On Hamiltonian perturbations of hyperbolic systems of conservation laws

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Abstract

We study the general structure of formal perturbative solutions to the Hamiltonian perturbations of spatially one-dimensional systems of hyperbolic PDEs $\mathbf{v}_t + [\phi(\mathbf{v})]_x = 0$. Under certain genericity assumptions it is proved that any *bihamiltonian* perturbation can be eliminated in all orders of the perturbative expansion by a change of coordinates on the infinite jet space depending *rationally* on the derivatives. The main tools is in constructing of the so-called quasi-Miura transformation of jet coordinates eliminating an arbitrary deformation of a semisimple bihamiltonian structure of hydrodynamic type (the quasitriviality theorem). We also describe, following [36], the invariants of such bihamiltonian structures with respect to the group of Miura-type transformations depending *polynomially* on the derivatives.

Contents

1 Introduction

Systems of evolutionary PDEs arising in many physical applications can be written in the form

$$w^i_t + V^i_j(w)w^j_x + \text{perturbation} = 0, \quad i = 1, \dots, n \tag{1.1}$$

where the perturbation may depend on higher derivatives. The dependent variables of the system

$$w = \big(w^1(x,t), \dots, w^n(x,t)\big)$$

are functions of one spatial variable x and the time t, summation over repeated indices will be assumed. $V^i_j(w)$ is a matrix of functions having *real distinct* eigenvalues. Therefore the system (1.1) can be considered as a perturbation of the hyperbolic system of first order quasilinear PDEs

$$v^i_t + V^i_j(v)v^j_x = 0, \quad i = 1, \dots, n \tag{1.2}$$

(it will be convenient to denote differently the dependent variables of the unperturbed system (1.2) and the perturbed one (1.1)). Recall (see, e.g., [9]) that the system (1.2) is called *hyperbolic* if the eigenvalues of the matrix $V^i_j(v)$ are all real and all n eigenvectors are linearly independent. In particular, *strictly hyperbolic* systems are those for which the eigenvalues are all real and pairwise distinct. An important particular class is the so-called *systems of conservation laws*

$$v^i_t + \partial_x \phi^i(v) = 0, \quad i = 1, \dots, n \tag{1.3}$$

where the dependent variables are chosen to be densities of conserved quantities; the functions $\phi^i(v)$ are the corresponding densities of fluxes (see, e.g., [9] regarding the physical applications of such systems). The relationships between solutions of the perturbed and unperturbed systems have been extensively studied for the case of *dissipative* perturbations of spatially one-dimensional systems of conservation laws (see, e.g., [6] and the references therein). Our strategic goal is the study of *Hamiltonian* perturbations of hyperbolic PDEs. Although many concrete examples of such perturbations have been studied (see, e.g., [46, 26, 19, 11, 34, 15, 32]), the general concepts and results are still missing.

Let us first explain how to recognize Hamiltonian systems among all systems of conservation laws. Recall [9] that the system of conservation laws (1.3) is *symmetrizable* in the sense of Friedrichs and Lax, Godunov, if there exists a constant symmetric positive definite matrix $\eta = (\eta_{ij})$ such that the matrix

$$\eta_{is}\frac{\partial \phi^s}{\partial v^j}$$

is symmetric,

$$\eta_{is}\frac{\partial \phi^s}{\partial v^j} = \eta_{js}\frac{\partial \phi^s}{\partial v^i}. \tag{1.4}$$

In case the symmetry (1.4) holds true but the symmetric matrix η is only nondegenerate but not necessarily positive definite one obtains *weakly symmetrizable* systems of conservation laws.

Lemma 1.1 *The system of conservation laws (1.3) is Hamiltonian if it is weakly symmetrizable.*

Proof Choosing the Poisson brackets in the constant form

$$\{v^i(x), v^j(y)\} = \eta^{ij}\delta'(x-y), \quad \left(\eta^{ij}\right) = \left(\eta_{ij}\right)^{-1}$$

and a local Hamiltonian

$$H = \int h(v(x))\,dx$$

one obtains the Hamiltonian system in the form (1.3) with

$$\phi^i(v) = \eta^{is}\frac{\partial h(v)}{\partial v^s}.$$

Both sides of (1.4) coincide then with the Hessian of the Hamiltonian density $h(v)$. The Lemma is proved. □

Recall that weakly symmetrizable systems of conservation laws enjoy the following important property: they possess two additional conservation laws

$$\partial_t p(v) + \partial_x q(v) = 0, \quad p = \frac{1}{2}\eta_{ij}v^i v^j, \quad q = v^i\frac{\partial h}{\partial v^i} - h(v) \tag{1.5}$$

$$\partial_t h(v) + \partial_x f(v) = 0, \quad f(v) = \frac{1}{2}\eta^{ij}\frac{\partial h}{\partial v^i}\frac{\partial h}{\partial v^j} \tag{1.6}$$

where $h(v)$ is the Hamiltonian density in the formulae above. For symmetrizable systems the function $p(v)$ is nonnegative.

