Universality of the double scaling limit in random matrix models

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Abstract

We study unitary random matrix ensembles in the critical case where the limiting mean eigenvalue density vanishes quadratically at an interior point of the support. We establish universality of the limits of the eigenvalue correlation kernel at such a critical point in a double scaling limit. The limiting kernels are constructed out of functions associated with the second Painlevé equation. This extends a result of Bleher and Its for the special case of a critical quartic potential.

The two main tools we use are equilibrium measures and Riemann-Hilbert problems. In our treatment of equilibrium measures we allow a negative density near the critical point, which enables us to treat all cases simultaneously. The asymptotic analysis of the Riemann-Hilbert problem is done with the Deift/Zhou steepest descent analysis. For the construction of a local parametrix at the critical point we introduce a modification of the approach of Baik, Deift, and Johansson so that we are able to satisfy the required jump properties exactly.

1 Introduction

We consider the unitary random matrix model

$$Z_{n,N}^{-1} \exp(-N \operatorname{Tr} V(M)) dM \tag{1.1}$$

defined on Hermitian $n \times n$ matrices M in a critical regime where the limiting mean density of eigenvalues vanishes at an interior point. It is a basic fact of random matrix theory [17, 34] that the eigenvalues of the random matrix ensemble (1.1) follow a determinantal point process with correlation kernel

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)}e^{-\frac{N}{2}V(y)}\sum_{k=0}^{n-1} p_{k,N}(x)p_{k,N}(y), \tag{1.2}$$

where $p_{k,N}$ denotes the kth degree orthonormal polynomial with respect to the weight $e^{-NV(x)}$ on $\mathbb{R}$.

We assume in this paper that the confining potential $V : \mathbb{R} \to \mathbb{R}$ in (1.1) is real analytic and that it satisfies the growth condition

$$\frac{V(x)}{\log(x^2+1)} \to +\infty \qquad \text{as } |x| \to +\infty. \tag{1.3}$$

These assumptions ensure that the mean eigenvalue density $\frac{1}{n}K_{n,N}(x,x)$ has a limit as $n, N \to \infty$, $n/N \to 1$, see e.g. [17], which we denote by $\psi_V(x)$. It is known that ψ_V is the density of the measure μ_V which minimizes the weighted energy

$$I_V(\mu) = \iint \log\frac{1}{|s-t|}d\mu(s)d\mu(t) + \int V(t)d\mu(t) \tag{1.4}$$

among all probability measure on $\mathbb{R}$. The measure μ_V is called the equilibrium measure in the external field V. The fact that V is real analytic ensures that the support $S_V = \mathrm{supp}(\mu_V)$ consists of a finite union of intervals [18].

It is a remarkable fact that local scaling limits of the kernel (1.2) depend only on the nature of the density ψ_V. This has been proved rigorously in the bulk of the spectrum for a quartic V in [9] and for general real analytic V in [20]. Indeed, if $\psi_V(x^*) > 0$, then

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x^*)}K_{n,n}\left(x^* + \frac{u}{n\psi_V(x^*)}, x^* + \frac{v}{n\psi_V(x^*)}\right) = K^{bulk}(u,v) \tag{1.5}$$

exists, and

$$K^{bulk}(u,v) = \frac{\sin\pi(u-v)}{\pi(u-v)}. \tag{1.6}$$

The scaling limits are different at special points of the spectrum. At edge points of the spectrum the density ψ_V typically vanishes like a square root, and then it is known that for some constant $c > 0$,

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}}K_{n,n}\left(x^* + \frac{u}{(cn)^{2/3}}, x^* + \frac{v}{(cn)^{2/3}}\right) = K^{edge}(u,v) \tag{1.7}$$

where

$$K^{edge}(u,v) = \frac{\mathrm{Ai}(u)\,\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\,\mathrm{Ai}(v)}{u-v} \tag{1.8}$$

and Ai is the Airy function. The Airy kernel is related to the Tracy-Widom distribution [39]. In (1.7) we have assumed that x^* is a right edge point. For a left edge point we change $u \mapsto -u$, $v \mapsto -v$ in the left-hand side of (1.7).

Other special points in the spectrum include

- Edge points where the density vanishes to a higher order. The possible edge point behaviors (at a right end point x^*) are

  $$\psi_V(x) = c(x^* - x)^{2k+\frac{1}{2}}(1 + o(1)) \quad \text{as } x \to x^*+ \tag{1.9}$$

  where $c > 0$ and k is a non-negative integer.

- Interior points where the density vanishes. Then

  $$\psi_V(x) = c(x - x^*)^{2k}(1 + o(1)) \quad \text{as } x \to x^* \tag{1.10}$$

  where $c > 0$ and k is a positive integer.

In these critical cases it is believed that the local scaling limit at x^* of the kernel only depends on the order of vanishing of the density at x^* [8].

The case where ψ_V vanishes quadratically at an interior point of S_V, that is, the case $k = 1$ in (1.10), was considered by Bleher and Its [10] for the case of a critical quartic potential $V(x) = \frac{g}{4}x^4 + \frac{t}{2}x^2$, with $g > 0$ and $t = t_c = -2\sqrt{g}$. Then

$$\psi_V(x) = \frac{1}{2\pi} g x^2 \sqrt{\frac{4}{\sqrt{g}} - x^2}, \qquad \text{for } x \in [-2g^{-1/4}, 2g^{-1/4}],$$

so that ψ_V vanishes quadratically at the origin. Bleher and Its consider the double scaling limit where t changes with n and tends to t_c as $n \to \infty$ in such a way that $n^{2/3}(t - t_c)$ remains constant. For the quartic potential this is equivalent to considering (1.1) where $n, N \to \infty$, $n/N \to 1$, such that

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) \tag{1.11}$$

exists. Bleher and Its gave a one-paremeter family $K^{crit}(u,v;s)$ of limiting kernels, depending on $s\in\mathbb{R}$, so that for some $c>0$,

$$\lim_{n,N\to\infty}\frac{1}{(cn)^{1/3}}K_{n,N}\left(\frac{u}{(cn)^{1/3}},\frac{v}{(cn)^{1/3}}\right)=K^{crit}(u,v;s) \tag{1.12}$$

where s is proportional to the value of the limit (1.11).

The critical kernels are expressed in terms of so-called ψ-functions associated with the Hastings-McLeod solution of the Painlevé II equation [28]. Consider as in [10] the linear differential equations for a 2-vector (or 2×2 matrix) $\Psi=\Psi(\zeta;s)$,

$$\frac{d}{d\zeta}\Psi=A\Psi,\qquad \frac{\partial}{\partial s}\Psi=B\Psi \tag{1.13}$$

where

$$A=A(\zeta;s)=\begin{pmatrix}4\zeta q & 4\zeta^2+s+2q^2+2r\\ -4\zeta^2-s-2q^2+2r & -4\zeta q\end{pmatrix}, \tag{1.14}$$

and

$$B=B(\zeta;s)=\begin{pmatrix}q & \zeta\\ -\zeta & -q\end{pmatrix}. \tag{1.15}$$

The compatibility condition for (1.13) is that $q=q(s)$ satisfies the Painlevé II equation $q''=sq+2q^3$ and that $r=r(s)=q'(s)$. We assume that $q(s)$ is the Hastings-McLeod solution of Painlevé II, which is characterized by the asymptotic condition

$$q(s)=\mathrm{Ai}(s)(1+o(1))\qquad \text{as } s\to+\infty.$$

The critical kernels op [10] are given by

$$K^{crit}(u,v;s)=\frac{\Phi^1(u;s)\Phi^2(v;s)-\Phi^2(u;s)\Phi^1(v;s)}{\pi(u-v)} \tag{1.16}$$

where $\begin{pmatrix}\Phi^1(\zeta;s)\\ \Phi^2(\zeta;s)\end{pmatrix}$ is the special solution to (1.13) which is real for real ζ, satisfies

$$\Phi^1(-\zeta;s)=\Phi^1(\zeta;s),\qquad \Phi^2(-\zeta;s)=-\Phi^2(\zeta;s)$$

and has asymptotics on the real line

$$\left.\begin{aligned}\Phi^1(\zeta;s) &= \cos\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1}),\\ \Phi^2(\zeta;s) &= -\sin\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1})\end{aligned}\right. \quad \text{as } \zeta \to \pm\infty.$$

If we put

$$\Phi_1 = \Phi^1 + i\Phi^2, \qquad \Phi_2 = \Phi^1 - i\Phi^2 \tag{1.17}$$

then

$$K^{crit}(u,v;s) = \frac{-\Phi_1(u;s)\Phi_2(v;s) + \Phi_2(u;s)\Phi_1(v;s)}{2\pi i(u-v)} \tag{1.18}$$

and $\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$ is a special solution of the differential equations

$$\frac{d}{d\zeta}\Psi(\zeta;s) = \begin{pmatrix}-4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir\\ 4\zeta q - 2ir & 4i\zeta^2 + i(s+2q)\end{pmatrix}\Psi(\zeta;s) \tag{1.19}$$

and

$$\frac{\partial}{\partial s}\Psi(\zeta;s) = \begin{pmatrix}-i\zeta & q\\ q & i\zeta\end{pmatrix}\Psi(\zeta;s). \tag{1.20}$$

The equations (1.19)-(1.20) for the ψ-functions correspond to the ones used by Flaschka and Newell [24] and we will also use those in what follows. The vector $\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$ is the unique solution of (1.19) with asymptotics

$$e^{i(\frac{4}{3}\zeta^3+s\zeta)}\begin{pmatrix}\Phi_1(\zeta;s)\\ \Phi_2(\zeta;s)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} + O\left(\zeta^{-1}\right) \tag{1.21}$$

as $\zeta \to \infty$ uniformly in $\varepsilon \le \arg\zeta \le \pi - \varepsilon$ for any $\varepsilon > 0$.

