ON NONLINEAR STEEPEST DESCENT

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ABSTRACT: We review some landmarks in the history of the nonlinear steepest descent method for the asymptotic evaluation of the solutions of Riemann-Hilbert factorization problems. We stress some recent results on the "non-self-adjoint" extension of the theory. In particular we consider the case of the semiclassical focusing NLS problem. We explain how the nonlinear steepest descent method gives rise to a maximin variational problem for Green potentials with external field in two dimensions and we announce results on existence and regularity of solutions to this variational problem.

0. INTRODUCTION

We are interested here in the asymptotic theory of Riemann-Hilbert problem factorizations associated to completeley integrable systems. This asymptotic method has been made rigorous and systematic in [DZ] where in fact the term "nonlinear steepest descent method" was first employed to stress the relation with the classical "steepest descent method" initiated by Riemann in the study of exponential integrals with a large phase parameter. Such exponential integrals appear in the solution of Cauchy problems for linear evolution equations, when one employs the method of Fourier transforms. In the case of nonlinear integrable equations, on the other hand, the nonlinear analog of the Fourier transform is the scattering transform and the inverse problem is now a Riemann-Hilbert factorization problem (as first noted in [S]). While in the "linear steepest descent method" the contour of integration has to be deformed to a union of contours of "steepest descent" which will make the explicit integration of the integral possible, in the case of the "nonlinear steepest descent method" one deforms the original Riemann-Hilbert factorization contour to appropriate steepest descent contours where the resulting Riemann-Hilbert problems are explicitly solvable.

In the linear case, if the phase and the critical points of the phase are real it may not be necessary to deform the integration contour. One has rather a Laplace integral problem on the contour given. When studying Riemann-Hilbert problems, the situation is more complicated and the analog of a real phase is here the self-adjointness of the underlying Lax operator. Then the spectrum is real and the Riemann-Hilbert contour is real. The needed "deformation contour" must stay near the real line and in fact there is a great degree of freedom of choice for the small pieces off the real line. On the other hand, things are different in the non-self-adjoint case. One novelty of the semiclassical problem for

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\partial_x^2\psi + |\psi|^2\psi = 0,$$
$$under\ \psi(x,0) = \psi_0(x),$$

studied in [KMM] is that, due to the non-self-adjointness of the underlying Lax operator, the "target contour" is very specific (if not unique) and by no means obvious. It is best characterized via the solution of a maximin energy problem, in fact it is an S-curve. The term "nonlinear steepest descent method" thus acquires full meaning.

1. LINEAR STATIONARY PHASE METHOD

Consider the Cauchy problem for the Airy quation

$$u_t + u_{xxx} = 0, u(x,0) = u_0(x). \tag{1}$$

It can be solved via Fourier transforms. Let

$$\hat{u}(\xi,t) = \int e^{-ix\xi} u(x,t)dx.$$

Then

$$\hat{u}_t(\xi,t) = -i\xi^3 \hat{u}(\xi,t),$$

so

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-i\xi^3 t}$$

and

$$u(x,t) = \int \hat{u}(\xi,0)e^{ix\xi - i\xi^3 t}d\xi. \tag{2}$$

To understand the long time asymptotic behavior of the formula (2) one needs to apply the stationary-phase/steepest-descent method. The principle is that the dominating contribution comes from the vicinity of the two stationary phase points $\xi_{1,2} = \pm(\frac{x}{3t})^{1/2}$. Through a local change of variables at each stationary phase point $\tau(\xi)$ such that $\tau(\xi_j) = 0$ we can calculate each contributing integral asymptotically to all orders with exponential error.

2. LINEAR STEEPEST DESCENT METHOD

We will consider a different example of an exponential integral; its domain must be deformed to unions of arcs of steepest descent (taken from [E]).

$$Ai(z) = \frac{1}{\pi}\int_0^\infty cos(\frac{s^3}{3} + zs)ds, \tag{3}$$

as $z \to \infty$. Set $s = z^{1/2}t$ and $x = z^{3/2}$.

$$Ai(x^{2/3}) = \frac{x^{1/3}}{2\pi}\int_{-\infty}^\infty exp(ix(\frac{t^3}{3} + t))dt. \tag{4}$$

The phase is $h(t) = i(\frac{t^3}{3} + t)$ and the zeros of $h(t) = i(t^2 + 1)$ are $\pm i$. To apply the stationary-phase/steepest-descent method, we have to deform the integral $off\ the\ real\ line$. The steepest descent paths are

$$Imh(t) = constant. \tag{5}$$

Set $\tau = \xi + i\eta$ to obtain

$$\xi(\xi^2 - 3\eta^2 + 3) = 0.$$

The curves of steepest descent are the imaginary axis and the two branches of a hyperbola. Let's denote by: C_1 the oriented branch from point i to $exp(i\pi/6)\cdot\infty$,

C_2 the oriented branch from point i to $exp(5i\pi/6)\cdot\infty$,

C_3 the oriented branch from point $-i$ to $exp(11i\pi/6)\cdot\infty$,

C_4 the oriented branch from point i to $exp(7i\pi/6)\cdot\infty$.

Clearly we can deform the integral in (4) to an integral supported in $C_1 - C_2$. So,

$$Ai(x^{2/3}) = \frac{x^{1/3}}{2\pi}(\int_{C_1} - \int_{C_2})exp(ix(\frac{t^3}{3} + t))dt = I_1 - I_2. \tag{6}$$

On each of these two integrals the phase is real. We have now two Laplace type integrals. Make the local change of variables

$$u = h(i) - h(t) = (t-i)^2 - \frac{1}{3}i(t-i)^3, \tag{7}$$

so that

$$u^{1/2} = (t-i)(1-\frac{1}{3}i(t-i))^{1/2}.$$

Invert

$$t - i = \Sigma_1^\infty \frac{i^{n-1}\Gamma(3n/2-1)}{n!\Gamma(n/2-1)3^{n-1}} u^{n/2}. \tag{8}$$

Now the integrals can be evaluated asymptotically to all orders:

$$I_j = exp(xh(i)) \int_0^\infty exp(-xu)\frac{dt}{du}du =$$
$$exp(2x/3)(-1)^j \int_0^\infty exp(-xu)\Sigma_1^\infty \frac{(-1)^n i^{n-1}\Gamma(3n/2-1)}{2(n-1)!\Gamma(n/2-1)3^{n-1}} u^{n/2-1}du =$$
$$\Sigma_1^\infty \frac{(-1)^n i^{n-1}\Gamma(3n/2-1)}{2(n-1)!\Gamma(n/2-1)3^{n-1}x^{n/2}}.$$

It follows from (6) that

$$Ai(z) = \frac{1}{2\pi z^{1/4}} exp(-\frac{2}{3}z^{3/2})\Sigma_0^\infty \frac{\Gamma(3m/2+1)}{(2m)!}(-9z^{3/2})^m. \tag{9}$$

These are uniform asymptotics as $z \to \infty, |argz| < \pi/3 - \epsilon$.

