EXACT SOLUTIONS OF INTEGRABLE 2D
CONTOUR DYNAMICS *

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Abstract

A class of exact solutions of the dispersionless Toda hierarchy constrained by a string equation is obtained. These solutions represent deformations of analytic curves with a finite number of nonzero harmonic moments. The corresponding \( \tau \)-functions are determined and the emergence of cusps is studied.

Key words: Contour dynamics, Toda hierarchy, conformal maps.

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1 Introduction

Integrable contour dynamics governed by the dispersionless Toda (dToda) hierarchy is a multifaceted subject. It underlies problems of complex analysis [1],[2], interface dynamics (Laplacian growth) [3], Quantum Hall effect [4] and associativity (WDVV) equations [5]. A common ingredient in many of its applications is the presence of random models of normal $N \times N$ matrices [1]-[4], [6],[7] with partition functions of the form

$$Z_N = \int dM dM^\dagger \exp(-\frac{1}{\hbar} \text{tr} W(M, M^\dagger)), \quad (1)$$

where

$$W(z, \bar{z}) = z\bar{z} + v_0 - \sum_{k \geq 1} (t_k z^k + \bar{t}_k \bar{z}^k). \quad (2)$$

In an appropriate large $N$ limit ($\hbar \to 0$, $s := \hbar N$ fixed), the eigenvalues of the matrices are distributed within a planar domain (support of eigenvalues) with sharp edges, which depends on the parameters $t := (s = \bar{s}, t_1, t_2 \ldots)$.

If the support of eigenvalues is a simply-connected bounded domain with boundary given by an analytic curve $\gamma$ ($z = z(p)$, $|p| = 1$), then $(s, t_1, t_2 \ldots)$ are harmonic moments of $\gamma$ and the curve evolves with $(t, \bar{t})$ according to the dToda hierarchy. Moreover, the corresponding $\tau$-function represents the quasiclassical limit of the partition function (1). A particularly interesting feature is that for almost all initial conditions the evolution of $\gamma$ leads to critical configurations in which cusp-like singularities develop. This behaviour is well-known in Laplacian growth [8] and random matrix theory [9].

In order to obtain solutions of the dToda hierarchy describing contour dynamics one must impose a string equation which leads to a particular type of Riemann-Hilbert problem [10]-[12]. In this paper we present a method for finding solutions in the form of Laurent polynomials

$$z = r p + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}}, \quad (3)$$

which describe dynamics of curves with a finite number of nonzero harmonic moments, namely $t_k = \bar{t}_k = 0$ for $k \geq K$. We exhibit examples for arbitrary $K$ and derive their corresponding $\tau$-functions. Furthermore the emergence of cusps is analytically studied.
2 dToda contour dynamics

Let \( z = z(p) \) be an invertible conformal map of the exterior of the unit circle to the exterior of a simply connected domain bounded by a simple analytic curve \( \gamma \) of the form

\[
\bar{z} = S(z),
\]

where bar stands for complex conjugation \( (z(\bar{p}) = z(p^{-1}) \) on \( \gamma \) and the Schwarz function \( S(z) \) is analytic in some domain containing \( \gamma \).

The map \( z(p) \) can be represented by a Laurent series

\[
z(p) = r p + \sum_{k=0}^{\infty} \frac{u_k}{p^k},
\]

with a real coefficient \( r \). The coefficients \( (r, u_0, u_1, \ldots) \) are functions of the harmonic moments \( t = (s = s, t_1, t_2, \ldots) \) of the exterior of \( \gamma \), which in turn can be introduced through the expansion of the Schwarz function

\[
S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{s}{z} + \sum_{k=1}^{\infty} \frac{v_k}{z^{k+1}},
\]

with \((v_1, v_2, \ldots)\) being functions dependent on \( t \). As a consequence of (1)-(3), it follows that \( z(p, t, \tilde{t}) \) solves the dToda hierarchy

\[
\partial_t z = \{H_k, z\}, \quad \partial_{\tilde{t}} z = -\{\tilde{H}_k, z\},
\]

\[
H_k := (z^k)_{\geq 1} + \frac{1}{2}(z^k)_0, \quad \tilde{H}_k := (\bar{z}^k)_{\leq -1} + \frac{1}{2}(\bar{z}^k)_0.
\]

where \( \{f, g\} := p (\partial_p f \partial_s g - \partial_p g \partial_s f) \), the function \( \bar{z}(p^{-1}) \) is defined by the Laurent series

\[
\bar{z}(p^{-1}) := \frac{r}{p} + \sum_{k=0}^{\infty} \tilde{u}_k p^k,
\]

and the symbols \((\ldots)_{\geq 1} \) \((\ldots)_{\leq -1}\) and \((\ldots)_0\) mean truncated Laurent series with only positive (negative) terms and the constant term, respectively. Furthermore, this solution satisfies the string equation
\{z(p), \bar{z}(p^{-1})\} = 1. \quad (9)

