}e^{-n\,\mathrm{tr}\,V_{s,t}(M)}dM. \tag{1.1}$$

Here, $Z_{n,s,t}$ is a normalization constant and the confining potential $V_{s,t}$ is a real analytic function, depending on two parameters $s,t\in\mathbb{R}$, satisfying the asymptotic condition,

$$\lim_{x\to\pm\infty}\frac{V_{s,t}(x)}{\log(x^2+1)}=+\infty, \qquad \text{uniformly for } s,t\in[-\delta_0,\delta_0] \text{ for some } \delta_0>0. \tag{1.2}$$

Then,

$$Z_{n,s,t}=\int_{\mathcal{H}_n}e^{-n\,\mathrm{tr}\,V_{s,t}(M)}dM$$

is convergent as $n\to\infty$ so that the random matrix model is well-defined.

It is well-known, see e.g. [25], that an important role in the study of the unitary random matrix ensemble (1.1) is played by the following scalar 2-point (correlation) kernel,

$$K_n^{(s,t)}(x,y)=e^{-\frac{n}{2}V_{s,t}(x)}e^{-\frac{n}{2}V_{s,t}(y)}\sum_{k=0}^{n-1}p_k^{(n,s,t)}(x)p_k^{(n,s,t)}(y), \tag{1.3}$$

constructed out of the orthonormal polynomials

$$p_k^{(n,s,t)}(x) = \kappa_k^{(n,s,t)} x^k + \cdots, \qquad \kappa_k^{(n,s,t)} > 0,$$

with respect to the varying weights $e^{-nV_{s,t}}$ on $\mathbb{R}$. Indeed, the correlations between the eigenvalues of M can be written in terms of the correlation kernel. More precisely, the m-point correlation function $\mathcal{R}_{n,m}^{(s,t)}$ satisfies [25],

$$\mathcal{R}_{n,m}^{(s,t)}(x_1, \ldots, x_m) = \det\left(K_n^{(s,t)}(x_i, x_j)\right)_{1\le i,j\le m}. \tag{1.4}$$

Further, the limiting mean eigenvalue distribution $\mu_{s,t}$ has a density $\rho_{s,t}$ which can be retrieved from the correlation kernel as follows,

$$\rho_{s,t}(x) = \lim_{n\to\infty} \frac{1}{n} K_n^{(s,t)}(x,x). \tag{1.5}$$

The limiting mean eigenvalue distribution $\mu_{s,t}$ equals [10] the equilibrium measure in external field $V_{s,t}$. This is the unique measure minimizing the logarithmic energy [28]

$$I_{V_{s,t}}(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V_{s,t}(y)d\mu(y), \tag{1.6}$$

among all probability measures μ on $\mathbb{R}$. Furthermore, there exists a real analytic function $q_{s,t}$, such that [9],

$$\rho_{s,t}(x) = \frac{1}{\pi}\sqrt{q_{s,t}^-(x)}, \tag{1.7}$$

where $q_{s,t}^-$ denotes the negative part of $q_{s,t}$, i.e. $q_{s,t} = q_{s,t}^+ - q_{s,t}^-$, with $q_{s,t}^\pm \ge 0$ and $q_{s,t}^+ q_{s,t}^- = 0$. Due to condition (1.2) we have that $q_{s,t}(x) \to +\infty$ as $x \to \pm\infty$, so that $\mu_{s,t}$ is supported on a finite union of intervals, which we denote by $\mathbb{S}_{s,t}$. It is known [28] that the equilibrium measure $\mu_{s,t}$ satisfies the following Euler-Lagrange variational conditions: there exists a constant $\kappa_{s,t} \in \mathbb{R}$ such that

$$2\int \log|x-u|d\mu_{s,t}(u) - V_{s,t}(x) = \kappa_{s,t}, \qquad \text{for } x \in \mathbb{S}_{s,t}, \tag{1.8}$$

$$2\int \log|x-u|d\mu_{s,t}(u) - V_{s,t}(x) \le \kappa_{s,t}, \qquad \text{for } x \in \mathbb{R}\setminus\mathbb{S}_{s,t}. \tag{1.9}$$

The external field $V_{s,t}$ is called regular if strict inequality in (1.9) holds, if the density $\rho_{s,t}$ does not vanish in the interior of the support $\mathbb{S}_{s,t}$, and if $q_{s,t}$ has a simple zero at each of the endpoints of the support $\mathbb{S}_{s,t}$. If one of these conditions is not valid, $V_{s,t}$ is called singular. The singular points x^* are classified as follows, see [10, 21]:

(i) $x^* \in \mathbb{R}\setminus\mathbb{S}_{s,t}$ is a type I singular point if equality in (1.9) holds. Then, x^* is a zero of $q_{s,t}^+$ of multiplicity $4m$ with $m \in \mathbb{N}$.

(ii) $x^* \in \mathbb{S}_{s,t}$ is a type II singular point if it is an interior point of $\mathbb{S}_{s,t}$ where the equilibrium density $\rho_{s,t}$ vanishes. Then, x^* is a zero of $q_{s,t}^-$ of multiplicity $4m$.

(iii) x^* is a type III singular point if it is an endpoint of the support $\mathbb{S}_{s,t}$ and a zero of $q_{s,t}$ of multiplicity larger than one. Then, x^* is a zero of $q_{s,t}$ of multiplicity $4m+1$, which means that $\rho_{s,t}(x) \sim c|x-x^*|^{(4m+1)/2}$.

In this paper, we consider external fields $V_{s,t}$ which are such that in the critical case $s = t = 0$, $V_0 = V_{0,0}$ has a type III singular (edge) point x^* with $m = 1$, i.e.

$$\rho_{0,0}(x) \sim c|x - x^*|^{5/2}, \qquad \text{as } x \to x^*. \tag{1.10}$$

Further, we take $V_{s,t}$ of the special form,

$$V_{s,t} = V_0 + sV_1 + tV_2, \tag{1.11}$$

where V_1 is an arbitrary real analytic function, while V_2 is real analytic and in addition satisfies some critical condition which we will specify in Section 1.4 below.

1.2 Universality in random matrix theory

Consider for now unitary random matrix ensembles $Z_n^{-1} e^{-n \operatorname{tr} V(M)} dM$ on the space of Hermitian $n\times n$ matrices M. Scaling limits of the associated correlation kernel K_n show universal behavior.

Near regular points, universality results have been established in [1, 8, 10, 11, 27]. For example, if x^* lies in the bulk of the spectrum (i.e. x^* is such that it lies in the interior of the support $\mathbb{S}$ of the equilibrium measure in external field V, and such that the equilibrium density ρ does not vanish at x^*) there is a constant c such that

$$\lim_{n\to\infty} \frac{1}{cn} K_n\left(x^* + \frac{u}{cn}, x^* + \frac{v}{cn}\right) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{1.12}$$

On the other hand, if x^* is a regular edge point of the spectrum (i.e. x^* is an endpoint of $\mathbb{S}$ and ρ vanishes like a square root at x^*), there is a constant c such that

$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_n\left(x^* + \frac{u}{cn^{2/3}}, x^* + \frac{v}{cn^{2/3}}\right) = \frac{\operatorname{Ai}(u)\operatorname{Ai}'(v) - \operatorname{Ai}(v)\operatorname{Ai}'(u)}{u-v}, \tag{1.13}$$

where Ai is the Airy function.

Near singular points, similar results hold. In those singular cases it is interesting to consider double scaling limits where the external field V depends on additional parameters. In [2, 5, 6, 29], an external field V was considered such that there is a type II singular (interior) point x^* with $m = 1$, i.e.

$$\rho(x) \sim c(x - x^*)^2, \qquad \text{as } x \to x^*.$$

If an additional parameter is included in the external field, $V_t = V/t$, one observes for t close to 1 the transition where two intervals in the support of the limiting mean density of eigenvalues merge to one interval through the critical case of a type II singular point. In the double scaling limit where $n \to \infty$ and $t \to 1$ in such a way that $c_0 n^{2/3}(t-1) \to s \in \mathbb{R}$ for some appropriately chosen constant c_0, there exists a constant c such that (for the associated correlation kernel $K_{n,t}$),

$$\lim \frac{1}{cn^{1/3}} K_{n,t}\left(x^* + \frac{u}{cn^{1/3}}, x^* + \frac{v}{cn^{1/3}}\right) = K^{\mathrm{crit,II}}(u, v; s).$$

Here, $K^{\mathrm{crit,II}}(u, v; s)$ is built out of functions associated with the Hastings-McLeod solution [18] of the second Painlevé equation.

The main purpose of this paper is to obtain, for the random matrix models in Section 1.1 above, a similar result near the type III singular (edge) point of V_0 with $m = 1$. We take a double scaling limit ($n \to \infty$ and $s, t \to 0$), and the limiting kernel $K^{\mathrm{crit,III}}$ will be built out of functions which are associated with a special solution of the fourth order analogue of the Painlevé I equation. The case of a type III singular (edge) point was also studied in the Physics literature [3, 4].

In addition, the techniques that we use to prove this allow us to determine the asymptotics (in a double scaling limit) of the recurrence coefficients in the three-term recurrence relation satisfied by the orthogonal polynomials $p_k^{(n,s,t)}$ with respect to the varying weights $e^{-nV_{s,t}}$ on $\mathbb{R}$.

1.3 Φ-functions associated with a special solution of the P_I^2 equation

We consider the following differential equation for $y = y(s,t)$, which we denote as the P_I^2 equation,

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right). \tag{1.14}$$

For $t = 0$, this equation is the second member in the Painlevé I hierarchy [20, 23]. The P_I^2 equation has been studied for example in [4, 19, 26] (for $t = 0$) and [7, 15] (for general t). The Lax pair for the P_I^2 equation is the linear system of differential equations

$$\frac{\partial \Psi}{\partial \zeta} = U\Psi, \qquad \frac{\partial \Psi}{\partial s} = W\Psi, \tag{1.15}$$

where

$$U = \frac{1}{240}\begin{pmatrix} -4y_s\zeta - (12yy_s + y_{sss}) & 8\zeta^2 + 8y\zeta + (12y^2 + 2y_{ss} - 120t) \\ U_{21} & 4y_s\zeta + (12yy_s + y_{sss}) \end{pmatrix}, \tag{1.16}$$

$$U_{21} = 8\zeta^3 - 8y\zeta^2 - (4y^2 + 2y_{ss} + 120t)\zeta + (16y^3 - 2y_s^2 + 4yy_{ss} + 240s), \tag{1.17}$$

and

$$W = \begin{pmatrix} 0 & 1 \\ \zeta - 2y & 0 \end{pmatrix}. \tag{1.18}$$

The system of differential equations (1.15)–(1.18) can only be solvable if $y = y(s,t)$ is a solution to the P_I^2 equation (1.14). For different solutions y, we have different Lax pairs.

We are interested in the special solution y which was studied in [4, 7, 15]. This solution $y = y(s,t)$ is characterized by the vanishing of its Stokes multipliers s_1, s_2, s_5, and s_6, see [19] for details. It was shown in [7] that y has no poles for real s and t, and that it has, for fixed $t \in \mathbb{R}$, the following asymptotic behavior,

$$y(s,t) = \mp(6|s|)^{1/3} \mp \frac{1}{3}6^{2/3}t|s|^{-1/3} + \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty. \tag{1.19}$$

It has been shown in [26, Appendix A] that for $t = 0$, y is uniquely determined by realness and asymptotic condition (1.19). For general t we are not aware of a similar result although it is supported by a conjecture of Dubrovin [15] that this should hold for general t. For $s,t \in \mathbb{R}$, the Lax pair (1.15)–(1.18) associated with this special choice of y has a unique solution $\left(\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right)$ for which the following limit holds, see [7, 19],

$$\zeta^{\frac{1}{4}\sigma_3}\begin{pmatrix}\Phi_1(\zeta;s,t)\\ \Phi_2(\zeta;s,t)\end{pmatrix} e^{\theta(\zeta;s,t)} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} e^{-\frac{1}{4}\pi i}, \qquad \text{as } \zeta \to \infty \text{ with } 0 < \operatorname{Arg}\zeta < 6\pi/7, \tag{1.20}$$

where $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ denotes the third Pauli-matrix, and where θ is given by

$$\theta(\zeta;s,t) = \frac{1}{105}\zeta^{7/2} - \frac{1}{3}t\zeta^{3/2} + s\zeta^{1/2}. \tag{1.21}$$

The functions Φ_1 and Φ_2 will appear below in the universal limiting correlation kernel near type III singular (edge) points of V_0 with $m = 1$.

1.4 Statement of results

We work under the following assumptions.