3. NONLINEAR STATIONARY PHASE METHOD [I,DZ]

Consider the nonlinear KdV equation

$$\begin{gathered} u_t - uu_x + u_{xxx} = 0, \\ u(x,0) = u_0(x). \end{gathered} \tag{10}$$

The analog of the Fourier transform is the scattering transform. Assume that $\int(1+x^2)|u_0(x)|dx < \infty$. The scattering data for the function $u_0(x)$ are the scattering coefficient $r(\xi)$ and the eigenvalues of the linear operator $L_0 = \frac{-d^2}{dx^2} + u_0(x)$. We think of ξ as a spectral variable. If u solves (10), then the eigenvalues of the linear operator $L = \frac{-d^2}{dx^2} + u(x)$ are constant with time, while the evolution of the scattering coefficient of u is given by

$$r_u(\xi, t) = r(\xi)exp(-i\xi^3 t). \tag{11}$$

One recovers the solution of (10) by solving the inverse scattering problem for $r_u(\xi,t)$ and the eigenvalues of L_0.

Let us for the moment concentrate on the analogous (but simpler in a sense) defocusing nonlinear Schrödinger equation

$$i\partial_t \psi + \frac{1}{2}\partial_x^2 \psi - |\psi|^2 \psi = 0, \qquad (12)$$
$$under \ \psi(x,0) = \psi_0(x),$$

where the initial data function is, say, Schwartz. This is a simpler situation because the corresponding linear operator has no eigenvalues. So let $r(\xi)$ be the scattering coefficient for the Dirac operator

$$L = \begin{pmatrix} i\partial_x & i\psi_0(x) \\ -i\psi_0^*(x) & -i\partial_x \end{pmatrix}.$$

It is a fact of direct scattering theory that r is also Schwartz, and that

$|r(\xi)| < 1, \ for \ all \ \xi \in \mathbb{R}$. The evolution of the scattering coefficient is given by

$$r_u(\xi, t) = r(\xi) exp(4i\xi^2 t).$$

The inverse scattering problem can be posed in terms of a Riemann-Hilbert factorization problem.

THEOREM. There exists a 2x2 matrix Q with analytic entries in the upper and lower open half-planes, such that the normal limits Q_+, Q_-, as ξ approaches the real line from above or below respectively, exist and satisfy

$$Q_+(\xi) = Q_-(\xi) \begin{pmatrix} 1 - |r(\xi)|^2 & -r^*(\xi)e^{-2i\xi x - 4i\xi^2 t} \\ r(\xi)e^{2i\xi x + 4i\xi^2 t} & 1 \end{pmatrix}, \ Im\xi = 0, \qquad (13)$$
$$and \ lim_{\xi \to \infty} Q(\xi) = I.$$

Furthermore the solution $\psi(x,t)$ of (12) is given by

$$\psi(x,t) = -2 lim_{\xi \to \infty} \xi Q_{12}. \qquad (14)$$

Thus, the initial value problem for (12) is reduced to a RH problem.

It was first realized by Its [I, IN], motivated by the study of the work of Jimbo, Miwa, Ueno, that the long time asymptotics for the solution of (10) can be extracted by reducing the problem (13) to a "local" RH problem located in a small neighborhood of the stationary phase point ξ_0 such that $\Theta'(\xi_0) = 0$ where $\Theta = \xi x + 2\xi^2 t$.

The deformation method has been made rigorous and systematic in [DZ]. Here are the basic ideas.

Suppose $\xi_0 = -\frac{x}{4t}$. Consider the region $\xi_0 < M$, some positive constant. Note the following factorizations.

$$\begin{pmatrix} 1-|r(\xi)|^2 & -r^*(\xi)e^{-2i\Theta} \\ r(\xi)e^{2i\Theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -r^*e^{-2i\Theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ re^{2i\Theta} & 1 \end{pmatrix} \; for \; \xi > \xi_0,$$

$$\begin{pmatrix} 1-|r(\xi)|^2 & -r^*(\xi)e^{-2i\Theta} \\ r(\xi)e^{2i\Theta} & 1 \end{pmatrix} = \begin{pmatrix} d_-^{-1} & 0 \\ \frac{rd_-^{-1}e^{-2i\Theta}}{1-|r|^2} & d_- \end{pmatrix} \begin{pmatrix} d_+ & \frac{-r^*d_+e^{2i\Theta}}{1-|r|^2} \\ 0 & d_+^{-1} \end{pmatrix} \; for \; \xi < \xi_0.$$

where d is a function analytic in $\mathbb{C} \setminus (-\infty, \xi_0]$ such that

$$d_+(\xi) = d_-(\xi)(1-|r(\xi)|^2) \qquad \text{for } -\infty < \xi \le \xi_0,$$

$$d_+(\xi) = d_-(\xi) \qquad \text{for } \xi > \xi_0 \tag{15}$$

$$d \to 1 \quad \text{as } \xi \to \infty.$$

The above factorizations suggest the following transformation. Consider an infinite cross centered at ξ_0 (see Figure 1 below). The actual angles between the four half-lines are not important, as long as they lie in the appropriate quadrants.