These properties can be proved through the twistor scheme of Takasaki-Takebe [3]. It uses Orlov-Schulman functions of the dToda hierarchy

\[
m = \sum_{k=1}^{\infty} k t_k \bar{z}^k + s + \sum_{k=1}^{\infty} \frac{v_k}{z^k},
\]

\[
\bar{m} = \sum_{k=1}^{\infty} k \bar{t}_k \bar{z}^k + s + \sum_{k=1}^{\infty} \frac{\bar{v}_k}{\bar{z}^k},
\]

and can be summarized as follows:

**Theorem** If \((z, m, \bar{z}, \bar{m})\) are functions of \((p, t, \bar{t})\) which admit expansions of the form \((5),(8),(10)\) and satisfy the equations

\[
\bar{z} = \frac{m}{\bar{z}}, \quad \bar{m} = m, \quad (11)
\]

then \((z, \bar{z})\) is a solution of the dToda hierarchy constrained by the string equation \((9)\).

### 3 Solutions

Equations \((11)\) are meaningful only when they are interpreted as a suitable Riemann-Hilbert problem on the complex plane of the variable \(p\). Thus \((z, m)\) must be analytic functions in a neighborhood \(D = \{ |p| > r \}\) of \(p = \infty\) and \((\bar{z}, \bar{m})\) must be analytic functions in a neighborhood \(D' = \{ |p| < r' \}\) of \(p = 0\). The statement of the Theorem holds provided \(A := D \cap D' \neq 0\).

We next prove that equations \((11)\) have solutions satisfying \((5),(8)\) and \((10)\) with

\[
t_k = 0, \quad k > K; \quad t_K \neq 0.
\]
In this way we assume

\[ m = \sum_{k=1}^{K} k t_k z^k + s + \sum_{k=1}^{\infty} \frac{v_k}{z^k}, \]

(12)

\[ \bar{m} = \sum_{k=1}^{K} k \bar{t}_k \bar{z}^k + s + \sum_{k=1}^{\infty} \frac{\bar{v}_k}{\bar{z}^k}. \]

Given two integers \( r_1 \leq r_2 \) we denote by \( V[r_1, r_2] \) the set of Laurent polynomials of the form

\[ c_{r_1} p^{r_1} + c_{r_1+1} p^{r_1+1} + \cdots + c_{r_2} p^{r_2}. \]

Let us look for solutions of (11) such that \( z \) and \( \bar{z} \) are meromorphic functions of \( p \) with possible poles at \( p = 0 \) and \( p = \infty \) only. Then, as a consequence of the assumptions (5), (8) and (12), from (11) it follows that

\[ z \in V[1 - K, 1], \quad \bar{z} \in V[-1, K - 1]. \]

(13)

The equation \( \bar{m} = m \) is equivalent to the system:

\[ \bar{m}_{\geq 1} = m_{\geq 1}, \]

(14)

\[ \bar{m}_0 = m_0, \]

(15)

\[ \bar{m}_{\leq -1} = m_{\leq -1}. \]

(16)

If we now set

\[ m = \bar{m} = \sum_{k=1}^{K} k t_k (z^k)_{\geq 1} + \bar{m}_0 + \sum_{k=1}^{K} k \bar{t}_k (\bar{z}^k)_{\leq -1}, \]

(17)

with

\[ \bar{m}_0 = s + \sum_{k=1}^{K} k \bar{t}_k (\bar{z}^k)_0. \]

it can be easily seen that \( \bar{m} \) has the required expansion of the form (12) provided \( z \) and \( \bar{z} \) satisfy (5) and (8). On the other hand, the expression (17)
for \( m \) has an expansion of the form (12) if the residue of \( \frac{m}{z} \) corresponding to its Laurent expansion in powers of \( z \) verifies

\[
\text{Res}(\frac{m}{z}, z) = s. \tag{18}
\]

Hence the problem reduces to finding \( z \) and \( \bar{z} \) satisfying (5),(8),(18) and

\[
z = \frac{m}{\bar{z}}. \tag{19}
\]

