DAMTP-2005-20

Paraconformal geometry of nth order ODEs, and exotic holonomy in dimension four

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Abstract

We characterise nth order ODEs for which the space of solutions M is equipped with a particular paraconformal structure in the sense of [1], that is a splitting of a tangent

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bundle as a symmetric tensor product of rank-two vector bundles. This leads to vanishing of $(n-2)$ invariants constructed out of the ODE.

If $n = 4$ the paraconformal structure is shown to be equivalent to the exotic $\mathcal{G}_3$ holonomy of Bryant. If $n = 4$ and M admits a torsion–free connection compatible with the paraconformal structure then the ODE is trivialisable by point transformations.

If $n = 2$ or 3 M admits an affine paraconformal connection with no torsion. In these cases additional constraints can be imposed on the ODE so that M admits a projective structure if $n = 2$, or an Einstein–Weyl structure if $n = 3$. The third order ODE can in this case be reconstructed from the Einstein–Weyl data.

1 Introduction

Consider a relation of the form

$$\Psi(x, y, \mathbf{t}) = 0 \tag{1.1}$$

between the variables $\mathbf{t} = (t_1, t_2, ...t_n)$ (local coordinates on an n-dimensional manifold M), and (x, y) (local coordinates on a two-dimensional manifold $\mathcal{Z}$, which we shall call the twistor space). For each fixed choice of (x, y) relation (1.1) defines a hypersurface in M. Conversely each choice of $\mathbf{t}$ defines a curve L_t in M. We can apply the implicit function theorem to (1.1), and regard L_t as a graph

$$x \longrightarrow (x, y = Z(x, \mathbf{t})). \tag{1.2}$$

Consider a system of algebraic equations consisting of $y = Z(x, \mathbf{t})$, and the first $(n-1)$ derivatives with respect to x. Solving this system for $\mathbf{t}$, and differentiating once more with respect to x yields

$$y^{(n)} := \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = F(x, y, y', ..., y^{(n-1)}), \tag{1.3}$$

where the explicit form of F is completely determined by (1.1).

Putting various geometric structures on M (which from now on will be identified with the space of solutions to the ODE (1.3)) imposes additional constraints on F. This idea goes back to Cartan [3], and his program of 'geometrising' ODEs. Extending Cartan's program to PDEs is possible, and underlies some approaches to general relativity [10], and other problems in mathematical physics [7].

A different approach based on twistor theory was suggested by Hitchin [11]. In this approach one works in a holomorphic category and $(x, y, \mathbf{t})$ are complex numbers. The graph (1.2) represents a compact holomorphic (i.e. rational) curve in $\mathcal{Z}$ with a prescribed normal bundle. The local differential geometry of M is encoded in a global embedding of the curve in $\mathcal{Z}$. The ODE (1.3) does not explicitly appear in the correspondence between M and $\mathcal{Z}$. The details of Hitchin's construction and its connection with the ODE approach have partially been worked out only for $n = 2$. In this case the exists an embedding of rational curve with a normal bundle $\mathcal{O}(1)$ in $\mathcal{Z}$ if and only if

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{11} - 4\frac{\mathrm{d}}{\mathrm{d}x}F_{01} - F_1\frac{\mathrm{d}}{\mathrm{d}x}F_{11} + 4F_1F_{01} - 3F_0F_{11} + 6F_{00} = 0, \tag{1.4}$$

where

$$F_0 = \frac{\partial F}{\partial y}, \qquad F_1 = \frac{\partial F}{\partial y'}, \qquad F_2 = \frac{\partial F}{\partial y''}, \qquad ..., \qquad F_{n-1} = \frac{\partial F}{\partial y^{(n-1)}},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \sum_{k=1}^{n-1} y^{(k)} \frac{\partial}{\partial y^{(k-1)}} + F \frac{\partial}{\partial y^{(n-1)}}.$$

The two-dimensional moduli space M of $\mathcal{O}(1)$ curves is in this case equipped with a projective structure, in a sense that the hyper-surfaces (curves) of constant (x, y) in (1.1) are geodesics of a torsion-free connection. Conversely, given a projective structure on M one defines $\mathcal{Z}$ as a quotient space of the foliation of $\mathbb{P}(TM)$ by the orbits of the geodesic flow. Each projective tangent space $\mathbb{P}(T_{\mathfrak{t}}M)$ maps to a rational curve with self-intersection number one in $\mathcal{Z}$.

The case $n = 3$ goes back to Cartan [3] and Chern [5], and was recently revisited in [15]. The conformal structure on M is defined by demanding that hyper-surfaces $\Sigma \subset M$ corresponding to points in $\mathcal{Z}$ are null. This conformal structure doesn't depend on $(x, y) \in \mathcal{Z}$, if $F(x, y, y', y'')$ satisfies a third–order differential constraint

$$\frac{1}{3} F_2 \frac{\mathrm{d}}{\mathrm{d}x} F_2 - \frac{1}{6} \frac{\mathrm{d}^2}{\mathrm{d}x^2} F_2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} F_1 - \frac{2}{27} (F_2)^3 - \frac{1}{3} F_2 F_1 - F_0 = 0. \tag{1.5}$$

This constraint has already appeared in the work Wünschmann [16]. One further condition

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} F_{22} - \frac{\mathrm{d}}{\mathrm{d}x} F_{12} + F_{02} = 0 \tag{1.6}$$

is imposed on F if hyper-surfaces Σ are totally geodesic with respect to some torsion-free connection D. The existence of a two-parameter family totally geodesic null hypersurface in M is equivalent to the vanishing of a trace-free part of symmetrised Ricci tensor of D. This is the Einstein–Weyl condition first introduced in [4]. The three-dimensional Einstein–Weyl spaces can therefore be obtained from a particular class of third-order ODEs (1.3). In the twistor approach [11] the moduli space of rational curves in $\mathcal{Z}$ with a normal bundle $\mathcal{O}(2)$ is automatically equipped with an EW structure, and all analytic EW structures locally arise in such way.

The only other case which has attracted some attention is $n = 4$. Bryant [2] has shown that there exist a correspondence between a class of fourth order ODEs, and exotic non-metric holonomies in dimension four. The conditions on F are only implicit in Bryant's work.

In Section 2 we shall generalise the Wünschmann condition (1.5) to $(n - 2)$ conditions in the case of nth order ODEs, and give an example of an ODE for which all these conditions are satisfied. We shall prove the following

Theorem 1.1 *Assume that space of solutions M to the nth order ODE (1.3) is equipped with an $SL(2,\mathbb{R})$ invariant paraconformal structure*

$$TM \cong \mathbb{R}^2 \odot \mathbb{R}^2 \odot ... \odot \mathbb{R}^2 = S^{n-1}(\mathbb{S}), \tag{1.7}$$

where $\mathbb{S}$ is a real two-dimensional vector bundle, and $\odot$ denotes symmetric tensor product. Then F satisfies $(n-2)$ conditions of the form

$$C_k\Big(F_i, \frac{\mathrm{d}F_i}{\mathrm{d}x}, ..., \frac{\mathrm{d}^{n-1}F_i}{\mathrm{d}x^{n-1}}\Big) = 0, \qquad i = 0, ..., n-1, \qquad k = 1, ...n-2. \tag{1.8}$$

Each expression C_k is a polynomial in the derivatives of F of order less than or equal to n, and equations (1.8) are invariant under point transformations on $\mathcal{Z}$.

