Poisson Pencils, Integrability, and Separation of Variables

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Abstract. In this paper we will review a recently introduced method for solving the Hamilton-Jacobi equations by the method of Separation of Variables. This method is based on the notion of pencil of Poisson brackets and on the bihamiltonian approach to integrable systems. We will discuss how separability conditions can be intrinsically characterized within such a geometrical set-up, the definition of the separation coordinates being encompassed in the bihamiltonian structure itself. We finally discuss these constructions studying in details a particular example, based on a generalization of the classical Toda Lattice.

Keywords: Hamilton-Jacobi Equations, Bihamiltonian Manifolds, Separation of Variables, Generalized Toda Lattices.

1. Introduction

The study of the separability of the Hamilton–Jacobi (HJ) equations associated with a given Hamiltonian function H is a very classical issue in Mechanics, dating back to the foundational works of Jacobi, Stäckel, Levi-Civita and others. It has recently received a strong renewed interests thanks to its applications to the theory of integrable PDEs of KdV type (namely, the theory of finite gap integration) and to the theory of quantum integrable systems (see, e.g., Dubrovin et. al. 1990; Sklyanin 1995).

As it is well known, the problem can be formulated as follows. Let (M,ω) be a $2n$ dimensional symplectic manifold, and let $(p_1,\ldots,p_n,q_1,\ldots,q_n)\equiv(\mathbf{p},\mathbf{q})$ be canonical coordinates (in $U\subset M$), i.e., $\omega_{|_U}=\sum_{i=1}^n dp_i\wedge dq_i$.

The (stationary) HJ equation reads

$$H(q_1,\ldots,q_n;\frac{\partial S}{\partial p_1},\ldots,\frac{\partial S}{\partial p_n})=E\,. \tag{1.1}$$

Definition 1.1. *A complete integral* $S(\mathbf{q};\alpha_1,\ldots,\alpha_n)$ *of HJ is a solution of (1.1), depending on n parameters* $(\alpha_1,\ldots,\alpha_n)$ *such that* $Det\frac{\partial^2 S}{\partial q_i\partial\alpha_j}\neq 0$
H is said to be separable *in the coordinates* $(\mathbf{p},\mathbf{q})$ *if HJ admits an additively separated complete integral, that is, a complete integral of the form*

$$S(\mathbf{q};\alpha_1,\ldots,\alpha_n)=\sum_{i=1}^n S_i(q_i;\alpha_1,\ldots,\alpha_n). \tag{1.2}$$

In this paper we will focus on an equivalent ‘Definition–Theorem’ of separability, originally due to Jacobi and recently widely used by Sklyanin and his collaborators. Let us consider an *integrable* Hamiltonian H, that is, let us suppose that, along with $H = H_1$ we have further $n-1$ mutually commuting integrals of the motion $H_2, \ldots H_n$, with $dH_1 \wedge \ldots \wedge dH_n \neq 0$.

Definition 1.2. *An integrable system* $(H_1, \ldots, H_n)$ *is separable in the coordinates* $(\mathbf{p}, \mathbf{q})$ *if there exist* n *non-trivial relations*

$$\Phi_i(q_i, p_i; H_1, \ldots, H_n) = 0, \quad i = 1, \ldots n\,, \tag{1.3}$$

connecting single pairs (q_i, p_i) *of canonical coordinates with the* n *Hamiltonians* H_i.

We called this a Definition–Theorem due to the fact that it is a *constructive* approach to separability, since the knowledge of the separation relations (1.3) allows one to reduce the probelm of finding a separated solution of HJ to quadratures. Indeed, one can solve the relations $\Phi_i(q_i, p_i; H_1, \ldots, H_n) = 0, \quad i = 1, \ldots n$ for the p_i to get $p_i = p_i(q_i; H_1, \ldots, H_n)$ and then define:

$$S(\mathbf{q}; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \int^{q_i} p_i(q_i'; H_1, \ldots, H_n)\big|_{H_i=\alpha_i} dq_i'\,. \tag{1.4}$$

This is by construction a separated solution of HJ; the fact that it is a complete integral is equivalent to the fact that the integrals of the motion depend non trivially on the momenta.

In intrinsic terms, one notices that, the equations $H_i = \alpha_i$, $i = 1, \ldots, n$ define a foliation $\mathcal{F}$ of M. The leaves of $\mathcal{F}$ are nothing but the (generalized) tori of the Arnol'd-Liouville theorem. The foliation is *Lagrangian*, that is, the restriction of the two-form ω to $\mathcal{F}$ vanishes. Hence the restriction to $\mathcal{F}$ of the Liouville form $\theta = \sum_{i=1}^{n} p_i dq_i$ to $\mathcal{F}$ is (locally) exact. Indeed, the function S defined by (1.4) is a (local) potential for the restriction of θ to $\mathcal{F}$. What is non intrinsic, and singles out the separation coordinates $(\mathbf{p}, \mathbf{q})$ is that the separation relations (1.3), which are another set of defining equations for the foliation $\mathcal{F}$, have the very special property of containing a single pair of canonical coordinates at a time. The problem to find such a system of coordinates and relations is the core of the theory of SoV. In particular, a natural question arises:

Is it possible to formulate intrinsic condition(s) on the Hamiltonians $(H_1, \ldots, H_n)$ *to* a priori *ensure separability in a set of canonical coordinates?*

Actually, this is the main issue studied by both the 'classical' Eisenhart-Benenti theory of separability of natural systems defined on cotangent bundles to Riemannian manifolds (M, g), as well as the 'modern' theory, due to the St. Petersburg and Montreal schools, of SoV for systems admitting a Lax representation. We notice that both such general approaches require the presence of *an additional structure* to solve the problem. Indeed, the Eisenhart-Benenti theory requires the existence of a conformal Killing tensor for the metric g, while the Lax theory requires – in addition to the knowledge of a Lax representation with spectral parameter for the Hamiltonian system under study – the existence of an r–matrix structure for such a Lax representation.

The method we will present/review in this paper, which has recently been exposed in the literature (see, e.g., Brouzet et.al. 1996; Morosi & Tondo 1997;2002, Błaszak 1998, Magri et. al. 2000; Falqui et. al. 2000a; 2000b Ibort et.al. 2000, Pedroni 2002,Falqui & Pedroni 2003, Bartocci et. al. 2003) can be seen as a kind of bridge between the classical and the modern points of view, putting an emphasis on the geometrical structures of the Hamiltonian theory. Its 'additional' structure is simply the requirement of the existence, on the symplectic manifold (M, ω) of a *second* Hamiltonian structure, compatible with the one defined by ω. Namely, the bihamiltonian structure on M will allow us:

1. To encompass the definition of a special set of coordinates, to be called *Darboux–Nijenhuis (DN)* coordinates, within a well defined geometrical object.
2. To formulate intrinsic (i.e., tensorial) conditions for the separability of a Hamiltonian integrable system, in the DN coordinates associated with the bihamiltonian structure.
3. To give recipes to characterize, find and handle sets of DN coordinates.

The paper is organized as follows. In §2 we will briefly introduce the notions of bihamiltonian geometry relevant for the paper. In particular, we will discuss the notion of DN coordinates, as well as methods to find them. In §3 we will discuss the main theorems of the bihamiltonian set–up for SoV, namely, the tensorial conditions ensuring separability of the HJ equations in DN coordinates.

Then we will discuss our constructions in a specific example, whose separability, to the best of our knowledge, has not yet been considered in the literature. It is a generalization of the periodic Toda lattice with four sites. In §4 we will recall its definition, and show how the "bihamiltonian recipe" for SoV can be applied to it. Although our constructions can be generalized to the generic N-site system, for the sake of concreteness and brevity we choose to consider the four-site system only, and sometimes rely on direct computations to prove some of its properties.

In the last subsection we will apply our geometrical scheme to study a specific reduction of this generalized Toda system, and find integrals of the motion which are not encompassed in the Lax representation. This result can be considered as a first step towards an alternative approach to the so–called chopping method of Deift et. al. 1984 for the full (non-periodic) Toda Lattice.

2. Some issues in the geometry of (bi)hamiltonian manifolds

We start this Section recalling some well known facts in the theory of Poisson manifolds (see, e.g., Vaisman 1996 for full details).