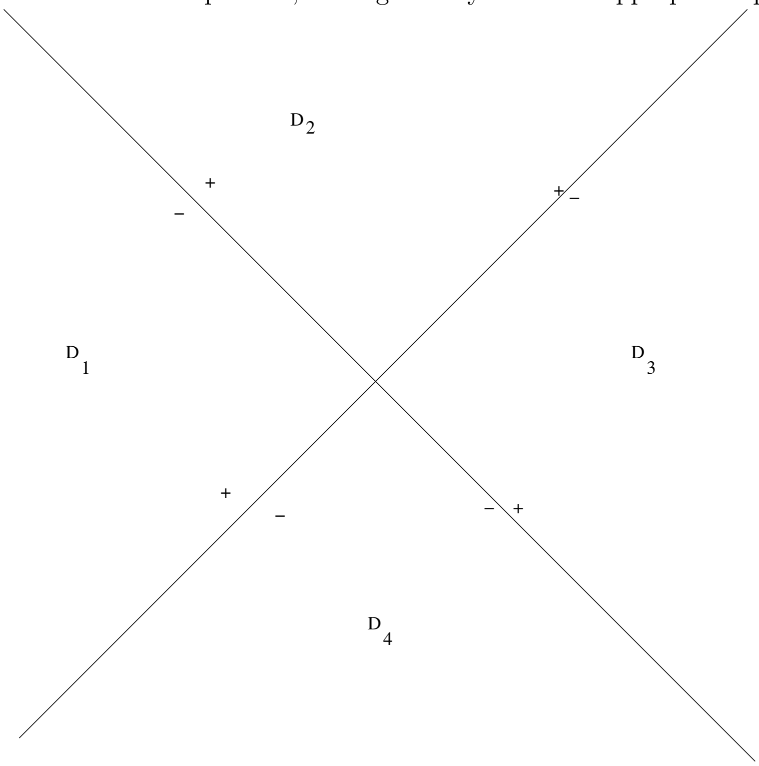

Figure 1. A cross centered at a stationary phase point.

Define a new matrix M by

$$
\begin{aligned}
&M = Q,\ \xi \in D_2 \cup D_4,\\
&M = \begin{pmatrix} d^{-1} & \frac{r^* d e^{2i\Theta}}{1-|r|^2} \\ 0 & d \end{pmatrix} Q,\ \xi \in D_1 \cap \{Im\xi > 0\},\\
&M = Q \begin{pmatrix} d^{-1} & 0 \\ \frac{r d^{-1} e^{-2i\Theta}}{1-|r|^2} & d \end{pmatrix},\ \xi \in D_1 \cap \{Im\xi < 0\},\\
&M = Q \begin{pmatrix} 1 & -r^* e^{-2i\Theta} \\ 0 & 1 \end{pmatrix}, \xi \in D_3 \cap \{Im\xi < 0\},\\
&\begin{pmatrix} 1 & 0 \\ -re^{2i\Theta} & 1 \end{pmatrix} Q, \xi \in D_3 \cap \{Im\xi > 0\}.
\end{aligned} \tag{16}
$$

It is immediate seen that there is no jump for M across the real axis. The jumps across the four halflines of the cross are

$$
\begin{gathered}
\begin{pmatrix} 1 & 0 \\ re^{2i\Theta} & 1 \end{pmatrix}, \begin{pmatrix} d & \frac{-r^* d e^{2i\Theta}}{1-|r|^2} \\ 0 & d^{-1} \end{pmatrix},\\
\begin{pmatrix} d^{-1} & 0 \\ \frac{r d^{-1} e^{-2i\Theta}}{1-|r|^2} & d \end{pmatrix}, \begin{pmatrix} 1 & -r^* e^{-2i\Theta} \\ 0 & 1 \end{pmatrix},
\end{gathered} \tag{17}
$$

in counterclockwise order starting at the first quadrant. Check Figure 1 for the orientation of the cross. The direction of contours is always compatible with $Re\xi$ increasing. "+" is to the left, "-" is to the right.

It is easy to see that the off-diagonal terms are exponentially small away from the center of the cross. So, they can be neglected asymptotically. One ends up with a Rieman-Hilbert problem on a cross centered at ξ_0. Apart from a small cross centered at ξ_0 the jumps are diagonal everywhere. In this sense, the dominating contribution to the solution of the Rieman-Hilbert problem comes from a small neighborhood of the stationary phase point. The Rieman-Hilbert problem can be solved explicitly via parabolic cylinder functions (following [I]) and the asymptotics for RH problem (13) and hence (11) are recovered.

The next step is a rescaling $\xi \to -\xi_0 + \xi(-t\xi_0)^{-1/2}$. The Riemann-Hilbert problem is then deformed to a new problem on an infinite cross, which can be explictly solved. In fact, after deforming the components of the cross back to the real line, it is equivalent to the following problem on the real line.

$$H_+(\xi) = H_-(\xi)exp(-i\xi^2\sigma_3)\begin{pmatrix} 1-|r(\xi_0)|^2 & -r^*(\xi_0) \\ r(\xi_0) & 1 \end{pmatrix} exp(i\xi^2\sigma_3), \quad H(\xi) \sim \xi^{i\nu\sigma_3}, \tag{18}$$

where ν is a constant depending only on ξ_0 and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli matrix.

Problem (18) can be solved explicitly, since (after conjugating) the jump is constant ([I]).

Even though [DZ] introduces the term "nonlinear steepest descent method" it is more accurate to say we have a nonlinear stationary phase method.

4. THE g-FUNCTION [DVZ94]

Back to the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x,0) = u_0(x). \tag{10}$$

Apart from the contributions of the (two) stationary phase points there is a new phenomenon here. A "collisionless shock" appears in the asymptotic region $x < 0$ and for $c_1^{-1} < -x/(3t)^{1/3}(\log t)^{2/3} < c_1, c_1 > 1$. The long time asymptotics are given by

$$u(x,t) \sim (-2x/3t)(A(\alpha) + B(\alpha)cn^2(2K(\alpha)\theta + \theta_0;\alpha)),$$

where $cn(.,\alpha)$ is the Jacobi cnoidal function of modulus α, and α in its turn is a function of some "slow" variable Z, $\theta \sim \log t$ is a "fast" variable, $K(\alpha)$ is the standard complete elliptic integral of the first kind, θ_0 depends on Z, and $A(\alpha), B(\alpha)$ are some explicit functions of α.

THEOREM [S]. There exists a row vector S with analytic entries in the upper and lower open half-planes, such that the normal limits S_+, S_-, as z approaches the real line from above or below respectively, exist and satisfy

$$S_+(z) = S_-(z)\begin{pmatrix} 1-|r(z)|^2 & -r^*(z)e^{-izx-4iz^3t} \\ r(\xi)e^{izx+4iz^3t} & 1 \end{pmatrix}, \ Imz = 0, \quad and\ lim_{z\to\infty}S(z) = (1,1). \tag{19}$$

Furthermore the solution $u(x,t)$ of (10) is given by

$$u(x,t) = -2i\frac{\partial}{\partial x}S_1^1, \tag{20}$$

where S_1^1 is the first entry of the residue of S at infinity. The stationary phase points are $\pm(-x/12t)^{1/2}$.