Assumptions 1.1

(i) We consider external fields $V_{s,t}$ of the form

$$V_{s,t} = V_0 + sV_1 + tV_2, \tag{1.22}$$

where V_0, V_1, and V_2 are real analytic and are such that there exists a $\delta_0 > 0$ such that the following holds

$$\lim_{|x|\to\infty} \frac{V_{s,t}(x)}{\log(x^2+1)} = +\infty, \qquad \text{uniformly for } s,t \in [-\delta_0, \delta_0]. \tag{1.23}$$

(ii) V_0 is such that the equilibrium measure ν_0 in external field V_0 is supported on one single interval $[a,b] \subset \mathbb{R}$, and b is a type III singular (edge) point of V_0 with $m = 1$. Then, ν_0 is of the form [9],

$$d\nu_0(x) = \frac{1}{2\pi} h_0(x) \sqrt{(b-x)(x-a)}\, \chi_{[a,b]}(x) dx, \tag{1.24}$$

with $\chi_{[a,b]}$ the indicator function of the set $[a,b]$, and with h_0 real analytic and satisfying,

$$h_0(b) = h_0'(b) = 0, \qquad \text{and} \qquad h_0''(b) > 0. \tag{1.25}$$

Furthermore, we assume that V_0 has no other singular points besides b. In particular, a is a regular (edge) point and we then have that

$$h_0(a) > 0. \tag{1.26}$$

(iii) V_2 is such that it satisfies the critical condition

$$\int_a^b \sqrt{\frac{u-a}{b-u}} V_2'(u) du = 0. \tag{1.27}$$

Throughout the rest of this paper we let $\mathcal{V}$ be the neighborhood of the real line where V_0, V_1, V_2, and h_0 are analytic.

Example 1.2 The assumptions above are valid for the particular example where V_0, V_1, and V_2 are given by,

$$V_0(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x, \qquad V_1(x) = x, \qquad V_2(x) = x^3 - 6x. \tag{1.28}$$

Then, the equilibrium measure ν_0 is supported on the interval $[-2,2]$ and given by

$$d\nu_0(x) = \frac{1}{10\pi}(x+2)^{1/2}(x-2)^{5/2}\chi_{[-2,2]}(x)dx. \tag{1.29}$$

It should be noted that a type III singular (edge) point cannot occur when V_0 is a polynomial of degree lower than 4.

Example 1.3 In the continuum limit of the Toda lattice [12], an external field of the form

$$V_{t_1,t_2}(x) = (1+t_1)(V_0(x)+t_2 x)$$

has been studied. This deformation of V_0 can be written in the form (1.22) (so that it is included in the class of external fields studied in this paper). Indeed, if we let $V_1(x) = x$ and $V_2(x) = V_0(x) + cx$, with c some constant chosen such that the critical condition (1.27) holds, then

$$V_{t_1,t_2} = V_0 + sV_1 + tV_2,$$

with $s = t_2 + t_1 t_2 - ct_1$ and $t = t_1$.

Remark 1.4 In Section 2 we will show that assumption (iii) is equivalent to the vanishing of the equilibrium density $\frac{d\nu_2(x)}{dx}$ at the right endpoint b, where ν_2 is the unique measure which minimizes $I_{V_2}(\nu)$, see (1.6), among all signed measures ν, supported on $[a,b]$ and having zero mass, $\nu([a,b]) = 0$.

Remark 1.5 The case where the left (instead of the right) endpoint of the support is singular can be transformed to our case by considering the external field $V_{s,t}(-x)$.

Remark 1.6 Without giving any mathematical details, we now describe the transitions that can occur for s and t near 0. First, if we let $t = 0$ and s vary around 0, one typically observes the transition from the regular one-interval case to the singular case and back to the regular one-interval case. Next, for $s = 0$ and t around 0, we can observe the transition from the regular one-interval case to the regular two-interval case. Finally, letting both s and t vary around 0, we can observe one of the above described transitions, or the critical transition where a type II singular point moves to the endpoint b, where it becomes a type III singular point before moving on as a type I singular point.

Further, to describe our results, we have to introduce constants c, c_1, and c_2,

$$c = \left(\frac{15}{2} h_0''(b)\sqrt{b-a}\right)^{2/7} > 0, \qquad c_1 = \frac{h_1(b)}{c^{1/2}(b-a)^{1/2}}, \qquad c_2 = -\frac{h_2'(b)}{c^{3/2}(b-a)^{1/2}}, \tag{1.30}$$

where h_0 is the real analytic function appearing in (1.24), and where the functions h_1 and h_2 are defined as,

$$h_j(x) = -\frac{1}{\pi} \fint_a^b \sqrt{(b-u)(u-a)}\, V_j'(u) \frac{du}{u-x}, \qquad \text{for } x \in [a,b] \text{ and } j = 1,2. \tag{1.31}$$

1.4.1 Universality of the double scaling limit

Our main result is the following.

Theorem 1.7 *Let $V_{s,t} = V_0 + sV_1 + tV_2$ be such that Assumptions* 1.1 *above are satisfied. We take a double scaling limit where we let $n \to \infty$ and at the same time $s, t \to 0$, in such a way that* $\lim n^{6/7} s$ *and* $\lim n^{4/7} t$ *exists, and put*

$$s_0 = c_1 \cdot \lim n^{6/7} s \in \mathbb{R}, \qquad t_0 = c_2 \cdot \lim n^{4/7} t \in \mathbb{R}, \tag{1.32}$$

where the constants c_1 and c_2 are defined by (1.30)*. Then, the 2-point kernel $K_n^{(s,t)}$ satisfies the following universality result,*

$$\lim \frac{1}{cn^{2/7}} K_n^{(s,t)}\left(b + \frac{u}{cn^{2/7}}, b + \frac{v}{cn^{2/7}}\right) = K^{\mathrm{crit,III}}(u,v;s_0,t_0), \tag{1.33}$$

uniformly for u, v in compact subsets of $\mathbb{R}$. Here, $K^{\text{crit,III}}$ is built out of the functions Φ_1 and Φ_2 defined in Section 1.3,

$$K^{\text{crit,III}}(u, v; s, t) = \frac{\Phi_1(u; s, t)\Phi_2(v; s, t) - \Phi_1(v; s, t)\Phi_2(u; s, t)}{-2\pi i(u - v)}. \tag{1.34}$$

Remark 1.8 Since $y(s, t)$ has no poles [7] for $s, t \in \mathbb{R}$, the kernel $K^{\text{crit,III}}(u, v; s, t)$ exists for all real u, v, s, and t. Furthermore, using a similar argument as in [7, Lemma 2.3 (ii)], one can show that $e^{\pi i/4}\Phi_1$ and $e^{\pi i/4}\Phi_2$ are real. It then follows that $K^{\text{crit,III}}(u, v; s, t)$ is real for real u, v, s, and t.

Remark 1.9 It is possible to give an integral formula for $K^{\text{crit,III}}$. Using the fact that $\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}$ satisfies the second differential equation of the Lax pair (1.15), we have that

$$\frac{\partial \Phi_1}{\partial s}(\zeta; s, t) = \Phi_2(\zeta; s, t), \qquad \text{and} \qquad \frac{\partial \Phi_2}{\partial s}(\zeta; s, t) = (\zeta - 2y(s, t))\Phi_1(\zeta; s, t).$$

Using (1.34) this yields,

$$\frac{\partial K^{\text{crit,III}}}{\partial s}(u, v; s, t) = \frac{1}{2\pi i}\Phi_1(u; s, t)\Phi_1(v; s, t).$$

Now, since $\lim_{s\to-\infty} K^{\text{crit,III}}(u, v; s, t) = 0$, which can be shown using a Deift/Zhou steepest descent method argument [13], it then follows that $K^{\text{crit,III}}$ has the following integral formula,

$$K^{\text{crit,III}}(u, v; s, t) = \frac{1}{2\pi i}\int_{-\infty}^{s} \Phi_1(u; \sigma, t)\Phi_1(v; \sigma, t)d\sigma. \tag{1.35}$$

Remark 1.10 Theorem 1.7 can be generalized to the case where the support of ν_0 (the equilibrium measure in external field V_0) consists of more than one interval. Then, the proof becomes much more technical, although the main ideas remain the same. We comment in Remark 3.8 on the modifications that have to be made in the multi-interval case.

1.4.2 Recurrence coefficients for orthogonal polynomials

It is well-known [30] that the orthonormal polynomials $p_k = p_k^{(n,s,t)}$ satisfy a three-term recurrence relation of the form,

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_k p_k(x) + a_k p_{k-1}(x), \tag{1.36}$$

where $a_k = a_k^{(n,s,t)} > 0$ and $b_k = b_k^{(n,s,t)} \in \mathbb{R}$ (we suppress the s and t dependence for brevity). In the generic case where V_0 has no singular points, the recurrence coefficients for $s = t = 0$ have the following asymptotics, see e.g. [2, 8],

$$a_n^{(n,0,0)} = \frac{b-a}{4} + \mathcal{O}(n^{-1}), \qquad b_n^{(n,0,0)} = \frac{b+a}{2} + \mathcal{O}(n^{-1}), \qquad \text{as } n \to \infty. \tag{1.37}$$

For singular potentials V_0, the constant terms in the expansions (1.37) remain the same, but the error terms behave differently [2, 6]. In our case of interest, where we have a type III singular (edge) point of V_0 with $m = 1$, the error term is of order $\mathcal{O}(n^{-2/7})$, and the coefficient of the $n^{-2/7}$ term is expressed in terms of the special solution y of the P_I^2 equation discussed in Section 1.3.

Theorem 1.11 *Let $V_{s,t}$ be such that Assumptions* 1.1 *above are satisfied. Consider the three-term recurrence relation* (1.36) *satisfied by the orthonormal polynomials $p_k = p_k^{(n,s,t)}$ with respect to the weight function $e^{-nV_{s,t}}$. Then, in the double scaling limit where $n \to \infty$ and $s, t \to 0$, in such a way that $\lim n^{6/7} s$ and $\lim n^{4/7} t$ exists, and put*

$$s_0 = c_1 \cdot \lim n^{6/7} s \in \mathbb{R}, \qquad t_0 = c_2 \cdot \lim n^{4/7} t \in \mathbb{R}, \tag{1.38}$$

with c_1 and c_2 given by (1.30)*, we have*

$$\begin{aligned} a_n^{(n,s,t)} &= \frac{b-a}{4} + \frac{1}{2c} y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}), \\ &= \frac{b-a}{4} + \frac{1}{2c} y(s_0, t_0) n^{-2/7} (1 + o(1)), \end{aligned} \tag{1.39}$$

and

$$\begin{aligned} b_n^{(n,s,t)} &= \frac{b+a}{2} + \frac{1}{c} y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}) \\ &= \frac{b+a}{2} + \frac{1}{c} y(s_0, t_0) n^{-2/7} (1 + o(1)), \end{aligned} \tag{1.40}$$

where the constant c is given by (1.30)*, and where y is the special solution of the P_I^2 equation discussed in Section* 1.3.

Remark 1.12 Note that the expansions of the recurrence coefficients are of the same form as the conjectured (by Dubrovin [15, Main Conjecture, Part 3], see also [14]) expansions for solutions of perturbed hyperbolic equations. Here, the perturbation parameter ϵ plays the role of $1/n$ in our context.

Remark 1.13 For polynomials which are orthogonal on certain complex contours, it can occur that the equilibrium density vanishes like a power 3/2. Asymptotics of the recurrence coefficients in this case were obtained in [16]. Here, a special solution of the Painlevé I equation occurs instead of a solution of the P_I^2 equation and the asymptotics are in powers of $n^{-1/5}$.

Observe further that in [16] there is no term of order $n^{-1/5}$ in the asymptotics. In (1.39) and (1.40) we see that there is no term of order $n^{-1/7}$. In the proof of Theorem 1.11 this term will drop out in a similar way as the $n^{-1/5}$-term in [16].

1.5 Outline of the rest of the paper

We prove our results by characterizing the orthogonal polynomials via the well-known 2×2 matrix valued Fokas-Its-Kitaev Riemann-Hilbert (RH) problem [17] and applying the Deift/Zhou steepest descent method [13] to analyze this RH problem asymptotically. This approach has been used many times before, see e.g. [5, 6, 8, 10, 11, 16, 22, 31, 32].

An important step in the Deift/Zhou steepest descent method is the construction of so-called g-functions associated with equilibrium measures. Those equilibrium measures will be constructed in Section 2. In order to deal with the deformations $V_{s,t}$ of V_0, we use modified equilibrium problems where we allow the measures to be negative, which was also done in [5, 6, 16]. Another modification of the equilibrium problem is that we choose the support of the equilibrium measure fixed, instead of allowing it to choose its own support.

In Section 3, we perform the Deift/Zhou steepest descent analysis to the RH problem Y for orthogonal polynomials. Via a series of transformations $Y \mapsto T \mapsto S \mapsto R$ we want to arrive at a RH problem for R which is normalized at infinity (i.e. $R(z) \to I$ as $z \to \infty$) and with jumps uniformly close to the identity matrix. Then, R itself is close to the identity matrix.

