A SCHEME RELATED TO THE BRAUER LOOP MODEL

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ABSTRACT. We introduce the **Brauer loop scheme** $E := \{M \in M_N(\mathbb{C}) : M \bullet M = 0\}$, where $\bullet$ is a certain degeneration of the ordinary matrix product. Its components of top dimension, $\lfloor N^2/2 \rfloor$, correspond to involutions $\pi \in S_N$ having one or no £xed points. In the case N even, this scheme contains the upper-upper scheme from [Knutson '04] as a union of $(N/2)!$ of its components. One of those is a degeneration of the *commuting variety* of pairs of commuting matrices.

The *Brauer loop model* is an integrable stochastic process studied in [de Gier–Nienhuis '04], based on earlier related work in [Martins–Nienhuis–Rietman '98], and some of the entries of its Perron–Frobenius eigenvector were observed (conjecturally) to equal the degrees of the components of the upper-upper scheme.

Our proof of this equality follows the program outlined in [Di Francesco–Zinn-Justin '04]. In that paper, the entries of the Perron–Frobenius eigenvector were generalized from numbers to polynomials, which allowed them to be calculated inductively using divided difference operators. We relate these polynomials to the *multi*degrees of the components of the Brauer loop scheme, de£ned using an evident torus action on E. As a consequence, we obtain a formula for the degree of the commuting variety, previously calculated up to 4×4 matrices.

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1. INTRODUCTION

1.1. **The scheme** E**.** Let N be a positive integer. Call a sequence $(i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$ **cyclically ordered**, written "$\circlearrowleft (i_1 \leq \ldots \leq i_k)$", if for some j, the rotated sequence $(i_j, i_{j+1}, \ldots, i_k, i_1, \ldots, i_{j-1})$ is weakly increasing, with the additional condition that all i's are equal if $i_1 = i_k$.

We de£ne a new product $\bullet$ on the space $M_N(\mathbb{C})$ of $N \times N$ matrices according to the rule

$$(P \bullet Q)_{ik} = \sum_{j:\ \circlearrowleft(i \leq j \leq k)} P_{ij} Q_{jk} \qquad i, k = 1, \ldots, N$$

This is a degeneration of the usual product, as we explain in section 2.3. Hence it is associative, and indeed has triple product

$$(P \bullet Q \bullet R)_{il} = \sum_{j,k:\ \circlearrowleft(i \leq j \leq k \leq l)} P_{ij} Q_{jk} R_{kl}.$$

With the usual addition, $(M_N(\mathbb{C}), \bullet)$ forms an algebra, and the identity matrix is again the unit. A matrix P possesses an inverse $P^{\bullet -1}$ for this product if and only if its diagonal entries are all non-zero. In particular, the set

$$U = \{M \in M_N(\mathbb{C}) : M_{ii} = 1, i = 1 \ldots N\}$$

forms a group under $\bullet$. We note that if R, R' are upper triangular, then $R \bullet R' = RR'$ and $R^{\bullet -1} = R^{-1}$. All these facts are easiest to see in the model of $\bullet$ presented in section 2.1.

The **cycling automorphism** $M'_{ij} := M_{i+1,j+1}$, where the indices are taken mod N, is an automorphism of both the ordinary and $\bullet$ multiplications. This automorphism is inner for the ordinary multiplication, but is an outer automorphism for $\bullet$. With this automorphism in mind, essentially every reference to i, j, etc. in this paper has an implicit "mod N".

We de£ne the **Brauer loop scheme** E to be the space of matrices $M \in M_N(\mathbb{C})$ that satisfy $M \bullet M = 0$, and have zero diagonal.[1] (The name will be explained in section 1.2.) In equations, we require

$$\begin{aligned} \sum_{\circlearrowleft(i \leq j \leq k)} M_{ij} M_{jk} &= 0 \qquad i, k = 1, \ldots, N, i \neq k \\ M_{ii} &= 0 \qquad i = 1, \ldots, N \end{aligned}$$

[1] These equations are not redundant: $M \bullet M = 0$ implies that $M_{ii}^2 = 0$ for each i, but not that $M_{ii} = 0$. This is an empty distinction on the set but an important distinction on the scheme, and one that affects the (multi)degree that will interest us later. A similar phenomenon occurs already with ordinary matrix multiplication (as we address in section 7): while any matrix with $M^2 = 0$ has zero trace, the linear trace condition can't be inferred algebraically from the quadratic conditions $M^2 = 0$.

The scheme E looks similar to the irreducible scheme $\{M : M^2 = 0\}$ (a precise relation is spelled out in section 7), and in particular has the same dimension $\lfloor N^2/2 \rfloor$. However, E is reducible, and we now describe its components of top dimension.

In what follows the parity of N will play a role, so write

$$N = 2n + r, \qquad r = 0 \text{ or } 1.$$

We will refer to involutions of $\{1, \ldots, N\}$ with r £xed points as **link patterns**, and draw them as chord diagrams in the disk. In particular, the 2-cycles of an involution will be referred to as chords, and a "crossing" in a link pattern is a pair of chords which cross each other when drawn as segments in the disk. There are $(N-1)!! := (N-1)(N-3)(N-5) \cdots (1+r)$ link patterns of size N.

The following is a combination of theorems 2 and 3.

Theorem. *For each $M \in E$, the nonzero elements of the diagonal of M^2 (with respect to* ordinary *multiplication) come in equal pairs. Put another way, there is a link pattern π such that $(M^2)_{ii} = (M^2)_{\pi(i)\pi(i)}$ for all i. In addition, $(M^2)_{ii} = 0$ if $\pi(i) = i$.*

Conversely, for each such π, the open subscheme

$$\left\{ M \in E : (M^2)_{ii} = (M^2)_{jj} \textit{ if and only if } j \in \{i, \pi(i)\} \right\}$$

is nonempty, irreducible, and of dimension $\lfloor N^2/2 \rfloor$.

Hence, each E_π de£ned by

$$E_\pi := \overline{\left\{ M \in E : (M^2)_{ii} = (M^2)_{jj} \text{ if and only if } j \in \{i, \pi(i)\} \right\}}$$

is a component of E. In fact we conjecture that $E = \cup_\pi E_\pi$. The closest we come to proving this, in theorems 3 and 4, is

Theorem. *If $E \neq \cup_\pi E_\pi$, where π runs over the set of link patterns, then $\dim(E \setminus \cup_\pi E_\pi) < \dim E$. Also, E is generically reduced along each E_π.*

Theorem 4 gives a different characterization of the $\{E_\pi\}$:

Theorem. *Let $\underline{\pi}$ denote the permutation matrix of a link pattern π, with the diagonal zeroed out if π has a £xed point (i.e. if N is odd). Then*

$$E_\pi = \overline{U \cdot \{t\underline{\pi} : t \textit{ diagonal}\}}$$

where U acts by $\bullet$-conjugation.

This lets us determine in theorem 5 some (and conjecturally, all) of the de£ning equations of the $\{E_\pi\}$.

The cycling automorphism acts on E, and on the set of link patterns by rotation. We will make use, too, of the action of the full symmetric group S_N on the set of link patterns by conjugation, even though S_N does not act on E. Denote by f_i the transposition $i \leftrightarrow i+1$ for $i = 1, \ldots, N$ (where $N + 1 \equiv 1$), and let $f_i \cdot \pi := f_i \circ \pi \circ f_i^{-1}$.

For each $i = 1 \ldots N$, there is an idempotent **Temperley–Lieb operator** e_i on the set of link patterns de£ned by

$$(e_i \cdot \pi)(j) = \begin{cases} i + (i+1) - j & \text{if } j = i \text{ or } j = i+1 \\ \pi(i) + \pi(i+1) - j & \text{if } j = \pi(i) \text{ or } j = \pi(i+1) \\ \pi(j) & \text{otherwise} \end{cases}$$

where all addition is mod N. Graphically, e_i connects the chords coming to $i, i+1$ to one another, and puts in a new chord connecting $i, i+1$.

Together, the $\{f_i\}$ and $\{e_i\}$ form a representation of the af£ne $O(1)$ **Brauer algebra**. (Actually, the f_i and e_i, $i = 1, \ldots, N-1$, which satisfy the relations of the usual $O(1)$ Brauer algebra, are enough to generate the whole algebra of operators.) The Brauer algebra is itself a degenerate point of the braid-monoid algebra.

1.2. **The Brauer loop model polynomials $\{\Psi_\pi\}$.** In [dGN] there is associated to each link pattern π a positive integer d_π, as follows. (They will at £rst only appear to be rationals.)

Consider a Markov process whose states are the set of link patterns. The transitions from a link pattern π are to $\{e_i \cdot \pi, f_i \cdot \pi\}$, where i is chosen with equal probability from $1 \ldots N$, and e_i, f_i are then chosen with probabilities 2/3 and 1/3 (see £gure 1). For the origin of this Markov process and its relation to standard quantum integrable models, see [MR, MNR].

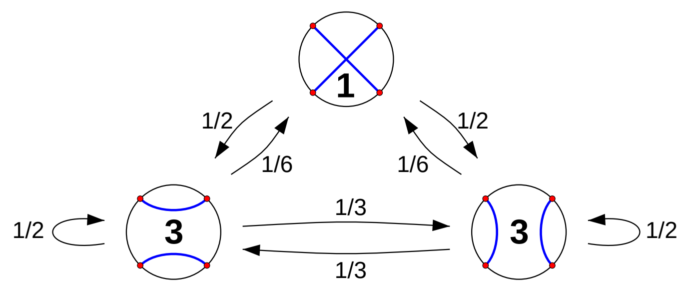

FIGURE 1. The de Gier–Nienhuis Markov process for $N = 4$. The edges are labeled with transition probabilities, and the nodes with the stationary distribution, rescaled to make the minimum value 1.

Many conjectures are stated in [dGN], among them that the least probable link patterns are those with the most crossings, and all other probabilities are *integer* multiples d_π of the least probable. Some of these d_π were also noticed to match the degrees of the components of the upper-upper scheme from [Kn]. (This scheme reappears here in section 5.)

This integrality, and much else, was proven in [DFZJ] by considering a slightly different Markov process, and generalizing the d_π to polynomials $\Psi_\pi \in \mathbb{Z}[z_1, \ldots, z_N]$. (In fact [DFZJ] only treats the case N even, but as indicated in [dGN] the case N odd is very similar.)

In this more general Markov process, the transition amplitudes are given by the so-called *transfer matrix*, one possible de£nition of which is

$$T' = \prod_{i=1}^{n} \prod_{j=1}^{n} \check{R}_{i+2j-2}(z_{2j-1} - z_{2i+2j-2}), \qquad \text{where } \check{R}_i(u) = a(u)1 + b(u)f_i + c(u)e_i$$

(for $N = 2n$; a similar expression exists for N odd) where the product is ordered and the indices meant mod N. Here $\check{R}_i(u)$ is a linear combination of $\mathbf{1}$, f_i, and e_i (their action on link patterns extended to turn them into linear operators), with coef£cients dependent on u still to be determined.

The stationary distribution can be encoded into a vector $\Psi = \sum_\pi \Psi_\pi \, \pi$ in the space of $\mathbb{Z}[z_1, \ldots, z_n]$-linear combinations of link patterns. Again, we scale Ψ to clear denominators, making the $\{\Psi_\pi\}$ polynomials with GCD = 1. The unnormalized probabilities d_π of the de Gier–Nienhuis process are recovered by setting all z's to zero.

We now explain the coef£cients we will use in $\check{R}_i$. Consider the equations

$$\check{R}_i(z_i - z_{i+1})\Psi = \tau_i \Psi \qquad i = 1, \ldots, N \tag{1}$$

where τ_i switches the variables z_i and z_{i+1}, i.e. $\tau_i F(\ldots, z_i, z_{i+1}, \ldots) = F(\ldots, z_{i+1}, z_i, \ldots)$. Since T' is a product of $\tau_i \check{R}_i(z_i - z_{i+1})$ operators, the equations (1) imply that $T'\Psi = \Psi$. More generally, products of $\tau_i \check{R}_i$, with the condition that the corresponding product of τ_i is the identity (which ensures that these operators have well-de£ned meaning once the z_i are specialized to complex numbers), generate a whole algebra of operators acting on $\mathbb{Z}[z_1, \ldots, z_N]$-linear combinations of link patterns. The "integrability" condition implies that this algebra must be commutative; this is achieved by requiring that the $\check{R}_i$ generate a representation of the af£ne $\hat{A}_{N-1}$ Weyl group, or equivalently that $\check{R}_i(u)\check{R}_i(-u) = I$ and $\check{R}_i(u)\check{R}_{i+1}(u+v)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(u+v)\check{R}_{i+1}(u)$ (the *Yang–Baxter equation*).

As shown in [NR], these two equations on the $\{\check{R}_i\}$ £x them uniquely (up to scaling of u, and multiplication by a function $\phi(u)$ satisfying $\phi(u)\phi(-u) = 1$) to be

$$\check{R}_i(u) = \Big((1-u)I + \frac{u}{2}(1-u)f_i + u e_i\Big) \Big/ \Big((1 - \frac{u}{2})(1+u)\Big). \tag{2}$$

These polynomials $\{\Psi_\pi\}$ are characterized by the following two properties (equations (3.11–14) and (3.19) of [DFZJ]):

- Recall that $N = 2n + r$, $r = 0, 1$, and de£ne $\pi_0(i) = i + n \bmod 2n$ for $i \le 2n$, and $\pi_0(N) = N$ if N is odd. So π_0 is a **maximally crossing link pattern**, and the only such if N is even. Then

$$\Psi_{\pi_0} = \prod_{\substack{i=1\ldots N \\ j:\ \circlearrowleft(i<j<i+n)}} (1 + z_i - z_j) \; \left(\prod_{i=n+1}^{N} (1 + z_i - z_{i+n})\right)^r \tag{3}$$

- If $\pi(i) \neq i+1$, then

$$\Psi_\pi + \Psi_{f_i \cdot \pi} = -\frac{2 + z_{i+1} - z_i}{1 + z_{i+1} - z_i} \, \partial_i \left((1 + z_{i+1} - z_i)\Psi_\pi\right) \tag{4}$$

  where $\partial_i F := (F - \tau_i F)/(z_i - z_{i+1})$.

The equations (4) allow one to express a component in terms of another with one more crossing (and, for N odd, to move the £xed point), until one reaches Ψ_{π_0}, which is given by (3). They are simply the components of equation (1) for which $\pi(i) \neq i+1$. Equation (3) is also required by the compatibility of the set of equations (1) and the coprimality of the Ψ_π, see [DFZJ].

The purpose of this paper is to give a geometric interpretation of these polynomials, extending and proving the observations of [dGN] (at $z_i \equiv 0$) and of [DFZJ]. Having one allows us to obtain a positivity property of the $\{\Psi_\pi\}$, stated below.