In [DVZ94] the following transformation is introduced

$$\hat{S}(z) = S((-x/12t)^{1/2}z)exp(-i\tau[4(z^3-3z)-g(z)]\sigma_3),$$

where g is a judicially chosen function that is analytic off a union of two intervals $[-b,-a],[a,b]$ such that $b>a>0$ are functions of the slow variable Z and such that

$$g(z) = 4z^3 - 12z + O(1/z)$$

at infinity, so that $\hat{S}$ is still the identity there. In fact,

$$g(z) = 12\int_b^z [(p^2-a^2)(p^2-b^2)]^{1/2}dp + 12\int_0^a [(p^2-a^2)(p^2-b^2)]^{1/2}dp.$$

The resulting Riemann-Hilbert problem has jump

$$v(z) = I,\ |z|>b,$$
$$\begin{pmatrix} 0 & e^{-24i\tau\int_0^a[(p^2-a^2)(p^2-b^2)]^{1/2}dp} \\ -e^{24i\tau\int_0^a[(p^2-a^2)(p^2-b^2)]^{1/2}dp} & 0 \end{pmatrix},\ a<z<b,$$
$$diag(const\cdot z^2,(const\cdot z^2)^{-1}),\ -a<z<a,$$
$$\begin{pmatrix} 0 & e^{-24i\tau\int_0^{-a}[(p^2-a^2)(p^2-b^2)]^{1/2}dp} \\ -e^{24i\tau\int_0^{-a}[(p^2-a^2)(p^2-b^2)]^{1/2}dp} & 0 \end{pmatrix},\ -b<z<-a.$$

The jump along $(-a,a)$ can be "conjugated away". One ends up with a RH problem on two slits. This can be solved in terms of genus 1 theta functions.

As is remarked in [DVZ94] the fact that the new RH problem is on two slits "is a new and essentially noninear feature of our nonlinear stationary phase method".

5. THE g-FUNCTION FOR THE ZERO DISPERSION KdV [DVZ97]

Consider now the KdV equation

$$
\begin{aligned}
&u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \\
&u(x,0) = u_0(x),
\end{aligned}
\tag{21}
$$

in the limit as $\epsilon \to 0$. Assume for simplicity, that the initial data are real analytic, positive and consist of a "hump" of unit height.

The associated RH problem is

$$
S_+(z) = S_-(z) \begin{pmatrix} 1-|r(z)|^2 & -r^*(z)e^{\frac{-izx-4iz^3t}{\epsilon}} \\ r(\xi)e^{\frac{izx+4iz^3t}{\epsilon}} & 1 \end{pmatrix},\ Imz = 0, \quad and\ lim_{z\to\infty} S(z) = (1,1).
\tag{22}
$$

The solution of (21) is recovered via

$$
u(x,t;\epsilon) = -2i\epsilon \frac{\partial}{\partial x} S_1^1 x, t; \epsilon).
\tag{23}
$$

The reflection coefficient r also depends on ϵ. In fact, the WKB approximation is

$$
\begin{aligned}
r(z) &\sim -ie^{\frac{-2i\rho(z)}{\epsilon}} \chi_[0,1](z) \\
1-|r(z)|^2 &\sim e^{\frac{-2\tau(z)}{\epsilon}},
\end{aligned}
$$

where

$$
\begin{aligned}
\rho(z) &= x_+ z + \int_{x_+}^{\infty} [z - (z^2 - u_0(x))^{1/2}]dx, \\
\tau(z) &= Re \int (u_0(x) - z^2)^{1/2} dx
\end{aligned}
$$

and $x_+(z)$ is the largest solution of $u_0(x_+) = z^2$.

[DKZ97] introduce a scalar g-function defined implicitly by the following conditions.

(i) g is analytic off the interval $[0,1]$ and vanishes at infinity.

(ii) "Finite gap ansatz" : There exists a finite set of disjoint open real intervals $I_j \in [0,1]$ such that the normal limits g_+, g_- of g exist along these intervals and, defining $h(z) = g_+(z) + g_-(z) - 2\rho + 4tz^3 + xz$, (iia) For $z \in \cup_j I_j$, we have $-\tau < (g_+ - g_-)/2i < 0$ and $h' = 0$.

(iib) For $z \in [0,1] \setminus \cup_j I_j$, we have $2i\tau = g_+ - g_-$ and $h' < 0$.

Unlike [DVZ94] where the g-function is defined explicitly, here it is only defined implicitly, via (i), (ii), (iia), (iib). In general (for any given data u_0) it is not true that the above conditions can be satisfied. It is conjectured however [DKM] that under the condition of analyticity a g-function satisfying the "finite gap ansatz" exists. (But see also [K00] for a proof of the "finite gap ansatz" in the analogous problem of the continuum Toda equations.)

Assuming the "finite gap ansatz" one can show that the RH problem reduces to one supported on the bands I_j with jumps of the form

$$\begin{pmatrix} 0 & -ie^{-ih(z)/\epsilon} \\ -ie^{ih(z)/\epsilon} & 0 \end{pmatrix}, \tag{24}$$

and in fact, because of (iib), $h(z)$ is a real constant on each band I_j. This RH problem can be solved explicitly via theta functions.

The g-function satisfying conditions (i), (ii), (iia), (iib) can be written as

$$g(z) = \int log(z - \eta) d\mu(\eta) \tag{25}$$

where μ is a continuous measure supported in $\cup_j I_j$. In a sense, the reduction of the given RH problem to an explicilty solvable one depends on the existence of a particular measure. Conditions (i), (ii), (iia), (iib) turn out to be equivalent to a maximization problem for logarithmic potentials under a particular external field depending on $x, t, u_0(x)$ over positive measures with an upper constraint. This is, of course, the celebrated Lax-Levermore variational problem [LL].

We end this section by noting that the methods of [DVZ97] have been used in the problem of asymptotics of orthogonal polynomials and the related problem of universality in the distribution of the eigenvalues of large hermitian random matrices [DKMVZ].

6. NONLINEAR STEEPEST DESCENT METHOD: SEMICLASSICAL FOCUSING NLS [KMM]

Consider now the focusing NLS equation.

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\partial_x^2\psi + |\psi|^2\psi = 0, \qquad (26)$$
$$under\ \psi(x,0) = \psi_0(x)$$

and consider the limit $\hbar \to 0$. The associated linear operator is the Dirac operator

$$L = \begin{pmatrix} ih\partial_x & -i\psi_0(x) \\ -i\psi_0^*(x) & -ih\partial_x \end{pmatrix}, \qquad (27)$$

which is non-self-adjoint. This is a major difference from the problem (12). We shall see that the deformation of the semiclassical RH problem cannot be confined to a small neighborhood of the real axis but is instead fully two-dimensional.