Before discussing our results, we like to point out an integral formula for the kernel $K^{crit}(u,v;s)$. If we take a derivative of (1.16) with respect to s and use (1.13) and (1.15), we get after some calculations

$$\frac{d}{ds}K^{crit}(u,v;s) = \frac{1}{\pi}\left[\Phi^1(u;s)\Phi^1(v;s) + \Phi^2(u;s)\Phi^2(v;s)\right].$$

Using the Deift/Zhou steepest-descent method for $s \to -\infty$ as done in [23], one can show that $K^{crit}(u,v;s) \to 0$ as $s \to -\infty$, so that we get

$$K^{crit}(u,v;s) = \frac{1}{\pi}\int_{-\infty}^{s}\left[\Phi^1(u;\sigma)\Phi^1(v;\sigma) + \Phi^2(u;\sigma)\Phi^2(v;\sigma)\right]d\sigma. \tag{1.22}$$

Since $\Phi^1(\zeta; s)$ and $\Phi^2(\zeta; s)$ are real for real ζ, formula (1.22) clearly shows that $K^{crit}(u, u; s) > 0$, as it should be.

2 Statement of results

It is the aim of this paper to show that the kernel $K^{crit}(u, v; s)$ is a universal limit. Whenever the limiting mean eigenvalue density ψ_V vanishes quadratically at an interior point, the correlation kernel $K_{n,N}$ has a double scaling limit given by (1.12).

In our Theorem 2.1 below, we use the equilibrium measure ω_S of a compact set $S \subset \mathbb{R}$. This is the unique probability measure on S that minimizes the logarithmic energy

$$I(\mu) = \iint \log \frac{1}{|s-t|} d\mu(s) d\mu(t) \tag{2.1}$$

among all Borel probability measures μ on S. If S is a single interval $[a, b]$, then ω_S has a density w_S given by

$$w_S(x) = \frac{1}{\pi\sqrt{(b-x)(x-a)}}, \qquad x \in (a, b).$$

If S is a finite union of disjoint intervals, say $S = \bigcup_{j=1}^n [a_j, b_j]$ with $a_j < b_j < a_{j+1}$. Then ω_S has density

$$w_S(x) = \frac{|p(x)|}{\pi\sqrt{\prod_{j=1}^n |(b_j - x)(x - a_j)|}}, \qquad x \in \bigcup_{j=1}^n (a_j, b_j), \tag{2.2}$$

where $p(x)$ is a monic polynomial of degree $n-1$ with exactly one zero in each of the gaps (b_j, a_{j+1}), $j = 1, \ldots, n-1$, see e.g. [37, Lemma 4.4.1]. Note that (2.2) has an extension to an analytic function in $\mathbb{C} \setminus (\mathbb{R} \setminus S^o)$, where $S^o = \bigcup_j (a_j, b_j)$, which is a fact that we will use in what follows.

The following is our main result.

Theorem 2.1 *Let V be real analytic on $\mathbb{R}$ such that $\lim_{x\to\pm\infty} \frac{V(x)}{\log(x^2+1)} = +\infty$. Let ψ_V be the density of the equilibrium measure in the external field, and let x^* be an interior point of $S_V = \operatorname{supp}(\psi_V)$ which is such that*

$$\psi_V(x^*) = \psi_V'(x^*) = 0, \qquad \psi_V''(x^*) > 0.$$

Let $n, N \to \infty$ such that the limit

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N}-1\right) = L$$

exists with $L \in \mathbb{R}$. Let $K_{n,N}$ be the correlation kernel (1.2) *for the eigenvalues of the random matrix model* (1.1)*. Then there exist constants $c > 0$ and $s \in \mathbb{R}$ such that*

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(x^* + \frac{u}{(cn)^{1/3}}, x^* + \frac{v}{(cn)^{1/3}}\right) = K^{crit}(u, v; s) \qquad (2.3)$$

uniformly for u, v in compact subsets of $\mathbb{R}$.

Explicit formulas for the constants c and s are

$$c = \frac{\pi\psi_V''(x^*)}{8} \qquad (2.4)$$

and

$$s = L\frac{\pi}{c^{1/3}} w_{S_V}(x^*), \qquad (2.5)$$

where w_{S_V} is the density of the equilibrium measure of S_V.

As noted before, Bleher and Its [10] proved (2.3) for the case of a critical quartic V. See [11] for a rigorous expansion of the free energy in this critical case.

Remark 2.2 The random matrix model (1.1) may be generalized to include a spectral singularity at the origin

$$Z_{n,N}^{-1} |\det M|^{2\alpha} \exp(-N \operatorname{Tr} V(M)) dM, \qquad \alpha > -1/2. \qquad (2.6)$$

If $\psi_V(0) > 0$ and $n = N \to \infty$, then the scaled limit of the correlation kernels is a Bessel kernel which involves Bessel functions of order $\alpha \pm \frac{1}{2}$, see [2, 33]. In the multicritical case where ψ_V vanishes quadratically at 0, an analogue of Theorem 2.1 is valid. In work in progress [15], we are considering the double scaling limit of (2.6) and we show that the limiting kernels are expressed in terms of the ψ-functions associated with a special solution of the general Painlevé II equation

$$q'' = sq + 2q^3 - \alpha.$$

The main ingredients in the proof of Theorem 2.1 are equilibrium measures and Riemann-Hilbert problems. We give some comments on both.

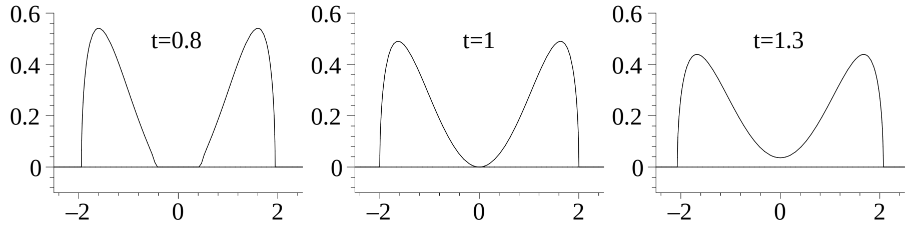

Figure 1: The density of μ_t for $V(x) = \frac{x^4}{4} - x^2$ with t equal to 0.8, 1 and 1.3, respectively.

Equilibrium measures. Recall that the equilibrium measure in external field V minimizes (1.4). We need to know how the equilibrium measure μ_V in the external field changes as a result of a change in V. The particular modification we consider here is

$$V_t = \frac{1}{t} V, \qquad t > 0,$$

so that $V_1 = V$. Let us put

$$\mu_t = \mu_{V_t}.$$

Then it is known that $t\mu_t$ and S_t are increasing as a function of t, see e.g. [16, 36, 38]. We also have the Buyarov-Rakhmanov formula [14]

$$\mu_t = \frac{1}{t} \int_0^t \omega_{\mathrm{supp}(\mu_\tau)} d\tau, \tag{2.7}$$

which expresses the equilibrium measure in the external field as an average of equilibrium measures of sets. A consequence of (2.7) is that

$$\left.\frac{d}{dt} (t\mu_t)\right|_{t=1} = \omega_{S_V}, \tag{2.8}$$

which partly explains why the equilibrium measure ω_{S_V} plays a role in the formula (2.5) for s.

For the case of interest in this paper we have that ψ_V vanishes at $x = x^*$. Then for $t > 1$, there is a positive density at x^*, while for $t < 1$, x^* is out of the support of μ_t. For t slightly less than 1, there is a gap in $\mathrm{supp}(\mu_t)$, see Figure 1. An asymptotic analysis based on the equilibrium measure μ_t would require a discussion of the two different situations $t > 1$ and $t < 1$, as is done in [3].

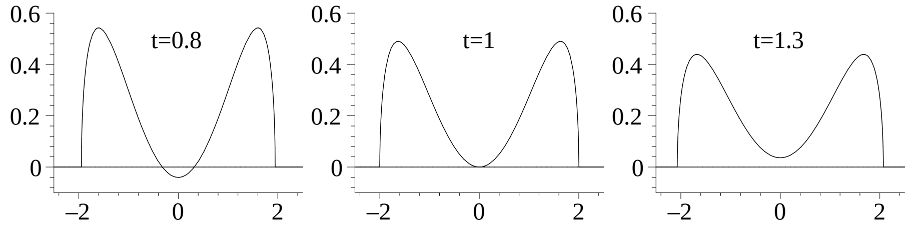

Figure 2: The density of ν_t for $V(x) = \frac{x^4}{4} - x^2$ with t equal to 0.8, 1 and 1.3, respectively (compare with Figure 1).

Therefore we found it convenient to introduce a modification of the equilibrium problem in external field, which will enable us to treat both cases simultaneously. The modification we make is that we do not require the measure to be non-negative in a neighborhood of the point x^*. For a sufficiently small $\delta_0 > 0$, we consider the problem to minimize

$$I_{V_t}(\nu) = \iint \log \frac{1}{|x-y|} d\nu(x) d\nu(y) + \frac{1}{t} \int V(x) d\nu(x) \tag{2.9}$$

among all signed measures $\nu = \nu^+ - \nu^-$ on $\mathbb{R}$, where $\nu^\pm$ are nonnegative measures, such that

$$\int d\nu = 1, \qquad \text{and} \qquad \operatorname{supp}(\nu^-) \subset [x^* - \delta_0, x^* + \delta_0]. \tag{2.10}$$

We denote the minimizer by ν_t and we let $S_t = \operatorname{supp}(\nu_t)$.