1.3. **Degrees and multidegrees.** Since E (and each E_π) is invariant under rescaling, it is the af£ne cone over a projective scheme, and thus has a well-de£ned degree. The **degree** $\deg_W X \in \mathbb{N}$ of an af£ne cone X in a vector space W is characterized by three properties:

1. If $X = W = \{0\}$, then $\deg_W X = 1$.
2. If the scheme X has top-dimensional components X_i, where $m_i > 0$ denotes the multiplicity of X_i in X, then $\deg_W X = \sum_i m_i \deg_W X_i$. This lets one reduce from the case of schemes to the case of varieties (reduced irreducible schemes).
3. If X is a variety, and H is a hyperplane in W, then $\deg_W X = \deg_H(X \cap H)$. (Note that $X \cap H$ may be neither reduced nor irreducible.)

From these it is easy to see that the degree is a nonnegative integer (and only zero if X is empty); more work is necessary to know that it is well-de£ned, but of course this is standard.

Theorem. *For each link pattern π the probability of the state π in the de Gier–Nienhuis Markov process is proportional to the degree of the component E_π. More precisely,* $\deg E_\pi = d_\pi$.

This was conjectured in [dGN] for those π satisfying $i \le n \implies \pi(i) > n$ (see section 5 for the meaning of this condition). Most elements of a proof in that special case were given in [DFZJ], by going beyond degrees to the more re£ned *multidegrees* of the components $\{E_\pi\}$ of E (our reference for multidegrees is [MS]). These are de£ned using an additional torus action on E, the conjugation action by invertible diagonal matrices, with respect to either $\bullet$ or ordinary multiplication (the action is the same). Let T denote the $(N+1)$-dimensional torus

$$T := \text{the rescaling } \mathbb{C}^\times \times \text{ the invertible diagonal matrices in } M_N(\mathbb{C}).$$

If we denote the usual basis for T's weight lattice by $(a, z_1, \ldots, z_N)$, then the weights of the T-action on $M_N(\mathbb{C})$ are $\{a + z_i - z_j\}$, $i, j = 1 \ldots N$.

When T acts on a vector space W preserving a subscheme X, one has an associated homogeneous **multidegree** $\mathrm{mdeg}_W X \in \mathbb{Z}[a, z_1, \ldots, z_N]$. It is also known as the equivariant cohomology class, and the equivariant Chow class [Br].[2] The multidegree is characterized by similar axioms to the degree, except that the third one is split into two cases:

3′. Assume X is a variety, and H is a T-invariant hyperplane in W.
   (a) If $X \not\subset H$, then $\mathrm{mdeg}_W X = \mathrm{mdeg}_H(X \cap H)$.
   (b) If $X \subset H$, then $\mathrm{mdeg}_W X = (\mathrm{mdeg}_H X) \cdot (\text{the weight of } T \text{ on } W/H)$.

From these it is easy to see that the multidegree is a positive sum of monomials in the weights of T on W, and is homogeneous of degree $\mathrm{codim}_W X$. Also, for our action of T on $M_N(\mathbb{C})$,

$$\deg_{M_N(\mathbb{C})} X = (\mathrm{mdeg}_{M_N(\mathbb{C})} X)|_{z_i \equiv 0, a=1}.$$

We will occasionally use a more general version of (3′b), in which X and Y are T-invariant varieties in W whose intersection has the expected dimension. Then $\mathrm{mdeg}_W(X \cap Y) = (\mathrm{mdeg}_W X)(\mathrm{mdeg}_W Y)$. The most commonly used case is when Y is a T-invariant hypersurface, and $\mathrm{mdeg}\, Y$ is the weight of its de£ning equation.

[2] It is closely related to the equivariant multiplicity [Ro], which is best thought of homologically rather than cohomologically. The equivariant multiplicity was introduced in [Jo] to study *orbital varieties*, the components of the intersection of a nilpotent orbit with the upper triangular matrices. In this paper we degenerate the nilpotent orbit $\{M^2 = 0\}$ to get components, rather than intersect with anything.

In this paper our interest is in the multidegrees of the $\{E_\pi\}$, which all live in the subspace $M_N(\mathbb{C})_{\Delta=0} \leq M_N(\mathbb{C})$ of matrices with zero diagonal. Hereafter we will drop the subscript on mdeg, and assume it to be $M_N(\mathbb{C})_{\Delta=0}$, with weights $\{a + z_i - z_j\}, i, j = 1 \ldots N, i \neq j$.

We can now state our main result (theorem 6):

Theorem. *For all link patterns* π*, the multidegree of* E_π *is the homogenized component* Ψ_π*:*

$$\mathrm{mdeg}\, E_\pi|_{a=1} = \Psi_\pi$$

The previous theorem is recovered by setting $z_i \equiv 0$.

We can give a geometric meaning to theorem 5 in [DFZJ] (generalized beyond N even):

Corollary 1. *The multidegree of* E *is the Pfaffian*

$$\mathrm{mdeg}\, E = \mathrm{Pf}\left(\frac{z_i - z_j}{a - (z_i - z_j)^2}\right)_{1 \leq i,j \leq N} \times \prod_{1 \leq i < j \leq N} \frac{a - (z_i - z_j)^2}{z_i - z_j}$$

where $\mathrm{Pf}\, A = \frac{1}{2^n n!} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^n A_{\sigma(2i-1)\,\sigma(2i)}$.

Its ordinary degree is the determinant $\det\left[\binom{2i+2j+1}{2i}\right]_{0 \leq i,j \leq n-1} = 1, 7, 307, \ldots$ *for* N *even,* $\det\left[\binom{2i+2j+3}{2i+1}\right]_{0 \leq i,j \leq n-1} = 3, 55, 6153, \ldots$ *for* N *odd.*

Since these multidegrees are very difficult to calculate directly, the reader may wonder what insight has been gained about the $\{\Psi_\pi\}$.

Theorem. *Each coefficient* Ψ_π *can be written as a sum, with positive coefficients, of products of distinct factors* $\{1 + z_i - z_j\}$*, where* $i \neq j$ *run over* $1, \ldots, N$.

Proof. In general, if T acts on V with weights $\lambda_1, \ldots, \lambda_m$, the multidegree of a T-invariant subscheme $X \subseteq V$ can be written as $p(\lambda_1, \ldots, \lambda_m)$, where p is a polynomial with positive coefficients, each monomial squarefree.

In the case at hand, $X = E_\pi$ and $V = M_N(\mathbb{C})_{\Delta=0}$, hence has weights $\{a + z_i - z_j\}$ for $i \neq j$. ☐

This positivity seems difficult to prove directly from equations (3) and (4), in much the same way that the divided-difference definition of Schubert polynomials does not make it easy to see that they have positive coefficients.

The most interesting case of the de Gier–Nienhuis conjecture was the first observed: for $\pi(i) = N + 1 - i$, N even, the coefficient d_π is the degree of the *commuting variety* of $n \times n$ matrices. The recursion relations provide a formula, albeit rather involved, for this quantity (theorem 8):

Theorem. *The degree of the commuting variety* $C = \{(X, Y) \in M_n(\mathbb{C}) : XY = YX\}$ *is*

$$\deg C = \left(\theta_1\ \theta_2\theta_1\ \cdots\ \theta_i\theta_{i-1}\cdots\theta_2\theta_1\ \cdots\ \theta_{n-1}\cdots\theta_2\theta_1 \prod_{i=1}^{n}(1+z_i)^{i-1}(1-z_i)^{n-i}\right)\Big|_{z_i \equiv 0}$$

where $\theta_i = -2\partial_i - \tau_i$*. (For computational purposes, note that one can and should immediately specialize* z_i *to* 0 *after the last application of* θ_{i-1}*, for each* $i = 1 \ldots n$*.)*

In this way the connection is useful in the other direction. In [dGN] the authors used their Markov process to compute the degree of the commuting variety (or at that point, a number conjecturally equal) up through 8×8 matrices.

Alternatively, one can use the formula above, or rather, a slight simplication of it via a change of variable proposed in [DFZJ, section 6.2], as an ef£cient algorithm for the computation of these numbers, and allowing us to go further. Here are the degrees through 11×11 matrices:

$$\deg C = 1,\ 3,\ 31,\ 1145,\ 154881,\ 77899563,\ 147226330175,\ 1053765855157617,$$
$$28736455088578690945,\ 3000127124463666294963283,\ 1203831304687539089648950490463, \ldots$$

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Throughout the paper, we use the notation

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}$$

where P is a property that may be true or false. For example, $\delta_{ab} = [a = b]$. We use e^{ij} to indicate the matrix with 1 in entry (i, j) and 0 elsewhere, so

$$(e^{ij})_{kl} = [i = k \text{ and } j = l].$$

2. Models of $(M_N(\mathbb{C}), \bullet)$

Let $M_\leq$ denote the upper triangle of a matrix M, so $(M_\leq)_{ij} = \begin{cases} M_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j. \end{cases}$ We will later use $M_>, M_<$ for the strict lower and upper triangles.

2.1. **The semidirect product model.** We can study the multiplication $(M_N(\mathbb{C}), \bullet)$ in terms of ordinary matrix multiplication, at the expense of making the cyclic invariance less obvious.

Let $R_N(\mathbb{C})$ denote the algebra of upper triangular matrices (on which the usual product and the $\bullet$ product coincide) and $M_N(\mathbb{C})/R_N(\mathbb{C})$ the evident quotient bimodule for $R_N(\mathbb{C})$. Then the semidirect product $R_N(\mathbb{C}) \times M_N(\mathbb{C})/R_N(\mathbb{C})$ carries the multiplication

$$(R, L) \cdot (V, M) := (RV, RM + LV).$$

Our algebra $(M_N(\mathbb{C}), \bullet)$ is isomorphic to this semidirect product, via the map

$$\begin{array}{rcl} (M_N(\mathbb{C}), \bullet) & \to & R_N(\mathbb{C}) \times (M_N(\mathbb{C})/R_N(\mathbb{C})) \\ M & \mapsto & (M_\leq, M + R_N(\mathbb{C})). \end{array}$$

An element (R, L) of this semidirect product is invertible (with inverse $(R^{-1}, -R^{-1}LR^{-1})$) if and only if R is an invertible upper triangular matrix, which of course is equivalent to having all its diagonal elements be nonzero. There is no condition on L.

The group of units $(M_N(\mathbb{C}), \bullet)^\times$ in this algebra is therefore also a semidirect product

$$(M_N(\mathbb{C}), \bullet)^\times \cong B \ltimes M_N(\mathbb{C})/R_N(\mathbb{C}),$$

where B denotes the group of invertible upper triangular matrices, and $M_N(\mathbb{C})/R_N(\mathbb{C})$ the vector space considered as an abelian group. Hence $(M_N(\mathbb{C}), \bullet)^\times$ is solvable, with the diagonal matrices serving as a maximal torus, and the group U as the unipotent radical.

In these $R_N(\mathbb{C}) \times M_N(\mathbb{C})/R_N(\mathbb{C})$ coordinates, the scheme E and the action take the form

$$\begin{aligned} E &\cong \left\{(R, L) : R^2 = 0, \operatorname{diag}(R) = 0, RL + LR \in R_N(\mathbb{C})\right\} \\ \operatorname{Ad}(A, 0) \cdot (R, L) &= (AUA^{-1}, ALA^{-1}) \\ \operatorname{Ad}(1, M) \cdot (R, L) &= (R, L + [M, R]) \end{aligned}$$

where we remember to always interpret the second entry as being in the quotient space $M_N(\mathbb{C})/R_N(\mathbb{C})$.

2.2. **The periodic strip model.** Let $R_\mathbb{Z}(\mathbb{C})$ denote the space of upper triangular matrices M, where the indices in M_{ij} run over $\mathbb{Z}$. Despite the in£nitude, each sum

$$(AB)_{ik} = \sum_j A_{ij} B_{jk}, \quad i \le j \le k$$

de£ning the product is £nite.

Let $S \in R_\mathbb{Z}(\mathbb{C})$ denote the shift matrix $S_{ij} = \delta_{i,j-1}$. To specify an element of the quotient ring $R_\mathbb{Z}(\mathbb{C})/\langle S^N \rangle$, one can use the matrix entries L_{ij}, $0 \le j - i < N$, as those with $0 > j - i$ are zero by triangularity and with $j - i \ge N$ are rendered ambiguous by the quotient. These ring elements can be pictured as in£nite diagonal strips of width N, on and above the main diagonal.

Proposition 1. *There is an injective ring homomomorphism*

$$\Phi : (M_N(\mathbb{C}), \bullet) \to R_\mathbb{Z}(\mathbb{C})/\langle S^N \rangle$$

given by $\Phi(M)_{ij} = M_{i \bmod N, j \bmod N}$ *for* $0 \le j - i < N$. *The image is the space of periodic strips, i.e.* $\Phi(M)_{ij} = \Phi(M)_{i+N,j+N}$ $\forall 0 \le j - i < N$.

Proof. The only claim worth commenting on is the ring homomorphism. Let i, k satisfy $0 \le k - i < N$. Since $\Phi(M)$ is periodic, we will assume $1 \le i \le N$ as well. Then there are two cases, depending on $k \le N$ or $k > N$. If $k \le N$,

$$\begin{aligned} \Phi(P \bullet Q)_{ik} = (P \bullet Q)_{ik} &= \sum_{j, i \le j \le k} P_{ij} Q_{jk} = \sum_{j, i \le j \le k} \Phi(P)_{ij} \Phi(Q)_{jk} \\ &= (\Phi(P)\Phi(Q))_{ik}. \end{aligned}$$

Whereas if $k > N$,

$$\begin{aligned} \Phi(P \bullet Q)_{ik} &= (P \bullet Q)_{i,k-N} = \sum_{j, i \le j \le N} P_{ij} Q_{j,k-N} + \sum_{j, 1 \le j \le k-N} P_{ij} Q_{j,k-N} \\ &= \sum_{j, i \le j \le N} \Phi(P)_{ij} \Phi(Q)_{jk} + \sum_{j, 1 \le j \le k-N} \Phi(P)_{i,j+N} \Phi(Q)_{j+N,k} \\ &= \sum_{j, i \le j \le k} \Phi(P)_{ij} \Phi(Q)_{jk} = (\Phi(P)\Phi(Q))_{ik}. \end{aligned}$$

□

This model has the bene£t of making the cyclic invariance obvious, and is the easiest to calculate with visually. To connect it with the semidirect product model, a pair (R, L) corresponds to the strip

$$\begin{matrix} \ddots & L & & & \\ & R & L & & \\ & & R & L & \\ & & & R & L \\ & & & & \ddots \end{matrix}$$

Such representations as periodic in£nite matrices have been considered in the context of loop algebras [KR]. This leads us naturally to the next model:

2.3. **The af£ne GL_n model.** Consider the ring of matrices $M_N(\mathbb{C}[t])$ over the polynomial ring $\mathbb{C}[t]$. Let $R_N(\mathbb{C}[t])$ denote the $\mathbb{C}[t]$-subalgebra generated by $\{e^{i,i+1}\}$ for $1 \leq i < N$, and $t\, e^{N,1}$. (These are the simple root spaces of the af£ne Lie algebra $\widehat{\mathfrak{gl}_N}$, leading to the name of this model.) Then the following is straightforward from the semidirect product model:

Proposition 2. *The map* $(M_N(\mathbb{C}), \bullet) \to R_N(\mathbb{C}[t])/(tR_N(\mathbb{C}[t]))$ *taking* M *to* $M_\leq + tM_>$ *is an isomorphism.*

We can regard $R_N(\mathbb{C}[t])$ as a family of algebra structures on $M_N(\mathbb{C})$ indexed by $t \in \mathbb{C}$, where the £ber $t = 1$ is ordinary multiplication and $t = 0$ is $\bullet$. There is an associated ¤at family whose t-£ber is the space of matrices that square to zero under the t-multiplication. We investigate this family in section 7, where we show that the ¤at limit as $t \to 0$ is supported on the top-dimensional components of E, and contains each component with multiplicity $2^{\lceil N/2 \rceil}$.