By unfolding the series of transformations we then get the asymptotics of Y. The key step in this method will be the local analysis near the endpoints a and b. Near the regular endpoint a, we construct (in Section 3.5) a parametrix built out of Airy functions. Due to the modified equilibrium measures, which have a fixed support, we also need to make a technical modification in the construction of the Airy parametrix, compared with the parametrix as used e.g. in [8]. To construct the local parametrix near the singular endpoint b (in Section 3.6) we use a model RH problem associated with the special solution y of the P_I^2 equation as discussed in Section 1.3.

The results of Section 3 will be used in Section 4 to prove the universality result for the correlation kernel (see Theorem 1.7) and in Section 5 to determine the asymptotics of the recurrence coefficients (see Theorem 1.11).

2 Equilibrium measures

We consider external fields $V_{s,t} = V_0 + sV_1 + tV_2$ which satisfy Assumptions 1.1 in the beginning of Section 1.4. In order to perform the Deift/Zhou steepest descent analysis to the RH problem for orthogonal polynomials one would expect to use the equilibrium measure $\mu_{s,t}$ in external field $V_{s,t}$ minimizing $I_{V_{s,t}}(\mu)$, see (1.6), among all probability measures μ on $\mathbb{R}$. However, as in [5, 6, 16] it will be more convenient to use modified equilibrium measures $\nu_{s,t}$ which we allow to be negative. Furthermore, unlike in [5, 6, 16], we take the support of the measures $\nu_{s,t}$ to be fixed instead of letting it depend on s and t.

The aim of this section is to find measures $\nu_{s,t}$ (depending on the parameters $s,t \in \mathbb{R}$) supported on the interval $[a,b] \subset \mathbb{R}$ (where $[a,b]$ is the support of the equilibrium measure ν_0 in external field V_0), such that $\nu_{s,t}([a,b]) = 1$, and such that they satisfy the following condition: there exist $\ell_{s,t} \in \mathbb{R}$ such that for every $\delta > 0$ there are $\varepsilon, \kappa > 0$ sufficiently small such that for $s,t \in [-\varepsilon, \varepsilon]$,

$$2\int \log|x-u|d\nu_{s,t}(u) - V_{s,t}(x) = \ell_{s,t}, \qquad \text{for } x \in [a,b]. \tag{2.1}$$

$$2\int \log|x-u|d\nu_{s,t}(u) - V_{s,t}(x) < \ell_{s,t} - \kappa, \qquad \text{for } x \in \mathbb{R} \setminus [a-\delta, b+\delta]. \tag{2.2}$$

We seek $\nu_{s,t}$ in the following form,

$$\nu_{s,t} = \nu_0 + s\nu_1 + t\nu_2, \tag{2.3}$$

where ν_0 is the equilibrium measure in external field V_0 minimizing $I_{V_0}(\nu)$, see (1.6), among all probability measures ν on $\mathbb{R}$. From Assumption 1.1 (ii) we know that ν_0 can be written as follows

$$d\nu_0(x) = \psi_{0,+}(x)\chi_{[a,b]}(x)dx, \tag{2.4}$$

where $\chi_{[a,b]}$ is the indicator function of the set $[a,b]$, and where $\psi_{0,+}$ is the +boundary value of the function

$$\psi_0(z) = \frac{1}{2\pi i}R(z)h_0(z), \qquad \text{for } z \in \mathcal{V} \setminus [a,b], \tag{2.5}$$

with h_0 analytic in the neighborhood $\mathcal{V}$ of the real line, and with

$$R(z) = \big((z-a)(z-b)\big)^{1/2}, \qquad \text{for } z \in \mathbb{C} \setminus [a,b]. \tag{2.6}$$

Here, we take the principal branch of the square root so that R is analytic in $\mathbb{C} \setminus [a,b]$. Further, since a is a regular (edge) point and since b is a type III singular (edge) point with $m = 1$, we have, cf. (1.25) and (1.26),

$$h_0(a) > 0, \qquad h_0(b) = h_0'(b) = 0, \qquad \text{and} \qquad h_0''(b) > 0. \tag{2.7}$$

Since V_0 is assumed to have no other singular points besides b, we know (cf. (1.8) and (1.9)) that ν_0 satisfies the following condition: there exists $\ell_0 \in \mathbb{R}$ such that

$$2\int \log|x-u|d\nu_0(u) - V_0(x) = \ell_0, \qquad \text{for } x \in [a,b], \tag{2.8}$$

$$2\int \log|x-u|d\nu_0(u) - V_0(x) < \ell_0, \qquad \text{for } x \in \mathbb{R}\setminus[a,b]. \tag{2.9}$$

We will now construct the two measures ν_1 and ν_2. In order to do this we introduce the following auxiliary (analytic) functions,

$$h_j(z) = \frac{1}{2\pi i}\oint_\gamma R(\xi)V_j'(\xi)\frac{d\xi}{\xi - z}, \qquad \text{for } z \in \mathcal{V} \text{ and } j = 1,2, \tag{2.10}$$

where γ is a positively oriented contour in $\mathcal{V}$ with $[a,b]$ and z in its interior, and where R is given by (2.6). Observe that, using the fractional residue theorem, one has,

$$h_j(x) = -\frac{1}{\pi i}\fint_a^b R_+(u)V_j'(u)\frac{du}{u-x}, \qquad \text{for } x \in [a,b], \tag{2.11}$$

where the integral is a Cauchy principal value integral. So, h_j is real on $[a,b]$. Observe that by Assumption 1.1 (iii) and (2.11),

$$h_2(b) = 0. \tag{2.12}$$

Lemma 2.1 *Define two signed measures ν_1 and ν_2 supported on $[a,b]$ as*

$$d\nu_j(x) = \psi_{j,+}(x)\chi_{[a,b]}dx, \qquad j = 1,2, \tag{2.13}$$

where $\chi_{[a,b]}$ is the indicator function of the set $[a,b]$, and where $\psi_{j,+}$ is the +boundary value of the function

$$\psi_j(z) = \frac{1}{2\pi i}\frac{h_j(z)}{R(z)}, \qquad \text{for } z \in \mathcal{V}\setminus[a,b]. \tag{2.14}$$

Here, h_j is given by (2.10), see also (2.11) for its expression on $[a,b]$, and R is given by (2.6). Then, ν_j has zero mass, i.e.

$$\nu_j([a,b]) = \int_a^b \psi_{j,+}(u)du = 0, \tag{2.15}$$

and there exist constants $\ell_j \in \mathbb{R}$ such that

$$2\int \log|x-u|d\nu_j(u) - V_j(x) = \ell_j, \qquad \text{for } x \in [a,b]. \tag{2.16}$$

Proof. Define, for $j = 1,2$, the auxiliary functions

$$F_j(z) = \frac{1}{2\pi i R(z)}\int_a^b R_+(u)V_j'(u)\frac{du}{u-z}, \qquad \text{for } z \in \mathbb{C}\setminus[a,b], \tag{2.17}$$

which, by standard techniques and by (2.10) and (2.14), are equal to

$$\begin{aligned} F_j(z) &= \frac{1}{2}V_j'(z) - \frac{1}{4\pi i R(z)}\oint_\gamma R(\xi)V_j'(\xi)\frac{d\xi}{\xi - z} \\ &= \frac{1}{2}V_j'(z) - \pi i\psi_j(z), \qquad \text{for } z \in \mathcal{V}\setminus[a,b], \end{aligned}$$

where γ is a positively oriented contour in $\mathcal{V}$ with $[a,b]$ and z in its interior. This, together with the fact that $\psi_{j,+} = -\psi_{j,-}$ on (a,b), yields

$$F_{j,+}(x) - F_{j,-}(x) = -2\pi i \psi_{j,+}(x), \qquad \text{for } x \in [a,b], \tag{2.18}$$

$$F_{j,+}(x) + F_{j,-}(x) = V_j'(x), \qquad \text{for } x \in [a,b]. \tag{2.19}$$

Since F_j is analytic in $\mathbb{C} \setminus [a,b]$ and since, by (2.17), $F_j(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, a standard complex analysis argument, shows that

$$\frac{1}{2\pi i}\int_a^b \frac{F_{j,+}(u) - F_{j,-}(u)}{u-z} ds = F_j(z), \qquad \text{for } z \in \mathbb{C} \setminus [a,b].$$

By (2.18), this yields,

$$F_j(z) = -\int_a^b \frac{\psi_{j,+}(u)}{u-z} du = -z^{-1}\int_a^b \psi_{j,+}(u) du + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty.$$

Comparing this with the fact that $F_j(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, we obtain $\int_a^b \psi_{j,+}(u)du = 0$, so that (2.15) is proven.

It remains to prove (2.16). It is straightforward to check that,

$$F_j(z) = -\int_a^b \frac{\psi_{j,+}(u)}{u-z} du = -\pi i \psi_j(z) + \frac{1}{2}\oint_\gamma \frac{\psi_j(\xi)}{\xi - z} d\xi, \qquad \text{for } z \in \mathcal{V} \setminus [a,b],$$

so that, using the fractional residue theorem,

$$F_{j,\pm}(x) = -\pi i \psi_{j,\pm}(x) - \fint_a^b \frac{\psi_{j,+}(u)}{u-x} du, \qquad \text{for } x \in [a,b].$$

From (2.19) and the fact that $\psi_{j,+} + \psi_{j,-} = 0$ on $[a,b]$ this yields,

$$\frac{d}{dx}\left(2\int \log|x-u| d\nu_j(u) + V_j(x)\right) = 2\fint_a^b \frac{\psi_{j,+}(u)}{u-x} du + \big(F_{j,+}(x) + F_{j,-}(x)\big) = 0. \tag{2.20}$$

This proves (2.16). □

Corollary 2.2 *Let $\nu_{s,t} = \nu_0 + s\nu_1 + t\nu_t$. Then, $d\nu_{s,t}(x) = \psi_{s,t,+}(x)\chi_{[a,b]}dx$, where*

$$\psi_{s,t} = \psi_0 + s\psi_1 + t\psi_2, \qquad \text{on } \mathcal{V} \setminus [a,b], \tag{2.21}$$

with ψ_0 given by (2.5) and ψ_1 and ψ_2 given by (2.14). So, $\nu_{s,t}$ is supported on $[a,b]$ and has mass one, i.e. $\nu_{s,t}([a,b]) = 1$. Further, there exist constants $\ell_{s,t} \in \mathbb{R}$ such that for any $\delta > 0$ there are $\varepsilon, \kappa > 0$ sufficiently small such that for $s,t \in [-\varepsilon, \varepsilon]$ the conditions (2.1) and (2.2) are satisfied.

Proof. Since $\nu_{s,t} = \nu_0 + s\nu_1 + t\nu_t$, from (2.15), and from the fact that $\nu_0([a,b]) = 1$ it is clear that $\nu_{s,t}([a,b]) = 1$. Next, with $\ell_{s,t} = \ell_0 + s\ell_1 + t\ell_2$, we have

$$2\int \log|x-u| d\nu_{s,t}(u) - V_{s,t}(x) - \ell_{s,t} = I_0(x) + sI_1(x) + tI_2(x) \tag{2.22}$$

where

$$I_j(x) = 2\int \log|x-u| d\nu_j(u) - V_j(x) - \ell_j, \qquad j = 1,2,3.$$

Then, condition (2.1) follows from (2.8) and (2.16). Now, by using (2.9) and the fact that $I_0(x) \to -\infty$ as $|x| \to \infty$, there exists $\kappa > 0$ such that

$$I_0 < -\frac{3}{2}\kappa, \qquad \text{on } \mathbb{R} \setminus [a-\delta, b+\delta]. \tag{2.23}$$

Further, one can check that I_1 and I_2 are bounded on $\mathbb{R} \setminus [a-\delta, b+\delta]$, and thus there exists $\varepsilon > 0$ such that for $s, t \in [-\varepsilon, \varepsilon]$,

$$sI_1 + tI_2 < \frac{1}{2}\kappa, \qquad \text{on } \mathbb{R} \setminus [a-\delta, b+\delta]. \tag{2.24}$$

Inserting (2.23) and (2.24) into (2.22) we obtain condition (2.2). □

Remark 2.3 The measure ν_1 (ν_2) is the equilibrium measure that minimizes $I_{V_1}(\nu)$ ($I_{V_2}(\nu)$) among all signed measures ν, supported on $[a,b]$ with $\nu([a,b]) = 0$. The measures $\nu_{s,t}$ on the other hand minimize $I_{V_{s,t}}(\nu)$ among all signed measures supported on $[a,b]$ with $\nu([a,b]) = 1$.

Observe that since ν_0 has a strictly positive density on (a,b) (since ν_0 has no type II singular points) we have for any $\delta > 0$ that $\nu_{s,t}$ is positive on $(a+\delta, b-\delta)$ for s,t sufficiently small.