In view of (13) we look for \( z \) and \( \bar{z} \) of the form

\[
z = r p + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}},
\]

\[
\bar{z} = \frac{r}{p} + \bar{u}_0 + \cdots + \bar{u}_{K-1} p^{K-1}. \tag{20}
\]

Now, in order to prevent \( z \) from having poles different from \( p = 0 \) and \( p = \infty \) we have to impose

\[
m(p_i) = 0, \tag{21}
\]

where \( p_i \) denote the \( K \) zeros of

\[
r + \bar{u}_0 p + \cdots + \bar{u}_{K-1} p^K = 0.
\]

In this way by using the expression (17) of \( m \), the only variables appearing in (19) are

\[
(p, t, \bar{t}, r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1}), \quad w_i := \bar{u}_i.
\]

Thus, by identifying coefficients of the powers \( p^i, i = 1 - K, \ldots, 1 \) we get \( K+1 \) equations which together with the \( K \) equations (21) determine the \( 2K+1 \) unknowns variables \( (r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1}) \) as functions of \( (t, \bar{t}) \). Moreover, provided \( r \) is a real coefficient, the equations (11) are invariant under the transformation

\[
T f(p) = f\left(\frac{\bar{1}}{p}\right).
\]
Hence if \((r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1})\) solves \((19)\) so does

\((r, w_0, \ldots, \bar{w}_{K-1}, \bar{u}_0, \ldots, \bar{u}_{K-1})\).

Therefore, if both solutions are close enough, they coincide and consequently

\(w_i = \bar{u}_i\), as required.

To complete our proof we must show that \((18)\) is satisfied too. To do that let us take two circles \(\gamma (|p| = r)\) and \(\gamma' (|p| = r')\) in the complex \(p\)-plane and denote by \(\Gamma\) and \(\Gamma'\) their images under the maps \(z = z(p)\) and \(\bar{z} = \bar{z}(1/p)\), respectively. Notice that due to \((5)\) and \((8)\), the curves \(\gamma\) and \(\gamma'\) have positive orientation if \(\gamma\) and \(\gamma'\) have positive and negative orientation, respectively. Then we have

\[
\text{Res}(\frac{m}{z}, z) - \text{Res}(\frac{\bar{m}}{\bar{z}}, \bar{z}) = \frac{1}{2i\pi} \oint_{\Gamma} \frac{m}{z} \, dz - \frac{1}{2i\pi} \oint_{\Gamma'} \frac{\bar{m}}{\bar{z}} \, d\bar{z}
\]

\[
= \frac{1}{2i\pi} \oint_{\gamma} \bar{z} \partial_p z \, d\bar{p} - \frac{1}{2i\pi} \oint_{\gamma'} z \partial_p z \, d\bar{p} = \frac{1}{2i\pi} \oint_{\gamma} \partial_p (\bar{zz}) \, d\bar{p} = 0,
\]

where we have taken into account that the integrands are analytic functions of \(p\) in \(\mathbb{C} - \{0\}\) and that \(\gamma\) and the opposite curve of \(\gamma'\) are homotopic with respect to \(\mathbb{C} - \{0\}\). Therefore, as we have already proved that \(\bar{m}\) has an expansion of the form \((12)\), we deduce

\[
\text{Res}(\frac{m}{z}, z) = \text{Res}(\frac{\bar{m}}{\bar{z}}, \bar{z}) = s,
\]

so that \((18)\) follows.

Let us illustrate the method with the case \(K = 2\). The polynomial \(p \bar{z}\) has two zeros at the points

\[
p_1 = \frac{-\bar{u}_0 + \sqrt{\bar{u}_0^2 - 4r \bar{u}_1}}{2 \bar{u}_1}, \quad p_2 = \frac{-\bar{u}_0 - \sqrt{\bar{u}_0^2 - 4r \bar{u}_1}}{2 \bar{u}_1},
\]

and from \((21)\) we get two equations which lead to

\[
-2r^2 t_2 \bar{u}_0^3 + 4r^3 t_2 \bar{u}_0 \bar{u}_1 + r^2 t_2 u_0 \bar{u}_0^2 \bar{u}_1 + 4r t_2 u_0 \bar{u}_0^2 \bar{u}_1 - r^2 t_1 \bar{u}_1^2 - 4r^2 t_2 u_0 \bar{u}_1^2

- s \bar{u}_0 \bar{u}_1^2 - \bar{t}_1 \bar{u}_0^2 \bar{u}_1^2 - 2 \bar{t}_2 \bar{u}_0^2 \bar{u}_1^2 + r \bar{t}_1 \bar{u}_1^3 = 0,
\]