There is another, manifestly cyclically invariant, way to degenerate the algebra $(M_N(\mathbb{C}), \times)$ to the algebra $(M_N(\mathbb{C}), \bullet)$. Let $t \cdot M$ be de£ned by

$$(t \cdot M)_{ij} = t^{(j-i) \bmod N} M_{ij}, \qquad (j - i) \bmod N \in [0, N)$$

and de£ne

$$M \times_t N := t^{-1} \cdot ((t \cdot M)(t \cdot N)).$$

So $M \times_1 N = MN$, and for $t \neq 0$ this multiplication is conjugate to the ordinary one. (If we left out the "$\bmod N$" part, it would be *equal* to the ordinary one.) Then it is easy to check that

$$\lim_{t \to 0} M \times_t N = M \bullet N.$$

3. Components of E and link patterns

3.1. **Decomposition of E in terms of involutions.** Recall that we use $\underline{\pi}$ to denote the permutation matrix of a permutation π, with the diagonal zeroed out. We care especially about involutions, because of Melnikov's theorem:

Theorem 1. [M] *Let* $B := R_N(\mathbb{C})^\times$ *denote the group of* $N \times N$ *invertible upper triangular matrices. The action by conjugation of* B *on the set* $\{X \in R_N(\mathbb{C}) : X^2 = 0\}$ *has £nitely many orbits, and each contains a unique partial permutation matrix.*

A partial permutation matrix is an element of this space if and only if it is $\underline{\pi}_<$ *for some involution* $\pi \in S_n$. *Hence the orbits are naturally indexed by involutions.*

For example, the identity matrix is an involution whose strict upper triangle vanishes, and the corresponding orbit consists only of the zero matrix. Using the semidirect product model, we easily obtain

Corollary 2. *If* $M \in E$*, then there exists a* $\bullet$*-invertible* P *and a link pattern* π *such that*

$$(P \bullet M \bullet P^{\bullet-1})_{\leq} = \underline{\pi}_{<}.$$

Theorem 2. *Let* $M \in E$*. Then the ordinary square* M^2 *(not* $M \bullet M$*) has diagonal entries which come in pairs, or put another way, there exists a link pattern* π *such that* $(M^2)_{ii} = (M^2)_{\pi(i)\pi(i)}$*for each* $i = 1, \ldots, N$*.*

Conversely, every link pattern π *is necessary: there exists* $M \in E$ *such that* $(M^2)_{ii} = (M^2)_{jj}$ *if and only if* $j \in \{i, \pi(i)\}$*.*

Proof. We £rst check that these diagonal elements are invariant under conjugation. If $M' = P \bullet M \bullet P^{\bullet-1}$,

$$(M'^2)_{ii} = \sum_{l=1}^{N} M'_{il} M'_{li} = \sum_{\substack{j,k,l,p,q \\ \circlearrowleft(i \leq j \leq k \leq l) \\ \circlearrowleft(l \leq p \leq q \leq i)}} P_{ij} M_{jk} P^{\bullet-1}_{kl} P_{lp} M_{pq} P^{\bullet-1}_{qi}$$

This can be visualized with i, j, k, l, p, q, i winding only once counterclockwise round a circle. If $p = k$, then $p = q = i = j = k$, and these terms contain a factor $M_{ii}^2 = 0$. The remaining terms have $\circlearrowleft (k \leq l \leq p)$, so that one can perform the summation over l:

$$(M'^2)_{ii} = \sum_{\substack{j,k,q \\ \circlearrowleft(i \leq j < k) \\ \circlearrowleft(k < q \leq i)}} P_{ij} M_{jk} M_{kq} P^{\bullet-1}_{qi}$$

Let us consider the summation at £xed j and q. If $i \neq j$ or $q \neq i$, one £nds $\circlearrowleft (j \leq k \leq q)$, and the sum over k is equal to $(M \bullet M)_{jq}$, which is zero for $M \in E$. There remain only the contributions at $i = j$ and $q = i$, which reduce to $(M'^2)_{ii} = (M^2)_{ii}$.

Now we use corollary 2 to reduce to the case that $M_{\leq} = \underline{\pi}_{<}$ for some involution π (not necessarily a link pattern). Then one easily computes

$$(M^2)_{ii} = \begin{cases} M_{i,\pi(i)} & \text{if } i > \pi(i) \\ M_{\pi(i),i} & \text{if } i < \pi(i) \\ 0 & \text{if } i = \pi(i) \end{cases}$$

and hence $(M^2)_{ii} = (M^2)_{\pi(i)\pi(i)}$.

To see that every link pattern π arises, let t be a generic diagonal matrix, and $M = \underline{\pi} t$. Then

$$(M^2)_{ii} = ((\underline{\pi} t)^2)_{ii} = \begin{cases} t_i t_{\pi(i)} & \text{if } i \neq \pi(i) \\ \\ 0 & \text{if } i = \pi(i). \end{cases}$$

By the genericity, $t_i t_{\pi(i)} \neq t_j t_{\pi(j)}$ unless $i = j$ or $i = \pi(j)$, and $t_i t_{\pi(i)} \neq 0$. Since π is a link pattern, there is at most one 0. So the only repetitions are the expected ones. □

Theorem 3. *The scheme* E *is* $\lfloor N^2/2 \rfloor$*-dimensional, and the top components correspond to link patterns. Moreover, for each link pattern* π *the scheme*

$$E_\pi := \overline{\left\{M \in E : (M^2)_{ii} = (M^2)_{jj} \text{ if and only if } j \in \{i, \pi(i)\}\right\}}$$

is irreducible.

Proof. We will give a £nite decomposition of E into irreducible pieces $\{F_\pi\}$ corresponding to involutions, with $\dim F_\pi = \frac{1}{2}(N^2$ minus the number of £xed points of $\pi)$. The closures of the F_π of largest dimension are de£nitely components, and there may be other, smaller components (but we conjecture not).

Consider the map $\rho : E \to R_N(\mathbb{C})$ given by $(R, L) \mapsto R$, in the semidirect product model. If we let B act on E by $\bullet$-conjugation (where we identify $B \cong \{(R, 0) : R \text{ invertible}\}$), and on $R_N(\mathbb{C})$ by ordinary conjugation, then this map ρ is B-equivariant. By theorem 1, the image is a £nite union of B-orbits, with the set $\{\underline{\pi}_< : \pi \text{ an involution}\}$ serving as orbit representatives.

For π an involution, let $F_\pi := \rho^{-1}(B \cdot \underline{\pi}_<)$, so E is the £nite disjoint union of the pieces $\{F_\pi\}$. Then restricted to F_π, the map ρ is a £ber bundle (since the image is a B-orbit), and it is enough to understand one £ber. In particular,

$$\dim F_\pi = \dim(B \cdot \underline{\pi}_<) + \dim \{L : L\underline{\pi}_< + \underline{\pi}_< L \in R_N(\mathbb{C})\}.$$

The dimension of the B-orbit was computed in [M, section 3.1] (where it is called $m + s$), but we will not make direct use of the slightly intricate formula given there.

Let $\operatorname{ad} X \cdot Y := XY - YX$. Consider the map $\operatorname{ad} \underline{\pi}_<$ on $R_N(\mathbb{C})$

$$e^{ij} \mapsto e^{i\pi(j)} \, [j < \pi(j)] - e^{\pi(i)j} \, [\pi(i) < i], \qquad i < j \tag{5}$$

whose image is the tangent space to $B \cdot \underline{\pi}_<$ at $\underline{\pi}_<$. It contains the subspace $(\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})_+$ where $R_N(\mathbb{C})_+$ is the *strictly* upper triangular matrices. This subspace $(\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})_+$ has codimension m in $(\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})$, where m is the number of 2-cycles in π, as easily seen by applying $\operatorname{ad} \underline{\pi}_<$ to the diagonal matrices.

Now consider the equations $\{L : \underline{\pi}_< L + L\underline{\pi}_< \in R_N(\mathbb{C})\}$. For each $i < j$, the lower triangle entry (j, i) must vanish:

$$(\underline{\pi}_< L + L\underline{\pi}_<)_{ji} = L_{\pi(j)i} \, [j < \pi(j)] + L_{j\pi(i)} \, [\pi(i) < i] = 0, \qquad i < j$$

Let $M_{ab} = L_{ab}$, times -1 if $\pi(b) < b$. Then these restrictions on L are equivalent to

$$M_{\pi(j)i} \, [j < \pi(j)] - M_{j\pi(i)} \, [\pi(i) < i] = 0, \qquad i < j. \tag{6}$$

(The signs only matter when both terms appear, and in this case it is easy to check that only the second one is negated.)

Since (5) and (6) have the same form, the space of matrices $\{M \in M_N(\mathbb{C})/R_N(\mathbb{C})\}$ satisfying these conditions (6) is exactly the perpendicular to the space $(\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})_+$ spanned by the image of (5), where "perpendicular" is de£ned with respect to the perfect pairing $\langle R, L \rangle := \operatorname{Tr}(RL)$ between $M_N(\mathbb{C})/R_N(\mathbb{C})$ and $R_N(\mathbb{C})_+$. Hence

$$\dim \, (\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})_+ + \dim \{L : \underline{\pi}_< L + L\underline{\pi}_< \in R_N(\mathbb{C})\} = \dim R_N(\mathbb{C})_+ = \frac{1}{2}(N^2 - N).$$

With m more from $(\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})/(\operatorname{ad} \underline{\pi}_<) \cdot R_N(\mathbb{C})_+$, the dimension of F_π is $\frac{1}{2}(N^2 - N) + m$. This is only maximized when $m = n$, i.e. π is a link pattern.

Finally, since F_π is a £ber bundle over the B-orbit $B \cdot \underline{\pi}_<$ with £ber a vector space $\{L : \underline{\pi}_< L + L\underline{\pi}_< \in R_N(\mathbb{C})\}$, it is irreducible. So each $\overline{F_\pi}$, for π a link pattern, contributes only one component of top dimension to E.

By the computation at the end of theorem 2, the set $\left\{M \in E : (M^2)_{ii} = (M^2)_{jj} \text{ if and only if } j \in \{i, \pi(i)\}\right\}$ is contained in F_π. Since they have the same dimension, this subset too is irreducible, as is its closure E_π. □

A similar technique was used in [Kn, lemma 1] to determine the components of the upper-upper scheme. In that case the dimension bound lets one prove that the upper-upper scheme is a complete intersection, hence has no lower-dimensional components. We conjecture that this equidimensionality also holds for the Brauer loop scheme E (which is not a complete intersection).

3.2. **Properties of the $\{E_\pi\}$ components.** In this section we show that the components $\{E_\pi\}$ are generically reduced, we parametrize them, and £nd some (and conjecturally, all) of their de£ning equations.

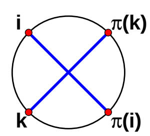
Theorem 4. *Each E_π is reduced at $\underline{\pi}t$ for t generic diagonal. Hence E_π is generically reduced.*

Proof. We do this by showing that the Zariski tangent space has the right dimension. The Zariski tangent space is the common kernel of the derivatives at $\underline{\pi}t$ of the de£ning equations for E.

The linear equations are handled by just working inside the (N^2-N)-dimensional space $M_N(\mathbb{C})_{\Delta=0}$. The derivative of $M \bullet M = 0$ is $P \mapsto P \bullet M + M \bullet P$. In the case at hand,

$$\begin{aligned}(L \bullet (\underline{\pi}t) + (\underline{\pi}t) \bullet L)_{ik} &= \sum_{j:\, \circlearrowleft(i,j,k)} (L_{ij}(\underline{\pi}t)_{jk} + (\underline{\pi}t)_{ij}L_{jk}) \\ &= L_{i\pi(k)}t_k[\circlearrowleft (i \le \pi(k) \le k)]] + t_{\pi(i)}L_{\pi(i)k}[\circlearrowleft (i \le \pi(i) \le k)].\end{aligned}$$

We require these to be zero for all i and k. Let us organize the equations as follows. If $i = k$ or $\pi(k)$ the equation is trivial. So we can assume that i and k belong to distinct orbits. Diagramatically, there are three ways for the orbits $\{i, \pi(i)\}, \{k, \pi(k)\}$ to relate:

(1) The chords $\{i, \pi(i)\}$ and $\{k, \pi(k)\}$ cross each other. In this case we can choose the labelling so that $\circlearrowleft (i < k < \pi(i) < \pi(k))$: 

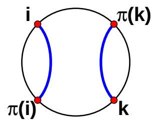
, and by inspection we £nd the following four equations:

$$\begin{aligned} t_i P_{ik} + t_k P_{\pi(i)\pi(k)} &= 0 \\ t_k P_{k\pi(i)} + t_{\pi(i)} P_{\pi(k)i} &= 0 \\ t_{\pi(i)} P_{\pi(i)\pi(k)} + t_{\pi(k)} P_{ik} &= 0 \\ t_{\pi(k)} P_{\pi(k)i} + t_i P_{k\pi(i)} &= 0 \end{aligned}$$

(all these equations are obtained from each other by rotation of 90°, which is the symmetry of the diagram). Generically, $t_i t_{\pi(i)} \neq t_k t_{\pi(k)}$ and we can in fact simplify this system to

$$P_{ik} = P_{k\pi(i)} = P_{\pi(i)\pi(k)} = P_{\pi(k)i} = 0$$

which shows that there are exactly four independent equations.

(2) The chords $\{i, \pi(i)\}$ and $\{k, \pi(k)\}$ do not cross each other, in which case we can choose $\circlearrowleft (i < \pi(i) < k < \pi(k))$: 

. We £nd again four equations, though of a different form:

$$\begin{aligned} t_i P_{ik} + t_k P_{\pi(i)\pi(k)} &= 0 \\ t_k P_{ki} + t_i P_{\pi(k)\pi(i)} &= 0 \\ t_{\pi(i)} P_{\pi(i)k} &= 0 \\ t_{\pi(k)} P_{\pi(k)i} &= 0 \end{aligned}$$

(note that they form groups of two, related by a rotation of 180° or equivalently exchange of i and k). They are generically (for non-zero t's) non-trivial and independent from each other.

(3) If one of the indices is a £xed point, one can assume that $\circlearrowleft (i < \pi(i) < k = \pi(k))$, in which case one £nds two equations:

$$P_{\pi(i)k} = P_{ki} = 0$$

(Not both i and k can be £xed, since $i \neq k$ and π is a link pattern.)