We note in passing that when seeking long time asymptotics for (26) with $\hbar = 1$ a collisionless shock phenomenon is also present; for x, t in the shock region the deformed RH problem is supported on a vertical imaginary slit. (See [K96].)

We rather focus on the semiclassical problem $\hbar \to 0$ which is more complicated.

For simplicity consider the very specific data $\psi_0(x) = Asechx$ where $A > 0$. Let $x_-(\eta) < x_+(\eta)$ be the two solutions of $sech^2(x) + \eta^2 = 0$. Also assume that $\hbar = A/N$ and consider the limit $N \to \infty$.

It is known that the reflection coefficient is identically zero and that the eigenvalues of L lie uniformly placed on the imaginary segment $[-iA, iA]$. In fact the eigenvalues are the points $\lambda_j = i\hbar(j + 1/2), j = 0, ..., N-1$ and their conjugates.

The associated RH problem is a meromorphic problem: to find a rational function with prescribed residues at the poles λ_j and their conjugates. It can be turned into a holomorphic problem by constructing two loops, one encircling the λ_j and one encircling their conjugates. We redefine the unknown 2x2 matrix inside the loops so that the poles vanish and thus arrive at a nontrivial jump across the two loops. (See [KMM] for details).

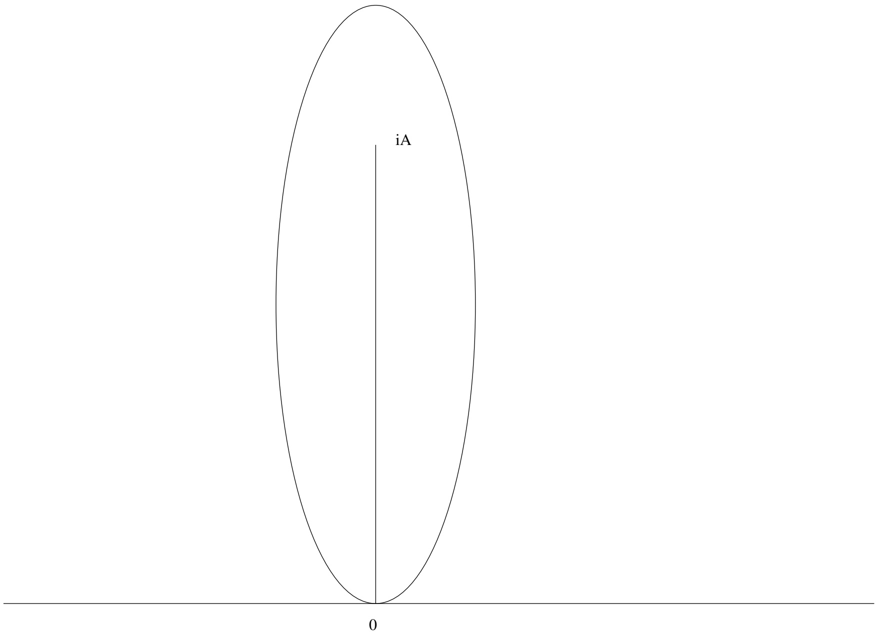

Figure 4. A loop C can be originally deformed anywhere as long as it stays away from the spike [0,iA]. The target contour however is an S−curve. The support of its equilibrium measure consists of finitely many analytic arcs, and C satisfies the S-property.

The loops can be deformed anywhere away from the poles (see Figure above). However, to be able to simplify the RH problem they have to be eventually located at a very specific position. The definition of a g-function will depend on the division of each loop into arcs, called "bands" and "gaps". There is also an associated measure supported on bands as in the KdV case. The analogous variational problem is not a maximization problem but rather a maximin problem. Here's the setting.

Let $\mathbb{H} = \{z : Imz > 0\}$, be the complex upper-half plane and $\bar{\mathbb{H}} = \{z : Imz \geq 0\} \cup \{\infty\}$ be the closure of $\mathbb{H}$. Let also $\mathbb{K} = \{z : Imz > 0\} \setminus \{z : Rez = 0, 0 < Imz \leq A\}$, where A is a positive constant. In the closure of this space, $\bar{\mathbb{K}}$, we consider the points ix_+ and ix_-, where $0 \leq x < A$ as distinct. In other words, we cut a slit in the upper half-plane along the segment $(0, iA)$ and distinguish between the two sides of the slit. The point infinity belongs to $\bar{\mathbb{K}}$, but not $\mathbb{K}$. Define $G(z;\eta)$ to be the Green's function for the upper half-plane

$$G(z;\eta) = log\frac{|z-\eta^*|}{|z-\eta|}$$

and let $d\mu^0(\eta)$ be the nonnegative measure $-id\eta$ on the segment $[0, iA]$ oriented from 0 to iA. The star denotes complex conjugation. Let the "external field" ϕ be defined by

$$\phi(z) = -\int G(z;\eta)d\mu^0(\eta) - Re(\pi(iA - z) + 2i(zx + z^2 t)), \tag{28}$$

where, without loss of generality $x > 0$.

Let $\mathbb{M}$ be the set of all positive Borel measures on $\bar{\mathbb{K}}$, such that both the free energy

$$E(\mu) = \int\int G(x, y)d\mu(x)d\mu(y), \ \ \mu \in \mathbb{M}$$

and $\int \phi d\mu$ are finite. Also, let

$$V^{\mu}(z) = \int G(z, x)d\mu(x), \ \ \mu \in \mathbb{M}.$$

be the Green's potential of the measure μ.

The weighted energy of the field ϕ is

$$E_\phi(\mu) = E(\mu) + 2\int \phi d\mu,$$

for any $\mu \in \mathbb{M}$.

Now, given any curve F in $\bar{\mathbb{K}}$, the equilibrium measure λ^F supported in F is defined by

$$E_\phi(\lambda^F) = min_{\mu \in M(F)} E_\phi(\mu),$$

where $M(F)$ is the set of measures in $\mathbb{M}$ which are supported in F, provided such a measure exists.