The class of Hamiltonian perturbations to be investigated will be written in the form

$$w_t^i + \left\{w^i(x), H\right\} = w_t^i + V_j^i(w)w_x^j + \sum_{k\geq 1}\epsilon^k U_k^i(w; w_x, \dots, w^{(k+1)}) = 0, \quad i = 1, \dots, n \tag{1.7}$$

where ϵ is the small parameter, $U_k^i(w; w_x, \dots, w^{(k+1)})$ are graded homogeneous polynomials[1] in the jet variables $w_x = (w_x^1, \dots, w_x^n)$, $w_{xx} = (w_{xx}^1, \dots, w_{xx}^n)$, $\dots$, $w^{(k+1)} = \left(w^{1,k+1}, \dots, w^{n,k+1}\right)$

$$\deg w^{i,m} = m, \quad i = 1, \dots, n, \; m > 0. \tag{1.8}$$

[1] A different class of perturbations, for the particular case of the KdV equation, was considered by Y.Kodama [30]. In his theory the terms of the perturbative expansion are polynomials also in w (here $n = 1$). The degree on the algebra of differential polynomials is defined by $\deg u^{(m)} = m + 2$, $m \geq 0$. Also some nonlocal terms appear in the Kodama's perturbation theory. Further developments of this method can be found in [31].

They arise, e.g., in the study of solutions slowly varying in the space-time directions [46]. The Hamiltonian are local functionals

$$H = \int \left[h^{[0]}(w) + \epsilon\, h^{[1]}(w; w_x) + \dots\right] dx, \tag{1.9}$$
$$\deg h^{[k]}(w; w_x, \dots, w^{(k)}) = k.$$

The Poisson brackets are assumed to be *local* in every order in ϵ, i.e. they are represented as follows

$$\{w^i(x), w^j(y)\} = \sum_{m\geq 0}\sum_{l=0}^{m+1} \epsilon^m A^{ij}_{m,l}(w; w_x, \dots, w^{(m+1-l)})\delta^{(l)}(x-y) \tag{1.10}$$

with polynomial in the derivatives coefficients

$$\deg A^{ij}_{m,l}(w; w_x, \dots, w^{(m+1-l)}) = m - l + 1 \tag{1.11}$$

We also assume that the coefficients of these differential polynomials are $\mathcal{C}^\infty$ smooth functions on a n-dimensional ball $w \in B \subset \mathbb{R}^n$. It is understood that the antisymmetry and the Jacobi identity for (1.10) hold true as identities for formal power series in ϵ. It can be readily seen that, for an arbitrary local Hamiltonian of the form (1.9) the evolutionary systems (1.7) has the needed form.

The leading term

$$\left\{w^i(x), w^j(y)\right\}^{[0]} := A^{ij}_{0,1}(w(x))\delta'(x-y) + A^{ij}_{0,0}(w(x); w_x(x))\delta(x-y) \tag{1.12}$$

is itself a Poisson bracket (the so-called *Poisson bracket of hydrodynamic type*, see [13]). We will always assume that

$$\det A^{ij}_{0,1}(w) \neq 0 \tag{1.13}$$

for all $w \in B \subset \mathbb{R}^n$. Redenote the coefficients of $\{w^i(x), w^j(y)\}^{[0]}$ as follows

$$g^{ij}(w) := A^{ij}_{0,1}(w), \quad Q^{ij}_k(w) w^k_x := A^{ij}_{0,0}(w; w_x) \tag{1.14}$$

(see (1.11). The coefficient $(g^{ij}(w))$ can be considered as a symmetric nondegenerate bilinear form on the cotangent spaces. The inverse matrix defines a metric

$$ds^2 = g_{ij}(w) dw^i dw^j, \quad (g_{ij}(w)) := \left(g^{ij}(w)\right)^{-1} \tag{1.15}$$

(not necessarily positive definite). Recall [13] that (1.12) - (1.14) defines a Poisson structure if and only if the metric is flat and $Q^{ij}_k(w) = -\sum_{l=1}^n g^{il}(w)\Gamma^j_{kl}(w)$ where Γ^j_{kl} are the Christoffel symbols of the Levi-Civita connection of the metric (1.15).