Then $\nu_t = \mu_t$ for $t \geq 1$, but for $t < 1$ there is a clear distinction between ν_t and μ_t, see Figure 2. However we still have the analogue of (2.8) (as we prove)

$$\frac{d}{dt} (t\nu_t) \bigg|_{t=1} = \omega_{S_V}, \tag{2.11}$$

What's more, we also have (2.11) at the level of densities, that is, if ψ_t denotes the density of ν_t, and if x is an interior point of S_V (in particular if $x = x^*$), then

$$\frac{d}{dt} (t\psi_t(x)) \bigg|_{t=1} = w_{S_V}(x), \tag{2.12}$$

where w_{S_V} is the density of the equilibrium measure of S_V, which is what we need for the proof of Theorem 2.1. Note that we do not have (2.12) for $x = x^*$ if ψ_t is the density of μ_t.

Riemann-Hilbert problem. The second main tool for the proof of Theorem 2.1 is the characterization of orthonormal polynomials by means of a Riemann-Hilbert problem, due to Fokas, Its, and Kitaev [25], and the subsequent asymptotic analysis of the Riemann-Hilbert problem by means of the Deift/Zhou steepest descent analysis of Riemann-Hilbert problems, introduced in [22], and further developed in [21, 19, 20], and other more recent papers. The Deift/Zhou steepest descent analysis of the Riemann-Hilbert problem consists of a sequence of explicit transformations, which result in a Riemann-Hilbert problem that is explicitly solvable in terms of Neumann series. (In fact, in this paper, we only need the first term of this series.) The critical point x^* needs special attention.

Of particular interest for us is the paper [3] by Baik, Deift, and Johansson on the length of the longest increasing subsequence of a random permutation of $\{1, 2, \ldots, n\}$. These authors show that the fluctuations of this random variable are distributed according to the Tracy-Widom distribution [39] in the limit as $n \to \infty$. One of the technical tools in this important paper is the asymptotic analysis of a Riemann-Hilbert problem on the unit circle which is related to an equilibrium measure (also on the unit circle) whose density vanishes at the point -1. This situation is comparable to ours. The authors of [3], see also subsequent papers [4, 6, 7, 5], construct a local parametrix near -1 with the aid of the ψ-functions associated with the Hastings-McLeod solution of Painlevé II. These ψ-functions satisfy a model Riemann-Hilbert problem and the local parametrix is constructed by appropriately mapping the model Riemann-Hilbert problem onto a neighborhood of -1 so that it satisfies certain desired jump properties approximately.

We follow the approach of [3] but we introduce a modification in the construction of the local parametrix so that it has the desired jump properties *exactly*, in contrast to [3] where the desired jump properties only hold *approximately*. The fact that we have the exact jump properties simplifies the arguments considerably and we feel that this is also a main contribution of the present paper.

Outline of the rest of the paper. In Section 3 we collect the necessary facts about equilibrium measures. In particular we study the modified equilibrium problem with external field in some detail. In Section 4 we discuss the Riemann-Hilbert problem satisfied by the ψ-functions. Here we follow [24]. Then in Section 5 we state the Riemann-Hilbert problem for orthogonal polynomials, discuss the relation with the correlation kernel $K_{n,N}$, and perform the transformations in the steepest descent analysis. Finally in Section 6 we give the proof of Theorem 2.1.

3 Equilibrium Measures

As explained in the previous section, we consider a modification of the equilibrium problem where we drop the non-negativity condition in a small neighborhood of x^*. We take $\delta_0 > 0$ sufficiently small so that

$$\psi_V(x) > 0 \qquad \text{for all } x \in [x^* - \delta_0, x^* + \delta_0] \setminus \{x^*\}. \tag{3.1}$$

and we use ν_t to denote the signed measure that minimizes (2.9) under the conditions (2.10). We define

$$S_t = \text{supp}(\nu_t). \tag{3.2}$$

The existence and uniqueness of ν_t follows as in [36].

Let

$$U^\nu(x) = \int \log \frac{1}{|x-y|} d\nu(y)$$

be the logarithmic potential of ν. Then standard arguments of potential theory [17, 36] show that ν_t is the unique signed measure satisfying (2.9) with the property that

$$2U^{\nu_t}(x) + \frac{1}{t}V(x) = \ell_t, \qquad x \in \text{supp}(\nu_t) \cup [x^* - \delta_0, x^* + \delta_0], \tag{3.3}$$

$$2U^{\nu_t}(x) + \frac{1}{t}V(x) \geq \ell_t, \qquad x \in \mathbb{R}. \tag{3.4}$$

for some constant ℓ_t.

As before we use μ_t to denote the equilibrium measure in the external field V_t. This is a probability measure that satisfies for some constant $\tilde{\ell}_t$,

$$2U^{\mu_t}(x) + \frac{1}{t}V(x) = \tilde{\ell}_t, \qquad x \in \text{supp}(\mu_t), \tag{3.5}$$

$$2U^{\mu_t}(x) + \frac{1}{t}V(x) \geq \tilde{\ell}_t, \qquad x \in \mathbb{R}. \tag{3.6}$$

It is known that $t\mu_t$ and $\text{supp}(\mu_t)$ are increasing with $t > 0$, see [14, 36]. For $t \geq 1$, we have $\text{supp}(\mu_V) \subset \text{supp}(\mu_t)$ which implies that $\nu_t = \mu_t$ in view of

(3.1) and the variational conditions (3.3)–(3.6). For $t < 1$, we have that x^* is outside the support of μ_t, and in fact the strict inequality in (3.6) holds for $x = x^*$ if $t < 1$.

In general there is the following inequality between μ_t and ν_t.

Lemma 3.1 *For every $t > 0$ we have*

$$\mu_t \leq \nu_t^+. \tag{3.7}$$

Proof. Let $\lambda = \nu_t - \mu_t$. From (3.3), (3.4), (3.5), and (3.6) it follows that

$$2U^\lambda(x) \leq \ell_t - \tilde{\ell}_t, \qquad \text{for } x \in S_t, \tag{3.8}$$

$$2U^\lambda(x) \geq \ell_t - \tilde{\ell}_t, \qquad \text{for } x \in \text{supp}(\mu_t). \tag{3.9}$$

The potential U^λ is subharmonic on $\mathbb{C} \setminus \text{supp}(\lambda^+)$ and since $\int d\lambda = 0$, it is subharmonic at infinity as well. By the maximum principle for subharmonic functions [35, 36], the maximum of U^λ is attained in $\text{supp}(\lambda^+)$ only. Since $\text{supp}(\lambda^+) \subset S_t$ we then have by (3.8) that equality in (3.9) holds for every $x \in \text{supp}(\mu_t)$, and so $\text{supp}(\mu_t) \subset \text{supp}(\lambda^+)$. This implies (3.7). □

It follows from (3.7) that $\text{supp}(\nu_t^-) \cap \text{supp}(\mu_t)$ is empty. Since for t slightly less than 1 a gap opens in $\text{supp}(\mu_t)$, which depends continuously on t, see [32], it follows that for any given $\delta \in (0, \delta_0)$, there is $t_0 < 1$ such that

$$\text{supp}(\nu_t^-) \subset [x^* - \delta, x^* + \delta], \qquad \text{for } t > t_0.$$

This shows that for $t < 1$ sufficiently close to 1, the definition of ν_t is independent of the choice of δ_0.

A very useful fact is that for real analytic V, say V is analytic in a neighborhood $\mathcal{V}$ of the real line, the measures μ_t have densities $\tilde{\psi}_t$ which can be expressed in terms of the negative part of an analytic function in $\mathcal{V}$. Indeed, if

$$\tilde{q}_t(z) = \left(\frac{V'(z)}{2t}\right)^2 - \frac{1}{t}\int \frac{V'(z) - V'(y)}{z - y} d\mu_t(y), \tag{3.10}$$

then it was shown in [18] that

$$\tilde{\psi}_t(x) = \frac{1}{\pi}\sqrt{\tilde{q}_t^-(x)}, \tag{3.11}$$

where $\tilde{q}_t^- = \max(0, -\tilde{q}_t)$ denotes the negative part of $\tilde{q}_t$. A consequence of (3.11) is that $\text{supp}(\mu_t)$ is the closure of the set where $\tilde{q}_t$ is negative.

The arguments of [18] can be readily extended to the signed measures ν_t, provided that $\mathrm{supp}(\nu_t^-) \subset [x^* - \delta, x^* + \delta]$ for some $\delta < \delta_0$. We define

$$q_t(z) = \left(\frac{V'(z)}{2t}\right)^2 - \frac{1}{t}\int \frac{V'(z) - V'(y)}{z - y} d\nu_t(y), \qquad z \in \mathcal{V}. \tag{3.12}$$

and then the following holds.

Proposition 3.2 *There exists $t_0 \in (0,1)$ such that for every $t > t_0$, the signed measure ν_t has a density ψ_t and*

$$q_t(x) = -\left[\pi\psi_t(x)\right]^2, \qquad \text{for } x \in S_t. \tag{3.13}$$

In addition we have

$$q_t(x) = \left[\int_{S_t} \frac{\psi_t(y)}{y - x} dy + \frac{V'(x)}{2}\right]^2 \qquad \text{for } x \in \mathbb{R} \setminus S_t. \tag{3.14}$$

Proof. This follows as in [18, Proposition 2.51]. □

Obviously, for $t \geq 1$ we have $q_t = \tilde{q}_t$. Since $\psi_1 = \psi_V$ has a double zero at x^*, we see from (3.13) that q_1 has a zero at x^* of order four. For t slightly bigger than 1, this fourth order zero splits into two double zeros in the complex plane away from the real axis.

For t slightly less than 1, there is a difference in the behavior of the zeros of q_t and $\tilde{q}_t$. Indeed, $\tilde{q}_t$ has two simple real zeros near x^* which are endpoints of the support of μ_t, and in addition there is a double real zero in between them. On the other hand, we have that q_t has two double real zeros near x^*, which are endpoints of the support of ν_t^-.