The conclusion is that each pair of chords contributes exactly 4 equations, and a chord plus a £xed point contributes 2 equations; thus, recalling that $N = 2n + r$ with $r = 0, 1$ the number of £xed points, a total of $4 \times n(n-1)/2 + 2 \times nr = 2n(n + r - 1)$ equations. Therefore the kernel is of dimension $N(N-1) - 2n(n+r-1) = 2n(n+r) + r(r-1)$. Setting $r = 0, 1$ we £nd the desired dimension $2n(n + r) = \lfloor N^2/2 \rfloor$. □

At this point we have three equivalent de£nitions of E_π:

- the closure of $\{M \in E : (M^2)_{ii} = (M^2)_{jj} \iff j \in \{i, \pi(i)\}\}$
- the closure of $\{M \in E : M_< \text{ is } B\text{-conjugate to } \underline{\pi}_<\}$
- the unique component of dimension $\lfloor N^2/2 \rfloor$ in $\{M \in E : (M^2)_{ii} = (M^2)_{\pi(i)\pi(i)}\}$.

This third de£nition is a £rst step in de£ning E_π by equations. To do better, we use yet another characterization of E_π.

Proposition 3. *Let π be a link pattern, and $\underline{\pi}$ its permutation matrix with the diagonal zeroed out. The irreducible set $U \cdot \{\underline{\pi}t, t \in T\}$ is dense in E_π.*

Proof. Since U and T are irreducible, so is $U \cdot \{\underline{\pi}t, t \in T\}$. Following the calculation at the end of theorem 2, we see $U \cdot \{\underline{\pi}t, t \in T\} \subseteq E_\pi$.

There are two steps. The £rst is to compute the dimension of a generic U-orbit $U \cdot (\underline{\pi}t)$. The second is to show that each U-orbit intersects the set of representatives $\{\underline{\pi}t\}$ in only one point, hence the dimension of $U \cdot \{\underline{\pi}t, t \in T\}$ is the dimension of $\{\underline{\pi}t, t \in T\}$ plus the dimension of a generic U-orbit.

We now compute the in£nitesimal stabilizer of U on $\underline{\pi}t$, where t is generic. The Lie algebra of U is $M_N(\mathbb{C})_{\Delta=0}$. The equation $\underline{\pi}t \bullet P = P \bullet \underline{\pi}t$ reads

$$t_{\pi(i)} P_{\pi(i)k} [\circlearrowleft (i \leq \pi(i) \leq k)] = P_{i\pi(k)} t_k [\circlearrowleft (i \leq \pi(k) \leq k)].$$

Note that these equations are exactly of the same form as those in the proof of theorem 4, up to a sign (much as went into equation (6)), and we shall not repeat the arguments that lead to the conclusion that P satisfies $2n(n+r-1)$ equations and therefore this is also the dimension of $U \cdot (\underline{\pi}t)$.

Next, assume that $P \bullet \underline{\pi}t = \underline{\pi}t' \bullet P$. For each $i = 1, \ldots, N$, the equation concerning entry $(i, \pi(i))$ reads $t_i = t'_i$. So each U-orbit contains a unique element of the form $\underline{\pi}t$.

Finally, noting that $\dim\{\underline{\pi}t, t \in T\} = 2n$, we compute $\dim U \cdot \{\underline{\pi}t, t \in T\} = 2n(n+r) = \lfloor N^2/2 \rfloor$. Since $U \cdot \{\underline{\pi}t, t \in T\} \subseteq E_\pi$ and has the same dimension, it is dense in E_π. □

Any equations satisfied by this dense open set are satisfied by all of E_π. We pay special attention to the linear equations, mostly in order to connect to proposition 1 of [DFZJ].

Proposition 4. *Assume the link pattern π has no chord connecting a pair of points between labels i and l (i.e. there are no j s.t. $\circlearrowleft (i \le j \le l), \circlearrowleft (i \le \pi(j) \le l)$). Then $M \in E_\pi$ implies $M_{il} = 0$.*

More generally, the periodic strip associated to M vanishes southwest of the (i, l) entry.

Proof. By the density, it is enough to check for $M = P \bullet (\underline{\pi}t) \bullet P^{\bullet -1}$ for some diagonal t. Write

$$M_{il} = \sum_{\circlearrowleft (i,j,k,l)} P_{ij}\, \underline{\pi}_{jk} t_k {P^{\bullet -1}}_{kl}$$

and notice that $k = \pi(j)$, $\circlearrowleft (i, j, k, l)$ contradicts the hypothesis on π. Therefore the sum is zero.

For the second conclusion, note that the hypothesis for the pair (i, l), plus $\circlearrowleft (i \le j \le k \le l)$, implies the hypothesis for the pair (j, k). □

Let $r_{ij}(M)$ denote the rank of the triangular matrix southwest of the (i, j) entry in the periodic strip model of M. In this language, the previous proposition asserted that $r_{il}(M) = 0$ for certain (i, l).

Theorem 5. *The variety E_π satisfies the following equations:*

1. *those defining E : $M \bullet M = 0$*
2. *those defining E_π : $(M^2)_{ii} = (M^2)_{\pi(i)\pi(i)}$*
3. *for any $M \in E_\pi$, and matrix entry (i, j), we have $r_{ij}(M) \le r_{ij}(\underline{\pi})$. In polynomial terms, this asserts the vanishing of all the minors of size $r_{ij}(\underline{\pi}) + 1$ in the submatrix southwest of entry (i, j) in the strip model.*

Proof. The first two are automatic. For the third, note that the action of U in the periodic strip model has a well-defined restriction to each southwest triangle, since U acts by north- and east-moving row and column operations. □

The third group of conditions appear in a similar context in [Fu], defining *matrix Schubert varieties*. They are highly interdependent, and Fulton defined the *essential set* of (i, j) whose rank conditions imply all the others. In the context at hand, the analogous set is defined as follows. Draw $\underline{\pi}$ in the strip model, and cross out all the boxes (i, j) directly north or directly east of each 1 entry in $\underline{\pi}$. The remaining set of boxes in the strip is the **diagram** of the link pattern π, and the northeast corners of each component of the diagram are the **essential set** of the diagram. Then it is easy to check that the rank conditions

$r_{ij}(M) \leq r_{ij}(\underline{\pi})$ for (i,j) not in the essential set are implied by those from the essential set. See £gure 2 for the possible diagrams in $N = 6$.

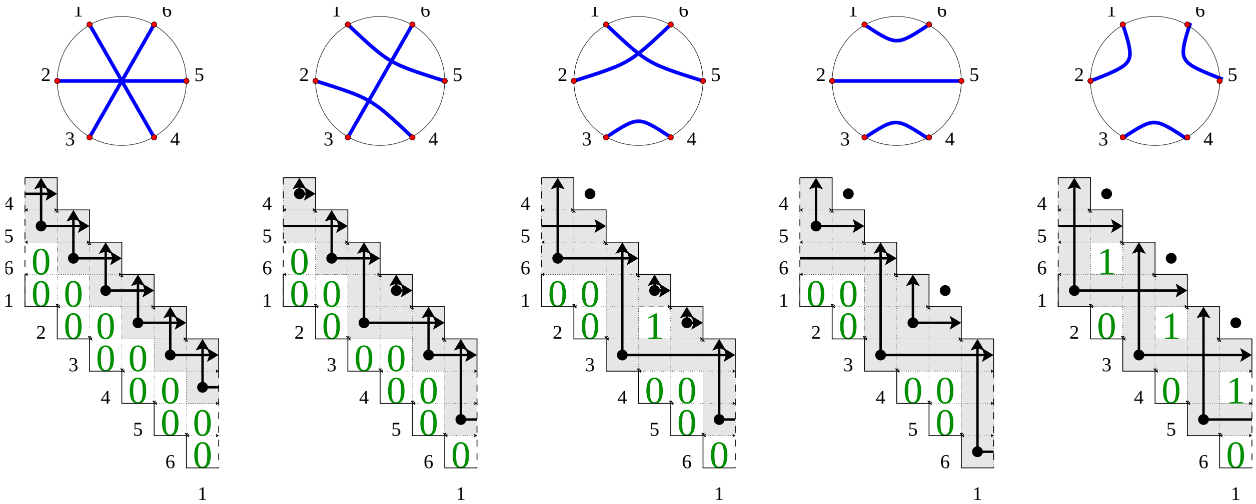


FIGURE 2. The diagrams for the link patterns of size 6, up to rotation. (We have left out the top diagonal $\{(i, i-1)\}$, whose irrelevance will be shown in lemma 2.) The rank function r_{ij} is written in the boxes of each diagram; for example $r_{ij} = 0$ implies that that matrix entry is actually 0.

Much the same sort of parametrization, and consequence for the de£ning equations, was used in [Kn] for the upper-upper scheme.

We conjecture that the equations in theorem 5 are *all* the equations de£ning the components. Because of the connection explained in section 5 between E and the upper-upper scheme, this would imply a similar conjecture in [Kn], which in turn would imply the well-known conjecture that the commuting scheme is reduced.

4. MULTIDEGREES AND THE BRAUER LOOP MODEL

Our goal in this section is the main result of the paper, the equality $(\mathrm{mdeg}\, E_\pi)|_{a=1} = \Psi_\pi$ relating E to the Brauer loop polynomials. The base case is easy:

Proposition 5. *De£ne* $\pi_0(i) = i + n \bmod 2n$ *for* $i \leq 2n$, *and* $\pi_0(N) = N$ *if N is odd. Then*

$$\mathrm{mdeg}\, E_{\pi_0} = \prod_{\substack{i=1\ldots N \\ j:\ \circlearrowleft(i<j<i+n)}} (a + z_i - z_j) \tag{7}$$

for N *even and*

$$\mathrm{mdeg}\, E_{\pi_0} = \prod_{\substack{i=1\ldots N \\ j:\ \circlearrowleft(i<j<i+n)}} (a + z_i - z_j) \left(\prod_{i=n+1}^{N} (a + z_i - z_{i+n}) \right)$$

for N *odd.*

Proof. Proposition 4 implies that $M_{il} = 0, l = i, \ldots, i+n-1 \mod N$ for N even and similarly for N odd. These linear equations imply $M \bullet M = 0$, and are of the right codimension ($2n(n-1)$ for N even, $2n^2$ for N odd) to de£ne a component of E, so they exactly de£ne E_{π_0}. The result follows from property (3'b) and property (1) for multidegrees. $\square$

4.1. **Geometry of divided difference operators.** The recursion relation (4) involves a divided difference operator ∂_i, so we take a moment to recall the geometry behind these, making no claims to originality.

Let L_i (for Levi subgroup) denote the group of invertible matrices with off-diagonal elements only in entries $(i, i+1), (i+1, i)$. Let B_i denote the subgroup in which the $(i+1, i)$ entry vanishes (so $B_i \leq B$, unless $i = N$).

Let X carry a left B_i-action, let V be a L_i-representation, and let $f : X \to V$ be B_i-equivariant. (In our case f will be an inclusion.) De£ne the *map* $-\partial_i f$ by

$$\begin{array}{rcl} -\partial_i f : L_i \times_{B_i} X & \to & V \\ (P, M) & \mapsto & P \cdot f(M) \end{array}$$

where $L_i \times_{B_i} X$ is the quotient of $L_i \times X$ by the diagonal action of B_i on the right of L_i and the left of X. (Note that the formula stated descends to this quotient.) One way to view $L_i \times_{B_i} X$ is as the associated X-bundle to the principal B_i-bundle over $L_i/B_i \cong \mathbb{P}^1$.

We now explain why this construction, based on the one of Bott and Samelson [BS], Demazure, and Hansen, is given such a suggestive name:

Lemma 1. *Let X be a variety in V invariant under B_i and rescaling, with multidegree $\mathrm{mdeg}_V X$. Then*

$$(-\partial_i f)_*(1) = -\partial_i \, \mathrm{mdeg}_V X$$

where $(-\partial_i f)_*$ denotes the pushforward map in equivariant cohomology.

If $-\partial_i f$ is generically 1:1, then

$$(-\partial_i f)_*(1) = \mathrm{mdeg}_V \, \overline{\mathrm{Im} - \partial_i f},$$

the multidegree of the closure of the image.

Proof. This proof requires more equivariant cohomology than just the multidegree technology used elsewhere in the paper. There are many references, such as [Br].

The space $L_i \times_{B_i} X$ equivariantly retracts to L_i/B_i (since it is an X-bundle, and X can be rescaled toward 0), hence is equivariantly formal. So the map

$$H^*_T(L_i \times_{B_i} X) \to H^*_T((L_i \times_{B_i} X)^T)$$

is an inclusion. This factors through

$$H^*_T(L_i \times_{B_i} X) \to H^*_T((B_i \times_{B_i} X) \; \cup \; (s_i B_i \times_{B_i} X))$$

since every T-£xed point in $L_i \times_{B_i} X$ lies over one of the two T-£xed points in L_i/B_i. Here s_i denotes the permutation matrix of the transposition $(i \leftrightarrow i+1)$.

Let $\alpha = z_i - z_{i+1}$. This is the weight on the tangent space $T_{s_i B_i/B_i}(L_i/B_i)$, and the negative of the weight on $T_{B_i/B_i}(L_i/B_i)$. Then we claim the following equality of elements of $H^*_T(L_i \times_{B_i} X) \otimes_{H^*_T} H^*_T[\alpha^{-1}]$, a localization of equivariant cohomology:

$$1 = \frac{[B_i \times_{B_i} X]}{-\alpha} + \frac{[s_i B_i \times_{B_i} X]}{\alpha}$$

This is proved by restricting to the two T-invariant £bers $B_i \times_{B_i} X$ and $s_i B_i \times_{B_i} X$. Being the preimages of the points B_i/B_i and $s_i B_i/B_i$, their normal bundles are trivial, with equivariant Euler classes $-\alpha$ and α.

When we apply $(-\partial_i f)_*$ to both sides of this equation, we get

$$(-\partial_i f)_*(1) = -\partial_i(f_*(1)) = -\partial_i \operatorname{mdeg}_V X,$$

as desired.

The second statement is standard. □

There are two differences between the usual Bott–Samelson construction and the one used here. One is that Bott–Samelson maps usually take values in a ¤ag manifold, rather than a vector space, but this is immaterial to the argument. The important one is that the input map f is traditionally equivariant under a *lower* triangular group B_-, so the tangent space to the basepoint $B_- \in G/B_-$ has weights that are *positive* roots. In our case we have invariance under the *upper* triangular B_i, leading us to the negative of the usual ∂_i.

