Exact Solutions of Integrable 2D Contour Dynamics *

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Abstract

A class of exact solutions of the dispersionless Toda hierarchy constrained by a string equation is obtained. These solutions represent deformations of analytic curves with a finite number of nonzero harmonic moments. The corresponding τ-functions are determined and the emergence of cusps is studied.

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1 Introduction

Integrable contour dynamics governed by the dispersionless Toda (dToda) hierarchy is a multifaceted subject. It underlies problems of complex analysis [1],[2], interface dynamics (Laplacian growth) [3] , Quantum Hall effect [4] and associativity (WDVV) equations [5]. A common ingredient in many of its applications is the presence of random models of normal $N \times N$ matrices [1]-[4], [6],[7] with partition functions of the form

$$Z_N = \int \mathrm{d}\, M \,\mathrm{d}\, M^\dagger \exp(-\frac{1}{\hbar}\,\mathrm{tr}\; W(M, M^\dagger)), \tag{1}$$

where

$$W(z, \bar{z}) = z\bar{z} + v_0 - \sum_{k\geq 1}(t_k\, z^k + \bar{t}_k\, \bar{z}^k). \tag{2}$$

In an appropriate large N limit ($\hbar \to 0$, $s := \hbar N$ fixed), the eigenvalues of the matrices are distributed within a planar domain (*support of eigenvalues*) with sharp edges, which depends on the parameters $t := (s = \bar{s},\, t_1, t_2 \ldots)$.

If the support of eigenvalues is a simply-connected bounded domain with boundary given by an analytic curve γ ($z = z(p)$, $|p| = 1$), then $(s, t_1, t_2 \ldots)$ are harmonic moments of γ and the curve evolves with $(t, \bar{t})$ according to the dToda hierarchy. Moreover, the corresponding τ-function represents the quasiclassical limit of the partition function (1). A particularly interesting feature is that for almost all initial conditions the evolution of γ leads to critical configurations in which cusp-like singularities develop. This behaviour is well-known in Laplacian growth [8] and random matrix theory [9].

In order to obtain solutions of the dToda hierarchy describing contour dynamics one must impose a string equation which leads to a particular type of Riemann-Hilbert problem [10]-[12]. In this paper we present a method for finding solutions in the form of Laurent polynomials

$$z = rp + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}}, \tag{3}$$

which describe dynamics of curves with a finite number of nonzero harmonic moments, namely $t_k = \bar{t}_k = 0$ for $k \geq K$. We exhibit examples for arbitrary K and derive their corresponding τ-functions. Furthermore the emergence of cusps is analytically studied.

2 dToda contour dynamics

Let $z = z(p)$ be an invertible conformal map of the exterior of the unit circle to the exterior of a simply connected domain bounded by a simple analytic curve γ of the form

$$\bar{z} = S(z), \tag{4}$$

where bar stands for complex conjugation ($z(\bar{p}) = z(p^{-1})$ on γ) and the *Schwarz function* $S(z)$ is analytic in some domain containing γ.

The map $z(p)$ can be represented by a Laurent series

$$z(p) = rp + \sum_{k=0}^{\infty} \frac{u_k}{p^k}, \tag{5}$$

with a real coefficient r. The coefficients $(r, u_0, u_1, \dots)$ are functions of the harmonic moments $t = (s = \bar{s},\ t_1, t_2 \dots)$ of the exterior of γ, which in turn can be introduced through the expansion of the Schwarz function

$$S(z) = \sum_{k=1}^{\infty} k\, t_k\, z^{k-1} + \frac{s}{z} + \sum_{k=1}^{\infty} \frac{v_k}{z^{k+1}}, \tag{6}$$

with $(v_1, v_2, \dots)$ being functions dependent on t. As a consequence of (1)-(3), it follows that $z(p, t, \bar{t})$ solves the dToda hierarchy

$$\begin{gathered} \partial_{t_k} z = \{H_k, z\}, \quad \partial_{\bar{t}_k} z = -\{\bar{H}_k, z\}, \\ H_k := (z^k)_{\geq 1} + \frac{1}{2}(z^k)_0, \quad \bar{H}_k := (\bar{z}^k)_{\leq -1} + \frac{1}{2}(\bar{z}^k)_0. \end{gathered} \tag{7}$$