3 Riemann-Hilbert analysis

3.1 RH problem for orthogonal polynomials

For each fixed n, s, and t, we consider the Fokas-Its-Kitaev Riemann-Hilbert problem [17] characterizing the orthogonal polynomials $p_k^{(n,s,t)}$ with respect to the weight functions $e^{-nV_{s,t}}$. We seek a 2×2 matrix-valued function $Y(z) = Y(z; n, s, t)$ (we suppress the n, s, and t dependence for brevity) that satisfies the following conditions.

RH problem for Y:

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Y possesses continuous boundary values for $x \in \mathbb{R}$ denoted by $Y_+(x)$ and $Y_-(x)$, where $Y_+(x)$ and $Y_-(x)$ denote the limiting values of $Y(z')$ as z' approaches x from above and below, respectively, and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV_{s,t}(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R}. \tag{3.1}$$

(c) Y has the following asymptotic behavior at infinity

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad \text{as } z \to \infty. \tag{3.2}$$

The unique solution of the RH problem is given by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \dfrac{\kappa_n^{-1}}{2\pi i} \displaystyle\int_{\mathbb{R}} \frac{p_n(u) e^{-nV_{s,t}(u)}}{u-z} du \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \displaystyle\int_{\mathbb{R}} \frac{p_{n-1}(u) e^{-nV_{s,t}(u)}}{u-z} du \end{pmatrix}, \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.3}$$

where $p_k = p_k^{(n,s,t)}$ is the k-th degree orthonormal polynomial with respect to the varying weight $e^{-nV_{s,t}}$, and where $\kappa_k = \kappa_k^{(n,s,t)} > 0$ is the leading coefficient of p_k. The solution (3.3) is due to Fokas, Its, and Kitaev [17], see also [8, 10, 11].

It is now possible to write the 2-point kernel $K_n^{(s,t)}$, see (1.3), in terms of Y. Indeed using the Christoffel-Darboux formula for orthogonal polynomials and the fact that $\det Y \equiv 1$ (which follows easily from (3.1), (3.2), and Liouville's theorem), we get

$$K_n^{(s,t)}(x,y) = e^{-\frac{n}{2}V_{s,t}(x)}e^{-\frac{n}{2}V_{s,t}(y)}\frac{1}{2\pi i(x-y)}\begin{pmatrix}0 & 1\end{pmatrix} Y_\pm^{-1}(y)Y_\pm(x)\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.4}$$

So, in order to prove Theorem 1.7, we need to analyze the RH problem for Y asymptotically. We do this by applying the Deift/Zhou steepest descent method [13] to this RH problem.

3.2 Normalization of the RH problem at infinity: $Y \mapsto T$

In order to normalize the RH problem for Y at infinity, the equilibrium measures $\nu_{s,t}$, introduced in Section 2 play a key role. Consider the log-transform $g_{s,t}$ of $\nu_{s,t}$,

$$g_{s,t}(z) = \int_a^b \log(z-u)d\nu_{s,t}(u), \qquad \text{for } z \in \mathbb{C}\setminus(-\infty,b]. \tag{3.5}$$

Here, we take the principal branch of the logarithm so that $g_{s,t}$ is analytic in $\mathbb{C}\setminus(-\infty,b]$. We now give properties of $g_{s,t}$ which are crucial in the following. From (3.5) and condition (2.1) it follows that

$$g_{s,t,+}(x) + g_{s,t,-}(x) - V_{s,t}(x) - \ell_{s,t} = 0, \qquad \text{for } x \in [a,b]. \tag{3.6}$$

Another crucial property is that

$$g_{s,t,+}(x) - g_{s,t,-}(x) = 2\pi i\int_x^b d\nu_{s,t}(u), \qquad \text{for } x \in \mathbb{R}, \tag{3.7}$$

so that since $\nu_{s,t}$ is supported on $[a,b]$ and has mass one (see Corollary 2.2),

$$g_{s,t,+}(x) - g_{s,t,-}(x) = \begin{cases} 2\pi i, & \text{for } x < a,\\ 0, & \text{for } x > b.\end{cases} \tag{3.8}$$

Now, we are ready to perform the first transformation $Y \mapsto T$. Define the matrix valued function T as

$$T(z) = e^{-\frac{1}{2}n\ell_{s,t}\sigma_3}Y(z)e^{-ng_{s,t}(z)\sigma_3}e^{\frac{1}{2}n\ell_{s,t}\sigma_3}, \qquad \text{for } z \in \mathbb{C}\setminus\mathbb{R}, \tag{3.9}$$

where $\ell_{s,t}$ is the constant that appears in the variational conditions (2.1) and (2.2), and where $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ denotes the third Pauli-matrix. Using (3.6), (3.8), the RH conditions for Y, and the fact that $g_{s,t}(z) = \log z + \mathcal{O}(1/z)$ as $z \to \infty$, it is straightforward to check that T is a solution to the following RH problem.

RH problem for T:

(a) $T : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $T_+(x) = T_-(x)v_T(x)$ for $x \in \mathbb{R}$, with

$$v_T = \begin{cases} \begin{pmatrix} e^{-n(g_{s,t,+}-g_{s,t,-})} & 1\\ 0 & e^{n(g_{s,t,+}-g_{s,t,-})}\end{pmatrix}, & \text{on } (a,b),\\ \begin{pmatrix} 1 & e^{n(g_{s,t,+}+g_{s,t,-}-V_{s,t}-\ell_{s,t})}\\ 0 & 1\end{pmatrix}, & \text{on } \mathbb{R}\setminus(a,b).\end{cases} \tag{3.10}$$

(c) $T(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

Remark 3.1 From (3.7) we see that the diagonal entries of v_T on (a,b) are rapidly oscillating for large n. Further, using condition (2.2) and (3.5), we see that $v_T - I$ decays exponentially on $\mathbb{R} \setminus [a-\delta, b+\delta]$.

3.3 Opening of the lens: $T \mapsto S$

Here, we will transform the oscillatory diagonal entries of the jump matrix v_T on (a,b) into exponentially decaying off-diagonal entries. This step is referred to as the opening of the lens.

Introduce a scalar function $\phi_{s,t}$ as,

$$\phi_{s,t}(z) = -\pi i \int_z^b \psi_{s,t}(\xi) d\xi, \qquad \text{for } z \in \mathcal{V} \setminus (-\infty, b], \tag{3.11}$$

where the path of integration does not cross the real line, and where $\psi_{s,t}$ is defined by (2.21).

The important feature of the function $\phi_{s,t}$ is that by (3.7), $\phi_{s,t,+}$ and $\phi_{s,t,-}$ are purely imaginary on (a,b) and satisfy,

$$-2\phi_{s,t,+}(x) = 2\phi_{s,t,-}(x) = 2\pi i \int_x^b d\nu_{s,t}(u) = g_{s,t,+}(x) - g_{s,t,-}(x), \quad \text{for } x \in (a,b), \tag{3.12}$$

which means that $-2\phi_{s,t}$ and $2\phi_{s,t}$ provide analytic extensions of $g_{s,t,+} - g_{s,t,-}$ into the upper half-plane and lower half-plane, respectively. Further, $2g_{s,t} + 2\phi_{s,t} - V_{s,t} - \ell_{s,t}$ is analytic in $\mathcal{V} \setminus (-\infty, b]$ and satisfies by (3.12) and (3.6),

$$2g_{s,t,\pm} + 2\phi_{s,t,\pm} - V_{s,t} - \ell_{s,t} = g_{s,t,+} + g_{s,t,-} - V_{s,t} - \ell_{s,t} = 0, \qquad \text{on } (a,b),$$

so that by the identity theorem,

$$2g_{s,t} - V_{s,t} - \ell_{s,t} = -2\phi_{s,t}, \qquad \text{on } \mathcal{V} \setminus (-\infty, a]. \tag{3.13}$$

Using (3.8) this yields,

$$\begin{aligned} g_{s,t,+} + g_{s,t,-} - V_{s,t} - \ell_{s,t} &= 2g_{s,t,-} - V_{s,t} - \ell_{s,t} + (g_{s,t,+} - g_{s,t,-}) \\ &= -2\phi_{s,t,-} + 2\pi i, \qquad \text{on } (-\infty, a). \end{aligned} \tag{3.14}$$

Inserting (3.12), (3.13), and (3.14) into (3.10), the jump matrix for T can be written in terms of $\phi_{s,t}$ as

$$v_T = \begin{cases} \begin{pmatrix} e^{2n\phi_{s,t,+}} & 1 \\ 0 & e^{2n\phi_{s,t,-}} \end{pmatrix}, & \text{on } (a,b), \\ \begin{pmatrix} 1 & e^{-2n\phi_{s,t,-}} \\ 0 & 1 \end{pmatrix}, & \text{on } \mathbb{R} \setminus (a,b). \end{cases} \tag{3.15}$$

It is straightforward to check, using the fact that $\phi_{s,t,+} + \phi_{s,t,-} = 0$ on (a,b), that v_T has on the interval (a,b) the following factorization,

$$v_T = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,-}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,+}} & 1 \end{pmatrix}, \qquad \text{on } (a,b), \tag{3.16}$$

and the opening of the lens is based on this factorization. Observe that, since $\operatorname{Re} \phi_{s,t,\pm}(x) = 0$ and $\operatorname{Im} \phi_{s,t,\pm}(x) = \mp \int_x^b d\nu_{s,t}(u)$ for $x \in (a,b)$ (see (3.12)), and since $\nu_{s,t}$ is positive on $(a+\delta, b-\delta)$ for

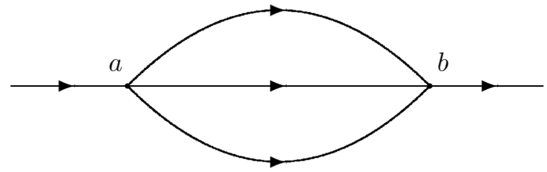

Figure 1: The lens Σ

$\delta > 0$ and s, t sufficiently small (see Remark 2.3), it follows (as in [8]) from the Cauchy-Riemann conditions that

$$\operatorname{Re}\phi_{s,t}(z) < 0, \qquad \text{for } |\operatorname{Im} z| \neq 0 \text{ small and } a + \delta < \operatorname{Re} z < b - \delta. \tag{3.17}$$

We deform the RH problem for T into a RH problem for S by opening a lens as shown in Figure 1, so that we obtain a contour Σ. For now, we choose the lens to be contained in $\mathcal{V}$, but we will specify later how we choose the lens exactly. Let

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lens.} \\ T(z)\begin{pmatrix} 1 & 0 \\ -e^{2n\phi_{s,t}(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the upper part of the lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t}(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the lower part of the lens.} \end{cases} \tag{3.18}$$

Then, using (3.16) and the RH conditions for T, one can check that S is the unique solution of the following RH problem.

RH problem for S:

(a) $S : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $S_+(z) = S_-(z)v_S(z)$ for $z \in \Sigma$, with

$$v_S = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (a,b), \\ \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t}} & 1 \end{pmatrix}, & \text{on } \Sigma \cap \mathbb{C}_\pm, \\ \begin{pmatrix} 1 & e^{-2n\phi_{s,t,-}} \\ 0 & 1 \end{pmatrix}, & \text{on } \mathbb{R} \setminus (a,b). \end{cases} \tag{3.19}$$

(c) $S(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

Remark 3.2 On the lips of the lens (away from a and b) and on $\mathbb{R} \setminus [a-\delta, b+\delta]$, it follows from (3.17) and (2.2) that the jump matrix for S converges exponentially fast to the identity matrix as $n \to \infty$. This convergence is uniform as long as we stay away from small disks surrounding the endpoints a and b. Near these endpoints we have to construct local parametrices.

3.4 Parametrix $P^{(\infty)}$ for the outside region

From Remark 3.2, we expect that the leading order asymptotics of Y will be determined by a solution $P^{(\infty)}$ of the following RH problem.

RH problem for $P^{(\infty)}$:

(a) $P^{(\infty)} : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}_+(x) = P^{(\infty)}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad$ for $x \in (a,b)$.

(c) $P^{(\infty)}(z) = I + \mathcal{O}(1/z), \qquad$ as $z \to \infty$.

It is well known, see for example [8, 11], that $P^{(\infty)}$ given by

$$P^{(\infty)}(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \left(\frac{z-b}{z-a} \right)^{\sigma_3/4} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}, \qquad \text{for } z \in \mathbb{C} \setminus [a,b], \tag{3.20}$$

is a solution to the above RH problem. Note that $P^{(\infty)}$ is independent of the parameters s, t and n.

3.5 Parametrix $P^{(a)}$ near the regular endpoint a

Here, we do the local analysis near the regular endpoint a. Let $U_{\delta,a} = \{z \in \mathbb{C} : |z-a| < \delta\}$ be a small disk with center a and radius $\delta > 0$ sufficiently small such that the disk lies in $\mathcal{V}$. We seek a 2×2 matrix valued function $P^{(a)}$ (depending on the parameters n, s, and t) in the disk $U_{\delta,a}$ with the same jumps as S and which matches with $P^{(\infty)}$ on the boundary $\partial U_{\delta,a}$ of the disk. We thus seek a 2×2 matrix valued function that satisfies the following RH problem.