The class of the Hamiltonians (1.9), Poisson brackets (1.10) and the evolutionary systems (1.7) is invariant with respect to *Miura-type transformations*

$$w^i \mapsto \tilde w^i = \Phi^i_0(w) + \sum_{k\geq 1} \epsilon^k \Phi^i_k(w; w_x, \dots, w^{(k)}), \quad i = 1, \dots, n \tag{1.16}$$
$$\deg \Phi^i_k(w; w_x, \dots, w^{(k)}) = k, \quad \det\left(\frac{\partial \Phi^i_0}{\partial w^j}\right) \neq 0.$$

As usual, the coefficients $\Phi^i_k(w; w_x, \dots, w^{(k)})$ are assumed to depend polynomially on the derivatives. Two Poisson brackets of the form (1.10) are called *equivalent* if they are related by a Miura-type transformation.

The group of Miura-type transformations is a natural extension of the group of local diffeomorphisms that plays an important role in the geometrical study of hyperbolic systems (see, e.g.,[44]).

An important result of [25] (see also [10, 16]) says that any Poisson bracket of the form (1.10) can be locally reduced by Miura-type transformations to the constant form

$$\left\{\tilde{w}^i(x), \tilde{w}^j(y)\right\} = \eta^{ij}\delta'(x-y), \quad \eta^{ij} = \text{const.} \tag{1.17}$$

We will denote the inverse matrix by the same symbol with lower indices

$$(\eta_{ij}) := \left(\eta^{ij}\right)^{-1}. \tag{1.18}$$

Connections of the theory of Hamiltonian systems (1.7) to the theory of systems of conservation laws is clear from the following statement.

Lemma 1.2 *By a change of dependent variables of the form (1.16) the Hamiltonian system (1.7) can be recast into the form of a system of conservation laws*

$$\begin{aligned} &\tilde{w}^i_t + \partial_x \psi^i(\tilde{w}; \tilde{w}_x, \dots; \epsilon) = 0, \quad i = 1, \dots, n \\ &\psi^i(\tilde{w}; \tilde{w}_x, \dots; \epsilon) = \sum_{k\geq 0} \epsilon^k \psi^i_k(\tilde{w}; \tilde{w}_x, \dots, \tilde{w}^{(k)}), \\ &\deg \psi^i_k(\tilde{w}; \tilde{w}_x, \dots, \tilde{w}^{(k)}) = k. \end{aligned} \tag{1.19}$$

The system of conservation laws (1.19) is Hamiltonian with respect to the Poisson bracket (1.17) iff *the right hand sides*

$$\psi_i := \eta_{ij}\psi^j(\tilde{w}; \tilde{w}_x, \dots; \epsilon) \tag{1.20}$$

satisfy

$$\frac{\partial \psi_i}{\partial \tilde{w}^{j,s}} = \sum_{t\geq s} (-1)^t \binom{t}{s} \partial_x^{t-s} \frac{\partial \psi_j}{\partial \tilde{w}^{i,t}} \tag{1.21}$$

for any i, $j = 1, \dots, n$, $s = 0, 1, \dots$.

In this paper we will investigate the structure of *formal* perturbative expansions of the solutions to (1.7)

$$w^i(x,t;\epsilon) = v^i(x,t) + \epsilon\, \delta_1 v^i(x,t) + \epsilon^2 \delta_2 v^i(x,t) + \dots \tag{1.22}$$

The leading term solves (1.2); the coefficients of the expansion $\delta_k v^i(x,t)$ are to be determined from linear PDEs with coefficients depending on v^i, $\delta_1 v^i$, …, $\delta_{k-1} v^i$ and their derivatives. Instead of developing this classical technique we propose a different approach that conceptually goes back to the Poincaré's treatment of perturbative expansions in the celestial mechanics. We will look for a transformation of the form

$$w^i = v^i + \sum_{k\geq 1} \epsilon^k \Phi_k^i(v; v_x, \dots, v^{(m_k)}), \quad i = 1, \dots, n \tag{1.23}$$

that maps *any* generic[2] solution $v^i(x,t)$ of the unperturbed system (1.2) to a solution $w^i(x,t;\epsilon)$ of the perturbed system. An important feature of such an approach to the perturbation theory is *locality*: changing the functions $v(x,t)$ for a given t only within a small neighborhood of the given point $x = x_0$ will keep unchanged the values of $w(x,t;\epsilon)$ outside this neighborhood. We call (1.23) *the reducing transformation* for the perturbed system (1.7).