In Section 3 we shall show that the paraconformal structure (1.7) exists on the moduli space of rational curves with normal bundle $\mathcal{O}(n-1)$ in a complex surface, which leads to a twistorial interpretation of the constraints on F.

In Section 4 we shall discuss the case $n = 4$, where the existence of the paraconformal structure is equivalent to the existence of the torsion–free connection with the $\mathcal{G}_3$ holonomy on the space of solutions to (1.3).

Theorem 1.2 *Let M be the space of solutions to the fourth order ODE*

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = F(x, y, y', y'', y''').$$

The following conditions are equivalent

1. *M admits the paraconformal structure (1.7).*

2. *M admits a torsion–free connection with holonomy $\mathcal{G}_3$*

3. *F satisfies a pair of third order PDEs*

$$\frac{11}{1600}(F_3)^4 - \frac{9}{50}(F_3)^2\frac{\mathrm{d}}{\mathrm{d}x}F_3 - \frac{1}{200}(F_3)^2 F_2 + \frac{21}{100}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_3\right)^2 + \frac{1}{50}\left(\frac{\mathrm{d}}{\mathrm{d}x}F_3\right)F_2 \tag{1.9}$$

$$-\frac{9}{100}(F_2)^2 + \frac{7}{20}F_3\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_3 - \frac{1}{5}\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_3 + \frac{3}{10}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_2 - \frac{1}{4}F_3\frac{\mathrm{d}}{\mathrm{d}x}F_2 - F_0 = 0,$$

$$\frac{9}{4}F_3\frac{\mathrm{d}}{\mathrm{d}x}F_3 - \frac{3}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_3 + 3\frac{\mathrm{d}}{\mathrm{d}x}F_2 - \frac{3}{8}(F_3)^3 - \frac{3}{2}F_2F_3 - 3F_1 = 0. \tag{1.10}$$

(differentiating the second condition w.r.t x and subtracting its constant multiple from the first one leads to a couple of 3rd order PDEs for F).

In Section 5 we shall study connections preserving the paraconformal structure, and show that they must necessarily have torsion if $n = 4$.

Theorem 1.3 *If M admits the paraconformal structure (1.7) and*

$$D : \Gamma(\mathbb{S}^k) \longrightarrow \Gamma(\mathbb{S}^k \otimes T^*M) = \Gamma(\mathbb{S}^{k+n-1}),$$

where $\mathbb{S}^k = \mathbb{S}^{\otimes k}$, $k = 0, 1, ..., n-1$, is a torsion–free connection preserving the paraconformal structure then additional constraints (5.5, A21–A25), with (P, Q) given by (2.5), need to be satisfied.

In particular the ODE (1.3) is trivalisable by point transformations if $n = 4$, i. e. it is equivalent to

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = 0.$$

If $n < 4$ then D always exists.

Finally in Section 6 we shall concentrate on the case $n = 3$, where the paraconformal structure in conformal in the usual sense, and the condition for F is the Wünschmann condition (1.5). We shall give an algorithm of determining the third order ODE satisfying (1.6), and (1.5) from a given Einstein–Weyl structure, based on the Lax formulation of the Einstein–Weyl conditions.

Most calculations leading to invariants like (1.9, 1.10) were preformed (or checked) using MAPLE. The resulting long expressions are usually unilluminating. They are nevertheless useful in constructing explicit examples like (2.6), and we have decided to include them in the paper. Readers who want to verify our calculations can obtain the MAPLE programs from us.

2 Wünschmann invariants

In this Section we shall establish Theorem 1.1, and give an example of ODE which leads to a paraconformal structure for any n. First we need to introduce the notation.

The isomorphism (1.7) identifies each tangent space $T_{\mathbf{t}}M$ with the space of homogeneous $(n-1)$th order polynomials in two variables

$$T \in TM \longrightarrow \mathbf{t} = t^{A_1 A_2 ... A_{n-1}} z_{A_1} z_{A_2} ... z_{A_{n-1}}, \qquad A_1, A_2, ..., A_{n-1} = 0, 1$$

where $z_{A_i} = (z_0, z_1) \in \mathbb{S} = \mathbb{R}^2$, and $t^{A_1A_2...A_{n-1}}$ is symmetric in its indices. The bold letters denote homogeneous polynomials. They can be tensored, and the resulting expressions need not be symmetric, e.g

$$\mathbf{r} \otimes \mathbf{s} = r^{A_1A_2...A_i} s^{B_1B_2...B_j} z_{A_1} z_{A_2} ... z_{A_i} \hat{z}_{B_1} \hat{z}_{B_2} ... \hat{z}_{B_j}.$$

The summation convention is used unless stated otherwise. Let $V_{n-1} \subset \mathbb{R}[z_0, z_1]$ be an n-dimensional space of homogeneous polynomials of degree $(n-1)$. Let $\mathbf{t} \in V_{n-1}$. The space V_{n-1} is an $SL(2,\mathbb{R})$ module, and the infinitesimal action of $SL(2,\mathbb{R})$ is given by $\mathbf{t} \rightarrow H(\mathbf{t})$, where

$$H = H_A{}^B z_B \frac{\partial}{\partial z_A} \in \mathbf{sl}(2,\mathbb{R}),$$

and $H_A{}^B$ is one of the following matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For each $p \geq 0$ define a linear, $SL(2,\mathbb{R})$ equivariant mapping $V_r \otimes V_s \longrightarrow V_{r+s-2p}$ given by

$$< \mathbf{t}, \mathbf{s} >_p = \varepsilon_{A_1B_1} \varepsilon_{A_2B_2} ... \varepsilon_{A_pB_p} \frac{\partial^p \mathbf{t}}{\partial z_{A_1} ... \partial z_{A_p}} \frac{\partial^p \mathbf{s}}{\partial z_{B_1} ... \partial z_{B_p}}, \tag{2.1}$$

where

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a symplectic form on $\mathbb{R}^2$. In particular $<,>_{n-1}$: $V_{n-1} \times V_{n-1} \rightarrow \mathbb{R}$ is symmetric or skew-symmetric (depending on n) linear form on V_{n-1}. For $m = 0, 1, ..., n-1$ define $C_m \subset V_{n-1}$ to be a two-dimensional *cone of order* m, given by all polynomials $\mathbf{t} = \mathbf{p}^m \mathbf{r}$, where $\mathbf{p} \in V_1$, and $\mathbf{r} \in V_{n-m-1}$.