where $\{f, g\} := p\,(\partial_p\, f\, \partial_s\, g - \partial_p\, g\, \partial_s\, f)$, the function $\bar{z}(p^{-1})$ is defined by the Laurent series

$$\bar{z}(p^{-1}) := \frac{r}{p} + \sum_{k=0}^{\infty} \bar{u}_k\, p^k, \tag{8}$$

and the symbols $(\dots)_{\geq 1}$ ($(\dots)_{\leq -1}$) and $(\dots)_0$ mean truncated Laurent series with only positive (negative) terms and the constant term, respectively. Furthermore, this solution satisfies the string equation

$$\{z(p), \bar{z}(p^{-1})\} = 1. \tag{9}$$

These properties can be proved through the twistor scheme of Takasaki-Takebe [3]. It uses Orlov-Schulman functions of the dToda hierarchy

$$\begin{aligned} m &= \sum_{k=1}^{\infty} k\, t_k\, z^k + s + \sum_{k=1}^{\infty} \frac{v_k}{z^k}, \\ \bar{m} &= \sum_{k=1}^{\infty} k\, \bar{t}_k\, \bar{z}^k + s + \sum_{k=1}^{\infty} \frac{\bar{v}_k}{\bar{z}^{\,k}}, \end{aligned} \tag{10}$$

and can be summarized as follows:

Theorem *If* $(z, m, \bar{z}, \bar{m})$ *are functions of* $(p, t, \bar{t})$ *which admit expansions of the form* (5),(8),(10) *and satisfy the equations*

$$\bar{z} = \frac{m}{z}, \quad \bar{m} = m, \tag{11}$$

then $(z, \bar{z})$ *is a solution of the dToda hierarchy constrained by the string equation* (9).

3 Solutions

Equations (11) are meaningful only when they are interpreted as a suitable Riemann-Hilbert problem on the complex plane of the variable p. Thus (z, m) must be analytic functions in a neighborhood $D = \{|p| > r\}$ of $p = \infty$ and $(\bar{z}, \bar{m})$ must be analytic functions in a neighborhood $D' = \{|p| < r'\}$ of $p = 0$. The statement of the Theorem holds provided $A := D \bigcap D' \neq \emptyset$.

We next prove that equations (11) have solutions satisfying (5),(8) and (10) with

$$t_k = 0, \quad k > K, \quad t_K \neq 0.$$

In this way we assume

$$
\begin{aligned}
m &= \sum_{k=1}^{K} k t_k z^k + s + \sum_{k=1}^{\infty} \frac{v_k}{z^k}, \\
\bar{m} &= \sum_{k=1}^{K} k\, \bar{t}_k \bar{z}^k + s + \sum_{k=1}^{\infty} \frac{\bar{v}_k}{\bar{z}^k}.
\end{aligned}
\tag{12}
$$

Given two integers $r_1 \leq r_2$ we denote by $\mathrm{V}[r_1, r_2]$ the set of Laurent polynomials of the form

$$
c_{r_1}\, p^{r_1} + c_{r_1+1}\, p^{r_1+1} + \cdots + c_{r_2}\, p^{r_2}.
$$

Let us look for solutions of (11) such that z and $\bar{z}$ are meromorphic functions of p with possible poles at $p = 0$ and $p = \infty$ only. Then, as a consequence of the assumptions (5),(8) and (12), from (11) it follows that

$$
z \in \mathrm{V}[1-K, 1], \quad \bar{z} \in \mathrm{V}[-1, K-1]. \tag{13}
$$

The equation $\bar{m} = m$ is equivalent to the system:

$$
\bar{m}_{\geq 1} = m_{\geq 1}, \tag{14}
$$

$$
\bar{m}_0 = m_0, \tag{15}
$$

$$
\bar{m}_{\leq -1} = m_{\leq -1}. \tag{16}
$$

If we now set

$$
m = \bar{m} = \sum_{k=1}^{K} k t_k (z^k)_{\geq 1} + \bar{m}_0 + \sum_{k=1}^{K} k\, \bar{t}_k (\bar{z}^k)_{\leq -1}, \tag{17}
$$