RH problem for $P^{(a)}$:

(a) $P^{(a)} : U_{\delta,a} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(a)}_+(z) = P^{(a)}_-(z) v_S(z)$ for $z \in \Sigma \cap U_{\delta,a}$, where v_S is given by (3.19).

(c) $P^{(a)}$ satisfies the matching condition

$$P^{(a)}(z) (P^{(\infty)})^{(-1)}(z) = I + \mathcal{O}(n^{-1/7}), \tag{3.21}$$

as $n \to \infty$ and $s, t \to 0$ such that (1.32) holds, uniformly for $z \in \partial U_{\delta,a} \setminus \Sigma$.

3.5.1 Airy model RH problem

We will construct $P^{(a)}$ by introducing an auxiliary 2×2 matrix valued function $A(\zeta; r)$ with jumps (in the variable ζ) on an oriented contour $\Gamma = \bigcup_j \Gamma_j$, shown in Figure 2, consisting of four straight rays

$$\Gamma_1 : \arg \zeta = 0, \qquad \Gamma_2 : \arg \zeta = \frac{6\pi}{7}, \qquad \Gamma_3 : \arg \zeta = \pi, \qquad \Gamma_4 : \arg \zeta = -\frac{6\pi}{7}.$$

These four rays divide the complex plane into four regions $\mathrm{I}, \mathrm{II}, \mathrm{III}$, and IV, also shown in Figure 2. Put

$$y_j = y_j(\zeta; r) = \omega^j \operatorname{Ai}(\omega^j (\zeta + r)), \qquad j = 0, 1, 2,$$

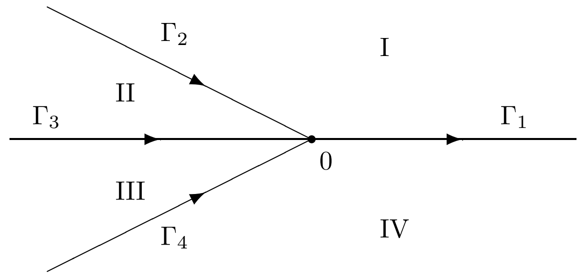

Figure 2: The oriented contour Γ. The four straight rays $\Gamma_1, \ldots, \Gamma_4$ divide the complex plane into four regions $\mathrm{I}, \mathrm{II}, \mathrm{III}$ and IV.

with $\omega = e^{\frac{2\pi i}{3}}$ and with Ai the Airy function, and let,

$$A(\zeta; r) = \sqrt{2\pi} e^{-\frac{\pi i}{4}} \times \begin{cases} \begin{pmatrix} y_0 & -y_2 \\ y_0' & -y_2' \end{pmatrix}, & \text{for } \zeta \in \mathrm{I}, \\ \begin{pmatrix} -y_1 & -y_2 \\ -y_1' & -y_2' \end{pmatrix}, & \text{for } \zeta \in \mathrm{II}, \\ \begin{pmatrix} -y_2 & y_1 \\ -y_2' & y_1' \end{pmatrix}, & \text{for } \zeta \in \mathrm{III}, \\ \begin{pmatrix} y_0 & y_1 \\ y_0' & y_1' \end{pmatrix}, & \text{for } \zeta \in \mathrm{IV}. \end{cases} \tag{3.22}$$

With y_j' we mean the derivative of y_j with respect to ζ. It is well-known, see e.g. [8, 11], that A satisfies the following RH problem.

RH problem for A:

(a) A is analytic for $\zeta \in \mathbb{C} \setminus \Gamma$ and for r in $\mathbb{C}$.

(b) A satisfies the following jump relations on Γ,

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3, \tag{3.23}$$

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{3.24}$$

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4. \tag{3.25}$$

(c) A has the following asymptotic behavior at infinity,

$$\begin{aligned} A(\zeta; r) &= (\zeta + r)^{-\frac{\sigma_3}{4}} N \left[I + \mathcal{O}\left((\zeta + r)^{-3/2} \right) \right] e^{-\frac{2}{3}(\zeta + r)^{3/2} \sigma_3} \\ &= \zeta^{-\frac{\sigma_3}{4}} N \left[I - \frac{1}{4} r^2 \zeta^{-1/2} \sigma_3 + \frac{1}{32} r^4 \zeta^{-1} I + \mathcal{O}(r^6 \zeta^{-3/2}) + \mathcal{O}(r \zeta^{-1}) \right] \\ &\qquad \times e^{-(\frac{2}{3}\zeta^{3/2} + r\zeta^{1/2})\sigma_3}, \end{aligned} \tag{3.26}$$

as $\zeta \to \infty$, uniformly for r such that

$$\operatorname{sgn}\big(\operatorname{Im}(\zeta + r)\big) = \operatorname{sgn}\big(\operatorname{Im}\zeta\big), \qquad \text{and} \qquad |r| < |\zeta|^{1/4}. \tag{3.27}$$

In (3.26), N is given by

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}. \tag{3.28}$$

3.5.2 Construction of $P^{(a)}$

We seek $P^{(a)}$ in the following form

$$P^{(a)}(z) = E^{(a)}(z)\sigma_3 A\left(n^{2/3} f_a(z); n^{2/3} r_{s,t}(z)\right) \sigma_3 e^{n\phi_{s,t}(z)\sigma_3}, \tag{3.29}$$

where $E^{(a)}$ is an invertible 2×2 matrix valued function analytic on $U_{\delta,a}$ and where f_a and $r_{s,t}$ are (scalar) analytic functions on $U_{\delta,a}$ which are real on $(a-\delta, a+\delta)$. In addition we take f_a to be a conformal map from $U_{\delta,a}$ onto a convex neighborhood $f_a(U_{\delta,a})$ of 0 such that $f_a(a) = 0$ and $f_a'(a) < 0$.

If those conditions are all satisfied, and if we open the lens Σ (recall that the lens was not yet fully specified) such that

$$f_a(\Sigma \cap (U_{\delta,a} \cap \mathbb{C}_+)) = \Gamma_4 \cap f_a(U_{\delta,a}), \qquad \text{and} \qquad f_a(\Sigma \cap (U_{\delta,a} \cap \mathbb{C}_-)) = \Gamma_2 \cap f_a(U_{\delta,a}),$$

then it is straightforward to verify, using (3.19) and (3.23)–(3.25), that $P^{(a)}$ defined by (3.29) satisfies conditions (a) and (b) of the RH problem for $P^{(a)}$.

Let

$$\begin{aligned} f_a(z) &= \left[\frac{3}{2}\left(-\pi i \int_z^a \psi_0(\xi)d\xi\right)(a-z)^{-3/2}\right]^{2/3}(a-z) \\ &= -\left(\frac{1}{2}h_0(a)\sqrt{b-a}\right)^{2/3}(z-a) + \mathcal{O}((z-a)^2), \qquad \text{as } z \to a, \end{aligned} \tag{3.30}$$

where we have used (2.5), and let

$$r_{s,t}(z) = \left(-\pi i \int_z^a (s\psi_1(\xi) + t\psi_2(\xi))d\xi\right) f_a(z)^{-1/2}. \tag{3.31}$$

Then, f_a is analytic with $f_a(a) = 0$ and $f_a'(a) < 0$, it is real on $(a-\delta, a+\delta)$, and it is a conformal mapping on $U_{\delta,a}$ provided $\delta > 0$ is sufficiently small. Further, it is straightforward to check that $r_{s,t}$ is analytic on $U_{\delta,a}$ and real on $(a-\delta, a+\delta)$, as well. Thus, f_a and $r_{s,t}$ satisfy the above conditions, so that $P^{(a)}$ defined by (3.29), with $E^{(a)}$ any invertible analytic matrix valued function, satisfies conditions (a) and (b) of the RH problem for $P^{(a)}$.

Remark 3.3 We can use any functions f_a and $r_{s,t}$, satisfying the conditions stated under equation (3.29), to construct the parametrix $P^{(a)}$. However, we have to choose them so as to compensate for the factor $e^{n\phi_{s,t}\sigma_3}$ in (3.29). Using (2.21), (3.11), and the fact that $\int_a^b \psi_{s,t+}(u)du = 1$ we have

$$e^{-n\left(\frac{2}{3}f_a(z)^{3/2} + r_{s,t}(z)f_a(z)^{1/2}\right)\sigma_3} = (-1)^n e^{-n\phi_{s,t}(z)\sigma_3}, \qquad \text{for } z \in U_{\delta,a} \setminus [a, a+\delta). \tag{3.32}$$

From this and (3.26) it is clear that our choice of f_a and $r_{s,t}$ will do the job.

It now remains to determine $E^{(a)}$ such that the matching condition (c) holds as well. In order to do this we make use of the following result.

Proposition 3.4 *Let $n \to \infty$ and $s, t \to 0$ such that* (1.32) *holds. Then,*

$$P^{(a)}(z) = (-1)^n E^{(a)}(z) \left(n^{2/3} f_a(z)\right)^{-\sigma_3/4} \sigma_3 N \sigma_3 \\ \times \left[I - \frac{(n^{4/7} r_{s,t}(z))^2}{4 f_a(z)^{1/2}} \sigma_3 \, n^{-1/7} + \frac{(n^{4/7} r_{s,t}(z))^4}{32 f_a(z)} I \, n^{-2/7} + \mathcal{O}(n^{-3/7}) \right]. \quad (3.33)$$

Proof. We will use the asymptotics (3.26) of A. In order to do this we have to check that condition (3.27) is satisfied for our choice of $\zeta = n^{2/3} f_a(z)$ and $r = n^{2/3} r_{s,t}(z)$.

Obviously $r_{s,t}(z) = \mathcal{O}(n^{-4/7})$ as $n \to \infty$ and $s, t \to 0$ such that (1.32) holds, uniformly for $z \in \partial U_{\delta,a}$. Then, it is straightforward to check that there exists $n_0 \in \mathbb{N}$ sufficiently large, and $\kappa_1, \kappa_2 > 0$ sufficiently small, such that

$$|n^{2/3} r_{s,t}(z)| < |n^{2/3} f_a(z)|^{1/4}, \quad (3.34)$$

for $z \in \partial U_{\delta,a}$ (for a possible smaller δ), for $n \geq n_0$, and for s and t such that $|c_1 n^{6/7} s - s_0| \leq \kappa_1$ and $|c_1 n^{4/7} t - t_0| \leq \kappa_2$.

Further, since f_a and $r_{s,t}$ are analytic near a and real valued on $(a-\delta, a+\delta)$ one can check that

$$\operatorname{Im} f_a(z) = f_a'(\operatorname{Re} z) \operatorname{Im} z + \mathcal{O}((\operatorname{Im} z)^2), \qquad \text{as } z \to a,$$

$$\operatorname{Im} r_{s,t}(z) = r_{s,t}'(\operatorname{Re} z) + \mathcal{O}((\operatorname{Im} z)^2), \qquad \text{as } z \to a.$$

Now, since $f_a'(a) \neq 0$ and $r_{s,t}'(\operatorname{Re} z) = \mathcal{O}(n^{-4/7})$ uniformly for $z \in U_{\delta,a}$ one then can find a constant $C > 0$ such that,

$$|\operatorname{Im} r_{s,t}(z)| < C |\operatorname{Im} z| < |\operatorname{Im} f_a(z)|,$$

for $z \in \partial U_{\delta,a} \setminus (a-\delta, a+\delta)$, for $n \geq n_0$, and for s and t such that $|c_1 n^{6/7} s - s_0| \leq \kappa_1$ and $|c_1 n^{4/7} t - t_0| \leq \kappa_2$ (for a possible smaller δ, κ_1 and κ_2, and for a possible larger n_0). This yields

$$\operatorname{sgn}(\operatorname{Im}(f_a(z) + r_{s,t}(z))) = \operatorname{sgn}(\operatorname{Im} f_a(z)). \quad (3.35)$$

We now have shown that condition (3.27) is satisfied so that we can use the asymptotic behavior (3.26) of A. Using (3.29), (3.26), (3.32), and the fact that $r_{s,t}(z) = \mathcal{O}(n^{-4/7})$ we obtain (3.33). □

From (3.33) and the fact that $r_{s,t} = \mathcal{O}(n^{-4/7})$ it is clear that (in order that the matching condition (c) is satisfied) we have to define $E^{(a)}$ by,

$$E^{(a)} = (-1)^n P^{(\infty)} \sigma_3 N^{-1} \sigma_3 (n^{2/3} f_a)^{\sigma_3/4}. \quad (3.36)$$

Obviously, $E^{(a)}$ is well-defined and analytic in $U_{\delta,a} \setminus (a, a+\delta)$. Further, using condition (b) of the RH problem for $P^{(\infty)}$, equation (3.28), and the fact that $f_{a,-}^{1/4} = i f_{a,+}^{1/4}$ on $(a, a+\delta)$, it is straightforward to check that $E^{(a)}$ has no jump on $(a, a+\delta)$. We then have that $E^{(a)}$ is analytic in $U_{\delta,a}$ except for a possible isolated singularity at a. However, $E^{(a)}$ has at most a square root singularity at a and hence it has to be a removable singularity. Further, since $\det P^{(\infty)} \equiv 1$ and $\det N = 1$ it is clear that $\det E^{(a)} \equiv 1$ and thus $E^{(a)}$ is invertible. This ends the construction of the parametrix near the regular endpoint.