4.2. The application to $\{E_\pi^{[i,i+1]}\}$.

Lemma 2. *Let $M \in M_N(\mathbb{C})_{\Delta=0}$, $\lambda \in \mathbb{C}$, and $M' = M + \lambda e^{i+1,i}$. Then $M \bullet M = M' \bullet M'$, and in particular, $M \in E$ if and only if $M' \in E$.*

Proof. If $e^{jk} \bullet e^{i+1,i} \neq 0$, then $k = i+1$ and $\circlearrowleft (j \leq i+1 \leq i)$, which forces $j = i+1 = k$. Similarly $e^{i+1,i} \bullet e^{jk} \neq 0$ implies $j = k$. Since M has zero diagonal there are no such terms to contribute to the square. □

This lemma 2 says that the entries on the top diagonal in the strip model are unconstrained. As such, we don't lose any information by setting them to 0, and we learn something about •-conjugation:

Lemma 3. *Let π be a link pattern. De£ne*

$$E_\pi^{[i,i+1]} := \{M \in E_\pi : M_{i+1,i} = 0\}.$$

Then B_i acts on $E_\pi^{[i,i+1]}$ by conjugation and •-conjugation, and the actions are the same. Also,

$$\operatorname{mdeg} E_\pi^{[i,i+1]} = (a + z_{i+1} - z_i) \operatorname{mdeg} E_\pi.$$

Proof. Plainly the diagonal matrices act on $E_\pi^{[i,i+1]}$ with both actions the same, so it remains to check the action of $e^{i,i+1} \in \mathrm{Lie}(B_i)$. If N is the •-commutator of $e^{i,i+1}$ and M, then

$$N_{jk} \;=\; [j=i]\, M_{i+1,k}\, [\circlearrowleft (i \leq i+1 \leq k)] - M_{ji}\, [i+1=k]\, [\circlearrowleft (j \leq i \leq i+1)]$$

whereas the ordinary commutator doesn't have the cyclic requirements. But $[\circlearrowleft (i \leq i+1 \leq k)]$ is 0 only for $i = k$, and that term can safely be added since $M_{i+1,i} = 0$ for $M \in E_\pi^{[i,i+1]}$. The other term is similar. So the •-commutator equals the ordinary commutator, hence the Lie algebra actions (and hence the Lie group actions) are the same.

Since B_i acts on E by •-conjugation, and B_i is connected, it acts on each component, such as E_π. So it remains to check that B_i preserves the subset $E_\pi^{[i,i+1]}$. For this we check the relevant matrix entry $N_{i+1,i}$ and see that is indeed zero.

To see the claim about multidegrees, let H be the space of matrices with vanishing $(i+1, i)$ entry (and vanishing diagonal). By lemma 2 and both parts of axiom (3′) of multidegrees,

$$\operatorname{mdeg} E_\pi^{[i,i+1]} = (a + z_{i+1} - z_i) \operatorname{mdeg}_H E_\pi^{[i,i+1]} = (a + z_{i+1} - z_i) \operatorname{mdeg} E_\pi.$$

□

We sequester some technicalities into a lemma whose proof we leave to the reader.

Lemma 4. *Let* $C, D \in M_N(\mathbb{C})_{\Delta=0}$. *Then for all* $j, m = 1 \ldots N$,

$$\begin{aligned} ((e^{i+1,i}C) \bullet D)_{jm} &= (e^{i+1,i}(C \bullet D))_{jm} + \delta_{i,m}\delta_{j,i+1}(CD)_{ii} \\ (C \bullet (De^{i+1,i}))_{jm} &= ((C \bullet D)e^{i+1,i})_{jm} + \delta_{i,m}\delta_{j,i+1}(CD)_{i+1,i+1} \\ ((Ce^{i+1,i}) \bullet D)_{jm} &= C_{j,i+1}D_{im}\,[\circlearrowleft (j \le i \le m)] \\ (C \bullet (e^{i+1,i}D))_{jm} &= C_{j,i+1}D_{im}\,[\circlearrowleft (j \le i+1 \le m)] \end{aligned}$$

If in addition $C_{i,i+1} = 0$, *then*

$$([e^{i+1,i}, C])^{\bullet 2}_{jm} = -\delta_{j,i+1}\delta_{i,m}(C^2)_{i,i+1}.$$

We have reached the heart of the paper:

Proposition 6. *If the link pattern* π *has no chord between* i *and* $i+1$, *then*

$$\text{mdeg}\, E_\pi + \text{mdeg}\, E_{f_i \cdot \pi} = -\frac{2a + z_{i+1} - z_i}{a + z_{i+1} - z_i}\, \partial_i\, \text{mdeg}\, E_\pi^{[i,i+1]}. \tag{8}$$

Proof. The outline is as follows. We will apply lemma 1 to the inclusion $f : E_\pi^{[i,i+1]} \to M_N(\mathbb{C})$, where L_i acts on $M_N(\mathbb{C})$ by ordinary conjugation. The image of $-\partial_i f$ is "almost" inside E; by imposing one new equation (with weight $2a + z_{i+1} - z_i$) the result Z is inside E. (There is an annoying technicality that we actually work not with $\overline{\text{Im} - \partial_i f}$ but the larger and equally good $Y := \overline{\text{Im} - \partial_i f} + \mathbb{C}e^{i+1,i}$.) Then we determine which components of E are contained in Z: they are the E_π and $E_{f_i \cdot \pi}$ components.

We begin with the map $-\partial_i f$. A generic element of L_i can be written as $(1 + \lambda e^{i+1,i})b$ for b an element of B_i. We know by lemma 3 that B_i acting by conjugation preserves $E_\pi^{[i,i+1]}$. To determine the closure of the image of $-\partial_i f$, it is therefore enough to look at the image of the dense open set

$$Q := \{(1 + \lambda e^{i+1,i}, M) : \lambda \in \mathbb{C}, M \in E_\pi^{[i,i+1]}\}.$$

Let $N = M + \lambda(e^{i+1,i}M - Me^{i+1,i})$ be in the image of Q. In particular,

$$N_{i+1,i} = M_{i+1,i} - \lambda(M_{i+1,i+1} - M_{ii}) = 0. \tag{9}$$

Then writing $N^{\bullet 2}$ for $N \bullet N$, we have

$$\begin{aligned} N^{\bullet 2} &= (M + \lambda[e^{i+1,i}, M])^{\bullet 2} \\ &= M \bullet M + \lambda([e^{i+1,i}, M] \bullet M + M \bullet [e^{i+1,i}, M]) + \lambda^2([e^{i+1,i}, M])^{\bullet 2} \end{aligned}$$

Using lemma 4, and $M \bullet M = 0$, we get

$$\begin{aligned} (N^{\bullet 2})_{jm} &= \lambda((e^{i+1,i}M) \bullet M - M \bullet (Me^{i+1,i}) + \lambda([e^{i+1,i}, M])^{\bullet 2} + M \bullet (e^{i+1,i}M) - (Me^{i+1,i}) \bullet M)_{jm} \\ &= \lambda(\delta_{i,m}\delta_{j,i+1}((M^2)_{ii} - (M^2)_{i+1,i+1} - \lambda(M^2)_{i,i+1}) \\ &\qquad + ([j \le i+1 \le m] - [j \le i \le m])\, M_{j,i+1}M_{i,m}) \\ &= \lambda(\delta_{i,m}\delta_{j,i+1}((M^2)_{ii} - (M^2)_{i+1,i+1} - \lambda(M^2)_{i,i+1}) + M_{j,i+1}M_{i,m}(\delta_{j,i+1} - \delta_{m,i})) \\ &= \lambda\delta_{i,m}\delta_{j,i+1}((M^2)_{ii} - (M^2)_{i+1,i+1} - \lambda(M^2)_{i,i+1}) \end{aligned}$$

where we used $M \in M_N(\mathbb{C})_{\Delta=0}$ to get from the third to the fourth line. So $N^{\bullet 2}$ is zero away from $(N^{\bullet 2})_{i+1,i}$.

In particular, $N \in E$ if and only if $\lambda = 0$ or $\lambda(M^2)_{i,i+1} = (M^2)_{ii} - (M^2)_{i+1,i+1}$. In the latter case,

$$\begin{aligned}
(N^2)_{ii} &= (\exp(\lambda e^{i+1,i})M\exp(-\lambda e^{i+1,i}))^2_{ii} \\
&= (\exp(\lambda e^{i+1,i})M^2\exp(-\lambda e^{i+1,i}))_{ii} \\
&= (M^2 + \lambda[e^{i+1,i}, M^2])_{ii} \\
&= (M^2)_{ii} + \lambda(e^{i+1,i}M^2 - M^2 e^{i+1,i})_{ii} \\
&= (M^2)_{ii} - \lambda(M^2)_{i,i+1} \\
&= (M^2)_{i+1,i+1}
\end{aligned}$$

Similarly $(N^2)_{i+1,i+1} = (M^2)_{ii}$, whereas $(N^2)_{jj} = (M^2)_{jj}$ for all other j. Hence the only top components of E that can appear in the image of $-\partial_i f$ are E_π and $E_{f_i \cdot \pi}$.

We use this same calculation to prove that $-\partial_i f$ is generically 1:1, as it is enough to prove it on Q. Assume that 1:1ness is violated:

$$\begin{aligned}
& -\partial_i f(1 + \mu e^{i+1,i}, M) = -\partial_i f(1 + \nu e^{i+1,i}, N) \\
\iff\ & \exp(\mu e^{i+1,i})M\exp(-\mu e^{i+1,i}) = \exp(\nu e^{i+1,i})N\exp(-\nu e^{i+1,i}) \\
\iff\ & \exp((\mu-\nu)e^{i+1,i})M\exp((\nu-\mu)e^{i+1,i}) = N \\
\implies\ & \exp((\mu-\nu)e^{i+1,i})M\exp((\nu-\mu)e^{i+1,i}) \in E_\pi
\end{aligned}$$

As we saw above, there are only two possibilities for $\mu - \nu$ for which this left hand side is even in E, much less E_π. If M is in the open set we used in theorem 3 to de£ne E_π, this $\exp((\mu-\nu)e^{i+1,i})M\exp((\nu-\mu)e^{i+1,i})$ is not in E_π unless $\mu - \nu = 0, M = N$.

We learn two things from $-\partial_i f$ being generically 1:1 :

- $\dim \overline{\mathrm{Im}\ -\partial_i f} = \dim(L_i \times_{B_i} E_\pi^{[i,i+1]})$, which in turn equals $1 + \dim E_\pi^{[i,i+1]} = \dim E_\pi = \dim E$. (In fact this only used $-\partial_i f$ being £nite:1.)
- by lemma 1, $\mathrm{mdeg}\, \overline{\mathrm{Im}\ -\partial_i f} = -\partial_i \mathrm{mdeg}\, E_\pi^{[i,i+1]}$.

Let $Y := \overline{\mathrm{Im}\ -\partial_i f} + \mathbb{C}e^{i+1,i}$. We note four properties of Y, the £rst two from equation (9):

- $\dim Y = \dim \overline{\mathrm{Im}\ -\partial_i f} + 1 = \dim E + 1$.
- $(a + z_{i+1} - z_i)\,\mathrm{mdeg}\, Y = \mathrm{mdeg}\, \overline{\mathrm{Im}\ -\partial_i f} = -\partial_i\, \mathrm{mdeg}\, E_\pi^{[i,i+1]}$ $= -\partial_i (a + z_{i+1} - z_i)\,\mathrm{mdeg}\, E_\pi$.
- By lemma 2 and calculation of $N^{\bullet 2}$, $M \in Y$ implies $(M \bullet M)_{jm} = 0$ unless $(j, m) = (i+1, i)$.
- Y is irreducible, since E_π hence $E_\pi^{[i,i+1]}$ hence $L_i \times_{B_i} E_\pi^{[i,i+1]}$ hence $\overline{\mathrm{Im}\ -\partial_i f}$ were.

Let Z denote the intersection of Y with the hypersurface $\{M : (M \bullet M)_{i+1,i} = 0\}$, so $\dim Z \geq \dim Y - 1$. Since Y satis£ed all but this one of E's de£ning equations, $Z \subseteq E$, so $\dim Z \leq \dim E = \dim Y - 1$. Hence $\dim Z = \dim E$, and

$$\mathrm{mdeg}\, Z = (2a + z_{i+1} - z_i)\mathrm{mdeg}\, Y$$

where $2a + z_{i+1} - z_i$ is the T-weight of the equation of that hypersurface. Note that this is the right-hand side of the equation we seek.

Since $Z \subseteq E$ and $\dim Z = \dim E$, the top-dimensional components of Z are a selection of the top-dimensional components of E. Since E is generically reduced on its top-dimensional components, Z is too.

Plainly $E_\pi \subseteq Z$, since $-\partial_i f$ restricted to $B_i \times_{B_i} E_\pi^{[i,i+1]}$ already has image $E_\pi^{[i,i+1]}$. We've already shown (by looking at the diagonal elements of the square) that the only other component of E that could appear in Z is $E_{f_i \cdot \pi}$.

At this point we have two possibilities for the top components of Z: just E_π, or $E_\pi \cup E_{f_i\cdot\pi}$. Assume (for contradiction) the £rst. Then we have $\operatorname{mdeg} Z = \operatorname{mdeg} E_\pi$, so

$$\begin{aligned} \operatorname{mdeg} E_\pi^{[i,i+1]} &= (a + z_{i+1} - z_i)\operatorname{mdeg} E_\pi = (a + z_{i+1} - z_i)\operatorname{mdeg} Z \\ &= (2a + z_{i+1} - z_i)(a + z_{i+1} - z_i)\operatorname{mdeg} Y = -(2a + z_{i+1} - z_i)\partial_i \operatorname{mdeg} E_\pi^{[i,i+1]} \end{aligned}$$

Apply ∂_i to both sides:

$$\partial_i \operatorname{mdeg} E_\pi^{[i,i+1]} = 2\partial_i \operatorname{mdeg} E_\pi^{[i,i+1]}$$

so $\operatorname{mdeg} E_\pi^{[i,i+1]} = 0$, which is false by the positivity of multidegrees.

Hence Z has two top components, E_π and $E_{f_i\cdot\pi}$, both generically reduced, so

$$\operatorname{mdeg} E_\pi + \operatorname{mdeg} E_{f_i\cdot\pi} = \operatorname{mdeg} Z = -\frac{2a + z_{i+1} - z_i}{a + z_{i+1} - z_i}\,\partial_i \operatorname{mdeg} E_\pi^{[i,i+1]}.$$

□

Theorem 6. *The multidegree of E_π is the homogenized component Ψ_π for all link patterns π:*

$$\operatorname{mdeg} E_\pi|_{a=1} = \Psi_\pi \tag{10}$$

Proof. Setting $a = 1$ in equations (7) and (8), one recovers the equations (3) and (4). As explained in [DFZJ] these characterize the Ψ_π uniquely, hence the equality (10). □

This has a corollary, for which it would be interesting to have a geometric proof along the lines of proposition 6.

Corollary 3. *If $\pi(i) = i + 1$, then*

$$-\partial_i\,(a + z_i - z_{i+1})\operatorname{mdeg} E_\pi^{[i,i+1]} = -2a\sum_\rho \partial_i \operatorname{mdeg} E_\rho^{[i,i+1]}$$

where the sum is taken over those $\rho \neq \pi$ such that $e_i \cdot \rho = \pi$ and the $i, i+1$ strands cross.

Proof. Let $u = z_i - z_{i+1}$. The π-component of the equations (1)

$$a(u)\Psi_\pi + b(u)\Psi_{f_i\cdot\pi} + c(u)\sum_{\rho:\ e_i\cdot\rho=\pi}\Psi_\rho = \tau_i\Psi_\pi$$

looks different depending on whether $\pi(i) = i + 1$ or $\pi(i) \neq i + 1$. The $\pi(i) \neq i + 1$ equations are simpler because the summation term vanishes, and can be rewritten as the equations (4).