Proof of Theorem 1.1. All elements of the spin space $\mathbb{S} = \mathbb{R}^2$ correspond to first order homogeneous polynomials in z_A. If $(p^0, p^1) \in \mathbb{R}^2$, then $\mathbf{p} = p^0 z_0 + p^1 z_1 \in V_1$. Let $\mathbf{p} \in V_1$, and let $\mathbf{q} = \mathbf{p}'$. The spin space $\mathbb{R}^2$ is two-dimensional therefore

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}x} = P\mathbf{p} + Q\mathbf{q} \tag{2.2}$$

for some $P, Q \in V_0$. Consider an element T of C_{n-1}, (a maximally null vector). The maximally null vectors correspond to polynomials $\mathbf{T} = \mathbf{p}^{n-1}$ with a repeated root of multiplicity $(n-1)$. Each solution $y = Z(x, \mathbf{t})$ defines a section of T^*M given by

$$E = \mathrm{d}_{x=\mathrm{const}} Z.$$

Assume that E is a maximally null one-form (an element of C_{n-1}), and construct a frame of n one-forms on T^*M given by

$$E, E', E'', ..., E^{(n-1)}.$$

The corresponding polynomials are of the form

$$\begin{aligned}
\mathbf{E} &= \mathbf{p}^{n-1} \\
\mathbf{E}' &= 0 + a_{11}\mathbf{p}^{n-2}\mathbf{q} \\
\mathbf{E}'' &= a_{20}\mathbf{p}^{n-1} + a_{21}\mathbf{p}^{n-2}\mathbf{q} + a_{22}\mathbf{p}^{(n-3)}\mathbf{q}^2 \\
&. \\
&. \\
&. \\
\mathbf{E}^{(n-1)} &= a_{(n-1)0}\mathbf{p}^{n-1} + a_{(n-1)1}\mathbf{p}^{n-2}\mathbf{q} + ... + a_{(n-1)(n-1)}\mathbf{q}^{n-1},
\end{aligned}$$

where

$$\mathbf{E}^{(i)} = \sum_{k=0}^{i} a_{ik}\mathbf{p}^{n-1-k}\mathbf{q}^k, \qquad i = 0, ..., n-1.$$

The upper triangular matrix (a_{ij}) can be computed using (2.2). It depends on P, Q and their derivatives with respect to x. The nth derivative of $\mathbf{E}$ with respect to x is given by

$$\mathbf{E}^{(n)} = a_{n0}\mathbf{p}^{n-1} + a_{n1}\mathbf{p}^{n-2}\mathbf{q} + ... + a_{n(n-1)}\mathbf{q}^{n-1}.$$

Remembering that $y = Z(x, \mathbf{t})$ is a solution to (1.3), and using a chain rule we express $\mathbf{E}^{(n)}$ as a linear combination of $\mathbf{E}, \mathbf{E}', ..., \mathbf{E}^{(n-1)}$ by

$$\mathbf{E}^{(n)} = \sum_{i=0}^{n-1} F_i \mathbf{E}^{(i)}. \tag{2.3}$$

This gives rise to

$$a_{nj} = \sum_{i=0}^{n-1} F_i a_{ij}. \tag{2.4}$$

Solving these n equations for P and Q yields

$$\begin{aligned}
Q &= \frac{2}{n(n-1)} F_{n-1} \\
P &= \frac{1}{n(n^2-1)} \Big(\frac{(3n-1)(n-2)}{n(n-1)} F_{n-1}^2 + 6F_{n-2} - 2(n-2)\frac{\mathrm{d}}{\mathrm{d}x} F_{n-1} \Big)
\end{aligned} \tag{2.5}$$

(the calculations leading to these formulae are presented in Appendix). The remaining equations imply vanishing of $(n-2)$ expressions constructed out of F. Each expression is a polynomial C_k in the derivatives of F of order less than or equal to n, of the form (1.8). This characterises a class of ODEs (1.3) such that their solution spaces admit a para-conformal structure (1.7).

It follows from the construction (and it may be checked if desired) that vanishing of these expressions is invariant under point transformations

$$\hat{x} = \hat{x}(x, y), \qquad \hat{y} = \hat{y}(x, y).$$

In fact more than that is true, and one can verify invariance under a larger pseudo-group of contact transformations.

□

If $n = 2$ the paraconformal structure always exists. If $n = 3$ the condition (1.8) is given by (1.5). If $n = 4$ we have two conditions given by (1.9) and (1.10). The polynomials corresponding to $n = 5$ are given in the Appendix.

There are no terms of the form F_{ij} in C_k, so the coefficients of the polynomials C_k are determined by looking at the special case of linear homogeneous equations, where

$$F = p_{n-1}(x)y^{(n-1)} + ... + p_0(x)y,$$

and $F_k = p_k(x)$.

2.1 Example

The general case of (2.4) regarded as an overdetermined system of PDEs for F appears to be intractable. To find some examples we seek $F = F(y^{(n-1)}, x)$. Let $z := y^{(n-1)}$. The ODE (1.3) reduces a 1st order ODE, and a sequence of quadratures

$$z' = F(z, x), \qquad y(x) = \int^x \int^{x_{n-1}} ... \int^{x_2} z(x_1)\mathrm{d}x_1\mathrm{d}x_2...\mathrm{d}x_{n-1}.$$

We use MAPLE to verify that all constraint equations (1.8) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{n-1} = \frac{1}{n}(F_{n-1})^2, \qquad \text{where now } \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + F\frac{\partial}{\partial z}.$$

We therefore need to solve

$$\frac{\partial^2 F}{\partial z \partial x} = \frac{1}{n}\Big(\frac{\partial F}{\partial z}\Big)^2 - F\frac{\partial^2 F}{\partial z^2}.$$

The Legendre transformation $s = F_z, G(s,x) = F(z(s,x),x) - sz(s,x)$ gives a linear equation for G. This transform can the be inverted, and the solution can be found for an arbitrary 'initial data' $F(z,0)$. To write down an explicit example make a further assumption that F is independent on x, which yields

$$F(z) = (az+b)^{\frac{n}{n-1}}.$$

Redefining $y(x)$ by a point transformation we can set $b = 0$, so the nth order ODE is

$$y^{(n)} = (ay^{(n-1)})^{\frac{n}{n-1}}. \tag{2.6}$$

The corresponding n-parameter family of solutions to (1.3) is readily found

$$y = c_0 + c_1 x + ... + c_{n-2}x^{n-2} - \frac{(n-1)^{(n-1)}}{a^n(n-2)!}\ln(x + c_{n-1})$$

If $n = 3$ the space of solutions to (1.3) is equipped with the NIL Einstein–Weyl structure [15] (see also Section 6.)

3 Twistor theory

We shall now show how, in the real analytic case, the paraconformal structure (1.7) on M can be encoded in a holomorphic geometry of rational curves embedded in a complex surface $\mathcal{Z}$. The nth order ODE on $\mathcal{Z}$ will then implicitly be given by the embedding $L \subset \mathcal{Z}$, provided that L has self-intersection number n.