with

$$
\bar{m}_0 = s + \sum_{k=1}^{K} k\, \bar{t}_k (\bar{z}^k)_0.
$$

it can be easily seen that $\bar{m}$ has the required expansion of the form (12) provided z and $\bar{z}$ satisfy (5) and (8) . On the other hand, the expression (17) for m has an expansion of the form (12) if the residue of $\dfrac{m}{z}$ corresponding to its Laurent expansion in powers of z verifies

$$Res(\frac{m}{z}, z) = s. \tag{18}$$

Hence the problem reduces to finding z and $\bar{z}$ satisfying (5),(8),(18) and

$$z = \frac{m}{\bar{z}}. \tag{19}$$

In view of (13) we look for z and $\bar{z}$ of the form

$$\begin{aligned} z &= rp + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}}, \\ \bar{z} &= \frac{r}{p} + \bar{u}_0 + \cdots + \bar{u}_{K-1} p^{K-1}. \end{aligned} \tag{20}$$

Now, in order to prevent z from having poles different from $p = 0$ and $p = \infty$ we have to impose

$$m(p_i) = 0, \tag{21}$$

where p_i denote the K zeros of

$$r + \bar{u}_0\, p + \cdots + \bar{u}_{K-1}\, p^K = 0.$$

In this way by using the expression (17) of m, the only variables appearing in (19) are

$$(p, t, \bar{t}, r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1}), \quad w_i := \bar{u}_i.$$

Thus, by identifying coefficients of the powers p^i, $i = 1 - K, \ldots, 1$ we get $K+1$ equations which together with the K equations (21) determine the $2K+1$ unknowns variables $(r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1})$ as functions of $(t, \bar{t})$. Moreover, provided r is a real coefficient, the equations (11) are invariant under the transformation

$$\mathrm{T} f(p) = \overline{f\Big(\frac{1}{\bar{p}}\Big)}.$$

Hence if $(r, u_0, \dots, u_{K-1}, w_0, \dots, w_{K-1})$ solves (19) so does

$$(r, \bar{w}_0, \dots, \bar{w}_{K-1}, \bar{u}_0, \dots, \bar{u}_{K-1}).$$

Therefore, if both solutions are close enough, they coincide and consequently $w_i = \bar{u}_i$, as required.

To complete our proof we must show that (18) is satisfied too. To do that let us take two circles γ ($|p| = r$) and γ' ($|p| = r'$) in the complex p-plane and denote by Γ and Γ' their images under the maps $z = z(p)$ and $\bar{z} = \bar{z}(1/p)$, respectively. Notice that due to (5) and (8), the curves Γ and Γ' have positive orientation if γ and γ' have positive and negative orientation, respectively. Then we have

$$Res(\frac{m}{z}, z) - Res(\frac{\bar{m}}{\bar{z}}, \bar{z}) = \frac{1}{2i\pi} \oint_{\Gamma} \frac{m}{z} \,\mathrm{d}\, z - \frac{1}{2i\pi} \oint_{\Gamma'} \frac{\bar{m}}{\bar{z}} \,\mathrm{d}\, \bar{z}$$

$$= \frac{1}{2i\pi} \oint_{\gamma} \bar{z} \partial_p z \;\mathrm{d}\, p - \frac{1}{2i\pi} \oint_{\gamma'} z \partial_p \bar{z} \;\mathrm{d}\, p = \frac{1}{2i\pi} \oint_{\gamma} \partial_p (\bar{z} z) \;\mathrm{d}\, p = 0,$$

where we have taken into account that the integrands are analytic functions of p in $\mathbb{C} - \{0\}$ and that γ and the opposite curve of γ' are homotopic with respect to $\mathbb{C} - \{0\}$. Therefore, as we have already proved that $\bar{m}$ has an expansion of the form (12), we deduce

$$Res(\frac{m}{z}, z) = Res(\frac{\bar{m}}{\bar{z}}, \bar{z}) = s,$$

so that (18) follows.