If however $\pi(i) = i + 1$, we rewrite as

$$\big(a(u) + b(u) + c(u)\big)\Psi_\pi + c(u)\sum_{\rho:\ e_i\cdot\rho=\pi,\rho\neq\pi}\Psi_\rho = \tau_i\Psi_\pi.$$

Substituting in the formula (2) for $\check{R}$, this becomes

$$\Psi_\pi + \frac{2u}{(2-u)(1+u)}\sum_{\rho:\ e_i\cdot\rho=\pi,\rho\neq\pi}\Psi_\rho = \tau_i\Psi_\pi$$

Hence

$$-\partial_i \Psi_\pi = \frac{1}{u}(\tau_i \Psi_\pi - \Psi_\pi) = \frac{2}{(2-u)(1+u)} \sum_{\rho:\ e_i\cdot\rho=\pi, \rho\neq\pi} \Psi_\rho.$$

The ρ-terms in the summation can be grouped into pairs $\{\rho, f_i \cdot \rho\}$. Since $\rho \neq \pi$, no $\rho = f_i \cdot \rho$, and $\rho(i) \neq i+1$. We can pick a preferred element of each pair by asking that the $(i, \rho(i))$ chord cross the $(i+1, \rho(i+1))$. Then the equation becomes

$$\begin{aligned} -\partial_i \Psi_\pi &= \frac{2}{(2-u)(1+u)} \sum_\rho (\Psi_\rho + \Psi_{f_i\cdot\rho}) \\ &= \frac{2}{(2-u)(1+u)} \sum_\rho -\frac{2-u}{1-u} \partial_i (1-u) \Psi_\rho \\ &= \frac{-2}{(1+u)(1-u)} \sum_\rho \partial_i (1-u) \Psi_\rho \end{aligned}$$

where the summation is over $\rho \neq \pi$, $e_i \cdot \rho = \pi$, and the $i, i+1$ chords of ρ cross. So

$$-\partial_i (1+u)(1-u)\Psi_\pi = -(1+u)(1-u)\partial_i \Psi_\pi = -2 \sum_\rho \partial_i (1-u) \Psi_\rho$$

Using theorem 6 we obtain the desired formula. □

There are many equivalent formulations, but the left side of the equation in this corollary has a geometric interpretation that seems the most suggestive: slice $E_\pi^{[i,i+1]}$ with the equation $M_{i,i+1} = 0$ (which was automatic in the other case, $\pi(i) \neq i+1$), then apply the Bott–Samelson construction, then impose $(M^2)_{i+1,i} = 0$ on the image of the Bott–Samelson map.

In the proof of proposition 6, we came close to giving a birational action of $\hat{S}_N$ on E. We intend to study this further in a future paper.

5. THE PERMUTATION SECTOR AND THE UPPER-UPPER SCHEME

In this section we work again in the (R, L) coordinate system on E.

De£ne the **permutation subspace** $M_N(\mathbb{C})_P$ to be the subspace of $\{(R, L)\}$ in which the upper triangular matrix R is supported in the northeast rectangle:

$$R_{ij} = 0 \quad \text{unless} \quad i \leq n,\ j \geq n+1$$

(recall that $N = 2n + r$, $r = 0, 1$). It is easy to check that $M_N(\mathbb{C})_P$ is invariant under •-conjugation by U.

Let X denote that northeast rectangle (or square, if N even), so X is an $n \times (n+r)$ matrix with

$$X_{ij} = R_{i,j+n}.$$

Similarly, let Y denote the transposed rectangle in L, so Y is an $(n+r) \times n$ matrix with

$$Y_{ij} = L_{i+n,j}.$$

De£ne the **permutation sector** $E_P \subseteq E$ to be the intersection $E \cap M_N(\mathbb{C})_P$. Then (as in lemma 2) the conditions on R and L are in fact only conditions on X and Y:

$$\forall (R, L) \in M_N(\mathbb{C})_P, \qquad (R, L) \in E_P \iff XY, YX \text{ are upper triangular square matrices.}$$

(Note that if N is odd, then YX is one size larger than XY.) In the case N even, this "upper-upper scheme" E_P was introduced in [Kn], and most of the next theorem proven. The case N odd was considered in [dGN].

Note that since E_P lives inside the linear subspace $M_N(\mathbb{C})_p$, its multidegree and that of its components are divisible by

$$\operatorname{mdeg} M_N(\mathbb{C})_p = \prod_{1\le i\le j\le n} (a + z_i - z_j) \prod_{n+1\le i\le j\le N} (a + z_i - z_j).$$

With these factors divided out, we recover the multidegrees relative to $M_N(\mathbb{C})_p$.

Let P denote the set of link patterns π such that $\forall i = 1, \dots, n$, $\pi(i) > n$. For $i > n$, this forces $\pi(i) \le n$ or (N odd) $\pi(i) = i$. For N even (the case considered in [DFZJ]), such π correspond in an obvious way to permutations of $\{1, \dots, n\}$.

Theorem 7. *The permutation sector* E_P *is a complete intersection, hence has multidegree*

$$\operatorname{mdeg} E_P = \prod_{1\le i\le j\le n} (a + z_i - z_j)(2a + z_j - z_i) \prod_{n+1\le i\le j\le N} (a + z_i - z_j)(2a + z_j - z_i)$$

as a subscheme of $M_N(\mathbb{C})_{\Delta=0}$.

Moreover, $E_P = \cup_{\pi\in P} E_\pi$*, and in particular is reduced.*

Proof. A complete intersection, by de£nition, is a scheme C whose codimension equals the number of de£ning equations. It is enough to check that $\operatorname{codim} C$ is at least this number of equations, as the inequality then implies the equality. There are

$$\left(\binom{\lfloor N/2\rfloor}{2} + \binom{\lceil N/2\rceil}{2}\right) + \binom{\lfloor N/2\rfloor}{2} + \binom{\lceil N/2\rceil}{2}$$

equations, for the vanishing of the two parts of R outside X, the strict lower triangle of XY, and the strict lower triangle of YX.

Since E_P is a subscheme of E, its codimension is at least that of E, namely $(N^2 - N) - \lfloor N^2/2\rfloor = \lceil N^2/2\rceil - N$. (We are computing codimension relative to the (N^2-N)-dimensional vector space $M_N(\mathbb{C})_{\Delta=0}$, and using theorem 4.) If $N = 2n$, then the number of de£ning equations and codimension are $4\binom{n}{2} \le 2n^2 - 2n$. If $N = 2n + 1$, then these two numbers are $2\binom{n}{2} + 2\binom{n+1}{2} \le 2n^2 + 2n + 1 - (2n + 1)$. In either case we get the desired inequality (with, of course, equality).

The multidegree of a complete intersection of T-invariant hypersurfaces is the product of the weights of the de£ning equations. This gives the stated multidegree for E_P.

We make use of two properties of complete intersections: they are equidimensional, and more speci£cally Cohen–Macaulay. Since E_P is equidimensional, its support is a union of components of E. Since E is generically reduced, so is E_P. Since E_P is Cohen–Macaulay and generically reduced, it is reduced (this was already proven in [Kn] in the case N even, via the same argument). So scheme-theoretically it is the union of some components of E.

Finally it remains to determine which components of E lie in E_P. By the proof of theorem 2, we know that for generic diagonal t, we have $\underline{\pi}t \in E_\pi$, $\underline{\pi}t \notin E_\rho$ for $\rho \ne \pi$. Since E and $M_N(\mathbb{C})_p$ are U-invariant, so is E_P, hence $U \cdot \{\underline{\pi}t\} \subseteq E$ for $\pi \in P$. Hence by proposition 3, $E_\pi \subseteq E_P$ if and only if $\underline{\pi}t \in E_P$. The vanishing conditions on E_P are then equivalent to $\pi \in P$. □

By the additivity of multidegrees, and theorem 6, we have the

Corollary 4. [DFZJ, for N even]

$$\sum_{\pi\in P} \Psi_\pi = \prod_{1\le i\le j\le n} (1+z_i-z_j)(2+z_j-z_i) \prod_{n+1\le i\le j\le N} (1+z_i-z_j)(2+z_j-z_i)$$

The N unrestricted, $z_i \equiv 0$ case was conjectured in [dGN].

Finally, we prove the original observation of [dGN], i.e. that the component Ψ_π for $\pi(i) = 2n+1-i$ provides the degree of the commuting scheme $C_n = \{(X,Y) \in M_n(\mathbb{C}) : XY = YX\}$. We strengthen this to a computation of the multidegree (for a new torus, as not all of T acts on C_n).

Theorem 8. *Let S be the product of $\mathbb{C}^\times$ and the diagonal matrices in $M_n(\mathbb{C})$. Then S acts on C_n by*

$$(\alpha, D)\cdot(X,Y) := (\alpha D X D^{-1}, \alpha D Y D^{-1}).$$

Let $N = 2n$. Let $\Phi_n = \operatorname{mdeg} M_N(\mathbb{C})_P = \prod_{1\le i\le j\le n}(a+z_i-z_j)\prod_{n+1\le i\le j\le N}(a+z_i-z_j)$. De£ne π_n by $\pi_n(i) := N+1-i$.

Then the S-multidegree of the commuting scheme inside $M_n(\mathbb{C})\times M_n(\mathbb{C})$ satis£es

$$S\text{−}\operatorname{mdeg}_{M_N(\mathbb{C})\times M_N(\mathbb{C})} C_n = (\operatorname{mdeg} E_{\pi_n}/\Phi_n)|_{z_i\equiv z_{n+i}},$$

where we denote the standard basis for S's weight lattice by $(a, z_1, \ldots, z_n)$.

Setting $a = 1, z_i \equiv 0$ on both sides, we get $\deg C_n = \deg E_{\pi_n}$.

Let $\Delta_n := (\operatorname{mdeg} E_{\pi_n}/\Phi_n)|_{z_{n+1}=\cdots=z_{2n}=0}$, so $\deg C_n = \Delta_n|_{a=1,z_1=\cdots=z_n=0}$. This Δ_n can be calculated as

$$\begin{aligned}
\Delta_n &= a^n\,\theta_1\,\theta_2\theta_1\,\cdots\,\theta_i\theta_{i-1}\cdots\theta_2\theta_1\,\cdots\,\theta_{n-1}\cdots\theta_2\theta_1\prod_{i=1}^{n}(a+z_i)^{i-1}(a-z_i)^{n-i}\\
&= a^n\,\theta_1\theta_2\cdots\theta_{n-1}\,\theta_1\theta_2\cdots\theta_{n-2}\,\cdots\,\theta_1\theta_2\,\theta_1\prod_{i=1}^{n}(a+z_i)^{i-1}(a-z_i)^{n-i}\\
&= a(\theta_1\cdots\theta_{n-1})(a+z_n)^{n-1}\prod_{i=1}^{n-1}(a-z_i)\,\Delta_{n-1}
\end{aligned}$$

where $\theta_i = -2a\partial_i - \tau_i$.

Proof. In [Kn], the equations of the commuting scheme C are S-equivariantly degenerated to those of

$$F_+ := \{(X,Y) : XY, YX \text{ upper triangular}, \operatorname{diag}(XY) = \text{ reverse of } \operatorname{diag}(YX)\}.$$

(The details of this family are unimportant here.) While this degeneration is conjectured in [Kn] to be ¤at, this is not proven. So a priori one only knows that the actual ¤at limit F of the commuting scheme is contained inside F_+.

This upper bound F_+ is contained in the upper-upper scheme, and it is easy to check that it contains one entire component F_- (corresponding to the reversal permutation) and only lower-dimensional parts of other components. Since the upper-upper scheme is generically reduced, F_+ is generically reduced along F_-. Putting these two facts together and applying axiom (2) of multidegrees, we see that F_+ and F_- have the same S-multidegree.

Since the upper-upper scheme has the same dimension, n^2+n, as the commuting scheme, $n^2+n \geq \dim F_+ \geq \dim F = n^2+n$.

Since F is a degeneration of the (irreducible) commuting scheme, it is equidimensional. The only component of F_+ of the right dimension is F_-, so $F \supseteq F_-$ and they are equal as sets. Since F_+ and F_- have the same S-multidegree, F trapped between them has the same S-multidegree as both. (If $F = F_-$, this is enough to prove that C_n is reduced, which is still unknown. In [Kn] it is further conjectured that $F_+ = F = F_-$.) This is also the S-multidegree of C_n, since C_n degenerates top F.

Our $(N+1)$-dimensional torus T acts on $M_n(\mathbb{C}) \times M_n(\mathbb{C})$, preserving F_+ and F_-, by

$$(\alpha, D_1, D_2) \cdot (X, Y) := (\alpha D_1 X D_2^{-1}, \alpha D_2 Y D_1^{-1})$$

where $D_1 := \operatorname{diag}(\zeta_1, \ldots, \zeta_n), D_2 := \operatorname{diag}(\zeta_{n+1}, \ldots, \zeta_{2n})$. This extends the action of S, which corresponds to the subtorus $D_1 = D_2$. Hence

$$\operatorname{mdeg}_{M_n(\mathbb{C})\times M_n(\mathbb{C})} F_-|_{z_i \equiv z_{n+i}} = S{-}\operatorname{mdeg}_{M_n(\mathbb{C})\times M_n(\mathbb{C})} F_-.$$

The component of E_P corresponding to the component F_- of the upper-upper scheme is E_π where $\pi(i) = 2n+1-i$. Hence

$$\operatorname{mdeg} E_{\pi_n} = \Phi_n \operatorname{mdeg}_{M_n(\mathbb{C})\times M_n(\mathbb{C})} F_-.$$

Combining these equations, we get

$$\begin{aligned} S{-}\operatorname{mdeg}_{M_n(\mathbb{C})\times M_n(\mathbb{C})} C_n &= S{-}\operatorname{mdeg}_{M_n(\mathbb{C})\times M_n(\mathbb{C})} F_- \\ &= \operatorname{mdeg}_{M_n(\mathbb{C})\times M_n(\mathbb{C})} F_-|_{z_i \equiv z_{n+i}} \\ &= (\operatorname{mdeg} E_{\pi_n}/\Phi_n)|_{z_i \equiv z_{n+i}}. \end{aligned}$$

To compute $\operatorname{mdeg} E_{\pi_n}/\Phi_n$, we apply equation (4) in order to produce the pattern π out of π_0 using transpositions f_i with i taking values in $\{1, \ldots, n-1\}$ only; the variables z_i, $i = n+1, \ldots, 2n$ do not appear in the divided difference operators used and can be set to zero from the start. Hence we can use the same calculation to compute Δ_n. (In fact Δ_n can also be interpreted as a multidegree, for the subtorus $D_2 = 1$, but this won't be relevant for us.) Finally, one can pull out a factor $\prod_{1\leq i<j\leq n}(a + z_i - z_j)$, turning the divided difference operator of equation (4) into θ_i (cf. [DFZJ, equation (4.20)]). The £rst two formulae (which only differ in the order of the $\{\theta_i\}$) follow. As explained in the introduction, the £rst is more practical.