Definition 2.1. *A Poisson manifold* $(M, \{\cdot, \cdot\})$ *is a manifold endowed with a Poisson bracket, that is a bilinear antisymmetric composition law defined on the space* $C^\infty(M)$ *satisfying:*

1. *The Leibnitz rule:* $\{fg, h\} = f\{g, h\} + g\{f, h\}$*;*
2. *The Jacobi identity* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

A Poisson bracket (or Poisson structure) can be equivalently described with a Poisson tensor, that is, with an application $P : T^*M \to TM$, smoothly varying with $m \in M$ defined by:

$$\{f, g\} = \langle df, Pdg \rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical pairing between T^*M and TM. In a given coordinate system $(x^1,\dots,x^n)$ on M, the Poisson tensor P associated with the Poisson bracket $\{\cdot,\cdot\}$ is represented as

$$P=\sum_{i,j} P^{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \qquad \text{with } P^{ij}=\{x^i,x^j\}.$$

The Jacobi identity is translated into a quadratic differential condition on the matrix P^{ij} known as vanishing Schouten bracket (see, for further details, e.g., Magri & Morosi 1984, Vaisman 1996), which in local coordinates reads

$$\sum_{s=1}^{n} P^{is}\frac{\partial P^{jk}}{\partial x_s}+P^{js}\frac{\partial P^{ki}}{\partial x_s}+P^{ks}\frac{\partial P^{ij}}{\partial x_s}=0,\ \forall\, i>j>k. \tag{2.1}$$

We notice that the existence of a Poisson structure does not give any restriction on the dimension of M. As an example, the reader should consider the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$, with Poisson brackets defined, thanks to the natural identification $\mathfrak{g}^{**}\simeq\mathfrak{g}$, by

$$\{f,g\}(X)=\langle X,[df(X),dg(X)]\rangle,\ X\in\mathfrak{g}^*,\ f,g\in C^\infty(\mathfrak{g}^*).$$

The most familiar instances of Poisson manifolds come from classical mechanics and are the cotangent bundles to smooth manifolds. As it is well known (see, e.g., Arnol'd 1989), they are equipped with a canonical Poisson structure associated with a symplectic, that is, closed (indeed exact) and non degenerate, two-form ω. A Poisson manifold $(M,\{\cdot,\cdot\})$, is called *symplectic* if the Poisson tensor P associated with $\{\cdot,\cdot\}$ is everywhere invertible. In such a case, the connection between the Poisson tensor P, the Poisson brackets $\{f,g\}$ and the symplectic two–form ω can be described in the following equivalent ways:

a) In local coordinates $(x_1,\dots,x_n)$ one has $\omega=\sum_{i,j}\omega_{ij}dx_i\wedge dx_j$, where the matrix ω_{ij} is the inverse matrix to P^{ij}.

b) It holds the equality
$$\{f,g\}=\omega(X_f,X_g)$$
where, e.g., $X_f=Pdf$ is the Hamiltonian vector field associated with f.

c) For every vector field Y and every function f, one has:
$$L_Y(f)\equiv\langle df,Y\rangle=\omega(Y,X_f)$$

The local structure of a Poisson manifold can be described as follows (see, e.g., Weinstein 1983 and Dubrovin et. al. 1990). Under suitable assumptions on the regularity of the Poisson brackets, one sees that in an open dense set of M the Poisson tensor has rank $r=2n$. There, M is foliated in regular Poisson submanifolds, called generic symplectic leaves, that are the (generic) common level sets of k functions $C_1,\dots,C_k$, called Casimir functions of P. The distinguished property of the Casimirs of P is that their Poisson bracket with any other function on M vanishes, or, equivalently, their differential lies in the kernel of P. The dimension of M is related with the integers n and k by $\dim M=k+2n$. As an example, we notice that, in the case of $M = \mathfrak{sl}(n)^*$, which we can identify with $\mathfrak{sl}(n)$ itself by means, say, of the trace form, the Casimir functions are $C_i = \mathrm{Tr}(X^{i+1}), i = 1, \ldots, n-1$, and the generic symplectic leaves consist of diagonalizable matrices with distinct eigenvalues that can be identified with the direct sum of the lower and upper nilpotent subalgebras.

Let us now come to the definition of bihamiltonian manifold.

Definition 2.2. *A manifold M is called a bihamiltonian manifold if it is endowed with* two *Poisson brackets $\{f,g\}$, $\{f,g\}'$, such that, for any $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ if M is complex), the linear combination*

$$\{f,g\}' - \lambda\{f,g\} \equiv \langle df, (P' - \lambda P)dg\rangle \tag{2.2}$$

defines a Poisson bracket. This property is known as the compatibility condition *between the two brackets.*

The expression (2.2) will be referred to as *pencil* of Poisson brackets, and the sum $P_\lambda = P' - \lambda P$ *pencil of Poisson tensors.* The most 'popular' property of bihamiltonian manifolds is contained in the following

Proposition 2.3. *Let f and f' two functions on a bihamiltonian manifold M, which satisfy the characteristic condition $Pdf = P'df'$. Then the Poisson brackets $\{f, f'\}$ and $\{f, f'\}'$ vanish.*

Proof. It consists of a one–line computation. Let us consider, e. g., $\{f, f'\}$:

$$\{f, f'\} = \langle df, Pdf'\rangle = -\langle df', Pdf\rangle = -\langle df', P'df'\rangle = 0.$$

□

Definition 2.4. *A vector field X that can be written as $X = Pdf = P'df'$ is called a* bihamiltonian vector field.

Corollary 2.5. *Let $f_k, k \in \mathbb{Z}$ be a sequence of (non-constant) functions satisfying*

$$Pdf_i = P'df_{i+1}. \tag{2.3}$$

Then $\{f_i, f_k\} = \{f_i, f_k\}' = 0, \quad \forall\, i, k \in \mathbb{Z}$.

Proof. Using twice equation (2.3) and the antisymmetry of the Poisson brackets we have, e.g.,

$$\begin{aligned}\{f_i, f_k\} &= \langle df_i, Pdf_k\rangle = \langle df_i, P'df_{k-1}\rangle = -\langle df_{k-1}, P'df_i\rangle = \\ &- \langle df_{k-1}, Pdf_{i+1}\rangle = \langle df_{i+1}, Pdf_{k-1}\rangle = \{f_{i+1}, f_{k-1}\}.\end{aligned}$$

Supposing $k > i$ and iterating this procedure $k - i$ times we get $\{f_i, f_k\} = \{f_k, f_i\}$.
□

Following Gel'fand & Zakharevich 2000, we can state the following

Amplification 2.6. *Let $f_n, n = 0, 1, \ldots$ and $g_n, n = 0, 1, \ldots$ two sequences of (non-constant) functions satisfying*

$$Pdf_i = P'df_{i+1}; Pdf_0 = 0, \quad Pdg_i = P'dg_{i+1}; Pdg_0 = 0. \tag{2.4}$$

Then, along with $\{f_n, f_m\} = \{f_n, f_m\}' = \{g_n, g_m\} = \{g_n, g_m\}' = 0$, it holds

$$\{f_n, g_m\} = \{f_n, g_m\}' = 0, \quad \forall\, n, m \in \mathbb{N}.$$

The family of vector fields associated with a sequence of functions satisfying the recursion relations (2.3) are customarily said to form a Lenard-Magri sequence. Lenard-Magri sequences that start from the null vector field, as in Amplification 2.6 are pictorially called *anchored* Lenard-Magri sequences. Notice that anchored Lenard sequences can occur in bihamiltonian manifold where at least one of the Poisson brackets is non-symplectic (indeed, e.g., df_0 is a non-trivial element of the kernel of P). We can compactly express equations (2.4) relative, say, to the sequence f_i, by considering the formal Laurent series $f(\lambda) = \sum_{i=0}^{\infty} f_i/\lambda^i$ via the equation

$$(P' - \lambda P)df(\lambda) = 0. \tag{2.5}$$

In analogy with the definition of Casimir of a Poisson bracket, Laurent series satisfying (2.5) are called Casimirs of the Poisson *pencil.* The reader should bear in mind that while Casimir functions for a single Poisson bracket are, in a sense, uninteresting functions, Casimirs of a *pencil* of Poisson bracket compactly encode non-trivial dynamics and constants of the motion.

For further use we notice that, as it often happens in the applications, if inside the family f_i we have an element f_n satisfying $P'df_n = 0$, we can form a *polynomial* Casimir of the pencil P_λ by the finite sum:

$$F(\lambda) = \lambda^n f_0 + \lambda^{n-1} f_1 + \cdots + f_n. \tag{2.6}$$

Anchored Lenard sequences may give rise to families of integrable systems. Let us see how this happens in the case of a $2n+1$-dimensional manifold endowed with a rank $2n$ pencil of Poisson tensors. Let us suppose that we have found a polynomial Casimir of the form (2.6), in which the $n+1$ functions $f_0, \ldots, f_n$ are functionally independent. Let $\mathcal{S}_c$ be a generic symplectic leaf of P corresponding to $f_0 = c$. The vector fields $X_{f_i}, i = 1, \ldots, n$ are tangent to $\mathcal{S}_c$, are Hamiltonian on $\mathcal{S}_c$ (w.r.t. P) and the restriction of the functions $f_1, \ldots, f_n$ provide n commuting integrals for each of them. In general it holds (Gel'fand & Zakharevich, 1993; 2000):

Proposition 2.7. *Let (M, P, P') a bihamiltonian manifold of dimension $d = 2n + k$, and let $dim(Ker(P' - \lambda P)) = k$, for generic values of λ. Let us suppose that $H^{(1)}(\lambda), \ldots, H^{(k)}(\lambda)$ are k polynomial Casimirs of the pencil P_λ of the form*

$$H^{(i)}(\lambda) = \lambda^{n_i} H_0^{(i)} + \lambda^{n-1} H_1^{(i)} + \cdots + H_{n_i}^{(i)}$$

such that the collection of differentials $\{dH_j^{(i)}\}_{i=1,\ldots,k}^{j=0,\ldots n_i}$ defines a $k+n$ dimensional distribution in T^*M. Then the vector fields defined by the anchored sequences associated with the $H^{(i)}$'s are integrable (in the Arnol'd-Liouville sense) on the generic symplectic leaves of P.