Let us illustrate the method with the case $K = 2$. The polynomial $p\,\bar{z}$ has two zeros at the points

$$p_1 = \frac{-\bar{u}_0 + \sqrt{\bar{u}_0^2 - 4\, r\, \bar{u}_1}}{2\, \bar{u}_1}, \quad p_2 = \frac{-\bar{u}_0 - \sqrt{\bar{u}_0^2 - 4\, r\, \bar{u}_1}}{2\, \bar{u}_1},$$

and from (21) we get two equations which lead to

$$-2\, r^2\, t_2\, \bar{u}_0^3 + 4\, r^3\, t_2\, \bar{u}_0\, \bar{u}_1 + r\, t_1\, \bar{u}_0^2\, \bar{u}_1 + 4\, r\, t_2\, u_0\, \bar{u}_0^2\, \bar{u}_1 - r^2\, t_1\, \bar{u}_1^2 - 4\, r^2\, t_2\, u_0\, \bar{u}_1^2$$
$$-s\, \bar{u}_0\, \bar{u}_1^2 - \bar{t}_1\, \bar{u}_0^2\, \bar{u}_1^2 - 2\, \bar{t}_2\, \bar{u}_0^3\, \bar{u}_1^2 + r\, \bar{t}_1\, \bar{u}_1^3 = 0, \tag{22}$$

$$-2\,r^3\,t_2\,\bar{u}_0^2 + 2\,r^4\,t_2\,\bar{u}_1 + r^2\,t_1\,\bar{u}_0\,\bar{u}_1 + 4\,r^2\,t_2\,u_0\,\bar{u}_0\,\bar{u}_1 - r\,s\,\bar{u}_1^2 - r\,\bar{t}_1\,\bar{u}_0\,\bar{u}_1^2$$
$$-2\,r\,\bar{t}_2\,\bar{u}_0^2\,\bar{u}_1^2 - 2\,r^2\,\bar{t}_2\,\bar{u}_1^3 = 0. \tag{23}$$

Identification of the powers of p in (19) implies

$$p: \quad -2\,r^2\,t_2 + r\,\bar{u}_1 = 0,$$
$$p^0: \quad 2\,r^2\,t_2\,\bar{u}_0 - r\,t_1\,\bar{u}_1 - 4\,r\,t_2\,u_0\,\bar{u}_1 + u_0\,\bar{u}_1^2 = 0,$$
$$p^{-1}: \quad -2\,r^2\,t_2\,\bar{u}_0^2 + 2\,r^3\,t_2\,\bar{u}_1 + r\,t_1\,\bar{u}_0\,\bar{u}_1 + 4\,r\,t_2\,u_0\,\bar{u}_0\,\bar{u}_1 - s\,\bar{u}_1^2 - \bar{t}_1\,\bar{u}_0\,\bar{u}_1^2$$
$$-2\,\bar{t}_2\,\bar{u}_0^2\,\bar{u}_1^2 - 4\,r\,\bar{t}_2\,\bar{u}_1^3 + u_1\,\bar{u}_1^3 = 0. \tag{24}$$

Then by solving equations (22)-(24) we get the solution:

$$z = \frac{p\,\sqrt{s}}{\sqrt{1 - 4\,t_2\,\bar{t}_2}} + \frac{2\,\sqrt{s}\,\bar{t}_2}{p\,\sqrt{1 - 4\,t_2\,\bar{t}_2}} - \frac{\bar{t}_1 + 2\,t_1\,\bar{t}_2}{-1 + 4\,t_2\,\bar{t}_2}, \tag{25}$$

which corresponds to the conformal map describing an *ellipse growing from a circle* [6]

Solutions for $K \geq 3$

Exact solutions associated to arbitrary values of K can be found from the previous scheme. However in order to avoid complicated expressions , we set

$$t_1 = t_2 = \cdots = t_{K-1} = \bar{t}_1 = \bar{t}_2 = \cdots = \bar{t}_{K-1} = 0.$$

and look for particular solutions satisfying

$$u_1 = u_2 = \cdots = u_{K-2} = \bar{u}_1 = \bar{u}_2 = \cdots = \bar{u}_{K-2} = 0,$$

or equivalently

$$z = r\,p + \frac{u_{K-1}}{p^{K-1}}, \quad \bar{z} = \frac{r}{p} + \bar{u}_{K-1} p^{K-1}. \tag{26}$$