In this section we regard (1.1) as a holomorphic relation between complex coordinates $(x, y, \mathbf{t})$. It gives us an $(n+1)$-complex dimensional hyper-surface (called the correspondence space) $\mathcal{F} \subset \mathcal{Z} \times M$ as a zero set of $\Psi : \mathcal{Z} \times M \to \mathbb{C}$. We shall examine a geometry of this correspondence space

$$\mathcal{F} = \{(z, \mathbf{t}) \in \mathcal{Z} \times M | z \in L_t\}$$

and a double fibration

$$M \xleftarrow{p} \mathcal{F} \xrightarrow{q} \mathcal{Z}. \tag{3.1}$$

The surface $\mathcal{Z}$ will eventually become a twistor space of (1.3). We shall however set up a more general correspondence, and consider Legendrian curves in three-dimensional twistor spaces.

Let Y be a complex three-fold with an embedded rational curve L with a normal bundle $N = \mathcal{O}(n-2) \oplus \mathcal{O}(n-2)$. We have $H^1(\mathbb{CP}^1, \mathcal{O}(n-2) \oplus \mathcal{O}(n-2)) = 0$, and so the moduli space of such curves in Y is a manifold $\mathcal{M}$ of dimension equal to

$$\dim H^0(\mathbb{CP}^1, \mathcal{O}(n-2) \oplus \mathcal{O}(n-2))) = 2n - 2.$$

Now we restrict our attention to a moduli space M of contact (Legendrian) curves with normal bundle N. The canonical line bundle of holomorphic three-forms on Y restricted to a curve L is

$$\kappa(Y) = T^*L \otimes \Lambda^2(N^*) = \mathcal{O}(2 - 2n),$$

since $T^*\mathbb{CP}^1 = \mathcal{O}(-2)$. From the general theory of contact structures it follows that the contact line bundle is given by ${L_c}^2 = \kappa(Y)$. Now pick a section τ of ${L_c}^* = \mathcal{O}(n-1)$ (a contact one-form), and contract it with a tangent vector V to a rational curve to get

$$V \lrcorner \tau \in \mathcal{O}(n-1) \otimes \mathcal{O}(-2).$$

The vanishing of this section (the Legendrian condition) gives $\dim H^0(\mathbb{CP}^1, \mathcal{O}(n-3)) = n - 2$ conditions on $\mathcal{M}$. Therefore the dimension of the moduli space M of Legendrian curves is

$$\dim M = (2n-2) - (n-2) = n.$$

This can be summarised by the double fibration picture

$$M \xleftarrow{\hat{p}} \hat{\mathcal{F}} \xrightarrow{\hat{q}} Y. \tag{3.2}$$

The curves $\hat{q}(\hat{p}^{-1}(\mathbf{t})) \cong \mathbb{CP}^1$ are Legendrian with respect to the contact form on Y. Merkulov's generalisation [12, 13] of Kodaira theorems guarantees that the moduli space M of Legendrian rational curves is stable under small deformations of Y.

Consider the special case $Y = \mathbb{P}(T\mathcal{Z})$. A rational curve L with normal bundle $\mathcal{O}(n-1)$ in $\mathcal{Z}$ has a natural lift $\hat{L}$ to Y, given by $z \in L \to (z, \dot{z} \in T_z L)$. The lifted curves are Legendrian with respect to the canonical contact structure on a projectivised tangent bundle. The double fibration (3.2) reduces to (3.1).

The existence of the complexiifed paraconformal structure (1.7) follows from the structure of the normal bundle. From Kodaira theory, since the appropriate obstruction groups vanish, we have

$$T_t M = \Gamma(L_t, N_t) = S^{n-1}(\mathbb{S}_t), \qquad \mathbb{S}_t = \mathbb{C}^2, \tag{3.3}$$

where N_t is the normal bundle to the rational curve $L_t = \mathbb{CP}^1$ in $\mathcal{Z}$ corresponding to the point $t \in M$. The nontrivial examples of ODEs satisfying all the constraints (1.8) can therefore be constructed by applying algebraic operations on a rational curve embedded in a total space of $\mathcal{O}(N)$ for N sufficiently large [9].

The correspondence space $\mathcal{F} = M \times \mathbb{CP}^1$ is equipped with a canonical $(n-1)$ dimensional distribution $\mathcal{D}$, such that $\mathcal{Z} = \mathcal{F}/\mathcal{D}$. The normal bundle to a rational curve $L_t := q(p^{-1}(\mathbf{t}))$ consists of vectors tangent to $\mathbf{t}$ (horizontally lifted to $T_{\mathbf{t},\lambda}\mathcal{F}$) modulo $\mathcal{D}$. Therefore we have a sequence of sheaves over $\mathbb{CP}^1$

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathbb{C}^n \longrightarrow \mathcal{O}(n-1) \longrightarrow 0.$$

The map $\mathbb{C}^n \longrightarrow \mathcal{O}(n-1)$ is given by $V^{A_1A_2...A_{n-1}} \longrightarrow V^{A_1A_2...A_{n-1}} z_{A_1} z_{A_2} ... z_{A_{n-1}}$. Its kernel consists of vectors of the form $z^{(A_1}\lambda^{A_2...A_{n-1})}$ with $\lambda^{A_2...A_{n-1}} \in \mathbb{C}^{n-1}$ varying. The twistor distribution is therefore $\mathcal{D} = O(-1) \otimes \mathrm{S}^{(n-2)}(\mathbb{C}^2)$. This distribution is the geodesic spray (5.3) if $n = 2$, or the Einstein–Weyl Lax pair (6.1) if $n = 3$.

4 Exotic $\mathcal{G}_3$ holonomy and fourth order ODEs

In this Section we shall make contact with Bryant's work [2] and show that if $n = 4$ the Wünshmann conditions (1.8) are equivalent to the existence of certain exotic holonomy on M.

Recall the notation introduced at the beginning of Section (2) and define $\mathcal{G}_k \subset GL(V_k)$ by

$$\mathcal{G}_k = \{g \in GL(V_k)|\ g(\mathbf{t}) \in C_k \ \text{ if } \ \mathbf{t} \in C_k\}.$$

Can $\mathcal{G}_{n-1}$ appear as a holonomy group of a torsion-free connection of an n–dimensional manifold? Bryant [2] has examined the Berger's criteria, and established that answer in 'no' if $n > 5$ (the case $n = 5$ is special, as five-dimensional representation of $SL(2,\mathbb{C})$ is a holonomy of a symmetric space $M = SL(3,\mathbb{C})/SL(2,\mathbb{C})$).

Let $<,>_2$ be given by (2.1), and let

$$g(X,X,X,X) :=<< X,X,>_2, < X,X,>_2>_2$$

be a $\mathcal{G}_3$ invariant quartic form on TM.

Definition 4.1 *The vector $X \in T_tM$ is null iff*

$$Q(X) := g(X,X,X,X) = 0.$$

The α–plane is a two–dimensional plane in T_tM spanned by vectors X, Y such that

$$Q(X + \lambda Y) = 0$$

for each value of a parameter λ.