(*a*) *Regular bihamiltonian manifolds and Darboux-Nijenhuis coordinates*

An important class of bihamiltonian manifold occurs when one element of the Poisson pencil (which without loss of generality we will assume to be P) is everywhere *invertible*, i.e., the Poisson bracket $\{\cdot, \cdot\}$ associated with P is symplectic. The possibility of defining the inverse to one of the Poisson tensors lead us to introduce a fundamental object in the bihamiltonian theory of SoV: the Nijenhuis (or Hereditary, or Recursion) operator

$$N = P' P^{-1},$$

(together with its dual $N^* = P^{-1}\,P'$). By definition, N (resp., N^*) is an endomorphism of the tangent bundle to M (resp., of the cotangent bundle). As a remarkable consequence of the compatibility of P and P' the Nijenhuis torsion of N, defined by its action on a pair of vector fields X, Y as

$$T(N)(X,Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X,Y]) \qquad (2.7)$$

identically vanishes (Magri & Morosi 1984). So, from the classical Frölicher–Nijenhuis theory, we know that its eigenspaces are *integrable* distributions. Such distributions will be the building blocks of the bihamiltonian set-up for SoV.

To explain this, we have to make some remarks and a genericity assumption. It can be shown that, owing to the antisymmetry of the Poisson tensors defining N, pointwise the eigenspaces of N are even dimensional. Througout the paper, we will assume that for generic points $m \in M$, N has the maximal number $n = \frac{1}{2}\dim(M)$ of different eigenvalues $\lambda_1, \ldots, \lambda_n$, so that the dimension of the eigenspace relative to any eigenvalue is 2. Otherwise stated, the characteristic polynomial of N is the square of the degree $n = \frac{1}{2}\dim(M)$ minimal polynomial $\Delta_N(\lambda)$, whose roots are pairwise distinct. We will call bihamiltonian manifolds endowed with a Poisson pencil with at least one of the elements of the Poisson pencil invertible, and such that the eigenvalues of the associated Nijenhuis tensor are maximally distinct, *regular bihamiltonian manifolds.*

Theorem 2.8. *On a regular bihamiltonian manifold there exist a class of coordinates (y_i, x_i), to be called* Darboux–Nijenhuis (DN) *coordinates, satisfying the two properties:*

(Darboux) *They are canonical, that is, $\{x_i, y_j\} = \delta_{ij}$, $\{x_i, x_j\} = \{y_i, y_j\}$=0.*

(Nijenhuis) *They diagonalize N, that is, $N = \sum_i \lambda_i \big(\dfrac{\partial}{\partial y_i} \otimes dy_i + \dfrac{\partial}{\partial x_i} \otimes dx_i\big)$.*

Th proof of this Theorem can be found in Magri 1990 and Gel'fand & Zakharevich 1993. Here we will sketch it and discuss its meaning. In words, the assertion states that DN coordinates are defined by the spectral properties of N, as follows. For all m in the open set U where the eigenvalues λ_i of N (which are the same as the eigenvalues of N^*) satisfy $\lambda_i \neq \lambda_j$, $i \neq j$, the cotangent space $T^*_m M$ admits the decomposition

$$T^*_m M = \oplus_{i=1}^n \mathcal{D}_{m,\lambda_i}, \quad \dim\, \mathcal{D}_{m,\lambda_i} = 2 \qquad (2.8)$$

into eigenspaces of N^*. Thanks to the vanishing of the torsion of N, each eigenspace $\mathcal{D}_{m,\lambda_i}$ is locally generated by differentials of pairs of independent functions (f_i, g_i). This means that the pointwise decomposition (2.8) holds (in $U' \subset U$) as

$$T^* M_{|_U'} = \oplus_{i=1}^n \mathcal{D}_{\lambda_i},$$

where $\mathcal{D}_{\lambda_i}$ is spanned by df_i and dg_i, with $N^* df_i = \lambda_i df_i$ and $N^* dg_i = \lambda_i dg_i$.

Functions whose differential belong to *different* summands $\mathcal{D}_{\lambda_i}$ are in involution with respect to the Poisson brackets defined both by P and P'. Indeed, suppose that f_1 and f_2 satisfy $N^* df_1 = \lambda_1 df_1, N^* df_2 = \lambda_2 df_2, \lambda_1 \neq \lambda_2$. The relation $N^* = P^{-1}P'$ implies that $P' df_1 = \lambda_1 P df_1,\ P' df_2 = \lambda_2 P df_2$. So,

$$\{f_1, f_2\}' = \begin{cases} \langle df_1, P' df_2\rangle = \lambda_2 \langle df_1, P df_2\rangle = \lambda_2\{f_1, f_2\} \\ -\langle df_2, P' df_1\rangle = -\lambda_1\langle df_2, P df_1\rangle = \lambda_1\{f_1, f_2\} \end{cases}$$

whence the assertion. It is equally straightforward to realize that the only non vanishing Poisson brackets have the form

$$\{f_i, g_i\} = F_i(f_i, g_i), \quad \{f_i, g_i\}' = F_i'(f_i, g_i), \quad i = 1, \ldots, n.$$

This means that, by quadratures, from the n pairs of functions $\{f_i, g_i\}_{i=1,\ldots,n}$ we can construct a set of canonical coordinates satisfying the Nijenhuis property of Theorem 2.8. Thus the class of coordinates where to frame the bihamiltonian set-up for SoV admits a clearcut and simple geometrical description. Admittedly, in the general case the computation of DN coordinates requires the integration of the two–dimensional distributions $\mathcal{D}_{\lambda_i}$ associated with the eigenvalues λ_i of N^*. Fortunately enough there are instances (that frequently occur in the applications) in which DN coordinates can be found in an easier way.

(b) *On Darboux-Nijenhuis coordinates*

In this subsection we will briefly discuss conditions and 'recipes' to algebraically find and/or characterize sets of Darboux-Nijenhuis coordinates on regular bihamiltonian manifolds. A very simplifying instance occurs whenever the eigenvalues λ_i of N (that are, in general, functions of the point $m \in M$) are *functionally independent.* It holds (see, e.g., Magri & Marsico 1996):

Proposition 2.9. *Let us define* $I_i := \frac{1}{2i} TrN^i$, $i = 1, \ldots, n$. *In the open set* U *where* $dI_1 \wedge \cdots \wedge \cdots dI_n \neq 0$ *the eigenvalues* λ_i , $i = 1, \ldots, n$ *are functionally independent, satisfy* $N^* d\lambda_i = \lambda_i d\lambda_i$, *and so may be used to construct a set of Darboux-Nijenhuis coordinates.*

Proof. We express the normalized traces I_i of the Nijenhuis tensor N in terms of its eigenvalues as $kI_k = \sum_{i=1}^n \lambda_i^k$. Hence $dI_k = \sum_{i=1}^n \lambda_i^{k-1} d\lambda_i$, that is, in matrix terms:

$$\begin{bmatrix} dI_1 \\ dI_2 \\ \vdots \\ dI_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} d\lambda_1 \\ d\lambda_2 \\ \vdots \\ d\lambda_n \end{bmatrix} \tag{2.9}$$

So we have

$$dI_1 \wedge \cdots \wedge dI_n = \prod_{i \neq j} (\lambda_i - \lambda_j) d\lambda_1 \wedge \cdots \wedge d\lambda_n,$$

i.e., on the open set where the traces of the powers of the Nijenhuis tensor are functionally independent, we have that the eigenvalues λ_i are different and functionally independent.

To proceed further we need to recall that (Magri & Morosi 1984) that the normalized traces I_i of the powers of Nijenhuis operator satisfy the recursion relation:

$$N^* dI_k = dI_{k+1}. \tag{2.10}$$

This can be proved as follows. At first one notices that (2.10) is equivalent to the relation

$$L_{NX}(I_k) = L_X(I_{k+1}), \ \forall \text{ vector field } X,$$

as it can be easily seen evaluating the equality of one-forms on a generic vector field X. Thanks to the Leibnitz property of the Lie derivative and the ciclicity of the trace, we see that

$$L_{NX}(I_k) = \mathrm{Tr}(L_{NX}(N)\cdot N^{k-1}) \quad \text{and } L_X(I_{k+1}) = \mathrm{Tr}(L_X(N)\cdot N^k). \qquad (2.11)$$

Since the vanishing of the Nijenhuis torsion of N implies that $L_{NX}(N) = N \cdot L_X(N)$, $\forall X$ the validity of (2.10) is proved..