The fact that $-q_t$ has only double zeros near x^* allows us to take an analytic square root, and in view of (3.13) we choose it so that for $z \in S_t$,

$$\psi_t(z) = \frac{1}{\pi}(-q_t(z))^{1/2}, \tag{3.15}$$

where the sign of the square root at $x = x^*$ is taken negative if $t < 1$ and positive if $t > 1$. The right-hand side of (3.15) has an analytic extension to a neighborhood of x^*, which is independent of the value of $t > t_0$. Thus ψ_t has an analytic extension to a fixed neighborhood of x^*, which will also be denoted by ψ_t.

Recall from the discussion before the statement of Theorem 2.1, that the density w_{S_V} of the equilibrium measure of S_V also has an analytic extension to a neighborhood of x^*, which we also denote by w_{S_V}. The remaining part of this section is devoted to the proof of the following proposition.

Proposition 3.3 *We have*

$$\lim_{t\to 1} \frac{t\psi_t(z) - \psi_1(z)}{t-1} = w_{S_V}(z) \tag{3.16}$$

uniformly for z in a neighborhood of x^*.

We start with a lemma which contains a weaker form of (3.16).

Lemma 3.4 *We have*

$$\frac{d}{dt}(t\nu_t)\bigg|_{t=1} = \omega_{S_V} \tag{3.17}$$

where ω_{S_V} is the equilibrium measure of S_V.

Proof. Buyarov and Rakhmanov [14] proved (for a very general class of V) that

$$\lim_{t\to 1-} \frac{t\mu_t - \mu_V}{t-1} = \omega_{S_V} \tag{3.18}$$

and

$$\lim_{t\to 1+} \frac{t\mu_t - \mu_V}{t-1} = \omega_{S_V^*} \tag{3.19}$$

where $\omega_{S_V^*}$ is the equilibrium measure of the set

$$S_V^* = \{x \in \mathbb{R} \mid 2U^{\mu_V}(x) + V(x) = \tilde{\ell}_1\},$$

see (3.5) with $t = 1$. For real analytic V, the sets S_V and S_V^* differ by an at most finite number of points, so that $\omega_{S_V} = \omega_{S_V^*}$. Thus by (3.18) and (3.19) we have

$$\lim_{t\to 1} \frac{t\mu_t - \mu_V}{t-1} = \omega_{S_V}. \tag{3.20}$$

Now write

$$\frac{t\nu_t - \nu_1}{t-1} = \frac{t\mu_t - \mu_V}{t-1} + \frac{t}{t-1}(\nu_t - \mu_t).$$

In view of (3.20) it suffices to prove that

$$\|\nu_t - \mu_t\| = o(t-1) \qquad \text{as } t \to 1, \tag{3.21}$$

where the norm denotes the total variation of a signed measure. We may assume that $t < 1$. Because of (3.7) we have

$$\|\nu_t - \mu_t\| = \|\nu_t^+ - \mu_t - \nu_t^-\| = \int d(\nu_t^+ - \mu_t) + \int d\nu_t^- = 2\int d\nu_t^-. \tag{3.22}$$

For $t < 1$, the support of ν_t^- is contained in an interval $[x^* - \delta(t), x^* + \delta(t)]$, with

$$\delta(t) = O\left((1-t)^{1/2}\right) \qquad \text{as } t \to 1-, \tag{3.23}$$

see [32, Lemma 8.1(iii)]. From (3.15) and the fact that q_t has two double zeros in $[x^* - \delta(t), x^* + \delta(t)]$ which are the end-points of the support of ν_t^-, we then easily get that

$$\frac{d}{dx}\nu_t^-(x) = O(\delta(t)^2) = O(1-t) \qquad \text{as } t \to 1- . \tag{3.24}$$

Combining (3.23) and (3.24), we find $\int d\nu_t^- = O\left((1-t)^{3/2}\right)$ as $t \to 1-$, which by (3.22) implies (3.21). This completes the proof of the lemma. □

We now give a characterization of ψ_V, which will be of use in the proof of Proposition 3.3.

Lemma 3.5 *For $x \in S_V$ we have*

$$\psi_V(x) = \frac{1}{2\pi^2}\frac{1}{w_{S_V}(x)}\int \frac{V'(x) - V'(y)}{x-y} d\omega_{S_V}(y), \tag{3.25}$$

where w_{S_V} is the density of ω_{S_V}.

Proof. By the Sokhotski-Plemelj formulas, see e.g. [27], we have that

$$F(z) = \int_{S_V} \frac{\psi_V(s)}{z-s} ds, \qquad \text{for } z \in \mathbb{C} \setminus S_V,$$

satisfies

$$F_+(x) + F_-(x) = V'(x), \qquad x \in S_V, \tag{3.26}$$

$$F_+(x) - F_-(x) = -2\pi i \psi_V(x), \qquad x \in S_V. \tag{3.27}$$

Let $S_V = \bigcup_{j=1}^n [a_j, b_j]$ and set

$$R(z) = \left(\prod_{j=1}^n (z-b_j)(z-a_j)\right)^{1/2}, \qquad z \in \mathbb{C} \setminus S_V,$$

where the square root is positive for $z > b_n$. Using (3.26) and the fact that $R_+(x) = -R_-(x)$ as $x \in S_V$, we see that

$$\frac{p(x)F_+(x)}{R_+(x)} - \frac{p(x)F_-(x)}{R_-(x)} = \frac{p(x)V'(x)}{R_+(x)} \tag{3.28}$$

for any polynomial p. Suppose p has degree at most $n-1$. Then (3.28) and the fact that $\frac{p(z)F(z)}{R(z)} \to 0$ as $z \to \infty$ imply that

$$\frac{p(z)F(z)}{R(z)} = \frac{1}{2\pi i} \int_{S_V} \frac{p(s)V'(s)}{R_+(s)} \frac{1}{s-z} ds, \qquad z \in \mathbb{C} \setminus S_V$$

which we can rewrite for $z \in \mathcal{V}$ as

$$\begin{aligned} \frac{p(z)F(z)}{R(z)} &= \frac{1}{2\pi i} \int_{S_V} \frac{V'(s)-V'(z)}{s-z} \frac{p(s)}{R_+(s)} ds + \frac{V'(z)}{2\pi i} \int_{S_V} \frac{p(s)}{R_+(s)} \frac{1}{s-z} ds \\ &= \frac{1}{2\pi i} \int_{S_V} \frac{V'(z)-V'(s)}{z-s} \frac{p(s)}{R_+(s)} ds + \frac{V'(z)}{2} \frac{p(z)}{R(z)} \end{aligned} \tag{3.29}$$

by contour integration. Now we use (3.29) and (3.27) to obtain

$$\psi_V(x) = \frac{F_+(x) - F_-(x)}{-2\pi i} = \frac{1}{2\pi^2} \frac{R_+(x)}{p(x)} \int_{S_V} \frac{V'(x)-V'(y)}{x-y} \frac{p(y)}{R_+(y)} dy,$$

and this holds for any polynomial p of degree at most $n-1$. Since we know by (2.2) that

$$w_{S_V}(x) = \frac{i}{\pi} \frac{p(x)}{R_+(x)}, \qquad x \in S_V,$$

for some monic polynomial p of degree $n-1$, we get (3.25). □

Now we can give the proof of Proposition 3.3.

Proof of Proposition 3.3. By (3.12) we have for $z \in \mathcal{V}$,

$$t^2 q_t(z) = \left(\frac{V'(z)}{2}\right)^2 - \int \frac{V'(z)-V'(y)}{z-y} d(t\nu_t)(y)$$

so that in view of (3.17) we find,

$$\left.\frac{d}{dt}(t^2 q_t(z))\right|_{t=1} = -\int \frac{V'(z)-V'(y)}{z-y} d\omega_{S_V}(y).$$

Thus by (3.15) for $z \neq x^*$ in a neighborhood of x^*,

$$\begin{aligned} \left.\frac{d}{dt}(t\psi_t(z))\right|_{t=1} &= \left.\frac{d}{dt}\frac{1}{\pi}(-t^2 q_t(z))^{1/2}\right|_{t=1} \\ &= \frac{1}{2\pi}(-q_V(z))^{-1/2} \int \frac{V'(z)-V'(y)}{z-y} d\omega_{S_V}(y) \\ &= \frac{1}{2\pi^2} \frac{1}{\psi_V(z)} \int \frac{V'(z)-V'(y)}{z-y} d\omega_{S_V}(y). \end{aligned} \tag{3.30}$$

By (3.25), the right-hand side of (3.30) is $\omega_{S_V}(z)$ in case $z \neq x^*$ is real, and by analytic continuation it continues to be $\omega_{S_V}(z)$ in a punctured neighborhood of x^*. Thus (3.16) holds for $z \neq x^*$ in a neighborhood of x^*, and then it easily follows for $z = x^*$ as well. □

4 Riemann-Hilbert Problem Associated with the Painlevé II Equation

In this section we recall the RH problem associated with the second Painlevé equation, see [24, 30, 23, 3, 29]. We transform this RH problem into a RH problem that will be used in the next section to construct a parametrix around x^*.

4.1 ψ-functions Associated with Painlevé II

Consider the differential equation of (1.19),

$$\frac{d}{d\zeta}\Psi(\zeta) = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir \\ 4\zeta q - 2ir & 4i\zeta^2 + i(s+2q^2) \end{pmatrix} \Psi(\zeta), \tag{4.1}$$

where Ψ is a 2×2 complex matrix-valued function and s, q and r are considered as parameters. All solutions of (4.1) are entire functions of ζ.