If we expand out the recursive formula given for Δ_n, we get

$$\Delta_n = a(\theta_1 \cdots \theta_{n-1})(a+z_n)^{n-1} \prod_{i=1}^{n-1}(a - z_i) a(\theta_1 \cdots \theta_{n-2})(a + z_{n-1})^{n-2} \prod_{i=1}^{n-2}(a - z_i)$$

To get from there to the second closed form, notice £rst that $\theta_i(pq) = p\theta_i(q)$ if p is symmetric in $\{z_i, z_{i+1}\}$. Hence we can pull each of the multiplicative factors $(a+z_m)^{m-1}\prod_{i=1}^{m-1}(a-z_i)$ to the right, past each θ_j, since they only encounter θ_j for $j < m-1$. Similarly, pull the isolated factors of a left.

$$\Delta_n = a^n(\theta_1\cdots\theta_{n-1})(\theta_1\cdots\theta_{n-2})\cdots(a+z_n)^{n-1}\left(\prod_{i=1}^{n-1}(a-z_i)\right)(a+z_{n-1})^{n-2}\left(\prod_{i=1}^{n-2}(a-z_i)\right)\cdots$$

Reordering the multiplicative factors, we get the second closed formula. □

6. FROM N TO N − 2 AS A GEOMETRIC VERTEX DECOMPOSITION

In this section we give a geometric interpretation of theorem 4 from [DFZJ]. Its proof is based on corollary 2.5 of [KMY], which reads

Corollary. *Let $X \subseteq H \times L$, where H has coordinates $x_1, \ldots, x_n$ and L has coordinate y. Assume that H, L are representations of a torus T, and X is a T-invariant subvariety. Let $w \in T^* = \mathrm{Sym}^1(T^*)$ be the weight of T on L, and $S \leq T$ the stabilizer of L, so the map $\mathrm{Sym}(T^*) \to \mathrm{Sym}(S^*)$ takes $p \mapsto p|_{w=0}$.*

Let the ideal I de£ning X be generated by $\{y^{d_i}q_i + r_i\}_{i=1}^m$, where $y^{d_i}q_i$ is sum of the terms in $y^{d_i}q_i + r_i$ divisible by the highest power of y. Let $J = \langle q_i \mid i = 1, \ldots, m\rangle$. Let $\Theta \subseteq H$ be the corresponding scheme. If we know that

- *Θ has only one component of dimension $\dim X - 1$*
- *that component is generically reduced*
- *X is not contained in a union of £nitely many translates of H*

then

$$(\mathrm{mdeg}_V X)|_{w=0} = (\mathrm{mdeg}_H \Theta)|_{w=0}.$$

Thanks to theorem 6 relating $\mathrm{mdeg}\, E_\pi$ to Ψ_π, the following is exactly theorem 4 from [DFZJ]. We explain after the theorem what new insight is available from a geometric proof.

Theorem 9. *Let π be a link pattern on $1 \ldots N$, and i such that $\pi(i) = i + 1$. We can associate to it a smaller link pattern π' on $1 \ldots i-1, i+2, \ldots N$.*

Then if we specialize $\mathrm{mdeg}\, E_\pi$ at $z_{i+1} = z_i + a$, we get

$$\mathrm{mdeg}\, E_\pi\big|_{z_{i+1}=z_i+a} = \left(\prod_{k\neq i,i+1}(a+z_{i+1}-z_k)(a+z_k-z_i)\right)\mathrm{mdeg}_{M_{N-2}(\mathbb{C})_{\Delta=0}} E_{\pi'}(\ldots, z_{i-1}, z_{i+2}, \ldots).$$

Proof. We use the notation of the corollary throughout. Let $H \times L = M_N(\mathbb{C})_{\Delta=0}$, where $H = \{M \in M_N(\mathbb{C})_{\Delta=0} : M_{i,i+1} = 0\}$ and $L = \{M \in M_N(\mathbb{C}) : M_{kl} \neq 0 \implies (k,l) = (i,i+1)\}$. Then $w = a + z_i - z_{i+1}$.

The equations $(M^2)_{ab} = \sum_{j:\, \circlearrowleft(a\leq j\leq b)} M_{aj}M_{jb}$ involve the entry $M_{i,i+1}$ if and only if $a = i$ or $b = i+1$, and their $y^{d_i}q_i$ terms are $M_{i,i+1}M_{i+1,c} = 0$ for $c \neq i$, $M_{ai}M_{i,i+1} = 0$ for $a \neq i+1$. This gives us some linear equations on Θ:

$$M_{i+1,c} = 0 \quad \text{for } c \neq i, \qquad M_{ai} = 0 \quad \text{for } a \neq i+1.$$

In particular, there are no equations on Θ involving the entries $M_{i,*}, M_{*,i+1}$ except the diagonal entries.

Using the equations from theorem 5, we can determine enough of the remaining equations on Θ to compute its multidegree: given $M \in \Theta$, if we let M' be M with its ith and $(i+1)$st rows and columns erased, then M' satis£es the equations from theorem 5 on $E_{\pi'}$. Applying axiom (3') of multidegrees, we get a linear factor for each vanishing $\{M_{i+1,c}, M_{ai}\}$, and the relation

$$\operatorname{mdeg}\Theta = \Bigg(\prod_{k\neq i,i+1}(a+z_{i+1}-z_k)(a+z_k-z_i)\Bigg)\operatorname{mdeg}_{M_{N-2}(\mathbb{C})_{\Delta=0}} E_{\pi'}(1,\dots,z_{i-1},z_{i+2},\dots,z_N).$$

Then apply the corollary. □

Put another way, the difference between $\operatorname{mdeg} E_\pi$ and the right-hand side of this equation is a multiple of $a+z_i-z_{i+1}$. With [KMY, theorem 2.4], we can give an interpretation of that multiple: it is the multidegree of the projection of E_π to the subspace $H = \{M \in M_N(\mathbb{C})_{\Delta=0} : M_{i,i+1} = 0\}$.

7. The flat limit D_0 of the nilpotent orbit $\{M : M^2 = 0\}$

In this section we elucidate the precise relation between E and the scheme $D_1 := \{M \in M_N(\mathbb{C}) : M^2 = 0\}$. We begin with some results about D_1.

Lemma 5. D_1 *is irreducible. For* N *even,* D_1 *is generically reduced; for* N *odd, it contains the underlying reduced scheme with multiplicity* 2.

Proof. The fact that D_1 is irreducible follows from Jordan canonical form. To check if D_1 is generically reduced we consider the point M with the following block structure:

$$M = \begin{pmatrix} 0_{(n+r)\times n} & 0_{(n+r)\times(n+r)} \\ \mathbf{1}_{n\times n} & 0_{n\times(n+r)} \end{pmatrix}$$

The Zariski tangent space is the kernel of $L \mapsto ML + LM$. For N even, this leads to the set of equations

$$L_{ij} = L_{i+n,j+n} \quad \text{and} \quad L_{i,j+n} = 0 \quad i,j = 1\dots n$$

hence the correct codimension of $2n^2$, which implies the generic reducedness of E_N. For N odd, we £nd this time

$$L_{ij} = L_{i+n+1,j+n+1} \quad i,j=1\dots n \quad \text{and} \quad L_{ij} = 0 \quad i = 1\dots n{+}1, j = n{+}1\dots N, (i,j) \neq (n{+}1,n{+}1)$$

hence a codimension of $n^2{+}(n{+}1)^2{-}1 = 2n(n{+}1)$ which is one less than the codimension of E_N. Note however that adding the extra equation $\operatorname{Tr} M = 0 \Rightarrow \operatorname{Tr} L = 0$ increases the codimension by 1 and makes E_N generically reduced. We now show that generically $(\operatorname{Tr} M)^2 = 0$, thus the multiplicity is 2.

This requires a bit more work, since we must go back to a generic M. We consider the following matrix P given by

$$P_{ij} = \begin{cases} \delta_{ij} & i \le n+1 \\ M_{i-n,j} & i > n+1. \end{cases}$$

Generically, $\det P$ is non-zero on D_1. (Otherwise, it would be identically zero since D_1 is irreducible, but it is easy to construct an $M \in D_1$ for which $\det P \neq 0$.) We therefore allow ourselves to invert $\det P$, and in particular to use the inverse matrix P^{-1}. Thanks to $M^2 = 0$, PMP^{-1} has a certain block structure which can be summarized as follows:

$$(PMP^{-1})_{ij} = \begin{cases} u_j & i = 1 \\ \delta_{i-n,j} & 1 < i \leq n+1 \\ 0 & i > n+1 \end{cases}$$

where the u_j are some polynomials of the M_{ij} and of $\det P^{-1}$ whose explicit form is not needed. Note that this is not quite the block structure of the M chosen in the beginning of the proof. This is because we have "missed" the fact that in odd dimension $\dim \operatorname{Im} M$ is generically one less than $\dim \operatorname{Ker} M$. We now write $(PMP^{-1})^2 = PM^2P^{-1} = 0$ and indeed £nd the extra condition that $u_1^2 = 0$. But $\operatorname{Tr} M = \operatorname{Tr}(PMP^{-1}) = u_1$, hence generically $(\operatorname{Tr} M)^2 = 0$. □

In fact the radical of D_1's ideal is generated by the entries of M^2 and M's characteristic polynomial [St, We].

Proposition 7. *The multidegree of the scheme* D_1 *is*

$$\begin{aligned} \operatorname{mdeg}_{M_N(\mathbb{C})} D_1 &= 2^r \prod_{i,j} (a + z_i - z_j) \sum_{\substack{S \subseteq \{1,\dots,N\} \\ |S|=n}} \prod_{s \in S, \bar{s} \notin S} ((a + z_s - z_{\bar{s}})(z_{\bar{s}} - z_s))^{-1} \\ &= 2^{n+r} a^N \prod_{i<j} \frac{(a - z_i + z_j)(a - z_j + z_i)}{(z_i - z_j)} \operatorname{Pf}\left(\frac{z_i - z_j}{(a + z_i - z_j)(a + z_j - z_i)} \right)_{1 \leq i,j \leq N} . \end{aligned}$$

Moreover, the sequence $\{\operatorname{mdeg} D_1^N\}$ *is characterized by the properties*

- $\operatorname{mdeg} D_1^0 = 1$, $\operatorname{mdeg} D_1^1 = 2a$
- $\operatorname{mdeg} D_1^N$ *is a symmetric polynomial in* $z_1, \dots, z_N$
-

$$\operatorname{mdeg}_{M_N(\mathbb{C})} D_1^N \Big|_{z_2 = z_1 + a} = 2a^2 \left(\prod_{k=3}^{N} (a + z_2 - z_k)(a + z_k - z_1) \right) \operatorname{mdeg}_{M_{N-2}(\mathbb{C})} D_1^{N-2}(z_3, \dots, z_N)$$

Proof. Let $Q = \{(V \in \operatorname{Gr}_n(\mathbb{C}^N), H \in \operatorname{Hom}(\mathbb{C}^N/V, V))\}$ be the vector bundle over the Grassmannian of n-planes in $\mathbb{C}^N$, where the £ber over V is the linear space of maps from $\mathbb{C}^N/V \to V$. (In fact Q is isomorphic to the cotangent bundle.) Then there is a generically $1:1$ map

$$\begin{aligned} \beta : Q &\to D_1 \\ (V, H) &\mapsto (\mathbb{C}^N \twoheadrightarrow \mathbb{C}^N/V \xrightarrow{H} V \hookrightarrow \mathbb{C}^N) \end{aligned}$$

which is equivariant with respect to an action of $\mathbb{C}^\times \times GL_N(\mathbb{C})$: let $\mathbb{C}^\times$ rescale the £bers of the bundle and rescale D_1, and $GL_N(\mathbb{C})$ act in the obvious ways. In particular β is equivariant for the action of our torus T.

The T-£xed points on Q are of the form $(V, \vec{0})$ where V is an n-dimensional coordinate subspace $\mathbb{C}^S$, using the coordinates $S \subseteq \{1, \dots, N\}$. The tangent space $T_{(V,\vec{0})}Q$ is isomorphic to $\operatorname{Hom}(V, \mathbb{C}^N/V) \oplus \operatorname{Hom}(\mathbb{C}^N/V, V)$, where the rescaling circle only acts on the second factor. The weights are $\{z_j - z_i\}, \{a + z_i - z_j\}$ where $i \in S, j \notin S$.

Via the same sort of equivariant localization arguments as in lemma 1, we obtain the formula

$$1 = \sum_S [(\mathbb{C}^S, 0)] \prod_{i\in S, j\notin S} ((a + z_i - z_j)(z_j - z_i))^{-1}$$

as a formula in (a localization of) $H^*_T(Q)$, where $[(\mathbb{C}^S, 0)]$ is the class of the point $(\mathbb{C}^S, 0) \in Q$. Pushing that into $M_N(\mathbb{C})$ using β_*, each [point] turns into the product of the weights on $M_N(\mathbb{C})$. Including the factor 2^r for the scheme structure, we get the desired formula and a close equivalent:

$$\begin{aligned}
\operatorname{mdeg}_{M_N(\mathbb{C})} D_1 &= 2^r \prod_{i,j} (a + z_i - z_j) \sum_{\substack{S\subseteq\{1,\ldots,N\}\\ |S|=n}} \prod_{i\in S, j\notin S} ((a + z_i - z_j)(z_j - z_i))^{-1} \\
&= 2^r \sum_{\substack{S\subseteq\{1,\ldots,N\}\\ |S|=n}} \prod_{i,j\in S}(a + z_i - z_j) \prod_{i,j\notin S}(a + z_i - z_j) \prod_{i\in S, j\notin S} \frac{a + z_j - z_i}{z_j - z_i}
\end{aligned}$$

The base cases are obvious, and the symmetry follows from the $GL_N(\mathbb{C})$ and hence S_N action. We will see the recurrence relation from the second version of the formula above. If $a + z_1 - z_2 = 0$, the only nonzero terms have $S \ni 1, S \not\ni 2$, so we can separate out the factors involving $1, 2$ and rewrite

$$\begin{aligned}
\prod_{i,j\in S}(a + z_i - z_j) &= a \prod_{i\in S\setminus 1} (z_2 - z_i)(a + z_i - z_1) \prod_{i,j\in S\setminus 1} (a + z_i - z_j) \\
\prod_{i,j\notin S}(a + z_i - z_j) &= a \prod_{j\notin S\cup 2} (a + z_2 - z_j)(z_j - z_1) \prod_{i,j\notin S\cup 2} (a + z_i - z_j) \\
\prod_{i\in S, j\notin S} \frac{a + z_j - z_i}{z_j - z_i} &= 2 \prod_{i\in S\setminus 1} \frac{a + z_2 - z_i}{z_2 - z_i} \prod_{j\notin S\cup 2} \frac{a + z_j - z_1}{z_j - z_1} \prod_{i\in S\setminus 1, j\notin S\cup 2} \frac{a + z_j - z_i}{z_j - z_i}
\end{aligned}$$

giving a total product of

$$\begin{aligned}
\operatorname{mdeg}_{M_N(\mathbb{C})} D_1^N &= 2a^2 \sum_{\substack{S\subseteq\{1,3,\ldots,N\}\\ S\ni 1, |S|=n}} \prod_{i\in S\setminus 1} (a + z_i - z_1)(a + z_2 - z_i) \prod_{j\notin S\cup 2} (a + z_2 - z_j)(a + z_j - z_1) \\
&\quad 2^r \prod_{i,j\in S\setminus 1} (a + z_i - z_j) \prod_{i,j\notin S\cup 2} (a + z_i - z_j) \prod_{i\in S\setminus 1, j\notin S\cup 2} \frac{a + z_j - z_i}{z_j - z_i} \\
&= 2a^2 \left(\prod_{i\neq 1,2} (a + z_i - z_1)(a + z_2 - z_i) \right) \operatorname{mdeg}_{M_N(\mathbb{C})} D_1^{N-2}(z_3, \ldots, z_N)
\end{aligned}$$

at $z_2 = z_1 + a$, as desired.