3.6 Parametrix $P^{(b)}$ near the critical endpoint b

Here, we do the local analysis near the critical endpoint b. Let $U_{\delta,b} = \{z \in \mathbb{C} : |z - b| < \delta\}$ be a small disk with center b and radius $\delta > 0$ sufficiently small such $U_{\delta,b}$ lies in $\mathcal{V}$ and such that the disks $U_{\delta,a}$ and $U_{\delta,b}$ do not intersect. We seek a 2×2 matrix valued function $P^{(b)}$ (depending on n, s and t) in the disk $U_{\delta,b}$ with the same jumps as S and with matches with $P^{(\infty)}$ on the boundary $\partial U_{\delta,b}$ of the disk. We thus seek a 2×2 matrix valued function that satisfies the following RH problem.

RH problem for $P^{(b)}$:

(a) $P^{(b)} : U_{\delta,b} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(b)}_+(z) = P^{(b)}_-(z) v_S(z)$ for $z \in U_{\delta,b} \cap \Sigma$, where v_S is given by (3.19).

(c) $P^{(b)}$ satisfies the matching condition

$$P^{(b)}(z)(P^{(\infty)})^{-1}(z) = I + \mathcal{O}(n^{-1/7}), \tag{3.37}$$

as $n \to \infty$ and $s, t \to 0$ such that (1.32) holds, uniformly for $z \in \partial U_{\delta,b} \setminus \Sigma$.

Due to the singular behavior of the equilibrium measure $d\nu_0(x)$ near b, see Assumptions 1.1 (ii), the Airy parametrix does not fit near b. Instead we use a different model RH problem associated with the P_I^2 equation (1.14).

3.6.1 Model RH problem for the P_I^2 equation

We construct $P^{(b)}$ by introducing the following model RH problem for the special solution y of the P_I^2 equation (1.14) as discussed in Section 1.3. This RH problem depends on two complex parameters s, t and has jumps on the oriented contour Γ as defined in Section 3.5, see Figure 2. We seek a 2×2 matrix valued function $\Psi(\zeta) = \Psi(\zeta; s, t)$ satisfying the following conditions.

RH problem for Ψ:

(a) Ψ is analytic for $\zeta \in \mathbb{C} \setminus \Gamma$.

(b) Ψ satisfies the following jump relations on Γ,

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3, \tag{3.38}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{3.39}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4. \tag{3.40}$$

(c) Ψ has the following behavior at infinity,

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N \left(I - h\sigma_3 \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-3/2}) \right) e^{-\theta(\zeta; s,t)\sigma_3}, \tag{3.41}$$

where $y = y(s, t)$ is the special solution of the P_I^2 equation (1.14) as discussed in Section 1.3, where $\frac{\partial h}{\partial s} = -y$, where N is given by (3.28), and where θ is given by (1.21).

Remark 3.5 Note that the only difference between the model RH problem for Airy functions and the one for P_{I}^2 lies in the asymptotic condition (c). In particular, in θ we have an extra factor $\zeta^{7/2}$.

If we fix $s_0, t_0 \in \mathbb{R}$, it was proven in [7, Lemma 2.3 and Proposition 2.5] that there exists a neighborhood $\mathcal{U}$ of s_0 and a neighborhood $\mathcal{W}$ of t_0 such that the RH problem for Ψ is (uniquely) solvable for all $(s,t) \in \mathcal{U} \times \mathcal{W}$. Furthermore, for $(s,t) \in \mathcal{U} \times \mathcal{W}$, Ψ is analytic both in s and t, and condition (c) holds uniformly for (s,t) in compact subsets of $\mathcal{U} \times \mathcal{W}$.

In [7, Section 2.3], the authors have shown that the solution Ψ of the RH problem for Ψ satisfies the Lax pair (1.15)–(1.18). From (1.20), (3.41), and (3.40) we then obtain

$$\begin{pmatrix} \Phi_1(\zeta;s,t) \\ \Phi_2(\zeta;s,t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \Psi_{11}(\zeta;s,t) \\ \Psi_{21}(\zeta;s,t) \end{pmatrix}, & \text{for } 0 < \operatorname{Arg}\zeta < 6\pi/7, \\ \begin{pmatrix} \Psi_{11}(\zeta;s,t) \\ \Psi_{21}(\zeta;s,t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } 6\pi/7 < \operatorname{Arg}\zeta < \pi. \end{cases} \tag{3.42}$$

3.6.2 Construction of $P^{(b)}$

We seek $P^{(b)}$ in the following form

$$P^{(b)}(z) = E^{(b)}(z)\Psi\left(n^{2/7}f_b(z); n^{6/7}sf_1(z), n^{4/7}tf_2(z)\right) e^{n\phi_{s,t}(z)\sigma_3}, \tag{3.43}$$

where $E^{(b)}$ is an invertible 2×2 matrix valued function analytic on $U_{\delta,b}$ and where f_b, f_1, and f_2 are (scalar) analytic functions on $U_{\delta,b}$ which are real on $(b-\delta, b+\delta)$. We take f_1 and f_2 to be such that $f_1(b) = c_1$ and $f_2(b) = c_2$ (where c_1 and c_2 are given by (1.30)). Then it is clear from (1.32) that for n sufficiently large and s and t sufficiently small,

$$n^{6/7}sf_1(z) \in \mathcal{U}, \qquad \text{and} \qquad n^{4/7}tf_2(z) \in \mathcal{W}, \qquad \text{for } z \in U_{\delta,b},$$

where $\mathcal{U}$ and $\mathcal{W}$ are the neighborhoods of s_0 and t_0 where Ψ exists. In addition we take f_b to be a conformal map from $U_{\delta,b}$ onto a convex neighborhood $f_b(U_{\delta,b})$ of 0 such that $f_b(b) = 0$ and $f_b'(b) > 0$.

If those conditions are all satisfied, and if we open the lens Σ (recall that the lens was not yet fully specified near b) such that

$$f_b(\Sigma \cap (U_{\delta,b} \cap \mathbb{C}_+)) = \Gamma_2 \cap f_b(U_{\delta,b}), \qquad \text{and} \qquad f_b(\Sigma \cap (U_{\delta,b} \cap \mathbb{C}_-)) = \Gamma_4 \cap f_b(U_{\delta,a}),$$

then it is straightforward to verify, using (3.19) and (3.38)–(3.40), that $P^{(b)}$ defined by (3.43) satisfies conditions (a) and (b) of the RH problem for $P^{(b)}$.

Let

$$f_b(z) = \left[105\left(-\pi i \int_z^b \psi_0(\xi)d\xi\right)(z-b)^{-7/2}\right]^{2/7}(z-b) = c(z-b) + \mathcal{O}(z-b)^2, \tag{3.44}$$

as $z \to 0$, where

$$c = \left(\frac{15}{2}h_0''(b)\sqrt{b-a}\right)^{2/7}.$$

To get the expansion of f_b near b we have used (2.5) and the facts that $h_0(b) = h_0'(b) = 0$ (see (2.7)). Further since $h_0''(b) > 0$ we have that $c > 0$. So, we have defined an analytic function

f_b with $f_b(b) = 0$ and $f_b'(b) = c > 0$, which is real on $(b-\delta, b+\delta)$, and which is a conformal mapping on $U_{\delta,b}$ provided $\delta > 0$ is sufficiently small.

Next, let f_1 and f_2 be defined by

$$f_1(z) = \left(-\pi i \int_z^b \psi_1(\xi) d\xi\right) f_b(z)^{-1/2}, \quad f_2(z) = -3\left(-\pi i \int_z^b \psi_2(\xi) d\xi\right) f_b(z)^{-3/2}. \tag{3.45}$$

Since f_b is a conformal mapping in $U_{\delta,b}$ it is clear from (2.14) and (2.6) that f_1 is analytic in $U_{\delta,b}$. To see that f_2 is analytic in $U_{\delta,b}$ as well, we also need to use the extra condition $h_2(b) = 0$ (see (2.12)). Further, f_1 and f_2 are real on $(b-\delta, b+\delta)$ and one can check that,

$$f_1(b) = \frac{h_1(b)}{c^{1/2}(b-a)^{1/2}} = c_1, \qquad f_2(b) = -\frac{h_2'(b)}{c^{3/2}(b-a)^{1/2}} = c_2. \tag{3.46}$$

Thus, f_b, f_1, and f_2 satisfy the above conditions, so that $P^{(b)}$ defined by (3.43), with $E^{(b)}$ any invertible analytic matrix valued function, satisfies conditions (a) and (b) of the RH problem for $P^{(b)}$.

Remark 3.6 As in Remark 3.3 we note that we could have also used different functions f_b, f_1, and f_2. However, we have to choose them so as to compensate for the factor $e^{n\phi_{s,t}\sigma_3}$ in (3.43). Using (1.21), (3.44), (3.45), (2.21), and (3.11) we have

$$\theta(n^{2/7} f_b(z); n^{6/7} s f_1(z), n^{4/7} t f_2(z)) = n\phi_{s,t}(z), \qquad \text{for } z \in U_{\delta,b} \setminus (b-\delta, b]. \tag{3.47}$$

From this and (3.41) it is clear that our choice of f_b, f_1, and f_2 will do the job.

It now remains to determine $E^{(b)}$ such that the matching condition (c) holds as well. In order to do this we make use of the following proposition (the analogon of Proposition 3.4).

Proposition 3.7 *Let $n \to \infty$ and $s, t \to 0$ such that* (1.32) *holds. Then,*

$$P^{(b)}(z) = E^{(b)}(z) \left(n^{2/7} f_b(z)\right)^{-\sigma_3/4} N \\ \times \left[I - h f_b(z)^{-1/2} \sigma_3 n^{-1/7} + \frac{1}{2}\begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} f_b(z)^{-1} n^{-2/7} + \mathcal{O}(n^{-3/7})\right]. \tag{3.48}$$

where we have used for brevity the notation

$$h = h(n^{6/7} s f_1(z), n^{4/7} t f_2(z)), \qquad \text{and} \qquad y = y(n^{6/7} s f_1(z), n^{4/7} t f_2(z)).$$

Proof. This follows easily from (3.43), (3.41), and (3.47) □

From (3.48) it is clear that (in order that the matching condition (c) is satisfied) we have to define $E^{(b)}$ by,

$$E^{(b)} = P^{(\infty)} N^{-1} \left(n^{2/7} f_b\right)^{\sigma_3/4}, \tag{3.49}$$

where N is given by (3.28) and where $P^{(\infty)}$ is the parametrix for the outside region, given by (3.20). Similarly as we have proven that $E^{(a)}$ is an invertible analytic matrix valued function in $U_{\delta,a}$, we can check that $E^{(b)}$ is invertible and analytic in $U_{\delta,b}$. This completes the construction of the parametrix near the singular endpoint.

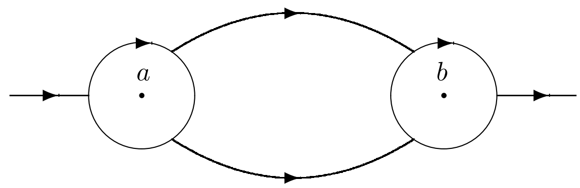

Figure 3: The contour Σ_R after the third and final transformation.

3.7 Final transformation: $S \mapsto R$

Having the parametrix $P^{(\infty)}$ for the outside region and the parametrices $P^{(a)}$ and $P^{(b)}$ near the endpoints a and b, we have all the ingredients to perform the final transformation of the RH problem. Define

$$R(z) = \begin{cases} S(z)\left(P^{(a)}\right)^{-1}(z), & \text{for } z \in U_{\delta,a} \setminus \Sigma, \\ S(z)\left(P^{(b)}\right)^{-1}(z), & \text{for } z \in U_{\delta,b} \setminus \Sigma, \\ S(z)\left(P^{(\infty)}\right)^{-1}(z), & \text{for } z \in \mathbb{C} \setminus (\Sigma \cup U_{\delta,a} \cup U_{\delta,b}). \end{cases} \tag{3.50}$$

Then, by construction of the parametrices, R has only jumps on the reduced system of contours Σ_R shown in Figure 3, and R satisfies the following RH problem. The circles around a and b are oriented clockwise.