For $j = 1, \ldots, 6$, let S_j be the sector

$$S_j = \{\zeta \in \mathbb{C} \mid \frac{2j-3}{6}\pi < \arg\zeta < \frac{2j-1}{6}\pi\}. \tag{4.2}$$

There exists a unique solution Ψ_j of equation (4.1) so that

$$\Psi_j(\zeta)e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = I + O(\zeta^{-1}) \tag{4.3}$$

as $\zeta \to \infty$ in the sector S_j. Here we use $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to denote the third Pauli matrix. There exist complex values a_j, $j = 1, \ldots, 6$ (called Stokes multipliers) so that

$$\Psi_{j+1}(\zeta) = \Psi_j(\zeta)A_j, \quad \text{for } j = 1, \ldots, 5, \qquad \Psi_1(\zeta) = \Psi_6(\zeta)A_6. \tag{4.4}$$

with

$$A_j = \begin{pmatrix} 1 & 0 \\ a_j & 1 \end{pmatrix} \qquad \text{if } j \text{ is odd,} \tag{4.5}$$

and

$$A_j = \begin{pmatrix} 1 & a_j \\ 0 & 1 \end{pmatrix} \qquad \text{if } j \text{ is even.} \tag{4.6}$$

Furthermore, we have

$$a_{j+3} = a_j \qquad \text{and} \qquad a_1 a_2 a_3 + a_1 + a_2 + a_3 = 0. \tag{4.7}$$

Now define the rays

$$\Gamma_j = \{\zeta \mid \arg \zeta = \frac{2j-1}{6}\pi\} \quad \text{for } j = 1, \ldots, 6, \tag{4.8}$$

oriented away from the origin, and the matrix-valued function

$$\Psi(\zeta) := \Psi_j(\zeta), \qquad \text{for } \zeta \in S_j. \tag{4.9}$$

Then Ψ satisfies the following RH problem

(a) Ψ is analytic on $\mathbb{C} \setminus \bigcup_{j=1}^{6} \Gamma_j$.

(b) $\Psi_+ = \Psi_- A_j$ on Γ_j,

(c) $\Psi(\zeta) e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = I + O(\zeta^{-1})$ as $\zeta \to \infty$,

(d) Ψ is bounded near 0.

The Stokes multipliers a_j depend on s, q and r. An isomonodromy deformation is a variation of these parameters such that the Stokes multipliers remain constant. Flaschka and Newell [24] showed that the isomonodromy deformations are given by the Painlevé II equation $q''(s) = sq(s) + 2q^3(s)$ and $r(s) = q'(s)$.

Any solution of the Painlevé II equation is a meromorphic function with an infinite number of poles. We write $\Psi(\zeta; s)$ for the Ψ function (4.9) with parameters s, $q = q(s)$, $r = r(s)$, where q is the Hastings-McLeod solution and $r(s) = q'(s)$. Let $\mathcal{P}$ be the set of poles of q. Then $\Psi(\zeta; s)$ is defined and analytic for $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{6} \Gamma_j$ and for $s \in \mathbb{C} \setminus \mathcal{P}$. It is known that there are no poles on the real line [28]. The Stokes multipliers corresponding to the Hastings-McLeod solution are

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = -1.$$

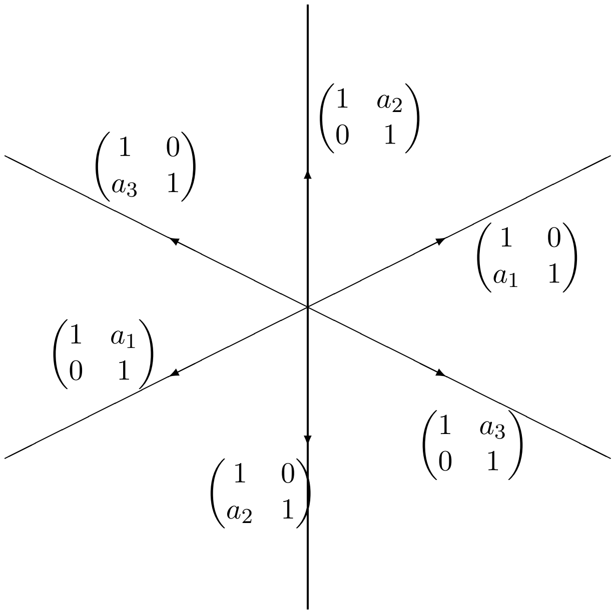

Figure 3: Jumps for $\Psi(\zeta; s)$.

Thus $\Psi(\zeta; s)$ is analytic across the imaginary axis. We reverse the orientation of Γ_3 and Γ_4, and we define

$$\Sigma_1 = \Gamma_1 \cup \Gamma_3, \qquad \text{and} \qquad \Sigma_2 = \Gamma_4 \cup \Gamma_6.$$

Then $\Psi(\zeta; s)$ solves the following RH problem, see also Figure 4.

(a) Ψ is analytic on $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$,

$$\text{(b1)} \quad \Psi_+ = \Psi_- \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} \text{ on } \Sigma_1, \tag{4.10}$$

$$\text{(b2)} \quad \Psi_+ = \Psi_- \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} \text{ on } \Sigma_2,$$

(c) $\Psi(\zeta; s)e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = I + O(\zeta^{-1})$ as $\zeta \to \infty$,

(d) Ψ is bounded near 0.

The RH problem has a solution if and only if $s \in \mathbb{C} \setminus \mathcal{P}$. The properties (c) and (d) are valid uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}$.

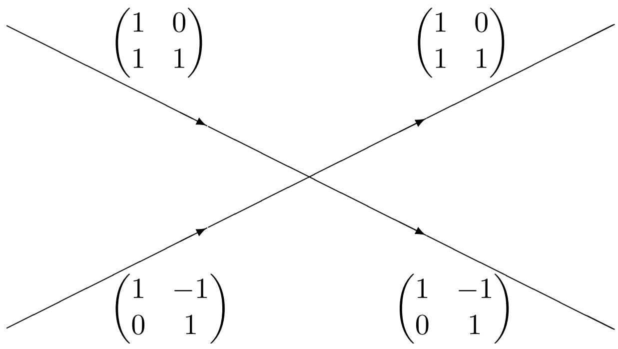

Figure 4: Jumps for $\Psi(\zeta; s)$.

4.2 Transformation of the RH problem

We modify the RH problem for Ψ so that it resembles the RH problem that we will need locally near x^*.

We introduce an additional parameter $\theta \in \mathbb{R}$ and define

$$M(\zeta; s, \theta) = \begin{cases} e^{i\theta\sigma_3}\Psi(\zeta; s)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}e^{-i\theta\sigma_3} & \text{for } \operatorname{Im}\zeta > 0, \\ e^{i\theta\sigma_3}\Psi(\zeta; s)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}e^{-i\theta\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } \operatorname{Im}\zeta < 0. \end{cases} \tag{4.11}$$

For any given $s \in \mathbb{C} \setminus \mathcal{P}$ and $\theta \in \mathbb{R}$, we then have that M is defined for $\zeta \in \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2)$, see Figure 5, and satisfies the following RH problem

(a) $M(\zeta; s, \theta)$ is analytic for $\zeta \in \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2)$ and $s \in \mathbb{C} \setminus \mathcal{P}$.

(b1) $M_+(\zeta; s, \theta) = M_-(\zeta; s, \theta) \begin{pmatrix} 1 & 0 \\ e^{2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} & 1 \end{pmatrix}$ for $\zeta \in \Sigma_1$,

(b2) $M_+(\zeta; s, \theta) = M_-(\zeta; s, \theta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $\zeta \in \mathbb{R}$,

(b3) $M_+(\zeta; s, \theta) = M_-(\zeta; s, \theta) \begin{pmatrix} 1 & 0 \\ e^{-2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} & 1 \end{pmatrix}$ for $\zeta \in \Sigma_2$.

(c1) $M(\zeta; s, \theta) = I + O(\zeta^{-1})$ as $\zeta \to \infty$ in the upper half plane, (4.12)

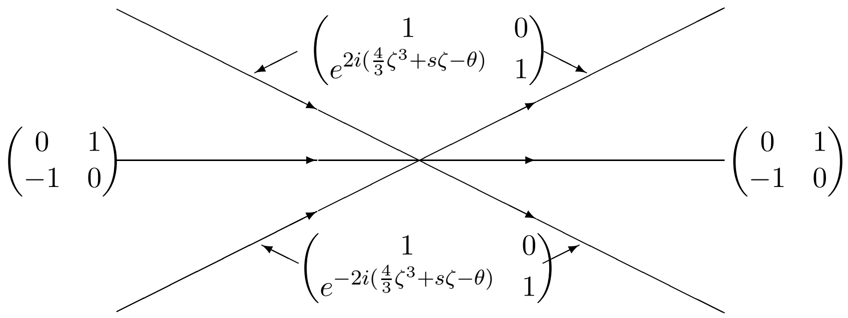

Figure 5: Jumps for $M(\zeta; s, \theta)$.

(c2) $M(\zeta; s, \theta) = (I + O(\zeta^{-1})) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as $\zeta \to \infty$ in the lower half plane,

$$\tag{4.13}$$

(d) $M(\zeta; s, \theta)$ is bounded for ζ near 0.

The properties (c1), (c2), and (d) hold uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}$ and for $\theta \in \mathbb{R}$.

5 Riemann-Hilbert Analysis

The proof of Theorem 2.1 is based on the steepest descent analysis of the Riemann-Hilbert problem for orthogonal polynomials.

Since the main point of the present discussion is the treatment of the critical point x^*, we will restrict ourselves to the one-interval case. We also assume that there are no other singular points besides x^*. Thus $S_V = [a, b]$ and ψ_V vanishes like a square root at a and b, and quadratically at x^* but at no other points of S_V. In addition, we assume that the inequality in the variational condition (3.6) is strict for $x \in \mathbb{R} \setminus [a, b]$. We then have that

$$S_t = \operatorname{supp}(\nu_t) = [a_t, b_t] \tag{5.1}$$

for certain $a_t < b_t$ for every t close to $t = 1$. Note that a_t is an increasing and b_t a decreasing function of t.