We now express the relations (2.10) in terms of the eigenvalues λ_i

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} N^*d\lambda_1 - \lambda_1 d\lambda_1 \\ N^*d\lambda_2 - \lambda_2 d\lambda_2 \\ \vdots \\ N^*d\lambda_n - \lambda_n d\lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (2.12)$$

Since the Vandermonde matrix in the LHS of this equation is by assumption invertible, we conclude that, for $i = 1, \ldots, n$, the relation $N^*d\lambda_i = \lambda_i d\lambda_i$ holds. □

This proposition can be rephrazed saying that "half" of the DN coordinates are algebraically provided by the Nijenhuis tensor itself. The remaining "half" $y_1, \ldots, y_n$ can always be found by quadratures. Actually, there is a condition leading to the algebraic solution of this problem too. To elucidate this, the following two considerations are crucial.

First of all one remarks that, thanks to the fact that, for DN coordinates, if $i \neq j$ one has $\{\lambda_i, y_j\} = 0$ one can replace the $n(n-1)/2$ equations $\{\lambda_i, y_j\} = \delta_{ij}$ with the n equations

$$\{\lambda_1 + \cdots + \lambda_n, y_j\} = 1, \; j = 1, \ldots, n,$$

that do not involve the individual coordinates λ_i but only their sum $\sum_{i=1}^n \lambda_i = I_1$, and the Hamiltonian vector field

$$Y = -PdI_1 = \sum_i \frac{\partial}{\partial y_i}.$$

The second argument goes as follows. Let us consider the distinguished functions I_k introduced in proposition2.9, and trade them for the coefficients p_i of the minimal polynomial

$$\Delta_N(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n$$

of N. The functions p_k and I_k are related by triangular Newton formulas

$$\begin{aligned} &I_1 = p_1; \quad I_2 = p_2 + \frac{1}{2}p_1^2; \quad I_3 = p_3 + p_2 p_1 + \frac{1}{3}p_1^3; \\ &I_4 = p_4 + p_1 p_3 + p_1^2 p_2 + \frac{1}{2}p_2^2 + \frac{1}{4}p_1^4; \quad I_5 = p_5 + \ldots. \end{aligned} \qquad (2.13)$$

As a consequence of the recursion relations (2.10), it can be easily shown that the p_i's satisfy the 'Frobenius' recursion relations

$$N^*dp_i = dp_{i+1} + p_i dp_1, \;\; \text{with } p_{n+1} \equiv 0. \qquad (2.14)$$

We can compactly write these relations as a single relation for the polynomial $\Delta_N(\lambda)$; indeed, a straightforward computation shows that they are equivalent to

$$N^* d\Delta_N(\lambda) = \lambda d\Delta_N(\lambda) + \Delta_N(\lambda) dp_1 \,. \tag{2.15}$$

Actually relations of this kind are very important for our purposes. Indeed, in Falqui & Pedroni 2003 we proved the following proposition:

Proposition 2.10. *Let $\Phi(\lambda)$ a smooth function defined on the manifold M, depending on an additional parameter λ. Suppose that there exists a one-form α_Φ such that*

$$N^* d\Phi(\lambda) = \lambda d\Phi(\lambda) + \Delta_N(\lambda)\alpha_\Phi \,. \tag{2.16}$$

Then, the n functions Φ_i obtained evaluating the "generating" function $\Phi(\lambda)$ for $\lambda = \lambda_i, i = 1, \ldots, n$ are Nijenhuis functions, that is, they satisfy $N^* d\Phi_i = \lambda_i d\Phi_i$.

Definition 2.11. *We will call a generating function $\Phi(\lambda)$ satisfying equation (2.16) a* Nijenhuis functions generator.

The relevance of the notion of Nijenhuis functions generator in the search for DN coordinates stems from the fact that Nijenhuis functions generators form an algebra $\mathcal{N}(M)$, *which is closed* under the action of the vector field $Y = -PdI_1$. In this way, knowing a set of Nijenhuis functions generators, we can obtain further elements of the algebra $\mathcal{N}(M)$ by repeated applications of the vector field Y. Clearly, in such an extended algebra, the characteristic equation

$$L_Y(\Psi(\lambda)) = 1 + \Delta_N(\lambda)\alpha_\psi$$

may be easier to be solved, thus yielding the missing Darboux-Nijenhuis coordinates coordinates y_i as $y_i = \Psi(\lambda_i)$. We will see an instance of this situation in the Example of § 4.

For an analysis of Darboux-Nijenhuis coordinates within the theory of multihamiltonian structure on loop algebras, see Falqui et. al 2000*b*, Degiovanni & Magnano, 2002, Harnad & Hurtubise 2002.

3. The Separability Conditions

As we have briefly recalled in §2, on a bihamiltonian manifold one is usually lead to consider bihamiltonian vector fields, that is vector fields X admitting the twofold Hamiltonian representation $X = Pdf = P'dg$. Let us now suppose that (M, P, P') be a regular bihamiltonian manifold of dimension $2n$, and that we were able to construct, by means of the Lenard–Magri iteration procedure a sequence of functions $H_1, H_2, \ldots$ satisfying $P'dH_i = PdH_{i+1}$. Let us also suppose that the first n of them be functionally independent. Then one easily shows that the all the further Hamiltonians $H_{n+1}, \ldots$ are functionally dependent from the first n. Indeed this follows from the fact that a regular Poisson manifold of dimension $2n$ cannot have more than n mutually commuting independent functions.

This means that, if we consider the Hamiltonian H_{n+1}, there must be a relation of the form

$$\psi(H_1, \ldots, H_n; H_{n+1}) = 0, \quad \text{with } \psi_{H_{n+1}} \equiv \frac{\partial \psi}{\partial H_{n+1}} \neq 0. \tag{3.1}$$

relating it with the independent Hamiltonians $H_i, i = 1, \ldots, n$.

Actually, the case of $H_i = I_i \equiv \frac{1}{2i}\mathrm{Tr}N^i$ is an instance of this situation. In fact, since by the Cayley–Hamilton theorem N annihilates its minimal polynomial, we have $N^n - \sum_{i=1}^n p_i N^{n-1} = 0$, yielding the relation

$$I_{n+1}/2(n+1) - \sum_{i=1}^{n} 2(n-i+1)p_i I_{n-i+1} = 0.$$

Differentiating equation (3.1) we see that, along with $P'dH_i = PdH_{i+1}, i = 1, \ldots, n-1$ it holds:

$$P'dH_n = PdH_{n+1} = \frac{-1}{\psi_{H_{n+1}}}\Big(\sum_{i=1}^{n} \frac{\partial \psi}{\partial H_i} PdH_i\Big), \tag{3.2}$$

that is, the vector field $X_{n+1} = PdH_{n+1} = P'dH_n$ is a linear combination of the vector fields $X_1 = PdH_1, \ldots X_n = PdH_n$.

This innocent looking observation is the clue for the bihamiltonian theory of SoV. Let $\{H_1, H_2, \ldots H_n\}$ any integrable system on M, that is, the H_is are mutually commuting (w.r.t.P) independent functions. We can construct a n-dimensional distribution, namely the distribution $\mathcal{D}_H$ spanned by the n mutually commuting vector fields $X_i = PdH_i$. This is nothing but the very classical tangent distribution to the invariant tori of the Liouville Arnol'd theory of integrable systems. Since M comes equipped with a second Poisson tensor P' we can as well consider the distribution $\mathcal{D}'_H$ generated by the Hamiltonians H_i under the action of P', that is, generated by the vector fields $X'_i = P'dH_i$. It holds:

Theorem 3.1. *Let $\{H_1, \ldots, H_n\}$ define, as explained above, an integrable system on a regular bihamiltonian manifold (M, P, P'). The Hamilton-Jacobi equations associated with any of the Hamiltonians H_i are separable in the DN coordinates x_i, y_i defined by $N = P'P^{-1}$ if and only if the distribution $\mathcal{D}'_H$ is contained in $\mathcal{D}_H$, or, equivalently, if the distribution $\mathcal{D}_H$ is invariant along N.*

Proof. We will first prove the equivalence of the invariance of $\mathcal{D}_H$ under N and the inclusion $\mathcal{D}'_H \subset \mathcal{D}_H$. To say that $\mathcal{D}'_H$ is contained in $\mathcal{D}_H$ is tantamount to saying that there exists a matrix F_{ij}, whose entries are, in general, functions defined on M such that, for $i = 1, \ldots, n$ it holds:

$$X'_i \equiv P'dH_i = \sum_j F_{ij} PdH_j = \sum_j F_{ij} X_j. \tag{3.3}$$

Writing $P' = NP$ we can translate these equalities into $NX_i = \sum_j F_{ij} X_j$ for all $i = 1, \ldots, n$.