The null vectors correspond to polynomials of the form $\mathbf{p}^2\mathbf{r}$. Vectors in an α–plane are then obtained by varying $\mathbf{r}$ and keeping $\mathbf{p}$ fixed.

Theorem 4.2 (Bryant [2]) *A four–dimensional manifold M admits a torsion free connection with holonomy $\mathcal{G}_3$ iff for every α–plane there exists a two–dimensional surface $\Sigma \subset M$ (called α–surface) tangent to this α–plane.*

The space of torsion–free $\mathcal{G}_3$ structures modulo diffeomorphisms depends on four arbitrary functions of three variables.

Bryant has also shown [2] that if M admits a $\mathcal{G}_3$ structure, then there exists a three–parameter family of α–surfaces. In the complexified category this family is parametrised by points of a complex three-fold Y, and the Legendrian $\mathcal{O}(2) \oplus \mathcal{O}(2)$ curves in Y correspond to points in M (compare this with the twistorial treatment in Section 3).

Proof of Theorem 1.2. The conditions (1.9, 1.10) are the invariants (1.8) with $n = 4$ which establishes the equivalence of (1) and (3).

To show (1) $\rightarrow$ (2) observe that the existence of the paraconformal structure (1.7) implies the existence of a symplectic structure ε up to scale on each spin space. This gives us the symmetric quartic form

$$g(X, X, X, X) = \varepsilon^{A_1B_1}\varepsilon^{A_2C_1}\varepsilon^{A_3D_1}\varepsilon^{B_2C_2}\varepsilon^{C_3D_3}\varepsilon^{D_3D_2}X_{A_1A_2A_3}X_{B_1B_2B_3}X_{C_1C_2C_3}X_{D_1D_2D_3}.$$

The quartic $Q(X) = 0$ selects null vectors, and α–planes. Theorem 4.2 asserts that these planes are integrable iff they come from a $\mathcal{G}_3$ structure. But they will always be integrable in the paraconformal case because of the following interpretation: Fixing a point in $Z \in \mathcal{Z}$ gives a three–dimensional surface in $N \subset M$, such that its normal ∇Z is a perfect cube. Fixing a point and a direction in $\mathcal{Z}$ gives an α–surface $\Sigma \subset N$ (think of Z, Z', Z'', Z''' as coordinates in M). This corresponds to fixing a point in $Y = P(T^*\mathcal{Z})$, and to Bryant's α–plane with normals Z, Z' which are gradients (so that it really is a surface).

It remains to demonstrate (2) $\rightarrow$ (1). Suppose we look directly for the quartic as

$$g = g_0 + g_1, \tag{4.1}$$

where g_0 is the form when the ODE is trivial ($F = 0$) and g_1 is a combination of all possible lower order terms (lower order in the sense of fewer derivatives of y):

$$\begin{aligned} g_0 &= 18\mathrm{d}y\, \mathrm{d}p\, \mathrm{d}q\, \mathrm{d}r - 9(\mathrm{d}y)^2\, (\mathrm{d}r)^2 + 3(\mathrm{d}p)^2\, (\mathrm{d}q)^2 - 8(\mathrm{d}p)^3\, \mathrm{d}r - 6\mathrm{d}y\, (\mathrm{d}q)^3, \\ g_1 &= \alpha(\mathrm{d}y)^4 + \beta(\mathrm{d}y)^3\, \mathrm{d}p + \gamma(\mathrm{d}y)^2\, (\mathrm{d}p)^2 + \delta(\mathrm{d}y)^3\, (\mathrm{d}q) + \epsilon\mathrm{d}y\, (\mathrm{d}p)^3 \\ &\quad + \xi(\mathrm{d}y)^2\, \mathrm{d}p\, \mathrm{d}q + \eta(\mathrm{d}y)^3\, \mathrm{d}r + \kappa(\mathrm{d}p)^4 + \gamma(\mathrm{d}y)\, (\mathrm{d}p)^2\, \mathrm{d}q + \mu(\mathrm{d}y)^2(\mathrm{d}q)^2 + \nu(\mathrm{d}y)^2\, \mathrm{d}p\, \mathrm{d}r \\ &\quad + \zeta(\mathrm{d}p)^3\, \mathrm{d}q + \pi\mathrm{d}y\, \mathrm{d}p\, (\mathrm{d}q)^2 + \theta\mathrm{d}y\, (\mathrm{d}p)^2\, \mathrm{d}r + \phi(\mathrm{d}y)^2\, \mathrm{d}q\, \mathrm{d}r, \end{aligned}$$

and $p = y', q = y'', r = y''', s = y'''' = F(x, y, p, q, r)$.

We need to fix 15 coefficients $(\alpha, ..., \phi)$. We impose

$$g' = \Lambda g$$

for some Λ and work systematically through the coefficients, fixing them in order (see Appendix for the details of this calculation). We solve equations (A1–A10) and (A11, A13). Now equation (A12) becomes (1.10). Then we solve (A14, A16). Now (A15) and (1.10) give (1.9).

The remaining conditions (A17–A20) give two more conditions on F, but we know these will be satisfied because of the paraconformal argument: once we impose (1.9,1.10) the quartic will exist and we can check that the coefficients we have found by a direct approach agree with what the paraconformal method has told us. Thus (1.9,1.10) are necessary and sufficient for integrability.

□

5 Torsion–free paraconformal connection

Let M admit a paraconformal structure, and let

$$D : \Gamma(\mathbb{S}^k) \longrightarrow \Gamma(\mathbb{S}^k \otimes T^*M) = \Gamma(\mathbb{S}^{k+n-1}), \qquad \text{where } \mathbb{S}^k = \mathbb{S}^{\otimes k}, \;\; k = 0, 1, ..., n-1$$

be a connection. In this section we shall show that if D preserves the paraconformal structure (1.7) then it neccesarily has torsion if $n > 3$.

Definition 5.1 *The connection D is called paraconformal if its action on elements of V_1 is given by*

$$D\mathbf{p} = U \otimes \mathbf{p} + V \otimes \mathbf{q}$$

for some $U, V \in \mathbb{S}^{n-1}$.