We will comment below on the modifications that have to be made in the multi-interval case, see Remark 6.1.

5.1 RH problem for Orthogonal Polynomials

For each n and N, we consider a Riemann-Hilbert problem, introduced by Fokas, Its and Kitaev [25, 26]. We will look for a 2×2 matrix-valued function $Y = Y_{n,N}$ (we drop the subscripts for simplicity) that satisfies the following conditions:

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$, (5.2)

(c) $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

Here $Y_+(x)$ (resp. $Y_-(x)$) denotes the limit as we approach $x \in \mathbb{R}$ from the upper (resp. lower) half-plane. The RH problem possesses a unique solution which is given by

$$Y(z) = \begin{pmatrix} \kappa_{n,N}^{-1} p_{n,N}(z) & \kappa_{n,N}^{-1} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{p_{n,N}(s) e^{-NV(s)}}{s-z} ds \\ -2\pi i \kappa_{n-1,N} p_{n-1,N}(z) & -\kappa_{n-1,N} \int_{\mathbb{R}} \frac{p_{n-1,N}(s) e^{-NV(s)}}{s-z} ds \end{pmatrix} \tag{5.3}$$

where $p_{n,N}(x) = \kappa_{n,N} x^n + \cdots$ denotes the orthonormal polynomial as before, see also [17, 31].

The correlation kernel (1.2) can be expressed directly in terms of the solution of the RH problem. Indeed, by the Christoffel-Darboux formula for orthogonal polynomials

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)} e^{-\frac{N}{2}V(y)} \frac{\kappa_{n-1,N}}{\kappa_{n,N}} \frac{p_{n,N}(x) p_{n-1,N}(y) - p_{n-1,N}(x) p_{n,N}(y)}{x-y}$$

which involves the orthogonal polynomials of degrees n and $n-1$ only. Using (5.3) and the fact that $\det Y(z) = 1$ for every $z \in \mathbb{C} \setminus \mathbb{R}$, we then get

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)} e^{-\frac{N}{2}V(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.4}$$

Applying the Deift-Zhou steepest-descent method to the RH problem will allow us to find the asymptotics of $K_{n,N}$ as given in Theorem 2.1.

We note that an expression like (5.4) was first given in the context of random matrices with external source [1, 12, 13] where the eigenvalue correlation kernel is given in terms of the solution of a 3×3 matrix valued RH problem, and the expression similar to (5.4) was found to be very convenient for asymptotic analysis. Also in the present 2×2-case we find it helpful to work with (5.4).

5.2 Normalization of the RH problem

In the first transformation we normalize the RH problem at infinity. A standard approach would be to use the equilibrium measure μ_t in external field V_t where $t = n/N$. If we would do this for the case where $t < 0$, we would have an equilibrium measure with a gap in the support around x^*, and an annoying consequence is that the equality in the variational conditions is not valid near x^*. For this reason we have introduced the signed measures ν_t in Section 2 and we will use these measures now to normalize the RH problem.

We let $t = n/N$ and assume t is sufficiently close to 1 so that (5.1) holds. As before we use

$$\psi_t := \frac{d\nu_t}{dx} \tag{5.5}$$

to denote the density of ν_t. Then we define the g-function

$$g_t(z) = \int \log(z-s)d\nu_t(s) = \int \log(z-s)\psi_t(s)ds, \tag{5.6}$$

where we take the branch cut of the logarithm along the negative real axis. Then the following properties of g_t are easy to check.

(i) $e^{g_t(z)}$ is analytic in $\mathbb{C} \setminus [a_t, b_t]$,

(ii) $e^{g_t(z)} = z + O(\frac{1}{z})$ as $z \to \infty$.

(iii) $g_{t+}(x) - g_{t-}(x) = 2\pi i \int_x^{b_t} \psi_t(s)ds$ for $x \in \mathbb{R}$,

(iv) $g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t \leq 0$ for $x \in \mathbb{R} \setminus [a_t, b_t]$,

(v) $g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t = 0$ for $x \in [a_t, b_t]$.

In terms of the analytic function q_t, see (3.12) and Proposition 3.2, we have

$$g_{t+}(x) - g_{t-}(x) = -2\int_{b_t}^{x} (q_t(s))_+^{1/2} ds, \qquad \text{for } x \in [a_t, b_t], \tag{5.7}$$

and

$$g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t = -2\int_{b_t}^{x} (q_t(s))^{1/2} ds \qquad \text{for } x > b_t, \tag{5.8}$$

$$g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t = -2\int_{a_t}^{x} (q_t(s))^{1/2} ds \qquad \text{for } x < a_t \tag{5.9}$$

where $(q_t(s))^{1/2}$ is analytic for $s \in \mathcal{V} \setminus [a_t, b_t]$ and that square root is taken which is positive for large real s. We define

$$\varphi_t(z) = \int_{b_t}^{z} (q_t(s))^{1/2} ds \tag{5.10}$$

and

$$\tilde{\varphi}_t(z) = \int_{a_t}^{z} (q_t(s))^{1/2} ds \tag{5.11}$$

which are defined and analytic in the neighborhood $\mathcal{V}$ of the real line where V is analytic with cuts along $(-\infty, b_t)$ and $(a_t, +\infty)$, respectively. Note that q_t has simple zeros in a_t and b_t and only double zeros in $\mathcal{V} \setminus [a_t, b_t]$, so that $q_t^{1/2}$ is indeed analytic there. It is possible that q_t has double real zeros, and that $q_t^{1/2}$ has sign changes in $(-\infty, a_t)$ or (b_t, ∞). However, by (5.8) and (5.9) we have that $\varphi_t(x) > 0$ for $x > b_t$ and $\tilde{\varphi}_t(x) > 0$ for $x < a_t$, since we are in a situation with strict inequality in (3.4) for $x > b_t$ and $x < a_t$.

We will now perform the first transformation of the RH problem: we take $t = n/N$ and define

$$T(z) = e^{\frac{n}{2}\ell_t\sigma_3} Y(z) e^{-ng_t(z)\sigma_3} e^{-\frac{n}{2}\ell_t\sigma_3} \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.12}$$

Using the jump condition (5.2) of Y and (5.12), we easily check that $T_+(x) = T_-(x)J_T(x)$ for $x \in \mathbb{R}$, where

$$\begin{aligned} J_T(x) &= e^{\frac{n}{2}\ell_t\sigma_3} e^{ng_{t-}(x)\sigma_3} \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} e^{-ng_{t+}(x)\sigma_3} e^{-\frac{n}{2}\ell_t\sigma_3} \\ &= \begin{pmatrix} e^{-n(g_{t+}(x)-g_{t-}(x))} & e^{n(g_{t+}(x)+g_{t-}(x)-\frac{1}{t}V(x)+\ell_t)} \\ 0 & e^{n(g_{t+}(x)-g_{t-}(x))}. \end{pmatrix} \end{aligned} \tag{5.13}$$

Because of the properties (5.7)–(5.9) and the definitions (5.10)–(5.11), we see that the jump matrix J_T has the following forms on the respective intervals (a_t, b_t), (b_t, ∞), and $(-\infty, a_t)$,

$$J_T = \begin{pmatrix} e^{2n\varphi_{t+}} & 1 \\ 0 & e^{2n\varphi_{t-}} \end{pmatrix} \quad \text{on } (a_t, b_t)$$
$$J_T = \begin{pmatrix} 1 & e^{-2n\varphi_t} \\ 0 & 1 \end{pmatrix} \quad \text{on } (b_t, \infty)$$
$$J_T = \begin{pmatrix} 1 & e^{-2n\tilde{\varphi}_t} \\ 0 & 1 \end{pmatrix} \quad \text{on } (-\infty, a_t).$$

Thus T is the unique solution of the RH problem

(a) T is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $T_+(x) = T_-(x) J_T(x) \quad$ for $x \in \mathbb{R}$,

(c) $T(z) = I + O(z^{-1}) \quad$ as $z \to \infty$.

5.3 Opening of the lens

The jump matrix J_T on the interval (a_t, b_t) has the factorization

$$\begin{aligned} J_T &= \begin{pmatrix} e^{2n\varphi_{t+}} & 1 \\ 0 & e^{2n\varphi_{t-}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_{t-}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_{t+}} & 1 \end{pmatrix}. \end{aligned} \tag{5.14}$$

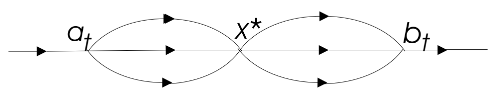

Figure 6: The contour after the opening of the lens.

Now we open lenses around the intervals (a_t, x^*) and (x^*, b_t) as shown in Figure 6. Let C_1 be the upper lips of the lenses and C_2 the lower lips, with orientation as in Figure 6. We open the lenses in such a way that they are fully contained in $\mathcal{V}$, the region of analyticity of V and q_t. In addition, we can take C_1 and C_2 in such a way that $\operatorname{Re}\varphi_t < 0$ on C_1 and C_2, with the exception of a neighborhood of x^* if $t < 1$. This effect is due to the fact that ν_t has a negative density near x^* if $t < 1$. However, if $t < 1$ increases to one, the exceptional neighborhood shrinks to a point. It follows that for any given $\delta > 0$, there is a constant $\gamma > 0$ such that for t sufficiently close to 1 we have

$$\operatorname{Re}\varphi_t(z) < -\gamma < 0$$

for all $z \in C_1 \cup C_2$ with $\min(|z - x^*|, |z - a|, |z - b|) > \delta$.