Under the previous assumptions and from (17) we have that

$$m = K t_K r^K p^K + s + K^2 \bar{t}_K r^{K-1} \bar{u}_{K-1} + \frac{K \bar{t}_K r^K}{p^K}. \tag{27}$$

Thus, we see that (21) leads us to a unique equation since from (27) it follows that m depends on p through p^K. Furthermore, if p_i satisfies $\bar{z}(p_i) = 0$, then

$$p_i^K = -\frac{r}{\bar{u}_{K-1}}.$$

Therefore, (21) becomes

$$s - \frac{K r^{K+1} t_K}{\bar{u}_{K-1}} + (K-1) K r^{K-1} \bar{t}_K \bar{u}_{K-1} = 0. \tag{28}$$

On the other hand, it is easy to see that

$$\frac{m}{\bar{z}} = \frac{K r^K t_K}{\bar{u}_{K-1}} p + \left(K^2 r^{K-1} \bar{t}_K + \frac{s\, \bar{u}_{K-1} - K r^{K+1} t_K}{\bar{u}_{K-1}^2} \right) \frac{1}{p^{K-1}},$$

consequently, by equating coefficients and taking (26) into account, we find that (19) leads to two equations only . More precisely

$$\begin{aligned} p: &\qquad r = \frac{K r^K t_K}{\bar{u}_{K-1}}, \\ p^{-(K-1)} &\qquad u_{K-1} = K^2 r^{K-1} \bar{t}_K + \frac{s\, \bar{u}_{K-1} - K r^{K+1} t_K}{\bar{u}_{K-1}^2}. \end{aligned} \tag{29}$$

Then, we get three equations for the three unknowns r, u_{K-1}, $\bar{u}_{K-1}$, which proves that there exits a solution of the form (26). In fact, by solving (28)-(29) we find that

$$z = r\, p + \frac{K\, \bar{t}_K\, r^{K-1}}{p^{K-1}}, \tag{30}$$

with r satisfying the implicit equation

$$K^2 (K-1) t_K \bar{t}_K r^{2(K-1)} - r^2 + s = 0. \tag{31}$$

Figures 1 and 2 show examples of the evolution of the curve $z(p)$, $|p| = 1$ as s grows and t_K is kept fixed.

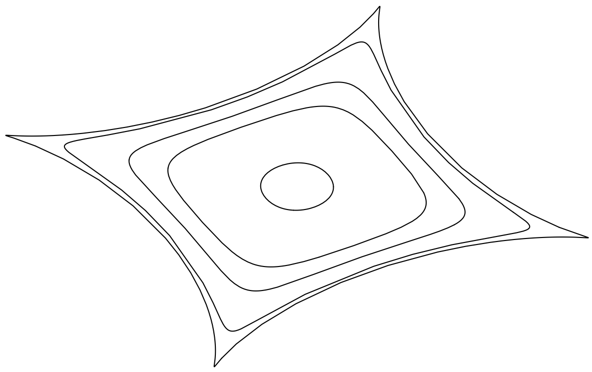

Figure 1: solution corresponding to $K = 4$

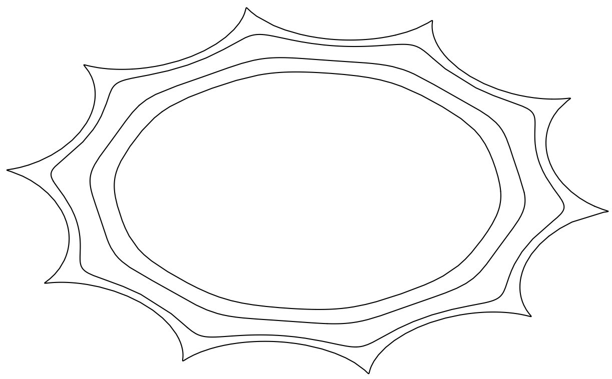

Figure 2: solution corresponding to $K = 10$

τ-functions

In [1] it was proved that there is a dToda τ-function associated to each analytic curve $z = z(p), \quad |p| = 1$, given by

$$2 \log \tau = -\frac{1}{2} s^2 + s\, v_0 - \frac{1}{2} \sum_{k \geq 1} (t_k\, v_k + \bar{t}_k\, \bar{v}_k), \tag{32}$$

where v_k are the coefficients of the expansion (6), and v_0 is determined by

$$\frac{\partial v_0}{\partial s} = \log r^2, \quad v_0 = \frac{\partial \log \tau}{\partial s}. \tag{33}$$