This definition implies that $D\mathbf{E} = (n-1)U \otimes \mathbf{E} + V \otimes \mathbf{E}'$. The torsion free condition becomes

$$U = A\mathbf{E} + B\mathbf{E}', \; V = (n-1)B\mathbf{E} + C\mathbf{E}'$$

for some $A, B, C \in V_0$. Demanding that the connection does not depend on (x, y) gives the consistency condition

$$(D\mathbf{p})'' = D(\mathbf{p}''), \tag{5.1}$$

which yields

$$\begin{aligned} 0 &= (D\mathbf{p})'' - D(P\mathbf{p} + Q\mathbf{q}) = (\frac{\mathrm{d}^2}{\mathrm{d}x^2} - Q\frac{\mathrm{d}}{\mathrm{d}x} - P)D\mathbf{p} - (DP) \otimes \mathbf{p} - (DQ) \otimes \mathbf{q} \\ &= (\alpha_1 \mathbf{E} + \alpha_2 \mathbf{E}' + \alpha_3 \mathbf{E}'' + \alpha_4 \mathbf{E}''') \otimes \mathbf{p} + (\beta_1 \mathbf{E} + \beta_2 \mathbf{E}' + \beta_3 \mathbf{E}'' + \beta_4 \mathbf{E}''') \otimes \mathbf{q} \\ &\quad -(\sum_{i=0}^{n-1} \mathbf{E}^{(i)} \frac{\partial P}{\partial y^{(i)}} \otimes \mathbf{p} + \sum_{i=0}^{n-1} \mathbf{E}^{(i)} \frac{\partial Q}{\partial y^{(i)}} \otimes \mathbf{q}). \end{aligned}$$

Coefficients of $\mathbf{E}^{(i)} \otimes \mathbf{p}$, and $\mathbf{E}^{(i)} \otimes \mathbf{q}$ have to vanish, which gives $2n$ conditions on F. Here (P, Q) are given by (2.5) and $\alpha_1, ..., \alpha_4, \beta_1, ..., \beta_4$ can be determined in terms of A, B, C, P, Q, and their derivatives.

$\mathbf{n} = \mathbf{2}$. In this case $\mathbf{E}''$ and $\mathbf{E}'''$ are determined in terms of $\mathbf{E}$ and $\mathbf{E}'$ according to (2.3), and therefore $\alpha_3, \alpha_4, \beta_3, \beta_4$ all vanish. There are no conditions on F arising from (5.1). However imposing the geodesic conditions on curves $Z = const$ yields (1.4). Let t^A and $z^A = \dot{t}^A$ be local coordinates on M and T_tM. The Christoffel symbols $\Gamma_{AB}^C = \Gamma_{AB}^C(\mathbf{t})$ of D are defined up to the projective equivalence

$$\Gamma_{AB}^C \sim \Gamma_{AB}^C + \delta_{(A}^C \omega_{B)}$$

for some $\omega_B = \omega_B(\mathbf{t})$. Let $t^C = t^C(\tau)$ be solutions to

$$\ddot{t}^C + \Gamma_{AB}^C \dot{t}^A \dot{t}^B = v\dot{t}^C, \qquad \dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{5.2}$$

where v is some function. These geodesic curves lift to the integral curves of the geodesic spray which is a projection of

$$L = z^A \frac{\partial}{\partial t^A} - \Gamma_{AB}^C z^A z^B \frac{\partial}{\partial z^C} \tag{5.3}$$

from TM to $\mathbb{P}(TM)$. Eliminating τ from (5.2) leads to a second order ODE for $t^1 = t^1(t^0)$ which is at most cubic in the first derivatives (the cubic term is given by $\varepsilon_{CD}\Gamma_{AB}^C z^A z^B z^D$, where z^A are homogeneous coordinates on $\mathbb{P}(TM)$, and $z^1/z^0 = \mathrm{d}t^1/\mathrm{d}t^0$). This ODE is dual to

(1.3) in the sense of Cartan. It could also be read off from the relation (1.1) by rewriting it as $t^1 = K(t^0, x, y)$, and eliminating (x, y) between K and its first two derivatives w.r.t t^0.

$\mathbf{n = 3}$. Now α_4 and β_4 vanish. The compatibility conditions in (5.1) fix A, B, C. Imposing the totally geodesic condition $C = 0$ on the null surfaces $Z = const$ gives the constraint (1.6). We shall come back to this case in the next Section.

$\mathbf{n \geq 4}$. The coefficients of $\mathbf{E}''' \otimes \mathbf{p}, \mathbf{E}''' \otimes \mathbf{q}$ and $\mathbf{E}'' \otimes \mathbf{p}$ fix (A, B, C) in terms of F and its derivatives and the coefficients of $\mathbf{E}'' \otimes \mathbf{q}, \mathbf{E}'$ and $\mathbf{E}$ give five equations (A21–A25) for F (the details are in the Appendix). In particular (A23) yields

$$\Big(\frac{6n-8}{n}\Big)F_{n-1}F_{(n-1)3} + (8-2n)\frac{\mathrm{d}}{\mathrm{d}x}F_{(n-1)3} - (2n-2)F_{(n-1)2} + 6F_{(n-2)3} = 0. \tag{5.4}$$

Proof of Theorem 1.3. If $n = 4$ the condition (5.4) reduces to

$$F_{33} = 0,$$

and the Wünschmann conditions (1.9, 1.10) now imply $A = B = C = 0$. Some computer algebra reduces (1.9, 1.10) to

$$F = \alpha(x) + \beta(x)y + \gamma(x)y' + \delta(x)y'' + \epsilon(x)y''',$$

where $\alpha(x), \delta(x), \epsilon(x)$ are arbitrary, and

$$\beta = \frac{11}{1600}\epsilon^4 - \frac{9}{50}\epsilon^2\epsilon' - \frac{1}{200}\epsilon^2\delta + \frac{21}{100}(\epsilon')^2 + \frac{1}{50}\epsilon'\delta - \frac{9}{100}\delta^2 + \frac{7}{20}\epsilon\delta'' - \frac{1}{5}\epsilon''' + \frac{3}{10}\delta'' - \frac{1}{4}\epsilon\delta'$$

$$\gamma = \frac{3}{4}\epsilon\epsilon' - \frac{1}{2}\epsilon'' + \delta' - \frac{1}{2}\delta\epsilon - \frac{1}{8}\epsilon^3.$$

We can however preform a point transformation (which is in fact fibre preserving)

$$y = a(x)\hat{y}(x) + b(x), \qquad x = c(\hat{x}),$$

and choose the functions (a, b, c) to set $\alpha = \delta = \epsilon = 0$. The resulting 4th order ODE (1.3) is therefore trivial up to point transformations.

□

If $n > 4$, then the coefficients of $\mathbf{E}^{(k)}$ with $k > 3$ give

$$\frac{\partial P}{\partial y^{(k)}} = \frac{\partial Q}{\partial y^{(k)}} = 0, \qquad k = 4, 5, ..., n-1. \tag{5.5}$$

The conditions (5.5) give the following

1. $n = 5$

$$F_{44} = 0, \tag{5.6}$$

2. $n = 6$

$$F_{55} = F_{45} = 3F_{44} - 8F_{53} = 0, \tag{5.7}$$

3. $n \geq 6$

$$F_{(n-1)k} = F_{(n-2)(k+1)} = 3F_{(n-2)4} - (n-2)F_{(n-1)3} = 0, \qquad k = 4, ..., n-1, \tag{5.8}$$

Equations (1.8) and (A21–A25) also have to be satisfied. If our constraints (1.8) are equivalent to the vanishing of the Doubrov invariants (given in the Theorem 1 of [6]) then (5.7, 5.8) and Theorem 2 in [6] imply that the ODE (1.3) is trivialisable by contact transformation (the case $n = 5$ requires more work using (A21–A25)). We have checked that our invariants coincide with those of Doubrov if $n = 3, 4$ and we have been able to show that if $n > 4$ then some of (1.8) coincide with some of the Doubrov's invariants, but the general proof is missing.