We define

$$S = \begin{cases} T & \text{outside the lenses,} \\ T\begin{pmatrix} 1 & 0 \\ -e^{2n\varphi_t} & 1 \end{pmatrix} & \text{in upper parts of the lenses,} \\ T\begin{pmatrix} 1 & 0 \\ e^{2n\varphi_t} & 1 \end{pmatrix} & \text{in lower parts of the lenses.} \end{cases} \tag{5.15}$$

The RH problem for T and the factorization (5.14) imply that S solves the following RH problem:

$$\begin{aligned} &\text{(a) } S \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup C_1 \cup C_2)\,, \\ &\text{(b) } S_+(z) = S_-(z) J_S(z) \quad \text{for } z \in \mathbb{R} \cup C_1 \cup C_2, \\ &\text{(c) } S(z) = I + O(z^{-1}) \quad \text{as } z \to \infty, \\ &\text{(d) } S \text{ remains bounded near } a_t,\ b_t, \text{ and } x^*, \end{aligned} \tag{5.16}$$

where J_S is given by

$$J_S = \begin{cases} \begin{pmatrix} 1 & e^{-2n\varphi_t} \\ 0 & 1 \end{pmatrix} & \text{on } (b_t, \infty), \\ \begin{pmatrix} 1 & e^{-2n\tilde{\varphi}_t} \\ 0 & 1 \end{pmatrix} & \text{on } (-\infty, a_t), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (a_t, b_t), \\ \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_t(z)} & 1 \end{pmatrix} & \text{on } C_1 \cup C_2, \end{cases} \tag{5.17}$$

The jump matrices on $\mathbb{R} \setminus (a_t, b_t)$ and on $C_1 \cup C_2$ tend to the identity matrix as $n \to \infty$ and $t \to 1$. Ignoring these jumps, we find the parametrix for the outside region. Uniform convergence breaks down in neighborhoods of a_t, b_t, and x^*, so that we will also need to construct local parametrices near those points.

5.4 Parametrix away from special points

The outside parametrix $S^\infty = S_t^\infty$ solves the following RH problem

(a) S^∞ is analytic in $\mathbb{C} \setminus [a_t, b_t]$,

(b) $S_+^\infty = S_-^\infty \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on (a_t, b_t),

(c) $S^\infty(z) = I + O(z^{-1})$ as $z \to \infty$.

As in [20] it has the solution

$$S_t^\infty(z) = \begin{pmatrix} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\beta(z)-\beta(z)^{-1}}{2i} \\ -\frac{\beta(z)-\beta(z)^{-1}}{2i} & \frac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix}, \qquad z \in \mathbb{C} \setminus [a_t, b_t], \tag{5.18}$$

where

$$\beta(z) = \beta_t(z) = \left(\frac{z - b_t}{z - a_t} \right)^{1/4}, \qquad z \in \mathbb{C} \setminus [a_t, b_t]. \tag{5.19}$$

Note that S_t^∞ depends on t.

5.5 Parametrix at edge points

At edge points a_t and b_t the density ψ_t vanishes like a square root. This allows the construction of local parametrices near a_t and b_t with the use of Airy functions. We will not give the details, see [17, 19, 20].

5.6 Parametrix at critical point

In a neighborhood $\Delta = \{z \in \mathbb{C} \mid |x^* - z| < \delta\}$ of x^*, we want to construct a parametrix P with the following properties

(a) P is analytic in $\Delta \setminus (\mathbb{R} \cup C_1 \cup C_2)$,

(b) $P_+ = P_- J_S(z)$ on $(\mathbb{R} \cup C_1 \cup C_2) \cap \Delta$,

(c) $P(z) = (I + O(n^{-1/3}))S_t^\infty(z)$ as $n \to \infty$, $t \to 1$, uniformly for $z \in \partial\Delta$,

(d) $P(z)$ remains bounded for z near x^*.

We seek a parametrix P near x^* in the form

$$P(z) = E_t(z) M\left(n^{1/3} f(z); n^{2/3} s_t(z), n\theta_t\right) \tag{5.20}$$

where E_t is analytic in Δ, f is a conformal map from Δ to a neighborhood of 0, s_t is analytic in Δ, and θ_t is a real constant. Recall that M is given by (4.11).

In view of the jump properties of M, we seek f, s_t, and θ_t so that for $z \in \Delta$,

$$i\left(\frac{4}{3}f(z)^3 + s_t(z)f(z) - \theta_t\right) = \begin{cases} \varphi_t(z) & \text{if } \operatorname{Im} z > 0, \\ -\varphi_t(z) & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{5.21}$$

Since $\varphi_{t+} = -\varphi_{t-}$, the right-hand side of (5.21) does indeed define an analytic function in Δ.

The conformal map f will not depend on t. Recall that q_V has a zero of order four at x^*, and that it is negative on $(a, b) \setminus \{x^*\}$. Thus $\int_{x^*}^z (-q_V(y))^{1/2} dy$ is analytic in Δ and has a zero of order three at x^*. We may assume there are no other zeros in Δ. Then we can take a third root and define

$$f(z) = \left[\frac{3}{4}\int_{x^*}^z (-q_V(y))^{1/2} dy\right]^{1/3}, \qquad z \in \Delta. \tag{5.22}$$

Then f is analytic in Δ with

$$\begin{aligned} f(z) &= \left(\frac{\pi\psi_V''(x^*)}{8}\right)^{1/3}(z - x^*) + O\left((z - x^*)^2\right) \\ &= c^{1/3}(z - x^*) + O\left((z - x^*)^2\right), \qquad \text{as } z \to x^* \end{aligned} \tag{5.23}$$

where c is given by (2.4). Taking smaller Δ if necessary, we then have that f is indeed a conformal map on Δ. Note that $f(x^*) = 0$ and that f is real and positive on $(x^*, x^* + \delta)$ and real and negative on $(x^* - \delta, x^*)$. We still have some freedom in opening the lenses. We take C_1 and C_2 in Δ so that f maps them to the rays where M has its jumps. That is, $C_1 \cap \Delta$ is mapped into $\arg\zeta = \pi/6$ and $\arg\zeta = 5\pi/6$, and $C_2 \cap \Delta$ is mapped into $\arg\zeta = -\pi/6$ and $\arg\zeta = -5\pi/6$.

Having f with $f(x^*) = 0$, we take $z = x^*$ in (5.21) and we see that we should take

$$\theta_t = i\varphi_{t+}(x^*) = -i\varphi_{t-}(x^*). \tag{5.24}$$

Then θ_t is real and it is also given by

$$\theta_t = -\int_{b_t}^{x^*} (-q_t(y))^{1/2} dy$$

and so

$$\theta_t \mp i\varphi_t(z) = \int_{x^*}^{z} (-q_t(y))^{1/2} dy, \qquad \text{for } \pm \operatorname{Im} z > 0. \tag{5.25}$$

Having f and θ_t we finally take $s_t(z)$ so that (5.21) holds, that is,

$$\begin{aligned} s_t(z) f(z) &= -\frac{4}{3} f(z)^3 + \theta_t \mp i\varphi_t(z) \\ &= \int_{x^*}^{z} \left((-q_t(y))^{1/2} - (-q_V(y))^{1/2} \right) dy \end{aligned} \tag{5.26}$$

where for the last line we used (5.22) and (5.25). Since the right-hand side of (5.26) is analytic in Δ and vanishes for $z = x^*$ we can divide by $f(z)$ (which has a simple zero at $z = x^*$) and obtain an analytic function s_t in Δ. Then $s_t(z)$ is real for real z, and $s_1(z) \equiv 0$.

So the above construction yields analytic functions f, s_t and a constant θ_t so that (5.21) holds. Then we can define the parametrix P by (5.20), provided that $n^{2/3} s_t(z)$ for $z \in \Delta$ stays away from the poles of the Hastings-McLeod solution of Painlevé II, since for $s \in \mathcal{P}$ we have that $M(\zeta; s, \theta)$ is not defined. For given values of n and t, we can take Δ small enough so that this is indeed the case. However, we want to let $n \to \infty$, $t \to 1$ so that $n^{2/3}(t-1) \to L$, and work with a neighborhood Δ that is independent of n and t, although it may depend on L.

Note that by (5.26) and (3.15)

$$s_t(z) f(z) = \pi \int_{x^*}^{z} (\psi_t(y) - \psi_V(y)) dy. \tag{5.27}$$

Because of (3.16) we have

$$\psi_t(y) - \psi_V(y) = (t-1)(w_{S_V}(y) - \psi_1(y)) + o(t-1),$$

as $t \to 1$, uniformly for y in a neighborhood of x^*. Using this in (5.27) we get that

$$\begin{aligned} s_t(z)\frac{f(z)}{z-x^*} &= \pi(t-1)\frac{1}{z-x^*}\int_{x^*}^{z}(w_{S_V}(y)-\psi_1(y))dy + o(t-1) \\ &= \pi(t-1)w_{S_V}(x^*) + (t-1)O(z-x^*) + o(t-1), \end{aligned} \tag{5.28}$$

where $o(t-1)$ is uniformly in z as $t \to 1$, and $O(z-x^*)$ is uniformly in t as $z \to x^*$. By (5.23) and (5.28) we then also have

$$s_t(z) = \pi c^{1/3}(t-1)w_{S_V}(x^*) + (t-1)O(z-x^*) + o(t-1), \tag{5.29}$$

where again $o(t-1)$ is uniformly in z as $t \to 1$, and $O(z-x^*)$ is uniformly in t as $z \to x^*$. Then if $n^{2/3}(t-1) \to L$ we get

$$\begin{aligned} n^{2/3}s_t(z) &= \pi c^{-1/3}Lw_{S_V}(x^*) + O(z-x^*) + o(1) \\ &= s + O(z-x^*) + o(1), \end{aligned} \tag{5.30}$$

where we used the definition (2.5) of the constant s. Since there are no poles on the real line, we can find a neighborhood Δ of x^* such that $n^{2/3}s_t(z) \notin \mathcal{P}$ for all $z \in \Delta$ if n is large enough and $n^{2/3}(t-1) \to L$. Note that Δ depends on L, but not on n and t. Then

$$M(n^{1/3}f(z); n^{2/3}s_t(z), n\theta_t)$$

is well-defined for z in a fixed neighborhood Δ of x^*.