The full proof of the fact that the invariance of $\mathcal{D}_H$ insures separability in DN coordinates can be found in Falqui & Pedroni 2003. It goes as follows.

At first we notice that the translation in terms of the codistribution $\mathcal{D}^*_H$ generated by the differentials of the Hamiltonians H_i of the invariance condition for $\mathcal{D}_H$ is the invariance condition $N^*\mathcal{D}^*_H \subset \mathcal{D}^*_H$. This can be easily seen applying to (3.3) the operator P^{-1}, to get $N^*dH_i = \sum_j F_{ij} dH_j$.

Since all the Poisson brackets $\{H_i, H_j\}$ vanish and M is a regular bihamiltonian manifold, the matrix F defined by (3.3) can be shown to have simple eigenvalues, that coincide with eigenvalues λ_i of N. So there exists a matrix S satisfying

$$SF = \Lambda S, \text{ where } \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

If we define the n one forms $\theta_i = \sum_j S_{ij} dH_j$, we get:

$$N^*\theta_i = \sum_j S_{ij} N^* dH_j = \sum_{j,k} S_{ij} F_{jk} dH_k = \sum_{j,k} \lambda_i \delta_{ij} S_{jk} dH_k = \lambda_i \theta_i. \qquad (3.4)$$

This means that θ_i is an eigenvector of N^* relative to λ_i; hence there must be functions F_i, G_i such that

$$\sum_j S_{ij} dH_j = F_i dx_i + G_i dy_i \qquad (3.5)$$

whence the existence of a separation relation $\Phi_i(x_i, y_i; H_1, \ldots, H_n)$ for all $i = 1, \ldots, n$. The converse statement can be trivially proved. □

We would like to stress that the separability condition of Theorem 3.1 is a *tensorial one*. That is, given a regular bihamiltonian manifold (M, P, P') this separability criterion can be checked in *any system of coordinates*, without the a–priori calculation of the DN coordinates themselves. Notice, also, that the validity of the statement does not (as it should be!) depend on the choice of mutually commuting integrals $\{H_1, \ldots, H_n\}$. That is, if we consider a “change of coordinates in the space of the actions”, that is we trade the H_i's for another complete set of integrals of the motion $K_i = K_i(H_1, \ldots, H_n)$, then the separability of the new Hamiltonians K_i will hold if and only if the separability of the original ones holds. Indeed, the dual distributions generated by the H_i's and the K_i's coincide.

A second remark is important and deserves to be explicitly spelled out. Although we have started our discussion considering the case of a family of bihamiltonian vector fields, that is the case of Lenard-Magri sequences, the hypotheses of Theorem 3.1 concern only the relations of the distributions generated respectively under the action of P and P' by the Hamiltonians H_i, *without any mention* of the fact that the generators of the distribution be bihamiltonian vector fields. Thus, although it might seem a somewhat odd statement, *the vector fields that are separable by means of the bihamiltonian approach are not necessarily bihamiltonian vector fields!*. It is also important to notice that it is not only a matter of choice of generators. Indeed, in Marsico 1996 it has been shown that the only bihamiltonian vector fields on a regular bihamiltonian manifold turn out to be associated with *separated functions of the eigenvalues of* N, that is functions of the form $H = \sum_{i=1}^n f_i(\lambda_i)$. This means that, in such a case, the distribution $\mathcal{D}_H$ coincides with that generated by the distinguished functions I_i. However, this is by far a very special example, that is, the range of applicability of the method is much wider than that, as it as already been quite widely shown in the literature.

The separation condition of Theorem 3.1 is based on the analysis of the behaviour of the characteristic distribution associated with an integrable system under the Nijenhuis tensor N. An equivalent criterion, based on the analysis of the Poisson brackets associated with the tensor P' can be formulated as follows.

Theorem 3.2. *Let $\{H = H_1, H_2, \dots H_n\}$ an integrable system defined on regular bihamiltonian manifold (M, P, P'). The Hamiltonians H_i are separable in the DN coordinates defined by $N = P'P^{-1}$ if and only if, along with the commutation relations $\{H_i, H_j\} = 0$ there hold also*

$$\{H_i, H_j\}' \equiv \langle dH_i, P'dH_j\rangle = 0, \quad \textit{for } i, j = 1, \dots, n. \tag{3.6}$$

Proof. The key formula is the relation between P, P' and N^*. Indeed, suppose that $\mathcal{D}_H^*$ be invariant along N^*. Then:

$$\{H_i, H_j\}' = \langle dH_i, P'dH_j\rangle = \langle dH_i, NPdH_j\rangle$$
$$\langle N^*dH_i, PdH_j\rangle = \sum_k F_{ik}\langle dH_k, PdH_j\rangle = \sum_k F_{ik}\{H_k, H_i\} = 0,$$

which, in view of Theorem 3.1 proves the statement in one direction. Now, let us suppose that (3.6) holds. Then, for every $ij = 1, \dots, n$ we have:

$$0 = \{H_i, H_j\}' = \langle dH_i, P'dH_j\rangle = \langle dH_i, NPdH_j\rangle = \langle N^*dH_i, PdH_j\rangle,$$

meaning that, for all $i = 1, \dots, n$, the one-form N^*dH_i belongs to the annihilator (w.r.t. P) of the distribution $\mathcal{D}_H$. Since such an annihilator coincides with $\mathcal{D}_H^*$, this means that, for all $i = 1, \dots, n$, $N^*dH_i \in \mathcal{D}_H^*$. □

This results leads to the following, somewhat daring, comparison. The Liouville-Arnol'd theorem on finite dimensional integrable Hamiltonian systems says that the geometrical structure underlying integrability of a Hamiltonian vector field defined on a symplectic manifold M, ω is a *Lagrangian* foliation of M. We can rephrase the content of Theorem 3.2 saying that the geometrical structure underlying the separability of a system defined on a regular bihamiltonian manifold(M, P, P') is a *bilagrangian* foliation of M.

We end our presentation of the bihamiltonian set-up for SoV with the following remark. Theorem 3.1 concerns only the existence of the separation relations. In principle, one could try to find these relations in concrete examples by actually diagonalizing the matrix F, and explicitly finding and integrating the relations (3.5). However, there is a very simple tensorial criterion which can be used to determine the *functional form* of the separation relations $\Phi_i(x_i, y_i; H_1, \dots, H_h)$, whose proof can be found in Falqui & Pedroni 2003.

Proposition 3.3. *Let $\{H_1, \dots, H_n\}$ be an integrable system defined on a regular bihamiltonian manifold, which is separable in the Darboux-Nijenhuis coordinates associated with $N = P^{-1}P'$. Consider the matrix F_{ij} fulfilling the relations (3.3). Then the separation relations are be* affine *in the Hamiltonians H, that is of the form*

$$\Phi_i(x_i, y_i; H_1, \dots, H_h) = \sum_j S_{ij}(x_i, y_i)H_j + U_i(x_i, y_i), \tag{3.7}$$

if and only if the matrix F satisfies the relation $N^*dF_{ij} = \sum_k F_{ik}dF_{kj}$.

The matrix S of (3.7) can be shown to be a suitably normalized matrix of eigenvectors of the matrix F. Its characteristic property is that the entries S_{ij} of the i–th row depend only on the pair (x_i, y_i) of Darboux-Nijenhuis coordinates. For this reason it can be called a Stäckel matrix.