Clearly, applying to (1.2) any transformation (1.23) *polynomial* in the derivatives (in every order in ϵ; in that case $m_k = k$) one obtains a perturbed system of the form (1.7). This is the case of *trivial* perturbations.

It is clear that solutions of trivial Hamiltonian perturbations share many properties of the solutions to the unperturbed hyperbolic PDEs (1.2). In particular, the trivial perturbation cannot balance the nonlinear effects in the hyperbolic system that typically cause gradient catastrophe of the solution.

Definition 1.3 *The system of PDEs (1.7) is called* quasitrivial *if it is not trivial but there exists a reducing transformation (1.23) with functions $\Phi_k^i(v; v_x, \dots, v^{(m_k)})$ depending* rationally *on the jet coordinates*

$$\begin{aligned} &w^{i,l}, \quad i = 1, \dots, n, \quad 1 \leq l \leq m_k \\ &\deg \Phi_k^i(v; v_x, \dots, v^{(m_k)}) = k, \quad k \geq 1. \end{aligned} \tag{1.24}$$

The first example of such a reducing transformation was found in [4] (see also [16]) for the KdV equation

$$w_t + w\, w_x + \frac{\epsilon^2}{12} w_{xxx} = 0 \tag{1.25}$$

(here $n = 1$):

$$w = v + \frac{\epsilon^2}{24} \partial_x^2 \log v_x + \epsilon^4 \partial_x^2 \left(\frac{v^{(4)}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xx}^3}{360\, v_x^4} \right) + O(\epsilon^6). \tag{1.26}$$

It is not an easy task to check cancellation of all the denominators even in this example! Because of the denominators the reducing deformation is defined only on the monotone solutions.

One of the main outputs of our paper is in proving quasitriviality of a large class of Hamiltonian perturbations of hyperbolic systems of conservation laws.

[2] We will later be more specific in describing the range of applicability of the transformations (1.23).

The systems in question are *bihamiltonian systems of PDEs.* That means that they can be represented in the Hamiltonian form in two different ways

$$w_t^i = \left\{H_1, w^i(x)\right\}_1 = \left\{H_2, w^i(x)\right\}_2, \quad i = 1, \dots, n \tag{1.27}$$

with two local Hamiltonians (see (1.9) above) and local *compatible* Poisson brackets $\{\ ,\ \}_1$, $\{\ ,\ \}_2$ of the form (1.10). Compatibility means that any linear combination

$$a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2$$

with arbitrary constant coefficients a_1, a_2 must be again a Poisson bracket.

The study of bihamiltonian structures was initiated by F. Magri [39] in his analysis of the so-called Lenard scheme of constructing the KdV integrals. I. Dorfman and I. Gelfand [24] and also A. Fokas and B. Fuchssteiner [22] discovered the connections between the bihamiltonian scheme and the theory of hereditary symmetries of integrable equations. However, it is not easy to apply these beautiful and simple ideas to the study of general bihamiltonian PDEs (see the discussion of the problems occured in [16]).

In this paper we will use a different approach to the study of bihamiltonian PDEs proposed in [16]. It is based on the careful study of the transformation properties of the bihamiltonian structures under the transformations of the form (1.23). Let us now proceed to the precise definitions and formulations of the results.

We will study bihamiltonian structures defined by compatible pairs of *local* Poisson brackets written in the form of ϵ-expansion

$$\begin{aligned}\{w^i(x), w^j(y)\}_a &= \{w^i(x), w^j(y)\}_a^{[0]} \\ &+ \sum_{m\geq 1}\sum_{l=0}^{m+1} \epsilon^m A^{ij}_{m,l;a}(w; w_x, \dots, w^{(m+1-l)})\delta^{(l)}(x-y),\ \ a = 1, 2,\end{aligned} \tag{1.28}$$

with polynomial in the derivatives coefficients

$$\deg A^{ij}_{m,l;a}(w; w_x, \dots, w^{(m+1-l)}) = m - l + 1. \tag{1.29}$$

The coefficients of these differential polynomials are $\mathcal{C}^\infty$ smooth functions on a n-dimensional ball $w \in B \subset \mathbb{R}^n$. Equivalence of bihamiltonian structures is defined with respect to Miura-type transformations.