The finite gap ansatz is equivalent to the existence of a so-called S-curve joining the points 0_+ and 0_- and lying entirely in $\bar{\mathbb{K}}$. (See Figure in the previous page.) By S-curve we mean an oriented curve F such that the equilibrium measure λ^F exists, its support consists of a finite union of analytic arcs and at any interior point of $supp\mu$ the so called S-property is satisfied

$$\frac{d}{dn_+}(\phi + V^{\lambda^F}) = \frac{d}{dn_-}(\phi + V^{\lambda^F}), \tag{29}$$

In the next section we will see that there is a C such that

$$E_\phi(\lambda^C) = max_{contoursF} E_\phi(\lambda^F) = max_{contoursF} min_{\mu\in M(F)} E_\phi(\mu), \tag{30}$$

and that the existence of an S-curve follows from the existence of a contour C maximizing the equilibrium measure. This is a non-trivial extension of the Lax-Levermore variational problem.

7. JUSTIFICATION OF THE NONLINEAR STEEPEST DESCENT METHOD

EXISTENCE THEOREM [KR]. For the external field given by (27), there exists a continuum $F \in \mathbb{F}$ such that the equilibrium measure λ^F exists and

$$E_\phi[F](= E_\phi(\lambda^F)) = max_{F\in\mathbb{F}} min_{\mu\in M(F)} E_\phi(\mu). \tag{31}$$

Assuming that the continuum F does not touch the linear segment $[0, iA]$ at more than a finite number of points, we have

REGULARITY THEOREM [KR]. The continuum F is in fact an S-curve.

REMARK. If the above assumption is dropped then a finite gap ansatz can still be proved, but one has to consider an infinite sheeted Riemann surface as the setting for the variational problem.

To prove the above theorems, we first introduce an appropriate topology on $\mathbb{F}$. We think of the closed upper half-plane $\bar{\mathbb{H}}$ as a compact space in the Riemann sphere. We thus choose to equip $\bar{\mathbb{H}}$ with the "chordal" distance, denoted by ρ_0, that is the distance between the images of z and ζ under the stereographic projection. This induces naturally a distance in $\bar{\mathbb{K}}$ (for example, $\rho_0(0_+, 0_-) = 2A$). We also denote by ρ_0 the induced distance between compact sets E, F in $\bar{\mathbb{K}}$: $\rho_0(E, F) = max_{z\in E} min_{\zeta\in F} \rho_0(z, \zeta)$. Then, we define the so-called Hausdorff metric on the set $I(\bar{\mathbb{K}})$ of closed non-empty subsets of $\bar{\mathbb{K}}$ as follows.

$$\rho_{\mathbb{K}}(A, B) = sup(\rho_0(A, B), \rho_0(B, A)).$$

LEMMA [KR]. The Hausdorff metric is indeed a metric. The set $I(\bar{\mathbb{K}})$ is compact and complete. Since $\mathbb{F}$ is a closed subset of $I(\bar{\mathbb{K}})$, $\mathbb{F}$ is also compact and complete.

Compactness of $\mathbb{F}$ is a necessary first ingredient to prove existence of a maximizing contour. The second ingredient is semicontinuity of energy.

We consider the functional that takes a given continuum F to the equilibrium energy on this continuum:

$$\mathbb{E} : F \to E_\psi[F] = E_\psi(\lambda^F) = \inf_{\mu \in M(F)} (E(\mu) + 2\int \psi d\mu). \tag{32}$$

In [KR] we show that for the external field given by (28), the energy functional defined in (32) is upper semicontinuous. Existence then follows easily.

To prove regularity the main ingredient is the following identity.

THEOREM [KR]. Let F be the maximizing continuum of and λ^F be the equilibrium measure. Let μ be the extension of λ^F to the lower complex plane via $\mu(z^*) = -\mu(z)$. Then

$$\begin{aligned} Re(\int \frac{d\mu(u)}{u-z} + V'(z))^2 = Re(V'(z))^2 - 2Re\int \frac{V'(z)-V'(u)}{z-u} d\mu(u) \\ +Re[\frac{1}{z^2}\int 2(u+z)V'(u)\ d\mu(u)]. \end{aligned} \tag{33}$$

Here V is the logarithmic potential of μ.

PROOF: By taking variations with respect to the equilibrium measure [KR].

From (33) it is easy to see that the support of the equilibrium measure of the maximizing continuum is characterized by

$$Re\int^z (R_\mu)^{1/2} dz = 0, \tag{34}$$

where

$$\begin{aligned} R_\mu(z) = (V'(z))^2 - 2\int_{supp\mu} \frac{V'(z)-V'(u)}{z-u} d\mu(u) \\ +\frac{1}{z^2}(\int_{supp\mu} 2(u+z)V'(u)\ d\mu(u)). \end{aligned}$$

Since $R_\mu(z)$ is a function analytic in $\mathbb{K}$, the locus defined by (34) is a union of arcs with endpoints at zeros of R_μ. Further analysis actually shows that R_μ has finitely many zeros.

The S-property follows easily from (33); see [KR]. This proves the Regularity Theorem.

8. CONCLUSION

Given the importance and the recent popularity of the "nonlinear steepest descent method" and the various different applications to such topics as soliton theory, orthogonal polynomials, solvable models in statistical mechanics, random matrices, combinatorics and representation theory, we believe that the present work offers an important contribution. In particular we expect that the results of this paper may be useful in the treatment of Riemann-Hilbert problems arising in the analysis of general complex or normal random matrices [WZ].

Although there are cases of non-self-adjoint problems where the target contour can be explictly computed without any recourse to a variational problem (which of course is always there; see e.g. [K96], [KSVW]), the most general global results (as in section 7) can only be justified by proving existence and regularity for a solution of a maximin variational problem in two dimensions.

APPENDIX. THE DESCRIPTION OF THE SEMICLASSICAL LIMIT OF THE FOCUSING NLS EQUATION UNDER THE FINITE GENUS ANSATZ

We present one of the main results of [KMM] on the semiclassical asymptotics for problem (26), under the assumption that the finite genus ansatz holds. In particular, we fix x, t and assume that the support of the maximizing measure of section 7 consists of a finite union of analytic arcs.