4. Example: a generalized Toda Lattice

In this final Section we will apply the general scheme outlined in the previous Sections to a specific model, with the aim of showing how the recipes discussed so far from a theoretical standpoint can be concretely applied. We will study a generalization of the four site Toda lattice, to be termed *$Toda^4_3$ model.* This system is a member of a family introduced in Kupershmidt 1985 as reductions of the discrete KP hierarchy. It can be described as follows. We consider on $M = \mathbb{C}^{12}$, endowed with global coordinates $\{b_i, a_i, c_i\}_{i=1..4}$ the Hamiltonian

$$H_{GT} = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - (a_1 + a_2 + a_3 + a_4), \tag{4.1}$$

and the linear Poisson tensor given by the matrix

$$P = \begin{bmatrix} \mathbf{0} & \mathbf{A_1} & \mathbf{C_1} \\ -\mathbf{A_1^T} & \mathbf{C_2} & \mathbf{0} \\ -\mathbf{C_1^T} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{with } \mathbf{A_1} = \begin{bmatrix} -a_1 & 0 & 0 & a_4 \\ a_1 & -a_2 & 0 & 0 \\ 0 & a_2 & -a_3 & 0 \\ 0 & 0 & a_3 & -a_4 \end{bmatrix}, \tag{4.2}$$

$$\mathbf{C_1} = \begin{bmatrix} -c_1 & 0 & c_3 & 0 \\ 0 & -c_2 & 0 & c_4 \\ c_1 & 0 & -c_3 & 0 \\ 0 & c_2 & 0 & -c_4 \end{bmatrix}, \quad \mathbf{C_2} = \begin{bmatrix} 0 & -c_1 & 0 & c_4 \\ c_1 & 0 & -c_2 & 0 \\ 0 & c_2 & 0 & -c_3 \\ -c_4 & 0 & c_3 & 0 \end{bmatrix}.$$

where we denoted by $\mathbf{0}$ the 4×4 matrix with vanishing entries..
Using (here and in the sequel) the cyclic identifications $a_{i+4} = a_i, \ldots$, the Hamiltonian vector field $X_{H_{GT}} = PdH_{GT}$ can be written as follows

$$\begin{bmatrix} \dot{b}_i \\ \dot{a}_i \\ \dot{c}_i \end{bmatrix} = \begin{bmatrix} a_{i-1} - a_i \\ a_i(b_{i+1} - b_i) + c_{i-1} - c_i \\ c_i(b_{i-2} - b_i) \end{bmatrix}, \quad i = 1, \ldots, 4, \tag{4.3}$$

The expert reader will notice that H_{GT} coincides with the Hamiltonian of the periodic four-site Toda lattice, written in Flaschka coordinates $b_i = p_i, a_i = \exp(q_i - q_{i+1})$. Indeed, on the hyperplane $M_T \simeq \mathbb{C}^8$ defined by $c_i = 0, i = 1, \ldots 4$ the vector field $X_{H_{GT}}$ defines the periodic Toda flow.

Proposition 4.1. *The Hamiltonian vector field $X_{H_{GT}}$ admits the Lax representation $\dot{L}(\mu) = [L(\mu), \Phi]$, where*

$$L(\mu) = \begin{bmatrix} b_1 & -\mu & \frac{c_3}{\mu^2} & \frac{a_4}{\mu} \\ \frac{a_1}{\mu} & b_2 & -\mu & \frac{c_4}{\mu^2} \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & b_3 & -\mu \\ -\mu & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & b_4 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 & \frac{c_3}{\mu^2} & \frac{a_4}{\mu} \\ \frac{a_1}{\mu} & 0 & 0 & \frac{c_4}{\mu^2} \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & 0 & 0 \\ 0 & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & 0 \end{bmatrix}. \tag{4.4}$$

The bihamiltonian aspects of this system have been discussed in Meucci 2001. In particular, it has been noticed that on M there exists a second Hamiltonian structure for the vector field $X_{H_{GT}}$. Namely one considers the bivector P' having the following form:

$$P' = \begin{bmatrix} \mathbf{A_2} & \mathbf{B_1} & \mathbf{C_3} \\ -\mathbf{B_1^T} & \mathbf{A_3} & \mathbf{C_4} \\ -\mathbf{C_3^T} & -\mathbf{C_4^T} & \mathbf{A_4} \end{bmatrix}, \text{ where } \mathbf{C_3} = \begin{bmatrix} -b_1c_1 & 0 & b_1c_3 & 0 \\ 0 & -b_2c_2 & 0 & b_2c_4 \\ c_1b_3 & 0 & -b_3c_3 & 0 \\ 0 & c_2b_4 & 0 & -b_4c_4 \end{bmatrix}, \tag{4.5}$$

$$\mathbf{A_2} = \begin{bmatrix} 0 & a_1 & 0 & -a_4 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ a_4 & 0 & -a_3 & 0 \end{bmatrix}, \quad \mathbf{B_1} = \begin{bmatrix} -b_1a_1 & c_1 & -c_3 & b_1a_4 \\ b_2a_1 & -b_2a_2 & c_2 & -c_4 \\ -c_1 & b_3a_2 & -b_3a_3 & c_3 \\ c_4 & -c_2 & b_4a_3 & -b_4a_4 \end{bmatrix},$$

$$\mathbf{A_3} = \begin{bmatrix} 0 & -b_2c_1 - a_1a_2 & 0 & b_1c_4 + a_1a_4 \\ b_2c_1 + a_1a_2 & 0 & -b_3c_2 - a_2a_3 & 0 \\ 0 & b_3c_2 + a_2a_3 & 0 & -b_4c_3 - a_3a_4 \\ -b_1c_4 - a_1a_4 & 0 & b_4c_3 + a_3a_4 & 0 \end{bmatrix},$$

$$\mathbf{C_4} = \begin{bmatrix} -a_1c_1 & -a_1c_2 & a_1c_3 & a_1c_4 \\ c_1a_2 & -a_2c_2 & -a_2c_3 & a_2c_4 \\ c_1a_3 & c_2a_3 & -a_3c_3 & -a_3c_4 \\ -c_1a_4 & c_2a_4 & c_3a_4 & -a_4c_4 \end{bmatrix}, \quad \mathbf{A_4} = \begin{bmatrix} 0 & -c_1c_2 & 0 & c_1c_4 \\ c_1c_2 & 0 & -c_2c_3 & 0 \\ 0 & c_2c_3 & 0 & -c_3c_4 \\ -c_1c_4 & 0 & c_3c_4 & 0 \end{bmatrix}.$$

It can be easily noticed that $X_{H_{GT}} = P'd(-\sum_{i=1}^4 b_i)$. More in general, we have the following

Proposition 4.2. *The pencil $P' - \lambda P$ is a pencil of Poisson brackets. The rank of the generic element of the pencil is eight. The characteristic polynomial $R(\lambda, \mu) = Det\big(\lambda\mathbf{1} - L(\mu)\big)$, written in terms of $\rho = \mu^4$ can be expanded as:*

$$R(\lambda, \rho) = \lambda^4 - \rho + H(\lambda) + (K(\lambda) - \lambda^2 J_1)/\rho + J_2/\rho^2. \tag{4.6}$$

The functions J_1 and J_2 are common Casimirs of P and P'. The polynomials $H(\lambda), K(\lambda)$ are polynomial Casimirs of the pencil $P_\lambda = P' - \lambda P$. They have the form

$$H(\lambda) = \lambda^3 H_0 - \lambda^2 H_1 + \lambda H_2 - H_3,\ K(\lambda) = K_0\lambda + K_1. \tag{4.7}$$

Explicitly, $J_1 = c_1c_3 + c_2c_4, J_2 = c_1c_2c_3c_4$, while the coefficients of $H(\lambda)$ and $K(\lambda)$ are given by:

$$H_0 = \sum_{i=1}^4 b_i,\ H_1 = \sum_{i>j=1}^4 b_ib_j + \sum_{i=1}^4 a_i,\ H_2 = \sum_{i=1}^4 \big(c_i + b_i(a_{i+1} + a_{1+2}) + b_ib_{i+1}b_{i+2}\big)$$

$$H_3 = \sum_{i=1}^4 b_ic_{i+1} + a_1a_3 + a_2a_4 + \text{cubic and quartic terms;}$$

$$K_0 = \sum_{i=1}^4 b_ic_{i-1}c_{i+1} - c_ia_{i-1}a_{i+2},\ K_1 = \sum_{i=1}^4 a_ic_{1+1}c_{1+2} + \text{quartic terms}$$

One can show that the eight functions $H_0, H_1, H_2, H_3, K_0, K_1, J_1, J_2$ are functionally independent and, thanks to the fact that they fill in Lenard sequences, are mutually in involution. The kernel of P is generated (at generic points $m \in M$) by the differentials of the four functions H_0, K_0, J_1, J_2. Hence, on the 8 dimensional manifold $\mathcal{S}_{\boldsymbol{\kappa}}$ defined by the equations $H_0 = \kappa_1, K_0 = \kappa_2, J_1 = \kappa_3, J_2 = \kappa_4$, that is, the generic symplectic leaf of P, the vector field $X_{H_{GT}}$ is completely integrable. To realize this we simply have to notice that H_{GT} can be expressed as $\frac{1}{2}H_0^2 - H_1$, and apply the properties of anchored Lenard Magri sequences collected in Proposition 2.7

(a) *Separation of Variables*

We will now show how to apply the ideas and recipes if the bihamiltonian set-up for SoV to the Toda_3^4 model introduced above. The first problem to deal with is that the Poisson tensor P' does not restrict to $\mathcal{S}_{\boldsymbol{\kappa}}$, but must be projected. This can be done as follows (see Degiovanni & Magnano 2002, Falqui & Pedroni 2002;2003, and Marciniak & Błaszak 2003 for details and the geometric background), by means of a kind of Dirac reduction process.