The leading terms of the bihamiltonian structure is itself a bihamiltonian structure of the form

$$\begin{gathered}\{w^i(x), w^j(y)\}_a^{[0]} = g_a^{ij}(w(x))\delta'(x-y) + \sum_{k=1}^n Q^{ij}_{a;k}(w(x))w_x^k\delta(x-y), \\ \det(g_a^{kl}) \neq 0 \quad \text{for generic points } w \in M, \quad i, j = 1, \dots, n,\ \ a = 1, 2\end{gathered} \tag{1.30}$$

(the bihamiltonian structure of the hydrodynamic type). We additionally assume that $\det(a_1 g_1^{kl}(w) + a_2 g_2^{kl}(w))$ does not vanish identically for $w \in M$ unless $a_1 = a_2 = 0$.

Definition 1.4 *The bihamiltonian structure (1.28) is called* semisimple *if the characteristic polynomial* $\det(g_2^{ij}(w) - \lambda g_1^{ij}(w))$ *in* λ *has* n *pairwise distinct real*[3] *roots* $\lambda_1(w), \dots, \lambda_n(w)$ *for any* $w \in B$.

The role of semisimplicity assumption can be illustrated by the following

Lemma 1.5 *Given a semisimple bihamiltonian structure* $\{\ ,\ \}_{1,2}^{[0]}$ *satisfying the above conditions, denote*

$$\lambda = u^1(w), \dots, \lambda = u^n(w)$$

the roots of the characteristic equation

$$\det\left(g_2^{ij}(w) - \lambda\, g_1^{ij}(w)\right) = 0. \tag{1.31}$$

The functions $u^1(w)$, $\dots$, $u^n(w)$ *satisfy*

$$\det\left(\frac{\partial u^i(w)}{\partial w^j}\right) \neq 0.$$

Using these functions as new local coordinates

$$w^i = q^i(u^1, \dots, u^n), \quad i = 1, \dots, n, \quad \det\left(\frac{\partial q^i(u)}{\partial u^j}\right) \neq 0 \tag{1.32}$$

reduces both the two flat metrics to the diagonal form

$$\frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l} g_1^{kl}(w) = \delta_{ij} f^i(u), \quad \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l} g_2^{kl}(w) = \delta_{ij} g^i(u) = \delta_{ij} u^i f^i(u). \tag{1.33}$$

The coefficients $Q_{a;k}^{ij}$ *in the coordinates* $(u^1, \dots, u^n)$ *read*

$$\sum_{k=1}^n Q_{1;k}^{ij} u_x^k = \frac{1}{2}\,\delta_{ij}\partial_x f^i + A^{ij}, \quad \sum_{k=1}^n Q_{2;k}^{ij} u_x^k = \frac{1}{2}\,\delta_{ij}\partial_x g^i + B^{ij}, \tag{1.34}$$

$$A^{ij} = \frac{1}{2}\left(\frac{f^i}{f^j} f_i^j u_x^j - \frac{f^j}{f^i} f_j^i u_x^i\right), \quad B^{ij} = \frac{1}{2}\left(\frac{u^i f^i}{f^j} f_i^j u_x^j - \frac{u^j f^j}{f^i} f_j^i u_x^i\right) \tag{1.35}$$

where $f_k^i = \frac{\partial f^i}{\partial u^k}$. *The leading term of any bihamiltonian system becomes diagonal in the coordinates* u^1, $\dots$, u^n,

$$u_t^i + V^i(u) u_x^i + \mathcal{O}(\epsilon) = 0, \quad i = 1, \dots, n. \tag{1.36}$$

Such coordinates are called *the canonical coordinates* of the semisimple bihamiltonian structure. They are defined up to a permutation. The functions $f_1(u)$, $\dots$, $f_n(u)$ satisfy a complicated system of nonlinear differential equations. The general solution

[3] One can relax the requirement of reality of the roots working with complex manifolds. In that case the coefficients must be analytic in w.

to this system depends on n^2 arbitrary functions of one variable. Integrability of this system has recently been proved in [17], [41]. For convenience of the reader we give a brief account of these results, following [17], in the Appendix below.

The following statement gives a simple criterion of a Hamiltonian system of conservation laws to be bihamiltonian.