First, we define the so-called g-function. Let C be the maximizing contour. A priori we seek a function satisfying

$$g(\lambda) \textit{ is independent of } \hbar.$$

$$g(\lambda) \textit{ is analytic for} \lambda \in \mathbb{C} \setminus (C \cup C^*).$$

$$g(\lambda) \to 0 \textit{ as } \lambda \to \infty.$$

$$g(\lambda) \textit{ assumes continuous boundary values from both sides of } C \cup C^*,$$

$$\textit{denoted by } g_+(g_-) \textit{ on the left (right) of } C \cup C^*.$$

$$g(\lambda^*) + g(\lambda)^* = 0 \textit{ for all } \lambda \in \mathbb{C} \setminus (C \cup C^*).$$

The assumptions above permit us to write g in terms of a measure ρ defined on the contour C. Indeed

$$g(\lambda) = \int_{C \cup C^*} log(\lambda - \eta)\rho(\eta)d\eta,$$

for an appropriate definition of the logarithm branch. The measure $\rho(\eta)d\eta$ is not uniquely defined by the conditions above; we actually choose $\rho(\eta)d\eta$ to be a maximizing measure as in section 7, doubled up according to

$$\rho(\eta^*) = (\rho(\eta))^*.$$

For $\lambda \in C$, define the functions

$$\begin{aligned} \theta(\lambda) &:= i(g_+(\lambda) - g_-(\lambda)), \\ \Phi(\lambda) := \int_0^{iA} log(\lambda - \eta)\rho^0(\eta)\, d\eta &+ \int_{-iA}^{0} log(\lambda - \eta)\rho^0(\eta^*)^*\, d\eta \\ +2i\lambda x + 2i\lambda^2 t + i\pi \int_{\lambda}^{iA} &\rho^0(\eta)\, d\eta - g_+(\lambda) - g_-(\lambda), \end{aligned} \tag{A.1}$$

where $\rho^0(\eta) = i$, the WKB density of eigenvalues introduced in section 6.

The finite genus ansatz implies that for each x, t there is a finite positive integer G such that the contour C can be divided into "bands" (the support of $\rho(\eta)d\eta$) and "gaps" (where $\rho = 0$). We denote these bands by I_j. More precisely, we define the analytic arcs $I_j, I_j^*, j = 1, ..., G/2$ as follows (they come in conjugate pairs). Let the points $\lambda_j,\ j = 0, ..., G$, in the open uper half-plane be the branch points of the function g. All such points lie on the contour C and we order them as $\lambda_0, \lambda_1, ..., \lambda_G$, according to the direction given to C. The points $\lambda_0^*, \lambda_1^*, ..., \lambda_G^*$ are their complex conjugates. Then let $I_0 = [0, \lambda_0]$ be the subarc of C joining points 0 and λ_0. Similarly, $I_j = [\lambda_{2j-1}, \lambda_{2j}],\ j = 1, ..., G/2$. The connected components of the set $\mathbb{C} \setminus \cup_j (I_j \cup I_j^*)$ are the so-called "gaps", for example the gap Γ_1 joins λ_0 to λ_1, etc.

It actually follows from the properties of ρ that the function $\theta(\lambda)$ defined on C is constant on each of the gaps Γ_j, taking a value which we will denote by θ_j, while the function Φ is constant on each of the bands, taking the value denoted by α_j on the band I_j.

The finite genus ansatz for the given fixed x, t implies that the asymptotics of the solution of (26) as $\hbar \to 0$ can be given by the next Theorem.

THEOREM A.1. Let x_0, t_0 be given. The solution $\psi(x,t)$ of (26) is asymptotically described (locally) as a slowly modulated $G+1$ phase wavetrain. Setting $x = x_0 + \hbar\hat{x}$ and $t = t_0 + \hbar\hat{t}$, so that x_0, t_0 are "slow" variables while $\hat{x}, \hat{t}$ are "fast" variables, there exist parameters

$a, U = (U_0, U_1, ..., U_G)^T$, $k = (k_0, k_1, ..., k_G)^T$, $w = (w_0, w_1, ..., w_G)^T$, $Y = (Y_0, Y_1, ..., Y_G)^T$, $Z = (Z_0, Z_1, ..., Z_G)^T$ depending on the slow variables x_0 and t_0 (but not $\hat{x}, \hat{t}$) such that

$$\begin{aligned}\psi(x,t) = \psi(x_0 + \hbar\hat{x}, t_0 + \hbar\hat{t}) \sim a(x_0,t_0) e^{iU_0(x_0,t_0)/\hbar} e^{i(k_0(x_0,t_0)\hat{x} - w_0(x_0,t_0)\hat{t})} \\ \cdot \frac{\Theta(Y(x_0,t_0) + iU(x_0,t_0)/\hbar + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}{\Theta(Z(x_0,t_0) + iU(x_0,t_0)/\hbar + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}.\end{aligned} \tag{A.2}$$

All parameters can be defined in terms of an underlying Riemann surface X. The moduli of X are given by $\lambda_j, \ j = 0, ..., G$ and their complex conjugates $\lambda_j^*, \ j = 0, ..., G$. The genus of X is G. The moduli of X vary slowly with x, t, i.e. they depend on x_0, t_0 but not $\hat{x}, \hat{t}$. For the exact formulae for the parameters as well as the definition of the theta functions we present the following construction.

The Riemann surface X is constructed by cutting two copies of the complex sphere along the slits $I_0 \cup I_0^*, I_j, I_j^*, j = 1...., G$, and pasting the "top" copy to the "bottom" copy along these very slits.

We define the homology cycles $a_j, b_j, \ j = 1, ..., G$ as follows. Cycle a_1 goes around the slit $I_0 \cup I_0^*$ joining λ_0 to λ_0^*, remaining on the top sheet, oriented counterclockwise, a_2 goes through the slits I_{-1} and I_1 starting from the top sheet, also oriented counterclockwise, a_3 goes around the slits $I_{-1}, I_0 \cup I_0^*, I_1$ remaining on the top sheet, oriented counterclockwise, etc. Cycle b_1 goes through I_0 and I_1 oriented counterclockwise, cycle b_2 goes through I_{-1} and I_1, also oriented counterclockwise, cycle b_3 goes through I_{-1} and I_2, and around the slits $I_{-1}, I_0 \cup I_0^*, I_1$, oriented counterclockwise, etc.

On X there is a complex G-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \ldots, G$ that can be written in the form

$$\nu_k(P) = \frac{\sum_{j=0}^{G-1} c_{kj}\lambda(P)^j}{R_X(P)}\, d\lambda(P)\,,$$

where $R_X(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to X: if P is on the first sheet of X then $R_X(P) = R(\lambda(P))$ and if P is on the second sheet of X then $R_X(P) = -R(\lambda(P))$. The coefficients c_{kj} are uniquely determined by the constraint that the differentials satisfy the normalization conditions:

$$\oint_{a_j} \nu_k(P) = 2\pi i\delta_{jk}.$$

From the normalized differentials, one defines a $G\times G$ matrix H (the period matrix) by the formula:

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

It is a consequence of the standard theory of Riemann surfaces that H is a symmetric matrix whose real part is negative definite.