We consider the vector fields $Z_1 = -\dfrac{\partial}{\partial b_4}$ and $Z_2 = \dfrac{\partial}{\partial a_4}$, notice that the matrix

$$G = \begin{pmatrix} L_{Z_1}(H_0) & L_{Z_1}(K_1) \\ L_{Z_2}(H_0) & L_{Z_2}(K_1) \end{pmatrix} = \begin{pmatrix} 1 & -c_1c_3 \\ 0 & -c_1a_3 - a_1c_2 \end{pmatrix}$$

is invertible, and form the bivector

$$\Delta = \sum_{i,j=1}^2 [G_{ij}^{-1}] Z_i \wedge X_1^j, \text{ where } X_1^1 = P'dH_0, X_1^2 = P'dK_0. \tag{4.8}$$

Lemma 4.3. *The modified bivector $Q = P' - \Delta$ defines a Poisson bracket, compatible with P; Q restricts to $\mathcal{S}_{\boldsymbol{\kappa}}$.*

Proof. The proof of the fact that $Q_\lambda = Q - \lambda P$ is a Poisson pencil follows (see Falqui & Pedroni 2002), from the equalities:

$$\begin{aligned} L_{Z_1}P = 0, \quad & L_{Z_1}P' = W_1^1 \wedge Z_1 - c_3\frac{\partial}{\partial a_3} \wedge Z_2 \\ L_{Z_2}P = (\frac{\partial}{\partial b_1} - \frac{\partial}{\partial a_1}) \wedge Z_2, \quad & L_{Z_2}P' = (b_4\frac{\partial}{\partial a_4} + \frac{\partial}{\partial b_1}) \wedge Z_1 + W_2^2 \wedge Z_2. \end{aligned} \tag{4.9}$$

with $W_1^1 = (a_3\frac{\partial}{\partial a_3} - a_4\frac{\partial}{\partial a_4} + c_2\frac{\partial}{\partial c_2} - c_4\frac{\partial}{\partial c_4})$ and

$$W_2^2 = (b_4\frac{\partial}{\partial b_4} - b_1\frac{\partial}{\partial b_1} - a_3\frac{\partial}{\partial a_3} - a_4\frac{\partial}{\partial a_4} - c_1\frac{\partial}{\partial c_1} + c_2\frac{\partial}{\partial c_2} + c_3\frac{\partial}{\partial c_3} - c_4\frac{\partial}{\partial c_4}),$$

as well as from the fact that

$$QdH_0 = QdK_0 = QdJ_1 = QdJ_2 = 0. \tag{4.10}$$

To show that (4.9) holds true is a matter of an explicit computation, while (4.10) follows from the definition of Q. In fact, the last two equations hold since J_1 and J_2 are Casimirs of P' invariant under Z_1 and Z_2, while for, e.g., H_0 one computes

$$\begin{aligned} QdH_0 = P'dH_0 - \Delta dH_0 &= X_1^1 - \sum_{i,j=1}^{2} \left([G_{ij}^{-1}]L_{Z_j}(H_0)\right) \cdot X_1^i \\ &= X_1^1 - \sum_{i,j=1}^{2} \left([G_{ij}^{-1}]G_{j1}\right) \cdot X_1^i = X_1^1 - \sum_i \delta_{i,1} \cdot X_1^i = 0, \end{aligned}$$

where the second equality follows from the fact that all the functions H_i, K_α, J_α are in involution w.r.t. P. □

Thanks to the above lemma, the generic symplectic leaf $\mathcal{S}_{\boldsymbol{\kappa}}$is endowed with the structure of a regular bihamiltonian manifold. It is easy to show that the non trivial Hamiltonians H_1, H_2, H_3, K_1 (more precisely, the restriction to $\mathcal{S}_{\boldsymbol{\kappa}}$of these Hamiltonians) satisfy the hypothesis of Theorem 3.1 w.r.t the (restriction to $\mathcal{S}_{\boldsymbol{\kappa}}$) of the pencil $Q - \lambda P$. Indeed we have:

$$QdH_i = P'dH_i - \sum_{i,j=1}^{2} [G_{ij}^{-1}]Z_i \wedge X_1^j(dH_i) = PdH_{i+1} - \sum_{i,j=1}^{2} [G_{ij}^{-1}]L_{Z_i}(H_i)X_1^j$$

(where we understand $H_4 = 0$) and

$$QdK_2 = P'dK_1 - \sum_{i,j=1}^{2} [G_{ij}^{-1}]Z_i \wedge X_1^j(dK_2) = \sum_{i,j=1}^{2} [G_{ij}^{-1}]L_{Z_i}(dK_1)X_1^j$$

So we proved that, for generic values of the Casimirs $\kappa_i, i = 1, \ldots, 4$, the system obtained by restriction of the Toda_3^4 flows on $\mathcal{S}_{\boldsymbol{\kappa}}$is separable in the DN coordinates associated with the restriction to $\mathcal{S}_{\boldsymbol{\kappa}}$of pencil $Q - \lambda P$. To finish our job we finally have to:
a) explicitly define the DN coordinates;
b) find the separation relations.

To solve the first problem, we will use the tools briefly described in §b. We rely on a result of Falqui & Pedroni 2003, as well as on explicit computations, to state the following

Proposition 4.4. *Let $\mathcal{G}(\lambda)$ be the matrix*

$$\mathcal{G} = \begin{pmatrix} L_{Z_1}H(\lambda) & L_{Z_2}H(\lambda) \\ L_{Z_1}K(\lambda) & L_{Z_2}K(\lambda) \end{pmatrix}; \tag{4.11}$$

The roots of the degree 4 *polynomial* $Det(\mathcal{G})$ *are the roots of the minimal polynomial* $\Delta(\lambda) = \lambda^4 - \sum_{i=1}^4 p_i\lambda^{4-i}$ *of the Nijenhuis tensor* $N = P^{-1}Q$ *associated with the regular Poisson pencil* Q_λ. *The coefficients* p_i *are well defined functions on the generic symplectic leaf* $\mathcal{S}_{\boldsymbol{\kappa}}$, *and are functionally independent. Furthermore, the ratios* $\rho(\lambda) = -\mathcal{G}_{2,2}/\mathcal{G}_{1,2}, \sigma(\lambda) = -\mathcal{G}_{2,1}/\mathcal{G}_{1,1}$ *are Nijenhuis function generators.*

Thus, one half of the Darboux-Nijenhuis coordinates will be given by the roots of $\mathrm{Det}(\mathcal{G})$. To find the remaining half we consider vector field $Y = -Pdp_1$ whose role has been discussed in §b. Since an explicit computations shows that $L_Y \log(\rho(\lambda)) = 1$, hence we can state the following

Proposition 4.5. *A set of Darboux-Nijenhuis coordinates for the restriction to the generic symplectic leaf* $\mathcal{S}_{\boldsymbol{\kappa}}$ *of the* Toda_3^4 *flows are given by the four roots* λ_i *of* $Det(\mathcal{G})(\lambda)$ *and by the values* μ_i *of the function* $\log(\rho(\lambda))$ *for* $\lambda = \lambda_i$, *where*

$$\rho(\lambda) = \frac{(-c_1a_3 - a_1c_2)\,\lambda + c_2a_1b_3 - a_1a_2a_3 + c_1b_2a_3 + c_1c_2}{c_1c_3\lambda + a_1a_2c_3 - c_1b_2c_3}$$

To find the separation relations we reconsider the Lax matrix (4.4), and notice that the pairs, e.g., $\lambda_i, \rho(\lambda_i)$ are common solutions to the system

$$\begin{aligned} &\rho\mathcal{G}_{1,1} + \mathcal{G}_{21} = 0 \\ &\rho\mathcal{G}_{1,2} + \mathcal{G}_{22} = 0. \end{aligned} \tag{4.12}$$

Since the Lie derivative of the matrix $\mathcal{L}(\lambda,\mu) = \lambda\mathbf{1} - L(\mu)$ along the vector fields Z_i are given by:

$$L_{Z_1}(\mathcal{L}(\lambda,\mu)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; L_{Z_1}(\mathcal{L}(\lambda,\mu)) = \begin{bmatrix} 0 & 0 & 0 & -\mu^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we see that the solutions of the system (4.12) are nothing but the solutions of the equations

$$[\mathcal{L}(\lambda,\mu)^\vee]_{44} = 0,\; [\mathcal{L}(\lambda,\mu)^\vee]_{41} = 0,$$

$\mathcal{L}(\lambda,\mu)^\vee$ being the classical adjoint to $\mathcal{L}(\lambda,\mu) = \lambda\mathbf{1} - L(\mu)$. Using standard arguments of linear algebra, and the results collected in Proposition 4.5, we can state

Proposition 4.6. *The separation relations connecting pairs of Darboux-Nijenhuis coordinates* λ_i, μ_i, *the Hamiltonians* H_1, H_2, H_3, K_2 *and the Casimirs* H_0, K_1, K_2, J_0 *are, on the generic symplectic leaf* $\mathcal{S}_{\boldsymbol{\kappa}}$, *given by the evaluation of the characteristic polynomial* $Det(\mathcal{L}(\lambda,\mu))$ *in* $\lambda = \lambda_i, \mu = \mu_i = \log(\rho(\lambda_i))$.