For the class of solutions (30) we have

$$2\,\log\tau = -\frac{1}{2}s^2 + s\,v_0 - \frac{1}{2}(t_K\,v_K + \bar{t}_K\,\bar{v}_K), \tag{34}$$

and from (11) and (30) it follows that

$$\begin{aligned} v_K = \frac{1}{2i\pi}\oint_\Gamma \bar{z}\,z^K\,\mathrm{d}\,z = & \frac{1}{2i\pi}\oint_\Gamma \bar{z}(p)z(p)^K\,(r-(K-1)\frac{u_{K-1}}{p^K})\,\mathrm{d}\,p \\ = & \frac{(r^2-s)(Ks-(K-2)r^2)}{2K(K-1)s}. \end{aligned} \tag{35}$$

On the other hand by differentiating (34) with respect to s and by taking into account (33) one finds

$$\begin{aligned} v_0 = & -s + s\log r^2 + \frac{(K-2)(K-1)K|t_K|^2(Ks-(K-2)r^2)r^{2K}}{2((K-1)^2K^2|t_K|^2r^{2K}-r^4)} \\ & + \frac{(K-2)(s-r^2)}{2(K-1)K}\Big(K + \frac{(K-2)r^4}{(K-1)^2K^2|t_K|^2r^{2K}-r^4}\Big). \end{aligned} \tag{36}$$

Thus, (34)-(36) and (31) characterize the τ-function of the curves determined by (30).

Cusps

The pictures of the curves associated with (30)-(31) show the presence of cusps at some value of s for each fixed value of t_K . Indeed by using the parametric equation $p = e^{i\,\theta}$, $(0 \leq \theta \leq 2\pi)$ for the unit circle, we have that cusps on the curve $z = z(p)$ appear at points where $z_\theta = 0$, $z_{\theta\theta} \neq 0$ and $z_{\theta\theta\theta}/z_{\theta\theta}$ has a nonzero imaginary part. Therefore a necessary condition for $p = p(\theta)$ is

$$\frac{\partial z}{\partial p}(p) = 0, \quad |p| = 1.$$

Thus from (30) we deduce

$$p^K = K(K-1)\bar{t}_K\,r^{K-2},$$

which together with the condition $|p| = 1$ requieres that

$$r = (K(K-1)|t_K|)^{-\frac{1}{K-2}}, \tag{37}$$

at some value $s = s(t_K)$. But according to (31) one finds that this happens at the value s_0 given by

$$s_0 = \frac{K-2}{K-1}(K(K-1)|t_K|)^{-\frac{2}{K-2}}, \tag{38}$$

which is the point at which the profile of both positive branches of r, as functions of s, develop an infinite slop (see figure 3).

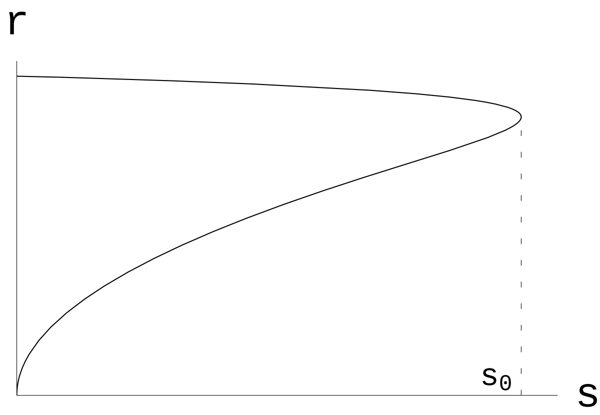

Figure 3: The positive branches of $r(s)$ for $K = 10$

Therefore, there are K cusps given by the roots

$$z_j = \frac{K}{K-1}\Big(\frac{r^2 - s_0}{K\, t_K}\Big)^{\frac{1}{K}}, \tag{39}$$

which emerge when s reaches the extreme value s_0 of the domain of existence of the two positive branches of r as a function of s.

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