In particular, we can define the theta function

$$\Theta(w) := \sum_{n\in\mathbb{Z}^G} \exp(\frac{1}{2}n^T Hn + n^T w),$$

where H is the period matrix associated to X. Since the real part of H is negative definite, the series converges.

We arbitrarily fix a base point P_0 on X. The Abel map $A : X \to Jac(X)$ is then defined componentwise as follows:

$$A_k(P; P_0) := \int_{P_0}^{P} \nu_k(P'),\ k = 1, \ldots, G,$$

where P' is an integration variable.

A particularly important element of the Jacobian is the Riemann constant vector K which is defined, modulo the lattice Λ, componentwise by

$$K_k := \pi i + \frac{H_{kk}}{2} - \frac{1}{2\pi i}\sum_{\substack{j=1\\ j\neq k}}^{G} \oint_{a_j} \left(\nu_j(P)\int_{P_0}^{P} \nu_k(P')\right),$$

where the index k varies between 1 and G.

Next, we will need to define a certain meromorphic differential on X. Let $\Omega(P)$ be holomorphic away from the points ∞_1 and ∞_2, where it has the behavior

$$\Omega(P) = dp(\lambda(P)) + \left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_1,$$
$$\Omega(P) = -dp(\lambda(P)) + O\left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_2,$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, j = 1, \dots, G.$$

Here p is a polynomial, defined as follows.

First, let us introduce the function $R(\lambda)$ defined by

$$R(\lambda)^2 = \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),$$

choosing the particular branch that is cut along the bands I_k^+ and I_k^- and satisfies

$$\lim_{\lambda\to\infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1,$$

This defines a real function, i.e. one that satisfies $R(\lambda^*) = R(\lambda)^*$. At the bands, we have $R_+(\lambda) = -R_-(\lambda)$, while $R(\lambda)$ is analytic in the gaps. Next, let's introduce the function $k(\lambda)$ defined by

$$k(\lambda) = \frac{1}{2\pi i}\sum_{n=1}^{G/2} \theta_n \int_{\Gamma_n^+\cup\Gamma_n^-} \frac{d\eta}{(\lambda-\eta)R(\eta)} + \frac{1}{2\pi i}\sum_{n=0}^{G/2} \int_{I_n^+\cup I_n^-} \frac{\alpha_n\ d\eta}{(\lambda-\eta)R_+(\eta)}.$$

Next let

$$H(\lambda) = k(\lambda)R(\lambda).$$

The function k satisfies the jump relations

$$k_+(\lambda) - k_-(\lambda) = -\frac{\theta_n}{R(\lambda)},\ \lambda \in \Gamma_n^+ \cup \Gamma_n^-$$
$$k_+(\lambda) - k_-(\lambda) = -\frac{\alpha_n}{R_+(\lambda)},\ \lambda \in I_n^+ \cup I_n^-,$$

and is otherwise analytic. It blows up like $(\lambda-\lambda_n)^{-1/2}$ near each endpoint, has continuous boundary values in between the endpoints, and vanishes like $1/\lambda$ for large

λ. It is the only such solution of the jump relations. The factor of $R(\lambda)$ renormalizes the singularities at the endpoints, so that, as desired, the boundary values of $H(\lambda)$ are bounded continuous functions. Near infinity, there is the asymptotic expansion:

$$\begin{aligned} H(\lambda) &= H_G\lambda^G + H_{G-1}\lambda^{G-1} + \cdots + H_1\lambda + H_0 + O(\lambda^{-1}) \\ &= p(\lambda) + O(\lambda^{-1}), \end{aligned} \tag{A.3}$$

where all coefficients H_j of the polynomial $p(\lambda)$ can be found explicitly by expanding $R(\lambda)$ and the Cauchy integral $k(\lambda)$ for large λ. It is easy to see from the reality of θ_j and α_j that $p(\lambda)$ is a polynomial with real coefficients.

Thus the polynomial $p(\lambda)$ is defined and hence the meromorphic differential $\Omega(P)$ is defined.

Let the vector $U \in \mathbb{C}^G$ be defined componentwise by

$$U_j := \oint_{b_j} \Omega(P).$$

Note that $\Omega(P)$ has no residues.

Let the vectors V_1, V_2 be defined componentwise by

$$\begin{aligned} V_{1,k} &= (A_k(\lambda^*_{1+}) + A_k(\lambda_{2+}) + A_k(\lambda^*_{3+}) + \cdots + A_k(\lambda_{G+})) + A_k(\infty) + \pi i + \frac{H_{kk}}{2}, \\ V_{2,k} &= (A_k(\lambda^*_{1+}) + A_k(\lambda_{2+}) + A_k(\lambda^*_{3+}) + \cdots + A_k(\lambda_{G+})) - A_k(\infty) + \pi i + \frac{H_{kk}}{2}, \end{aligned}$$

where $k = 1, ..., G$, and the $+$ index means that the integral for A is to be taken on the first sheet of X, with base point λ^0_+.

Finally, let

$$a = \frac{\Theta(Z)}{\Theta(Y)} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k),$$
$$k_n = \partial_x U_n,\ w_n = -\partial_t U_n,\ n = 0, \ldots, G,$$

where

$$Y = -A(\infty) - V_1,\ Z = A(\infty) - V_1,$$

and $U_0 = -(\theta_1 + \alpha_0)$ where θ_1 is the (constant in λ) value of the function θ in the gap Γ_1 and α_0 is the (constant) value of the function ϕ in the band I_0.

Now, the parameters appearing in formula (A.2) are completely described.

We simply note here that the U_i and hence the k_i and w_i are real. We also note that the denominator in (A.2) never vanishes (for any $x_0, t_0, \hat{x}, \hat{t}$).

REMARK. Theorem A.1 presents pointwise asymptotics in x, t. In [KMM], these are extended to uniform asymptotics in certain compact sets covering the x, t-plane. Error estimates are also given in [KMM].

REMARK. As mentioned above, we do not know if the support of the equilibrium measure of the maximizing continuum is unique. But the asymptotic formula (A.2) depends only on the endpoints λ_j of the analytic subarcs of the support. Since the asymptotic expression (A.2) has to be unique, it is easy to see that the endpoints also have to be unique. Different Riemann surfaces give different formulae (except of course in degenerate cases: a degenerate genus 2 surface can be a pinched genus 0 surface and so on).

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