In [DFZJ, theorem 5] it was shown that the symmetry, base case, and recurrence relation are enough to determine $\sum_\pi \Psi_\pi$, and to derive a Pfaffian formula. The recurrence relation here differs only in the factor $2a^2$, which does not affect the argument. This completes the proof.

It is perhaps interesting that there is a direct calculation leading to the Pfaffian formula for the multidegree of D_1. Here we use a slightly different, analytic, language to emphasize the connection to matrix models. We give the details of the calculation in the case N even.

The action of the torus T, and the moment map

$$\Phi : M_N(\mathbb{C}) \to \mathrm{Lie}\,(T)^*, \qquad M \mapsto \pi \sum_{i,j} |M_{ij}|^2 \,(a + z_i - z_j)$$

both restrict to D_1. Using the matrix $Z = \mathrm{diag}(z_1, \ldots, z_N)$, we can rewrite this as $\Phi(M) = \pi(a\mathrm{Tr}\, MM^\dagger + \mathrm{Tr}\, Z[M, M^\dagger])$.

Writing c for $\prod_{i,j}(a + z_i - z_j)$, a formal application of the push-pull formula leads to the formula

$$\mathrm{mdeg}\, D_1 = c \int_{M \in D_1} d\mu(M) \exp\left(-\pi(a\mathrm{Tr}\, MM^\dagger + \mathrm{Tr}\, Z[M, M^\dagger])\right)$$

where the measure $d\mu(M)$ on D_1 is derived from the ¤at metric $\sum_{i,j} |M_{ij}|^2$. It is not our intention to provide a rigorous justi£cation of the above, but we will show that it leads to the correct formula we have already justi£ed by other means.

There is a decomposition of $M \in E_N$ as $M = \Omega M' \Omega^\dagger$ where Ω is unitary and M' has the $n \times n$ block structure

$$M' = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$$

and X is a diagonal matrix: $X = \mathrm{diag}(x_1, \ldots, x_n)$ with $x_i \geq 0$. To £nd such a decomposition, £rst obtain the obvious block decomposition with X arbitrary ($\mathrm{Im}\, M \subset \mathrm{Ker}\, M$, $\dim \mathrm{Ker}\, M \geq n$), then use the standard fact that for any $n \times n$ complex matrix X there exist $n \times n$ unitary matrices V, W such that $VXW^\dagger$ is diagonal positive.

Noting that $\mathrm{Tr}\, MM^\dagger = \sum_{i=1}^n x_i^2$ we perform the change of variables in the integral. The measure in the new variables must be carefully computed by setting $\Omega = 1 + i d\Omega$ with $d\Omega = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ Hermitian, $x_i' = x_i + dx_i$, expanding the metric $\sum_{i,j} |M_{ij}|^2$ at £rst order in $d\Omega$ and dx_i, and £nally taking the square root of its determinant. The diagonal parts H_{11}, H_{22} contribute the usual factors $\prod_{i=1}^n x_i \prod_{i<j}(x_i^2 - x_j^2)^2$, but remarkably the part H_{12} contributes $\prod_{i=1}^n x_i^2 \prod_{i<j}(x_i^2 + x_j^2)^2$, so that this recombines into

$$\mathrm{mdeg}\, D_1 = c \int \frac{d\Omega}{(2\pi)^n n!} \prod_{i=1}^n dx_i x_i^3 \exp\left(-\pi a x_i^2\right) \Delta^2(x_i^4) \exp\left(-\pi \mathrm{Tr}\, Z\Omega(XX^\dagger - X^\dagger X)\Omega^\dagger\right)$$

where the factor $(2\pi)^n n!$ comes from the non-uniqueness of the decomposition, and $\Delta(\cdot)$ is the van der Monde determinant: $\Delta(x_i^4) = \prod_{i<j}(x_i^4 - x_j^4)$. The integral over the unitary group is the Harish Chandra–Itzykson–Zuber integral [HC, IZ] (see also [ZJZ]). The diagonal matrices Z and $XX^\dagger - X^\dagger X$ have entries respectively z_j, $j = 1, \ldots, N$, and $\pm x_i^2$, $i = 1, \ldots, n$; we write the latter as $XX^\dagger - X^\dagger X = \mathrm{diag}(\epsilon x_i^2)$, $(i, \epsilon) \in \{1, \ldots, n\} \times \{-1, +1\}$. We thus £nd

$$\mathrm{mdeg}\, D_1 = c\, \frac{(2\pi)^N}{(2\pi)^n n!} \int_0^\infty \prod_{i=1}^n dx_i x_i^3 \exp\left(-\pi a x_i^2\right) \Delta^2(x_i^4) \frac{\det\left(\exp\left(\epsilon \pi z_j x_i^2\right)\right)}{\Delta(z_j)\Delta(\epsilon x_i^2)}$$

$\Delta(\epsilon x_i^2) = \prod_{i<j}(x_i^2 - x_j^2)^2(x_i^2 + x_j^2)^2 \prod_i(2x_i^2)$, so that one can simplify and compute

$$
\begin{aligned}
\operatorname{mdeg} D_1 &= c\,\frac{\pi^n}{n!}\int_0^\infty \prod_{i=1}^n dx_i x_i \exp\left(-\pi a x_i^2\right)\frac{\det(\exp\left(\pi\epsilon z_j x_i^2\right))}{\Delta(z_j)} \\
&= c\,\frac{\pi^n}{n!}\sum_{\sigma\in S_N}(-1)^\sigma \frac{1}{\Delta(z_j)}\int_0^\infty \prod_{i=1}^n dx_i x_i \exp\left(-\pi x_i^2(a + z_{\sigma(2i-1)} - z_{\sigma(2i)})\right) \\
&= a^N \prod_{i<j}\frac{(a - z_i + z_j)(a - z_j + z_i)}{(z_i - z_j)}\operatorname{Pf}\left(\frac{1}{a + z_j - z_i} - \frac{1}{a + z_i - z_j}\right)
\end{aligned}
$$

Since $\frac{1}{a+z_j-z_i} - \frac{1}{a+z_i-z_j} = \frac{2(z_i-z_j)}{a-(z_i-z_j)^2}$, we obtain the desired expression; it differs from that of corollary 1 by a factor of $2^n a^N$. The power of a is simply due to the different embedding space ($M_N(\mathbb{C})$ versus $M_N(\mathbb{C})_{\Delta=0}$).

For N odd the result of the computation of the integral is strictly identical; however to obtain the multidegree of D_1 one must take into account the multiplicity 2, hence the factor 2^r. □

Theorem 10. *Let*

$$D_t := \{M : (M_{\leq} + tM_{>})^2 = 0\}, \qquad t \neq 0$$

so each $D_t \cong D_1 = \{M : M^2 = 0\}$. *Define* D_0 *to be the flat limit* $\lim_{t\to 0} D_t$. *Then the scheme* D_0 *is supported on* $\cup_\pi E_\pi$*, and contains each* E_π *with the same multiplicity* 2^{n+r}.

Proof. As explained in section 2.3, the limit of the set of equations $(M_{\leq} + tM_{>})^2 = 0$ as $t \to 0$ is the set $M \bullet M = 0$. However, these may not generate the limit ideal defining D_0. So we can only infer a containment (of schemes), $D_0 \subseteq \{M \in M_N(\mathbb{C}) : M \bullet M = 0\}$. While this latter scheme is bigger than E, it has the same support, so as sets $D_0 \subseteq E$.

Since D_1 is irreducible and hence equidimensional, the flat limit D_0 is also equidimensional, so it is supported on E's components of top dimension, $\cup_\pi E_\pi$. (Remember that we conjecture that E has no other components, but even if it does they're not in D_0.)

Consequently

$$\operatorname{mdeg} D_1 = \operatorname{mdeg} D_0 = \sum_\pi c_\pi \operatorname{mdeg} E_\pi$$

for some coefficients $\{c_\pi \in \mathbb{N}\}$, where c_π is the multiplicity of E_π in D_0.

However, we already know $\operatorname{mdeg} D_1$ from proposition 7 and $\sum_\pi \operatorname{mdeg} E_\pi$ from corollary 1, from which we see that taking $c_\pi \equiv 2^{n+r}$ gives a solution. To know it's the right one, it is enough to show that the polynomials $\{\operatorname{mdeg} E_\pi\}$ are linearly independent over $\mathbb{Z}$.

Let $\sum_\pi d_\pi \operatorname{mdeg} E_\pi = 0$ be a linear relation among them. By theorem 6, we also know $\sum_\pi d_\pi \Psi_\pi = 0$. Let ρ be a link pattern. By [DFZJ, lemma 2], the specialization of Ψ_π at $a = 0$, $z_i = z_{\rho(i)}$, $i = 1 \ldots N$ is nonzero if and only if $\pi = \rho$, allowing us to pick out the d_ρ term and show $d_\rho = 0$.

Hence the $\{\operatorname{mdeg} E_\pi\}$ are linearly independent, and the multiplicities are all 2^{n+r}. □

We conjecture that $D_0 = \{M \in M_N(\mathbb{C}) : M \bullet M = 0\}$ as schemes, which would imply our earlier conjecture that E is equidimensional, in that D_0 is the flat limit of a variety and hence equidimensional.

8. An additional circle action

Throughout this paper we claimed to be working with the action of an N+1-dimensional torus T on the scheme E. Since the 1-dimensional subtorus of T consisting of scalar matrices acts trivially, it is really more honest to consider this an action of the N-dimensional quotient torus. The corresponding statement for the multidegrees is that while we considered our multidegrees as polynomials in $a, z_1, \ldots, z_N$, they can all be written as polynomials in the N expressions $a, z_1 - z_2, \ldots, z_{N-1} - z_N$. (Of course, for the multidegrees of subschemes of $M_N(\mathbb{C})_{\Delta=0}$, we have an even better statement – they are polynomials with positive coef£cients in the weights $\{a + z_i - z_j : i \neq j\}$ of $M_N(\mathbb{C})_{\Delta=0}$.)

Abstractly, we should expect that $(M_N(\mathbb{C}), \bullet)$ has an extra degree of symmetry beyond that of $(M_N(\mathbb{C}), \times)$, in that $\bullet$ is the multiplication on the degenerate £ber of a 1-parameter family (see section 2.3). It is easy to write down this bigger action: de£ne

$$(\alpha, \beta_1, \ldots, \beta_N) \cdot e^{ik} := \alpha \left(\prod_{j:\circlearrowleft(i \leq j < k)} \beta_j \right) e^{ik}.$$

If $\beta_i = \zeta_i \zeta_{i+1}^{-1}$ for each i, then the action of $(\alpha, \beta_1, \ldots, \beta_N)$ is just conjugation by the diagonal matrix $\mathrm{diag}(\zeta_1, \ldots, \zeta_N)$, followed by rescaling by α. Hence this extends the T-action.

It also is easy to check that the action of the subgroup with $\alpha = 1$ preserves the product $\bullet$ on $M_N(\mathbb{C})$. First,

$$\begin{aligned} (1, \beta_1, \ldots, \beta_N) \cdot (e^{hj} \bullet e^{km}) &= (1, \beta_1, \ldots, \beta_N) \cdot (\delta_{jk}[\circlearrowleft (h \leq j \leq m)] e^{hm}) \\ &= \left(\prod_{i:\circlearrowleft(h \leq i < m)} \beta_i \right) \delta_{jk}[\circlearrowleft (h \leq j \leq m)] e^{hm}. \end{aligned}$$

Now notice that $\circlearrowleft (h \leq j \leq m)$ implies that

$$\prod_{i:\circlearrowleft(h \leq i < m)} \beta_i = \prod_{i:\circlearrowleft(h \leq i < j)} \beta_i \prod_{l:\circlearrowleft(j \leq l < m)} \beta_l$$

which is what we need to establish

$$(1, \beta_1, \ldots, \beta_N) \cdot (e^{hj} \bullet e^{km}) = \left((1, \beta_1, \ldots, \beta_N) \cdot e^{hj}\right) \bullet \left((1, \beta_1, \ldots, \beta_N) \cdot e^{km}\right)$$

when both sides are nonzero. Since this action with $\alpha = 1$ preserves $\bullet$, it preserves the scheme E, and α is just acting by rescaling E.

Call this bigger torus $\overline{T}$, and use $(a, b_1, \ldots, b_N)$ for the obvious basis of its weight lattice. Then we get the following equation on the $\overline{T}$– and T–multidegrees of an af£ne scheme $X \subseteq M_N(\mathbb{C})_{\Delta=0}$:

$$\overline{T}-\mathrm{mdeg}\ X|_{b_i = z_i - z_{i+1}} = T-\mathrm{mdeg}\ X$$

where each b_i has been specialized to $z_i - z_{i+1}$. The kernel of this specialization is generated by

$$b_{tot} := \sum_{i=1}^{N} b_i.$$

The proof of proposition 6, which applied equally well to $\overline{T}$-multidegrees, actually proves

$$\text{(11)} \qquad \operatorname{mdeg} E_\pi + \operatorname{mdeg} E_{f_i \cdot \pi} = -\frac{2a - b_i + b_{tot}}{a - b_i + b_{tot}}\, \partial_i \left((a - b_i + b_{tot}) \operatorname{mdeg} E_\pi \right).$$

because the $\overline{T}$-weights on the $(i, i+1)$ entries of M and M^2 are $a - b_i + b_{tot}$ and $2a - b_i + b_{tot}$. The base case, $\pi_0(i) = i + n \bmod 2n$ for $i \le 2n$, and $\pi_0(N) = N$ if N is odd, is also just as easy as in proposition 5:

$$\operatorname{mdeg} E_{\pi_0} = \prod_{\substack{i=1\ldots N \\ j:\, \circlearrowleft(i<j<i+n)}} \Big(a + \sum_{k:\, \circlearrowleft(i \le k < j)} b_k \Big) \left(\prod_{i=n+1}^{N} \Big(a + \sum_{k:\, \circlearrowleft(i \le k < i+n)} b_k \Big) \right)^r$$

Lacking a geometric proof of corollary 3, we don't know how to lift that formula to one about $\overline{T}$-multidegrees.

The passage from $\{z_i\}$ to $\{b_i\}$ apparently spoils the integrability of the corresponding stochastic process, so we do not know how to state an analogue of theorem 6.

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