\((22)\)
\[-2 r^3 t_2 \bar{u}_0^2 + 2 r^4 t_2 \bar{u}_1 + r^2 t_1 \bar{u}_0 \bar{u}_1 + 4 r^2 t_2 u_0 \bar{u}_0 \bar{u}_1 - r s \bar{u}_1^2 - r \bar{\bar{t}}_1 \bar{u}_0 \bar{u}_1^2 \]
\[-2 r \bar{\bar{t}}_2 \bar{u}_0^2 \bar{u}_1^2 - 2 r^2 \bar{\bar{t}}_2 \bar{u}_1^3 = 0. \quad (23)\]

Identification of the powers of $p$ in (19) implies
\[p : \quad -2 r^2 t_2 + r \bar{u}_1 = 0,\]
\[p^0 : \quad 2 r^2 t_2 \bar{u}_0 - r t_1 \bar{u}_1 - 4 r t_2 u_0 \bar{u}_1 + u_0 \bar{u}_1^2 = 0,\]
\[p^{-1} : \quad -2 r^2 t_2 \bar{u}_0^2 + 2 r^3 t_2 \bar{u}_1 + r t_1 \bar{u}_0 \bar{u}_1 + 4 r t_2 u_0 \bar{u}_0 \bar{u}_1 - s \bar{u}_1^2 - \bar{\bar{t}}_1 \bar{u}_0 \bar{u}_1^2 \]
\[= -2 \bar{\bar{t}}_2 \bar{u}_0^2 \bar{u}_1^2 - 4 r \bar{\bar{t}}_2 \bar{u}_1^3 + u_1 \bar{u}_1^3 = 0. \quad (24)\]

Then by solving equations (22)-(24) we get the solution:
\[z = \frac{p \sqrt{s}}{\sqrt{1 - 4 t_2 t_2}} + \frac{2 \sqrt{s} \bar{\bar{t}}_2}{p \sqrt{1 - 4 t_2 t_2}} - \frac{\bar{\bar{t}}_1 + 2 t_1 \bar{\bar{t}}_2}{-1 + 4 t_2 \bar{\bar{t}}_2}, \quad (25)\]
which corresponds to the conformal map describing an ellipse growing from a circle [6]

**Solutions for $K \geq 3$**

Exact solutions associated to arbitrary values of $K$ can be found from the previous scheme. However in order to avoid complicated expressions, we set
\[t_1 = t_2 = \cdots = t_{K-1} = \bar{\bar{t}}_1 = \bar{\bar{t}}_2 = \cdots = \bar{\bar{t}}_{K-1} = 0,\]
and look for particular solutions satisfying
\[u_1 = u_2 = \cdots = u_{K-2} = \bar{u}_1 = \bar{u}_2 = \cdots = \bar{u}_{K-2} = 0,\]
or equivalently
\[z = r p + \frac{u_{K-1}}{p^{K-1}}, \quad \bar{z} = \frac{r}{p} + \bar{u}_{K-1} p^{K-1}. \quad (26)\]
Under the previous assumptions and from (17) we have that

\[ m = Kt_K r^K p^K + s + K^2 \bar{t}_K r^{K-1} \bar{u}_{K-1} + \frac{K \bar{t}_K r^K}{p^K}. \]  

(27)

Thus, we see that (21) leads us to a unique equation since from (27) it follows that \( m \) depends on \( p \) through \( p^K \). Furthermore, if \( p_i \) satisfies \( \bar{z}(p_i) = 0 \), then

\[ p_i^K = -\frac{r}{\bar{u}_{K-1}}. \]

Therefore, (21) becomes

\[ s - \frac{K r^{K+1} t_K}{\bar{u}_{K-1}} + (K - 1) K r^{K-1} \bar{t}_K \bar{u}_{K-1} = 0. \]  

(28)

On the other hand, it is easy to see that

\[ \frac{m}{\bar{z}} = \frac{K r^K t_K}{\bar{u}_{K-1}} p + \left( K^2 r^{K-1} \bar{t}_K + \frac{s \bar{u}_{K-1} - K r^{K+1} t_K}{\bar{u}_{K-1}^2} \right) \frac{1}{p^{K-1}}, \]

consequently, by equating coefficients and taking (26) into account, we find that (19) leads to two equations only. More precisely

\[ p : \quad r = \frac{K r^K t_K}{\bar{u}_{K-1}}, \]  