We notice that, *a posteriori,* the separation coordinates for the Toda_3^4 system fall in the class described in, e.g., Sklyanin 1992; 1995, Kuznetsov et. al 1997, Adams et. al. 1993; 1997 and Dubrovin & Diener 1994. Namely, the DN coordinates that separate the Toda_3^4 system are Algebro-geometrical Darboux coordinates associated with the spectral curve (4.6), and fulfill the so-called Sklyanin's 'magical recipe'.

As a final remark, in connection with the discussion on the relation between the bihamiltonian property of an integrable vector field and the separability of the associated HJ equations of §3, we notice that the Hamiltonians H_1, H_2, H_3, K_2 are functionally independent from the coefficients of the minimal polynomial of the Nijenhuis tensor obtained from $Q - \lambda P$. So, this is a further instance of a system which is not bihamiltonian on a regular manifold, but turns out to be separable via the bihamiltonian method of SoV.

(*b*) *A remarkable subsystem: the open Toda$_3^4$ system*

In this last subsection we will discuss a remarkable reduction of the periodic Toda$_3^4$ system, leading to the corresponding generalization of the open (or non-periodic) one. In the manifold $M \simeq \mathbb{C}^{12}$ we consider the nine–dimensional submanifold M_0 defined by the equations:

$$a_4 = c_3 = c_4 = 0. \tag{4.13}$$

One can easily verify that the restriction $X^0_{H_{GT}}$ to M_0 of the vector field $X_{H_{GT}}$ is tangent to M_0. Also, the tensor P can be restricted to M_0; indeed, the expression of its restriction P_0 w.r.t. the natural coordinates $\{b_1, \dots, b_4, a_1, \dots, a_3, c_1, c_2\}$ of M_0 is obtained from (4.2) simply by removing the 9th, 11th, 12th rows and columns.

One can easily check that $X^0_{H_{GT}} = P_0 dH_{GT,0}$ with

$$H_{GT,0} = \frac{1}{2}(b_1^2 + b_2^2 + b_3 + b_4^2) - (a_1 + a_2 + a_3), \tag{4.14}$$

and recognize that this function is the Hamiltonian of the open Toda lattice. Also, a Lax pair for $X^0_{H_{GT}}$ is

$$L_0 = \begin{bmatrix} b_1 & -\mu & 0 & 0 \\ \frac{a_1}{\mu} & b_2 & -\mu & 0 \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & b_3 & -\mu \\ -\mu & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & b_4 \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{a_1}{\mu} & 0 & 0 & 0 \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & 0 & 0 \\ 0 & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & 0 \end{bmatrix}. \tag{4.15}$$

It should be clear from the form of the Lax pair that the vector field $X^0_{H_{GT}}$ on M_0 is an extension of the standard open Toda lattice towards the so-called full open Toda lattice, which is a system describing a flow on the lower Borel subgroup of $sl(N)$. The integrability of the full open Toda lattice was established in Deift et. al. 1984 (see, also, Ercolani et. al. 1993). The idea was to complement the integrals of the motion coming from the Lax representation with additional integrals obtained by means of the 'chopping method', whithin the group–theoretical point of view.

The need to supply the standard results of the Lax theory with further methods should be clear from the following considerations. The only Casimir function of P_0 is $h_0 = \sum_{i=1}^4 b_i$. Hence, its symplectic leaves $\mathcal{S}_\xi \subset (M_0)$ are the eight dimensional manifolds defined by $h_0 = \xi$, and $X^0_{H_{GT}}$ can be seen as a Hamiltonian system with *four* degrees of freedom.

The characteristic polynomial of the matrix L_0 is

$$\mathrm{Det}(\lambda \mathbf{1} - L_0(\mu)) = -\mu^4 + \lambda^4 - h_0\lambda^3 + h_1\lambda^2 - h_2\lambda + h_3, \tag{4.16}$$

that is, it provides us with only *three* non trivial Hamiltonians

$$h_1 = \sum_{i>j=1}^{4} b_i b_j + \sum_{i=1}^{3} a_i, \quad h_2 = \sum_{i>j>k=1}^{4} b_i b_j b_k + \sum_{i=1}^{3} a_i(b_{i+2} + b_{i+3}) + c_1 + c_2$$
$$h_3 = b_1 a_2 b_4 + a_1 b_3 b_4 + b_1 b_2 a_3 + b_1 c_2 + a_1 a_3 + c_1 b_4 + b_1 b_2 b_3 b_4. \tag{4.17}$$

We will now show how the tools we previously introduced can be used to geometrically prove the complete integrability of such a system and, moreover, yield the existence of an additional integral of the motion .

The main property is that, along with P, the tensor Q resticts to M_0. This can be proven as follows: one checks by direct inspection that this is true for P'; the assertion follows from the fact that the vector field X_1^2 (defined in (4.8)) vanishes on M_0, while Z_1 and X_1^1, which coincides with X_{GT}, are tangent to M_0 at the points of M_0.

Furthermore, we add two observations. The first one concerns the restriction $\mathcal{G}_0$ to M_0 of the matrix $\mathcal{G}$. It has the form

$$\mathcal{G}_0 = \begin{bmatrix} \mathcal{G}_{11}^0 & \lambda^2 - (b_2 + b_3)\lambda + b_2 b_3 + a_2 \\ 0 & -(c_1 a_3 + a_1 c_2)\lambda - a_1 a_2 a_3 + c_1 c_2 + c_1 b_2 a_3 + a_1 c_2 b_3 \end{bmatrix} \tag{4.18}$$

with $\mathcal{G}_{11}^0 = \lambda^3 - \pi_1 \lambda^2 - \pi_2 \lambda - \pi_3$ a degree three polynomial. Hence its determinant (that is, the minimal polynomial of the Nijenhuis tensor N_0 induced by the pair $Q_0 - \lambda P_0$ on $\mathcal{S}_\xi$) factors as $\mathcal{G}_{11}^0 \mathcal{G}_{22}^0$.

The second observation consists in the fact that the three surviving Hamiltonians h_1, h_2, h_3 given by (4.17) satisfy the conditions:

$$Q_0 dh_i = \sum_{j=1}^{3} F_{ij}^0 P_0 dh_i, \text{ with } F_{ij}^0 = \begin{bmatrix} \pi_1 & 1 & 0 \\ \pi_2 & 0 & 1 \\ \pi_3 & 0 & 0 \end{bmatrix}. \tag{4.19}$$

We notice that the functions $\pi_i, i = 1, \ldots, 3$ and the root

$$\lambda_4 = \frac{-a_1 a_2 a_3 + c_1 c_2 + a_1 c_2 b_3 + c_1 b_2 a_3}{c_1 a_3 + a_1 c_2} \tag{4.20}$$

of $\mathcal{G}_{22}^0$ are still functionally independent and hence (generically) different on $\mathcal{S}_\xi$.

Lemma 4.7. *Let σ any function satisfying $Q_0 d\sigma = \lambda_4 P_0 d\sigma$. Under the above hypotheses, the brackets $\{\sigma, h_i\}_{P_0}$ and $\{\sigma, h_i\}_{Q_0}$ vanish.*

Proof. Evaluating both sides of $Q_0 d\sigma = \lambda_4 P_0 d\sigma$ on the differentials (dh_1, dh_2, dh_3), and switching the action of the Poisson tensors on the dh_i's, we get

$$\langle d\sigma, Q_0 dh_i \rangle = \lambda_4 \langle d\sigma, P_0 dh_i \rangle, i = 1, \ldots, 3$$

Inserting (4.19) we get the equation $\sum_{j=1}^{3} \left(F_{ij}^0 - \lambda_4 \delta_{ij}\right) \langle d\sigma, P_0 dh_j \rangle = 0$. Since λ_4 is *not* an eigenvalue of F_{ij}^0, the Lemma is proved. □

So a fourth integral of the motion, that commutes with the Hamiltonian H^0_{GT} for the open Toda$^4_3$ lattice is given indeed by the distinguished root λ_4 of equation (4.20); this constructively proves the integrability of the system.

Finally, we notice that this method proves the existence of a *fifth* integral of the motion. Indeed, we know that, along with λ_4, there must exist another independent function μ_4, satisfying the hypotheses of Lemma 4.7 and functionally independent of λ_4 and of the h_i's. In such a comparatively low dimensional case, such a function can be explicitly found to be

$$\mu_4 = \frac{c_2\left(a_1b_2c_1 - a_2{a_1}^2 - {c_1}^2 - c_1b_3a_1\right)}{c_1\left(c_1a_3 + a_1c_2\right)\left({\lambda_4}^3 - \pi_1{\lambda_4}^2 - \pi_2\lambda_4 - \pi_3\right)}.$$

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