6 From Einstein–Weyl structures to third order ODEs

In three dimensions the existence a paraconformal structure (1.7) is equivalent to the existence of a conformal structure $[h]$ of signature $(+--)$. This is a well know fact based on representing vectors as symmetric matrices

$$X^a = (X^1, X^2, X^3) \longrightarrow X^{AB} = \begin{pmatrix} X^1 + X^2 & X^3 \\ X^3 & X^1 - X^2 \end{pmatrix} \in \Gamma(\mathbb{S} \otimes \mathbb{S}).$$

The matrices corresponding to null vectors (i.e $h(X, X) = 0, h \in [h]$) have vanishing determinant, and must have rank one. Therefore $X^{AB} = p^A p^B$ for such vectors.

Set $n = 3$, and assume that the 3rd order ODE (1.3) satisfies the Wünshmann condition (1.5). For each choice of (x, y) (1.1) defines a surface in M which is null w.r.t $[h]$. In the last section we have shown that if $n = 3$ the null surfaces $y = Z(t_a, x)$ are totally geodesic w.r.t some torsion–free connection D if $F(x, y, y', y'')$ satisfies the constraint (1.6), and it is well known [4, 11, 14] that the Einstein–Weyl (EW) equations are equivalent to the existence of a two dimensional family of surfaces $\Sigma \subset M$ which are null with respect to $[h]$, and totally geodesic with respect to D.

Let M be a 3-dimensional manifold with a torsion-free connection D, and a conformal structure $[h]$ of signature $(++-)$ which is compatible with D in a sense $Dh = \omega \otimes h$ for some one-form ω. Here $h \in [h]$ is a representative metric in a conformal class. If we change this representative by $h \rightarrow \psi^2 h$, then $\omega \rightarrow \omega + 2\mathrm{d}\ln\psi$, where ψ is a non-vanishing function on W. A triple $(M, [h], D)$ is called a Weyl structure. The conformally invariant Einstein–Weyl (EW) equations state that the symmetrised part of the Ricci tensor of D is proportional to the representative of $[h]$.

Given a third order ODE which satisfies (1.5, 1.6), the EW structure can be reconstructed following the steps described in [4, 15]. The problem of reconstructing the ODE starting from a given EW structure was left open in these references. We shall present a method which reduces the problem of finding the allowed ODE to a system of linear PDEs. First recall the Lax representation for EW equation [8]. Let X_1, X_2, X_3 be three independent vector fields on M, and let e_1, e_2, e_3 be the dual one-forms. Assume that

$$h = e_2 \otimes e_2 - 2(e_1 \otimes e_3 + e_3 \otimes e_1)$$

and some one–form ω give an EW structure. Let $X(\lambda) = X_1 - 2\lambda X_2 + \lambda^2 X_3$ where $\lambda \in \mathbb{CP}^1$ is a projective coordinate on the fibres of $\mathbb{S} \rightarrow M$. Then $h(X(\lambda), X(\lambda)) = 0$ for all $\lambda \in \mathbb{CP}^1$ so $X(\lambda)$ determines a sphere of null vectors. The vectors $X_1 - \lambda X_2$ and $X_2 - \lambda X_3$ form a basis of the orthogonal complement of $X(\lambda)$. For each $\lambda \in \mathbb{CP}^1$ they span a null two-surface. Therefore the Frobenius theorem implies that the horizontal lifts to $\mathbb{S}$

$$L_0 = X_1 - \lambda X_2 + l_0 \partial_\lambda, \qquad L_1 = X_2 - \lambda X_3 + l_1 \partial_\lambda \tag{6.1}$$

satisfy $[L_0, L_1] = aL_0 + \beta L_1$ for some α, β which are linear in λ. The functions l_0 and l_1 are third order in λ, because the Möbius transformations of $\mathbb{CP}^1$ are generated by vector fields quadratic in λ.

To find the third order ODE corresponding to $([h], D)$ we construct two independent solutions $x(t_a, \lambda), y(t_a, \lambda)$ to the pair of linear PDEs

$$L_0 f = L_1 f = 0,$$

and eliminate λ between x and y. This gives $y = Z(x, t_a)$. Now we follow the prescription given in the introduction to produce the third order ODE. Both invariants (1.5) and (1.6) will be satisfied as a consequence of the EW condition.

As an example, consider the Einstein–Weyl $(+--)$ structure on Thurston's Nil manifold $S^1 \times \mathbb{R}^2$ [14, 15] given by

$$h = \alpha^2(\mathrm{d}t_2 + t_1\mathrm{d}t_3)^2 - 4\mathrm{d}t_1\mathrm{d}t_3, \qquad \omega = \alpha^2(\mathrm{d}t_2 + t_1\mathrm{d}t_3).$$

Choose the Lax pair

$$L_0 = \partial_1 + \alpha^{-1}\lambda\partial_2, \qquad L_1 = -\alpha^{-1}\partial_2 - \lambda(\partial_3 - t_1\partial_2) + \alpha\lambda\partial_\lambda$$

so that $[L_0, L_1] = 0$. We find a kernel of (L_0, L_1) to be

$$x = \lambda + \alpha t_3, \qquad y = \lambda t_1 - \alpha t_2 - \alpha^{-1}\ln\lambda$$

so that the totally geodesic surfaces are given by $y = Z(x, t_a)$ with

$$Z(x, t_a) = (x - \alpha t_3)t_1 - \alpha t_2 - \alpha^{-1}\ln(x - \alpha t_3).$$

The resulting third order ODE is

$$y''' = 2\sqrt{\alpha}(y'')^{3/2}$$

which is a special case of our general example (2.6) with $n = 3$.

Acknowledgements

We thank Robert Bryant for useful discussions, and Michael Eastwood for bringing our attention to the reference [6].

Appendix

Determining P and Q. The first step is to calculate recursive formulae for a_{ij}. This yields $a_{00} = 1$, and

$$\begin{aligned}
a_{ik} &= 0, \qquad \text{for} \qquad k > i, \\
a_{(i+1)k} &= (a_{ik})' + kQa_{ik} + (n-k)a_{i(k-1)} + (k+1)a_{i(k+1)}P, \qquad 0 \le k \le i-1, \\
a_{(i+1)i} &= (a_{ii})' + iQa_{ii} + (n-i)a_{i(i-1)}, \\
a_{(i+1)(i+1)} &= (n-i-1)a_{ii}.
\end{aligned}$$

These relations give

$$a_{ii} = \frac{(n-1)!}{(n-i-1)!}, \qquad a_{(i+1)i} = \frac{(i+1)i}{2} Q a_{ii}, \qquad a_{n(n-2)} = \alpha P + \beta Q^2 + \gamma Q',$$

where

$$\alpha = \frac{n(n+1)!}{6}, \qquad \beta = \frac{n!}{24}(3n-5)(n-1)(n-2), \qquad \gamma = \frac{n!}{6}(n-1)(n-2).$$

Solving the last two equations in (2.4) corresponding to $j = n-1$, and $j = n$ for P and Q yields (2.5).