(29)

\[ p^{-(K-1)} u_{K-1} = K^2 r^{K-1} \bar{t}_K + \frac{s \bar{u}_{K-1} - K r^{K+1} t_K}{\bar{u}_{K-1}^2}. \]

Then, we get three equations for the three unknowns \( r, u_{K-1}, \bar{u}_{K-1} \), which proves that there exists a solution of the form (26). In fact, by solving (28)-(29) we find that

\[ z = r p + \frac{K \bar{t}_K r^{K-1}}{p^{K-1}}, \]  

(30)

with \( r \) satisfying the implicit equation

\[ K^2 (K - 1) t_K \bar{t}_K r^{2(K-1)} - r^2 + s = 0. \]  

(31)

Figures 1 and 2 show examples of the evolution of the curve \( z(p) \), \(|p| = 1\) as \( s \) grows and \( t_K \) is kept fixed.
In [1] it was proved that there is a dToda $\tau$-function associated to each analytic curve $z = z(p), \ |p| = 1$, given by

$$2 \log \tau = -\frac{1}{2}s^2 + s v_0 - \frac{1}{2} \sum_{k \geq 1} (t_k v_k + \bar{t}_k \bar{v}_k), \quad (32)$$

where $v_k$ are the coefficients of the expansion (6), and $v_0$ is determined by

$$\frac{\partial v_0}{\partial s} = \log r^2, \quad v_0 = \frac{\partial \log \tau}{\partial s}. \quad (33)$$
For the class of solutions (30) we have
\[
2 \log \tau = -\frac{1}{2} s^2 + s v_0 - \frac{1}{2} (t_K v_K + \bar{t}_K \bar{v}_K),
\] (34)
and from (11) and (30) it follows that
\[
v_K = \frac{1}{2i\pi} \oint_{\Gamma} \bar{z} z^K \, d\bar{z} = \frac{1}{2i\pi} \oint_{\Gamma} \bar{z}(p)z(p)^K (r - (K - 1) \frac{t_K - 1}{p^K}) \, dp
\]
\[
= \frac{(r^2 - s)(Ks - (K - 2)t^2)}{2K(K - 1)s}.
\] (35)

On the other hand by differentiating (34) with respect to \(s\) and by taking into account (33) one finds
\[
v_0 = -s + s \log r^2 + \frac{(K - 2)(K - 1)K|t_K|^2(Ks - (K - 2)t^2)r^{2K}}{2((K - 1)^2K^2|t_K|^2r^{2K} - r^4)}
\]
\[
+ \frac{(K - 2)(s - r^2)}{2(K - 1)K} \left( K + \frac{(K - 2)r^4}{(K - 1)^2K^2|t_K|^2r^{2K} - r^4} \right).
\] (36)

Thus, (34)-(36) and (31) characterize the \(\tau\)-function of the curves determined by (30).

**Cusps**

The pictures of the curves associated with (30)-(31) show the presence of cusps at some value of \(s\) for each fixed value of \(t_K\). Indeed by using the parametric equation \(p = e^{i\theta}, (0 \leq \theta \leq 2\pi)\) for the unit circle, we have that cusps on the curve \(z = z(p)\) appear at points where \(z_0 = 0, z_{\theta_0} \neq 0\) and \(z_{\theta_0}/z_{\theta_0}\) has a nonzero imaginary part. Therefore a necessary condition for \(p = p(\theta)\) is
\[
\frac{\partial z}{\partial p}(p) = 0, \quad |p| = 1.
\]

Thus from (30) we deduce
\[
p^K = K(K - 1)\bar{t}_K r^{K - 2};
\]
which together with the condition $|p| = 1$ requires that

$$r = (K(K - 1)|t_K|)^{-\frac{1}{K-2}},$$

(37)

at some value $s = s(t_K)$. But according to (31) one finds that this happens at the value $s_0$ given by

$$s_0 = \frac{K - 2}{K - 1}(K(K - 1)|t_K|)^{-\frac{2}{K-2}},$$

(38)

which is the point at which the profile of both positive branches of $r$, as functions of $s$, develop an infinite slope (see figure 3).

![Figure 3: The positive branches of $r(s)$ for $K = 10$](image)

Therefore, there are $K$ cusps given by the roots

$$z_j = \frac{K}{K - 1}\left(\frac{r^2 - s_0}{K t_K}\right)^{\frac{1}{K}},$$

(39)

which emerge when $s$ reaches the extreme value $s_0$ of the domain of existence of the two positive branches of $r$ as a function of $s$.

References