Finally, we define E_t in such a way that the matching condition at $\partial\Delta$ is satisfied. We do this by defining

$$E_t(z) = \begin{cases} S_t^\infty(z) & \text{for } \operatorname{Im} z > 0, \\ S_t^\infty(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \operatorname{Im} z \leq 0, \end{cases} \tag{5.31}$$

so that $E_t(z)$ is analytic near x^* by the jump property (5.16) of S_t^∞. Because of the asymptotic behavior of $M(\zeta; s, \theta)$ in (4.12) and (4.13) (which is valid uniformly for s away from the poles and for $\theta \in \mathbb{R}$) we have the matching condition

$$P(z) = S_t^\infty(z)(I + O(n^{-1/3})) = (I + O(n^{-1/3}))S_t^\infty(z),$$

uniformly for $z \in \partial\Delta$. This completes the construction of the parametrix P in the neighborhood of x^*.

5.7 Third Transformation

We set

$$R(z) = \begin{cases} S(z)P^{-1}(z) & \text{for } z \text{ in disks around } x^*,\ a, \text{ and } b, \\ S(z)(S_t^\infty)^{-1}(z) & \text{for } z \text{ outside the disks.} \end{cases} \tag{5.32}$$

Since S and P have the same jumps inside each of the disks, and S and S_t^∞ have the same jumps on $[a_t, b_t]$, R has jumps on a contour Γ as shown in Figure 7.

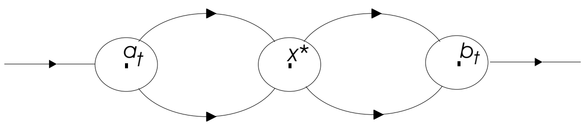

Figure 7: The contour Γ after the third and final transformation.

R solves the following RH problem:

(a) R is analytic in $\mathbb{C} \setminus \Gamma$,

(b) $R_+(z) = R_-(z)J_R(z)$ as $z \in \Gamma$,

(c) $R(z) = I + O(z^{-1})$ as $z \to \infty$,

(d) R is bounded,

for certain jump matrices J_R, which as $n \to \infty$, $t \to 1$ so that $n^{2/3}(t-1) \to L$, satisfy

$$J_R(z) = \begin{cases} I + O(n^{-1}) & \text{for } z \text{ in circles around } a \text{ and } b \\ I + O(n^{-1/3}) & \text{for } z \in \partial\Delta, \\ I + O(e^{-\gamma n}) & \text{for some fixed } \gamma > 0 \text{ elsewhere on } \Gamma. \end{cases}$$

As in [17, 19] it now follows that

$$\|R(z) - I\|_\infty = O(n^{-1/3}) \tag{5.33}$$

uniformly for $z \in \mathbb{C} \setminus \Gamma$.

6 Proof of Theorem 2.1

Now we are ready for the proof of Theorem 2.1. We start with the expression (5.4) for the kernel $K_{n,N}$. After the transformation (5.12), we find that for $x, y \in (a_t, b_t)$,

$$K_{n,N}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{-n\varphi_{t+}(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ 0 \end{pmatrix}. \tag{6.1}$$

Using formula (5.15) for S in the upper parts of the lenses, we get for $x, y \in (a_t, b_t)$,

$$K_{n,N}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -e^{n\varphi_{t+}(y)} & e^{n\varphi_{t+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ e^{n\varphi_{t+}(x)} \end{pmatrix} \tag{6.2}$$

Assume that x and y are inside the disk Δ around x^*. Then by (5.32), (5.20), (4.11), and (5.21) we get

$$S_+(x) = R(x) E_t(x) e^{in\theta_t \sigma_3} \Psi_+(n^{1/3} f(x); n^{2/3} s_t(x)) e^{n\varphi_{t+}(x)\sigma_3},$$

and so

$$S_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ e^{n\varphi_{t+}(x)} \end{pmatrix} = R(x) E_t(x) e^{in\theta_t \sigma_3} \Psi(n^{1/3} f(x); n^{2/3} s_t(x)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.3}$$

Similarly

$$\begin{aligned} &\begin{pmatrix} -e^{n\varphi_{t+}(y)} & e^{n\varphi_{t+}(y)} \end{pmatrix} S_+^{-1}(y) \\ &= \begin{pmatrix} -1 & 1 \end{pmatrix} \Psi^{-1}(n^{1/3} f(y); n^{2/3} s_t(y)) e^{-in\theta_t \sigma_3} E_t^{-1}(y) R^{-1}(y). \end{aligned} \tag{6.4}$$

Now we fix u and v and take

$$x = x^* + \frac{u}{(cn)^{1/3}} \quad \text{and} \quad y = x^* + \frac{v}{(cn)^{1/3}}, \tag{6.5}$$

so that for n large enough, x and y are inside the disk around x^*, so that (6.2), (6.3), and (6.4) hold. Then it follows from (5.23) and (6.5) that

$$n^{1/3} f(x) \to u, \quad \text{and} \quad n^{1/3} f(y) \to v \qquad \text{as } n \to \infty. \tag{6.6}$$

From (5.30) and (6.5) we get

$$n^{2/3} s_t(x) \to s, \quad \text{and} \quad n^{2/3} s_t(y) \to s \tag{6.7}$$

as $n \to \infty$, $t \to 1$ such that $n^{2/3}(t-1) \to L$. Furthermore, from (5.33), (6.5), and the fact that R is analytic near x^*, we get

$$R^{-1}(y)R(x) = I + O\left(\frac{x-y}{n^{1/3}}\right) = I + O\left(\frac{u-v}{n^{2/3}}\right). \tag{6.8}$$

From (5.18), (5.31), and (6.5) we easily get

$$E_t^{-1}(y)E_t(x) = I + O\left(\frac{u-v}{n^{1/3}}\right) \tag{6.9}$$

as $n \to \infty$. The constants implied by the O-symbols in (6.8) and (6.9) are independent of u and v, when u and v are restricted to a compact subset of $\mathbb{R}$. Combining (6.8) and (6.9) we get that

$$e^{-in\theta_t\sigma_3}E_t^{-1}(y)R^{-1}(y)R(x)E_t(x)e^{in\theta_t\sigma_3} = I + O\left(\frac{u-v}{n^{1/3}}\right), \tag{6.10}$$

since θ_t is real.

Then multiplying (6.4) and (6.3) and letting $n \to \infty$, $t \to 1$ such that $n^{2/3}(t-1) \to L$, we get by using (6.2), (6.5), (6.6), (6.7), and (6.10) that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(x^* + \frac{u}{(cn)^{1/3}}, x^* + \frac{v}{(cn)^{1/3}}\right) \\
&= \frac{1}{2\pi i(u-v)} \lim_{n\to\infty} \begin{pmatrix} -e^{n\varphi_{t+}(y)} & e^{n\varphi_{t+}(y)} \end{pmatrix} S_+^{-1}(y)S_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ e^{n\varphi_{t+}(x)} \end{pmatrix} \\
&= \frac{1}{2\pi i(u-v)} \begin{pmatrix} -1 & 1 \end{pmatrix} \Psi^{-1}(v;s)\Psi(u;s) \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned} \tag{6.11}$$

uniformly for u and v in compact subsets of $\mathbb{R}$.

By (1.21), (4.3) and (4.9) we have that

$$\begin{pmatrix} \Phi_1(\zeta;s) \\ \Phi_2(\zeta;s) \end{pmatrix} = \Psi(\zeta;s)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{for } \zeta \in S_2 \cup S_3.$$

In view of the jump (4.10) satisfied by $\Psi(\zeta;s)$ on Σ_1 we have that

$$\begin{pmatrix} \Phi_1(\zeta;s) \\ \Phi_2(\zeta;s) \end{pmatrix} = \Psi(\zeta;s)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } \zeta \in S_1 \cup S_4, \tag{6.12}$$

Since $\det \Psi \equiv 1$, we also get (after simple calculation)

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \Psi^{-1}(\zeta; s) = \begin{pmatrix} -\Phi_2(\zeta; s) & \Phi_1(\zeta; s) \end{pmatrix} \quad \text{for } \zeta \in S_1 \cup S_4. \tag{6.13}$$

Note that $S_1 \cup S_4$ includes the full real line, so that we can take $\zeta = u$ in (6.12) and $\zeta = v$ in (6.13) which we use in (6.11) to obtain (2.3). This completes the proof of Theorem 2.1.

Remark 6.1 The above proof of Theorem 2.1 was given under the assumption that S_V consists of one interval, and that there are no other singular points except for x^*. Here singular point refers to the classification of [20] according to which there are three types of non-regular behavior for a real-analytic external field V, see also [32]. The singular points of type I are points in $\mathbb{R} \setminus S_V$ where equality in the variational inequality (3.6) with $t = 1$ holds. Singular points of type II are interior points of S_V where the density ψ_V vanishes, and singular points of type III are edge points of S_V where ψ_V vanishes to higher order than a square root. Here we indicate briefly the modifications that have to be made if these assumptions are not satisfied.

If S_V consists of more than one interval, then the main complication is that the construction of the outside parametrix S_t^∞ is more complicated, since it uses Θ-functions as in [20, Lemma 4.3]. It should be noted that S_t^∞ will also depend on n. As a result it will follow that E_t as defined in (5.31) also depends on n. However this will not effect the asymptotic behavior (6.9) as $n \to \infty$, so that the above proof goes through.

In case there are other singular points, we have to construct special local parametrices around each of them. For our purposes, we only need to know the existence of those parametrices, which is established in [20, Section 5].

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