RH problem for R:

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $R_+(z) = R_-(z) v_R(z)$ for $z \in \Sigma_R$, with

$$v_R = \begin{cases} P^{(a)}\left(P^{(\infty)}\right)^{-1}, & \text{on } \partial U_{\delta,a}, \\ P^{(b)}\left(P^{(\infty)}\right)^{-1}, & \text{on } \partial U_{\delta,b}, \\ P^{(\infty)} v_S \left(P^{(\infty)}\right)^{-1}, & \text{on the rest of } \Sigma_R. \end{cases} \tag{3.51}$$

(c) $R(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

(d) R remains bounded near the intersection points of Σ_R.

As $n \to \infty$ and $s, t \to 0$ such that (1.32) holds, we have by construction of the parametrices that the jump matrix for R is close to the identity matrix, both in L^2 and L^∞- sense on Σ_R,

$$v_R(z) = \begin{cases} I + \mathcal{O}(n^{-1/7}), & \text{on } \partial U_{\delta,a} \cup \partial U_{\delta,b}, \\ I + \mathcal{O}(e^{-\gamma n}), & \text{on the rest of } \Sigma_R, \end{cases} \tag{3.52}$$

with $\gamma > 0$ some fixed constant. Then, arguments as in [10, 11] guarantee that R itself is close to the identity matrix,

$$R(z) = I + \mathcal{O}(n^{-1/7}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R, \tag{3.53}$$

as $n \to \infty$ and $s, t \to 0$ such that (1.32) holds. This completes the Deift/Zhou steepest descent analysis.

Remark 3.8 The Deift/Zhou steepest descent method can be generalized to the case where the support of ν_0 consists of more than one interval. However, there are two (technical) differences. First, in the multi-interval case, the equilibrium measures ν_1 and ν_2 have densities which are more complicated than in the one-interval case, but it remains possible to give explicit formulae. Consequently, condition (1.27), which expresses the requirement that the density of ν_2 vanishes at the singular endpoint, has to be modified. Further, the construction of the outside parametrix $P^{(\infty)}$ is more complicated, since it uses Θ-functions as in [10, Lemma 4.3]. With these modifications the asymptotic analysis can be carried through in the multi-interval case.

3.8 Asymptotics of R

For the purpose of proving the universality result for the kernel $K_n^{(s,t)}$ (Theorem 1.7) it is enough to unfold the series of transformations $Y \mapsto T \mapsto S \mapsto R$ and to use (3.53). This will be done in the next section. However, in order to determine the asymptotics of the recurrence coefficients $a_n^{(n,s,t)}$ and $b_n^{(n,s,t)}$ (Theorem 1.11) we need to expand the $\mathcal{O}(n^{-1/7})$ term in (3.53).

We show that the jump matrix v_R for R has an expansion of the form,

$$v_R(z) = I + \frac{\Delta_1(z)}{n^{1/7}} + \frac{\Delta_2(z)}{n^{2/7}} + \mathcal{O}(n^{-3/7}), \tag{3.54}$$

as $n \to \infty$ and $s,t \to 0$ such that (1.32) holds, uniformly for $z \in \Sigma_R$, and we will explicitly determine Δ_1 and Δ_2. On $\Sigma_R \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b})$, the jump matrix is the identity matrix plus an exponentially small term, so that

$$\Delta_1(z) = 0, \qquad \Delta_2(z) = 0, \qquad \text{for } z \in \Sigma_R \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b}). \tag{3.55}$$

Now, from (3.51), (3.33), and (3.48) we obtain (3.54) with,

$$\Delta_1(z) = -\frac{1}{4}\left(n^{4/7} r_{s,t}(z)\right)^2 f_a(z)^{-1/2} P^{(\infty)}(z)\sigma_3 P^{(\infty)}(z)^{-1}, \qquad \text{for } z \in \partial U_{\delta,a}, \tag{3.56}$$

$$\Delta_1(z) = -h f_0(z)^{-1/2} P^{(\infty)}(z)\sigma_3 P^{(\infty)}(z)^{-1}, \qquad \text{for } z \in \partial U_{\delta,b}, \tag{3.57}$$

and

$$\Delta_2(z) = \frac{1}{32}\left(n^{4/7} r_{s,t}(z)\right)^4 f_a(z)^{-1} I, \qquad \text{for } z \in \partial U_{\delta,a}, \tag{3.58}$$

$$\Delta_2(z) = \frac{1}{2} f_0(z)^{-1} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix}, \qquad \text{for } z \in \partial U_{\delta,b}, \tag{3.59}$$

where we have used for brevity

$$h = h(n^{6/7} s f_1(z), n^{4/7} t f_2(z)), \qquad y = y(n^{6/7} s f_1(z), n^{4/7} t f_2(z)).$$

Observe that Δ_1 and Δ_2 have an extension to an analytic function in a punctured neighborhood of a and a punctured neighborhood of b with simple poles at a and b.

As in [11, Theorem 7.10] we obtain from (3.54) that R satisfies,

$$R(z) = I + \frac{R^{(1)}(z)}{n^{1/7}} + \frac{R^{(2)}(z)}{n^{2/7}} + \mathcal{O}(n^{-3/7}), \tag{3.60}$$

as $n \to \infty$ and $s,t \to 0$ such that (1.32) holds, which is valid uniformly for $z \in \mathbb{C} \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b})$. We have that

$$R^{(1)} \text{ and } R^{(2)} \text{ are analytic on } \mathbb{C} \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b}), \tag{3.61}$$

$$R^{(1)}(z) = \mathcal{O}(1/z), \qquad R^{(2)}(z) = \mathcal{O}(1/z), \qquad \text{as } z \to \infty. \tag{3.62}$$

We will now compute the functions $R^{(1)}$ and $R^{(2)}$ explicitly.

Determination of $R^{(1)}$:

Expanding the jump relation $R_+ = R_- v_R$ using (3.54) and (3.60), and collecting the terms with $n^{-1/7}$ we find

$$R_+^{(1)}(z) = R_-^{(1)}(z) + \Delta_1(z), \qquad \text{for } z \in \partial U_{\delta,a} \cup \partial U_{\delta,b}.$$

This together with (3.61) and (3.62) gives an additive RH problem for $R^{(1)}$. Recall that Δ_1 is analytic in a neighborhood of $z=a$ and $z=b$ except for simple poles at a and b. So,

$$\Delta_1(z) = \frac{A^{(1)}}{z-a} + \mathcal{O}(1), \qquad \text{as } z \to a, \qquad \Delta_1(z) = \frac{B^{(1)}}{z-b} + \mathcal{O}(1), \qquad \text{as } z \to b,$$

for certain matrices $A^{(1)}$ and $B^{(1)}$. We then see by inspection that

$$R^{(1)}(z) = \begin{cases} \dfrac{A^{(1)}}{z-a} + \dfrac{B^{(1)}}{z-b}, & \text{for } z \in \mathbb{C} \setminus (\overline{U}_{\delta,a} \cup \overline{U}_{\delta,b}), \\[2ex] \dfrac{A^{(1)}}{z-a} + \dfrac{B^{(1)}}{z-b} - \Delta_1(z), & \text{for } z \in U_{\delta,a} \cup U_{\delta,b}, \end{cases} \tag{3.63}$$

solves the additive RH problem for $R^{(1)}$. In now remains to determine $A^{(1)}$ and $B^{(1)}$. This can be done by expanding the formulas (3.56) and (3.57) near $z=a$ and $z=b$, respectively. We then find after a straightforward calculation (using also the fact that $f_1(b) = c_1$ and $f_2(b) = c_2$, see (3.46),

$$A^{(1)} = \frac{1}{8}\sqrt{b-a}\,(n^{4/7} r_{s,t}(a))^2 (-f_a'(a))^{-1/2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \tag{3.64}$$

$$B^{(1)} = \frac{1}{2} h \sqrt{b-a}\, f_b'(b)^{-1/2} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \tag{3.65}$$

where we used h to denote $h(c_1 n^{6/7} s, c_2 n^{4/7} t)$ for brevity.

Determination of $R^{(2)}$:

Next, expanding the jump relation $R_+ = R_- v_R$ using (3.54) and (3.60), and collecting the terms with $n^{-2/7}$ we find

$$R_+^{(2)}(z) = R_-^{(2)}(z) + R_-^{(1)}(z)\Delta_1(z) + \Delta_2(z), \qquad \text{for } z \in \partial U_{\delta,a} \cup \partial U_{\delta,b}.$$

This together with (3.61) and (3.62) gives an additive RH problem for $R^{(2)}$. Since $R_-^{(1)}$ is the boundary value of the restriction of $R^{(1)}$ to the disks $U_{\delta,a}$ and $U_{\delta,b}$ and since Δ_1 and Δ_2 are analytic in a neighborhood of a and b, except for simple poles at a and b, we have

$$R^{(1)}(z)\Delta_1(z) + \Delta_2(z) = \frac{A^{(2)}}{z-a} + \mathcal{O}(1), \qquad \text{as } z \to a,$$

$$R^{(1)}(z)\Delta_1(z) + \Delta_2(z) = \frac{B^{(2)}}{z-b} + \mathcal{O}(1), \qquad \text{as } z \to b,$$

for certain matrices $A^{(2)}$ and $B^{(2)}$. As in the determination of $R^{(1)}$ we then see by inspection that

$$R^{(2)}(z) = \begin{cases} \dfrac{A^{(2)}}{z-a} + \dfrac{B^{(2)}}{z-b}, & \text{for } z \in \mathbb{C} \setminus (\overline{U}_{\delta,a} \cup \overline{U}_{\delta,b}), \\[2ex] \dfrac{A^{(2)}}{z-a} + \dfrac{B^{(2)}}{z-b} - R^{(1)}(z)\Delta_1(z) - \Delta_2(z), & \text{for } z \in U_{\delta,a} \cup U_{\delta,b}, \end{cases} \tag{3.66}$$

solves the additive RH problem for $R^{(2)}$. The determination of $A^{(2)}$ and $B^{(2)}$ is more complicated than the determination of $A^{(1)}$ and $B^{(1)}$. It involves $R^{(1)}(a)$ and $R^{(1)}(b)$ for which we need to determine also the constant terms in the expansions of Δ_1 near $z = a$ and $z = b$. After a straightforward (but rather long calculation) we find,

$$A^{(2)} = \frac{(n^{4/7} r_{s,t}(a))^4}{32(-f_a'(a))} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \frac{(n^{4/7} r_{s,t}(a))^2 h}{8(-f_a'(a))^{1/2} f_b'(b)^{1/2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \tag{3.67}$$

$$B^{(2)} = \frac{y + h^2}{2 f_b'(b)} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \frac{(n^{4/7} r_{s,t}(a))^2 h}{8(-f_a'(a))^{1/2} f_b'(b)^{1/2}} \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}, \tag{3.68}$$

where we used h and y to denote $h(c_1 n^{6/7} s, c_2 n^{4/7} t)$ and $y(c_1 n^{6/7} s, c_2 n^{4/7} t)$ for brevity.

4 Universality of the double scaling limit

Here, we will prove the univarsality result for the 2-point correlation kernel $K_n^{(s,t)}$. We do this by using the expression (3.4) for $K_n^{(s,t)}$ in terms of Y and by unfolding the series of transformations $Y \mapsto T \mapsto S \mapsto R$.

Proof of Theorem 1.7. From equations (3.4), (3.9), and (3.13), the reader can verify that the 2-point kernel $K_n^{(s,t)}$ can be written as, cf. [5, 6],

$$K_n^{(s,t)}(x,y) = e^{-n\phi_{s,t,+}(x)} e^{-n\phi_{s,t,+}(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } x, y \in \mathbb{R}.$$

From (3.18) and the fact that $S_+ = RP_+^{(b)}$ on $(b-\delta, b+\delta)$, see (3.50), we have

$$T_+ = \begin{cases} RP_+^{(b)}, & \text{on } (b, b+\delta), \\ RP_+^{(b)} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,+}} & 1 \end{pmatrix}, & \text{on } (b-\delta, b). \end{cases}$$

Inserting this in the previous equation for $K_n^{(s,t)}$ we arrive at,

$$K_n^{(s,t)}(x,y) = e^{-n\phi_{s,t,+}(x)} e^{-n\phi_{s,t,+}(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \widehat{P}^{-1}(y) R^{-1}(y) R(x) \widehat{P}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.1}$$

for $x \in (b-\delta, b+\delta)$, where

$$\widehat{P} = \begin{cases} P_+^{(b)}, & \text{on } (b, b+\delta), \\ P_+^{(b)} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,+}} & 1 \end{pmatrix}, & \text{on } (b-\delta, b). \end{cases} \tag{4.2}$$

Further, we define

$$\widehat{\Psi} = \begin{cases} \Psi_+ & \text{on } \mathbb{R}_+, \\ \Psi_+ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on } \mathbb{R}_-, \end{cases} \tag{4.3}$$

where Ψ is the solution of the RH problem for Ψ, see Section 3.6. By (3.42), we have that $\widehat{\Psi}_{11} = \Phi_1$ and $\widehat{\Psi}_{21} = \Phi_2$. Using (3.43), (4.2), and (4.3) a straightforward calculation yields,

$$\widehat{P}(x) = E^{(b)}(x) \widehat{\Psi}\left(n^{2/7} f_b(x); n^{6/7} s f_1(x), n^{4/7} t f_2(x)\right) e^{n\phi_{s,t,+}(x)\sigma_3}, \qquad \text{for } x \in (b-\delta, b+\delta).$$

Inserting this into (4.1) we then obtain,

$$K_n^{(s,t)}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \widehat{\Psi}^{-1}\left(n^{2/7} f_b(y); n^{6/7} s f_1(y), n^{4/7} t f_2(y)\right)$$
$$\times (E^{(b)})^{-1}(y) R^{-1}(y) R(x) E^{(b)}(x) \widehat{\Psi}\left(n^{2/7} f_b(x); n^{6/7} s f_1(x), n^{4/7} t f_2(y)\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (4.4)$$

for $x \in (b-\delta, b+\delta)$.