Conditions for F. We shall give explicit forms of (1.8) for $n = 5$

$$\begin{aligned}
0 \;=\; & -\frac{1}{5}\frac{d^4}{dx^4}F_4 - \frac{2}{25}F_3\frac{d^2}{dx^2}F_4 - \frac{7}{25}\left(\frac{d}{dx}F_4\right)\frac{d}{dx}F_3 - \frac{28}{3125}(F_4)^5 + \frac{16}{25}\left(\frac{d}{dx}F_4\right)\frac{d^2}{dx^2}F_4 \\
& -F_0 + \frac{8}{25}F_4\frac{d^3}{dx^3}F_4 - \frac{1}{5}F_2F_3 - \frac{7}{100}F_2(F_4)^2 + \frac{1}{5}F_2\frac{d}{dx}F_4 - \frac{11}{125}F_4(F_3)^2 \\
& -\frac{141}{2500}F_3(F_4)^3 + \frac{137}{1250}(F_4)^3\frac{d}{dx}F_4 - \frac{9}{25}F_4\left(\frac{d}{dx}F_4\right)^2 - \frac{9}{50}F_4\frac{d^2}{dx^2}F_3 \\
& -\frac{103}{500}(F_4)^2\frac{d^2}{dx^2}F_4 + \frac{101}{1000}(F_4)^2\frac{d}{dx}F_3 + \frac{7}{50}F_3\frac{d}{dx}F_3 + \frac{1}{5}\frac{d^3}{dx^3}F_3 + \frac{28}{125}F_4F_3\frac{d}{dx}F_4 \\
0 \;=\; & -12\,F_2 - \frac{36}{5}F_3F_4 - \frac{48}{25}(F_4)^3 + \frac{72}{5}F_4\frac{d}{dx}F_4 - 12\frac{d^2}{dx^2}F_4 + 18\frac{d}{dx}F_3 \\
0 \;=\; & \frac{102}{25}F_4\frac{d^2}{dx^2}F_4 + \frac{18}{5}\frac{d^2}{dx^2}F_3 - \frac{16}{5}\frac{d^3}{dx^3}F_4 + \frac{68}{25}\left(\frac{d}{dx}F_4\right)^2 \\
& -\frac{4}{625}(F_4)^4 - \frac{16}{25}(F_3)^2 - \frac{34}{125}F_3(F_4)^2 + \frac{4}{25}F_3\frac{d}{dx}F_4 - \frac{172}{125}(F_4)^2\frac{d}{dx}F_4 \\
& -4\,F_1 - \frac{2}{5}F_2F_4 - \frac{9}{5}F_4\frac{d}{dx}F_3 = 0.
\end{aligned}$$

The conditions for $n > 5$ can be written down using recursive relations and MAPLE, but the resulting formulae are very long, and inconclusive.

Calculations leading to a proof of Theorem 1.2. We collect the terms in $g' - \Lambda g = 0$ by order of derivatives (eg $dy\, dp\, dr$ has order $4 = 0 + 1 + 3$).
6th order

$$18F_3 + 2\pi + 2\theta + 2\phi - 18\Lambda = 0 \tag{A1}$$

$$-18F_3 + \phi + 9\Lambda = 0 \tag{A2}$$

$$\pi + 6\Lambda = 0 \tag{A3}$$

$$-8F_3 + \zeta + \theta + 8\Lambda = 0 \tag{A4}$$

$$3\zeta + \pi - 3\Lambda = 0. \tag{A5}$$

5th order

$$-18F_2 + 2\mu + \nu + \phi' + \phi F_3 - \Lambda\phi = 0 \tag{A6}$$

$$18F_2 + 2\lambda + 2\mu + \pi' - \Lambda\pi = 0 \tag{A7}$$

$$\lambda + 2\nu + \theta F_3 + \theta' - \Lambda\theta = 0 \tag{A8}$$

$$-8F_2 + 4\kappa + \lambda + \zeta' - \Lambda\zeta = 0. \tag{A9}$$

4th order

$$-18F_1 + \xi + 3\eta + \nu F_3 + \nu' - \Lambda\nu = 0 \tag{A10}$$

$$\xi + \phi F_2 + \mu' - \Lambda\mu = 0 \tag{A11}$$

$$18F_1 + 3\epsilon + 2\xi + \theta F_2 + \lambda' - \Lambda\lambda = 0 \tag{A12}$$

$$-8F_1 + \epsilon + \kappa' - \Lambda\kappa = 0. \tag{A13}$$

3rd order

$$-18F_0 + \delta + \eta F_3 + \eta' - \Lambda\eta = 0 \tag{A14}$$

$$18F_0 + 2\gamma + 3\delta + \nu F_2 + \phi F_1 + \xi' - \Lambda\xi = 0 \tag{A15}$$

$$-8F_0 + 2\gamma + \theta F_1 + \epsilon' - \Lambda\epsilon = 0. \tag{A16}$$

2nd order

$$3\beta + \nu F_1 + \theta F_0 + \gamma' - \Lambda\gamma = 0 \tag{A17}$$

$$\beta + \eta F_2 + \phi F_0 + \delta' - \Lambda\delta = 0. \tag{A18}$$

1st order

$$4\alpha + \eta F_1 + \nu F_0 + \beta' - \Lambda\beta = 0. \tag{A19}$$

0th order

$$\eta F_0 + \alpha' - \Lambda\alpha = 0. \tag{A20}$$

Conditions for F leading to a proof of Theorem 1.3 .

$$B = P_3, \qquad C = Q_3, \qquad A = P_2 + QP_3 - 2Q_3P - 2\frac{\mathrm{d}}{\mathrm{d}x}P_3,$$

where P, Q are given by (2.5).

$$2\frac{\mathrm{d}A}{\mathrm{d}x} + Q_0 + (n-1)\frac{\mathrm{d}}{\mathrm{d}x}(BQ + \frac{\mathrm{d}}{\mathrm{d}x}B) = 0 \tag{A21}$$

$$\frac{d^2}{dx^2}C - 2A - Q_1 + \frac{d}{dx}(CQ) + (n-1)QB + 2n\frac{dB}{dx} = 0 \tag{A22}$$

$$CQ + 2\frac{dC}{dx} - Q_2 + (n+1)B = 0 \tag{A23}$$

$$\frac{d^2A}{dx^2} - Q\frac{dA}{dx} - P_0 + (n-1)\frac{d}{dx}(BP) + (n-1)P\frac{dB}{dx} = 0 \tag{A24}$$

$$2\frac{dA}{dx} - P_1 - QA + \frac{d^2B}{dx^2} - Q\frac{dB}{dx} + \frac{d}{dx}(CP) + P\frac{dC}{dx} + 2(n-1)BP = 0. \tag{A25}$$

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