Now, we introduce for the sake of brevity some notation. Let

$$u_n = b + \frac{u}{cn^{2/7}}, \quad \text{and} \quad v_n = b + \frac{v}{cn^{2/7}}, \quad \text{with } c = f_b'(b) = \left(\tfrac{15}{2} h_0''(b)\sqrt{b-a}\right)^{2/7}. \quad (4.5)$$

We then have,

$$\lim_{n\to\infty} n^{2/7} f_b(u_n) = u, \qquad \text{and} \qquad \lim_{n\to\infty} n^{2/7} f_b(v_n) = v. \quad (4.6)$$

Furthermore, since $f_1(b) = c_1$ and $f_2(b) = c_2$ (see (3.46)) we have in the limit as $n \to \infty$ and $s, t \to 0$ such that (1.32) holds,

$$\lim n^{6/7} s f_1(u_n) = s_0, \qquad \lim n^{6/7} s f_1(v_n) = s_0, \quad (4.7)$$
$$\lim n^{4/7} t f_2(u_n) = t_0, \qquad \lim n^{4/7} t f_2(v_n) = t_0. \quad (4.8)$$

Now, a similar argument as in [24] shows that

$$\lim E_b^{-1}(v_n) R(v_n)^{-1} R(u_n) E_b(u_n) = I. \quad (4.9)$$

Inserting (4.6)–(4.9) into (4.4) and using the fact that $\widehat{\Psi}_{11} = \Phi_1$ and $\widehat{\Psi}_{21} = \Phi_2$ it is then straightforward to obtain

$$\lim \frac{1}{cn^{2/7}} K_n^{(s,t)}(u_n, v_n)$$
$$= \frac{1}{2\pi i(u-v)} \begin{pmatrix} 0 & 1 \end{pmatrix} \widehat{\Psi}^{-1}(v; s_0, t_0) \widehat{\Psi}(u; s_0, t_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{1}{-2\pi i(u-v)} \left(\Phi_1(u; s_0, t_0)\Phi_2(v; s_0, t_0) - \Phi_1(v; s_0, t_0)\Phi_2(u; s_0, t_0)\right), \quad (4.10)$$

where we take the limit $n \to \infty$ and $s, t \to 0$ such that (1.32) holds. This completes the proof of Theorem 1.7. □

5 Asymptotics of the recurrence coefficients

We will now determine the asymptotics of $a_n^{(n,s,t)}$ and $b_n^{(n,s,t)}$ as $n \to \infty$ and $s, t \to 0$ such that (1.32) holds. In order to do this, we make use of the following result, see e.g. [8, 11]. Let Y be the unique solution of the RH problem for Y. There exist 2×2 constant (independent of z but depending on n, s and t) matrices Y_1 and Y_2 such that

$$Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \quad (5.1)$$

and

$$a_n^{(n,s,t)} = \sqrt{(Y_1)_{12}(Y_1)_{21}}, \qquad b_n^{(n,s,t)} = (Y_1)_{11} + \frac{(Y_2)_{12}}{(Y_1)_{12}}. \quad (5.2)$$

We need to determine the constant matrices Y_1 and Y_2. For large $|z|$ it follows from (3.9), (3.18), and (3.50), that

$$Y(z) = e^{\frac{1}{2}n\ell_{s,t}\sigma_3} R(z) P^{(\infty)}(z) e^{ng_{s,t}(z)\sigma_3} e^{-\frac{1}{2}n\ell_{s,t}\sigma_3}. \tag{5.3}$$

So, in order to determine Y_1 and Y_2 we need the asymptotic behavior of $P^{(\infty)}(z)$, $e^{ng_{s,t}(z)\sigma_3}$, and $R(z)$ as $z \to \infty$.

Asymptotic behavior of $P^{(\infty)}(z)$ as $z \to \infty$:

Expanding the factor $((z-b)/(z-a))^{\sigma_3/4}$ in (3.20) at $z = \infty$ it is clear that,

$$P^{(\infty)}(z) = I + \frac{P_1^{(\infty)}}{z} + \frac{P_2^{(\infty)}}{z} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{5.4}$$

with

$$P_1^{(\infty)} = \frac{i}{4}(b-a)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P_2^{(\infty)} = \frac{i}{8}(b^2-a^2)\begin{pmatrix} * & 1 \\ -1 & * \end{pmatrix}. \tag{5.5}$$

Asymptotic behavior of $e^{ng_{s,t}(z)\sigma_3}$ as $z \to \infty$:

By (3.5) we have

$$e^{ng_{s,t}(z)\sigma_3}\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{G_1}{z} + \frac{G_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{5.6}$$

with

$$G_1 = -n\int_a^b u d\nu_{s,t}(u)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. \tag{5.7}$$

Asymptotic behavior of $R(z)$ as $z \to \infty$:

As in [11] the matrix valued function R has the following asymptotic behavior at infinity,

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty. \tag{5.8}$$

The compatibility with (3.60), (3.63), and (3.66) yields that

$$R_1 = \left(A^{(1)} + B^{(1)}\right)n^{-1/7} + \left(A^{(2)} + B^{(2)}\right)n^{-2/7} + \mathcal{O}(n^{-3/7}), \tag{5.9}$$

$$R_2 = \left(aA^{(1)} + bB^{(1)}\right)n^{-1/7} + \left(aA^{(2)} + bB^{(2)}\right)n^{-2/7} + \mathcal{O}(n^{-3/7}), \tag{5.10}$$

as $n \to \infty$ and $s,t \to 0$ such that (1.32) holds. Here, $A^{(1)}$, $B^{(1)}$, $A^{(2)}$, and $B^{(2)}$ are given by (3.64), (3.65), (3.67), and (3.68), respectively.

Now, we are ready to determine the asymptotics of the recurrence coefficients.

Proof of Theorem 1.11. Note that by (5.3), (5.4), (5.6) and (5.8),

$$Y_1 = e^{\frac{1}{2}n\ell_{s,t}\sigma_3}\left[P_1^{(\infty)} + G_1 + R_1\right]e^{-\frac{1}{2}n\ell_{s,t}\sigma_3} \tag{5.11}$$

and

$$Y_2 = e^{\frac{1}{2}n\ell_{s,t}\sigma_3}\left[P_2^{(\infty)} + G_2 + R_2 + R_1P_1^{(\infty)} + \left(P_1^{(\infty)} + R_1\right)G_1\right]e^{-\frac{1}{2}n\ell_{s,t}\sigma_3} \tag{5.12}$$

We start with the recurrence coefficient $a_n^{(n,s,t)}$. Inserting (5.11) into (5.2), and using the facts that $(P_1^{(\infty)})_{12} = -(P_1^{(\infty)})_{21} = i(b-a)/4$ (by (5.5)) and $(G_1)_{12} = (G_1)_{21} = 0$ (by (5.7)), we obtain

$$a_n^{(n,s,t)} = \left[\left(\frac{b-a}{4}\right)^2 + i\frac{b-a}{4}\left((R_1)_{21} - (R_1)_{12}\right) + (R_1)_{12}(R_1)_{21}\right]^{1/2}. \tag{5.13}$$

Now, from the formula (5.9) for R_1 and the formulas (3.64), (3.65), (3.67), and (3.68) for $A^{(1)}, B^{(1)}, A^{(2)}$, and $B^{(2)}$, we have

$$(R_1)_{21} - (R_1)_{12} = -i\left[\frac{y}{f_b'(b)} + \left(\frac{(n^{4/7}r_{s,t}(a))^2}{4(-f_a'(a))^{1/2}} + \frac{h}{f_b'(b)}\right)^2\right] n^{-2/7} + \mathcal{O}(n^{-3/7}),$$

and

$$(R_1)_{12}(R_1)_{21} = -\frac{b-a}{4}\left(\frac{(n^{4/7}r_{s,t}(a))^2}{4(-f_a'(a))^{1/2}} + \frac{h}{f_b'(b)}\right)^2 n^{-2/7} + \mathcal{O}(n^{-3/7}).$$

Note that we have used y to denote $y(c_1 n^{6/7}s, c_2 n^{4/7}t)$ for brevity. Inserting the latter two equations into (5.13) and using the fact that $f_b'(b) = c$ (by (3.44)) we then obtain (1.39).

We will now consider the recurrence coefficient $b_n^{(n,s,t)}$. Inserting (5.11) and (5.12) into (5.2), and using the facts that $(P_1^{(\infty)})_{11} = (P_1^{(\infty)})_{22} = 0$, $(G_1)_{12} = (G_2)_{12} = 0$, $(G_1)_{11} + (G_1)_{22} = 0$, and $R_1 = \mathcal{O}(n^{-1/7})$, we obtain

$$\begin{aligned} b_n^{(n,s,t)} &= (R_1)_{11} + \frac{(P_2^{(\infty)})_{12} + (R_1)_{11}(P_1^{(\infty)})_{12} + (R_2)_{12}}{(P_1^{(\infty)})_{12} + (R_1)_{12}} \\ &= (R_1)_{11} + \left[\frac{(P_2^{(\infty)})_{12}}{(P_1^{(\infty)})_{12}} + (R_1)_{11} + \frac{(R_2)_{12}}{(P_1^{(\infty)})_{12}}\right] \\ &\qquad \times \left[1 - \frac{(R_1)_{12}}{(P_1^{(\infty)})_{12}} + \left(\frac{(R_1)_{12}}{(P_1^{(\infty)})_{12}}\right)^2 + \mathcal{O}(n^{-3/7})\right]. \end{aligned}$$

Since $(P_1^{(\infty)})_{12} = i(b-a)/4$, $(P_2^{(\infty)})_{12} = i(b^2-a^2)/8$, $R_1 = \mathcal{O}(n^{-1/7})$, and $R_2 = \mathcal{O}(n^{-1/7})$ we then obtain after a straightforward calculation and combining terms,

$$\begin{aligned} b_n^{(n,s,t)} = \frac{b+a}{2} &+ \left(2(R_1)_{11} + 2i\frac{b+a}{b-a}(R_1)_{12} - \frac{4i}{b-a}(R_2)_{12}\right)\left(1 + \frac{4i}{b-a}(R_1)_{12}\right) \\ &- \frac{4i}{b-a}(R_1)_{11}(R_1)_{12} + \mathcal{O}(n^{-3/7}). \end{aligned} \tag{5.14}$$

Now, from (5.9), (5.10), (3.64), (3.65), (3.67), and (3.68) we have

$$\begin{aligned} 2(R_1)_{11} + 2i\frac{b+a}{b-a}(R_1)_{12} - \frac{4i}{b-a}(R_2)_{12} &= 2i\left[A_{12}^{(2)} - B_{12}^{(2)}\right] n^{-2/7} + \mathcal{O}(n^{-3/7}) \\ &= \left(\frac{y}{f_b'(b)} + \frac{h^2}{f_b'(b)} - \frac{(n^{4/7}r_{s,t}(a))^4}{16(-f_a'(a))}\right) n^{-2/7} + \mathcal{O}(n^{-3/7}), \end{aligned}$$

and

$$(R_1)_{11}(R_1)_{12} = -i\left[\left(A_{12}^{(1)}\right)^2 - \left(B_{12}^{(1)}\right)^2\right] n^{-2/7} + \mathcal{O}(n^{-3/7})$$

$$= -i\frac{b-a}{4}\left(\frac{h^2}{f_b'(b)} - \frac{(n^{4/7}r_{s,t}(a))^4}{16(-f_a'(a))}\right) n^{-2/7} + \mathcal{O}(n^{-3/7}).$$

Inserting the latter two equations into (5.14) and using the facts that $(R_1)_{12} = \mathcal{O}(n^{-1/7})$ and $f_b'(b) = c$ we obtain (1.40). So, the theorem is proven. □

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