Lemma 1.6 *Let us consider a strictly hyperbolic system (1.2) Hamiltonian with respect to the Poisson bracket* $\{\ ,\ \}_1^{[0]}$. *This system is bihamiltonian with respect to the semisimple Poisson pencil* $\{\ ,\ \}_{1,2}^{[0]}$ iff *it becomes diagonal in the canonical coordinates for the Poisson pencil:*

$$\frac{\partial u^i}{\partial v^k}\frac{\partial v^l}{\partial u^j}V_l^k(v) = V^i(u)\delta_j^i. \tag{1.37}$$

Observe that the canonical coordinates are *Riemann invariants* (see, e.g., [46]) for the leading term of the system of PDEs (1.36). The coefficients $V^i(u)$ in the gas dynamics are called *characteristic velocities* [46]. In particular the semisimplicity assumption implies hyperbolicity of the leading term of the Hamiltonian systems.

Definition 1.7 ([16]) *The bihamiltonian structure (1.28) is said to be* trivial *if it can be obtained from the leading term*

$$\{v^i(x), v^j(y)\}_a^{[0]} = g_a^{ij}(v(x))\delta'(x-y) + Q_{a;k}^{ij}(v(x))\delta(x-y), \quad a = 1,\, 2 \tag{1.38}$$

by a Miura-type transformation

$$w^i = v^i + \sum_{k\geq 1}\epsilon^k F_k^i(v; v_x, \dots, v^{(k)}), \tag{1.39}$$

$$\deg F_k^i(v; v_x, \dots, v^{(k)}) = k, \quad i = 1, \dots, n$$

where the coefficients $F_k^i(v; v_x, \dots, v^{(k)})$ *are graded homogeneous polynomials in the derivatives. It is called* quasitrivial *if it is not trivial and there exists a transformation*

$$w^i = v^i + \sum_{k\geq 1}\epsilon^k F_k^i(v; v_x, \dots, v^{(m_k)}) \tag{1.40}$$

reducing (1.28) to (1.38) but the functions F_k *depend* rationally *on the jet coordinates* $v^{i,m}$, $m \geq 1$ *with*

$$\deg F_k = k, \quad k \geq 1 \tag{1.41}$$

and m_k *are some positive integers. If such a transformation (1.39) or (1.40) exists, it is called a* reducing transformation *of the bihamiltonian structure (1.28).*

A transformation of the form (1.40) is called a *quasi-Miura transformation.*

We are now in a position to formulate the main result of the present paper.

Quasitriviality Theorem *For any semisimple bihamiltonian structure (1.28) there exists a reducing transformation of the form (1.40). The coefficients F_k^i have the form*

$$F_k^i(v; v_x, \dots, v^{(m_k)}) \in C^\infty(B)\left[v_x, \dots, v^{(m_k)}\right]\left[\left(u_x^1 u_x^2 \dots u_x^n\right)^{-1}\right] \qquad (1.42)$$
$$m_k \le \left[\frac{3\,k}{2}\right].$$

Here $u^i = u^i(v)$ are the canonical coordinates (1.33).

Using this theorem we achieve the goal of constructing the reducing transformation for a bihamiltonian system (1.27), (1.9):

Corollary 1.8 *The reducing transformation for the bihamiltonian structure (1.28) is also a reducing transformation for any bihamiltonian system (1.27).*

Another corollary says that the solution of any system of bihamiltonian PDEs of the above form can be reduced to solving *linear PDEs*. Let us first rewrite the reducing transformation in the canonical coordinates

$$\tilde u^i = u^i + \sum_{k\ge 1} \epsilon^k G_k^i(u; u_x, \dots, u^{(m_k)}), \quad i = 1, \dots, n. \qquad (1.43)$$

Let $W^i(u;\epsilon)$, $i = 1, \dots, n$ be an arbitrary solution to the linear system

$$\frac{\partial W^i}{\partial u^j} = \frac{\partial V^i/\partial u^j}{V^i - V^j}\left(W^i - W^j\right), \quad i \ne j \qquad (1.44)$$

in the class of formal power series in ϵ. In this system the functions $V^i(u)$ are eigenvalues of the matrix $V_j^i(w)$, cf. (1.36). Let us assume that the system of equations

$$x = V^i(u)t + W^i(u; \epsilon = 0), \quad i = 1, \dots, n \qquad (1.45)$$

has a solution $(x, t, u) = (x_0, t_0, u_0)$ such that

$$\det\left(t\,\partial V^i(u)/\partial u^j + \partial W^i(u;\epsilon)/\partial u^j\right)_{t=t_0,\, u=u_0,\, \epsilon=0} \ne 0. \qquad (1.46)$$

For (x,t) sufficiently closed to (x_0, t_0) denote $u(x,t) = (u^1(x,t), \dots, u^n(x,t))$ the unique solution to the equations (1.45) such that

$$u(x_0, t_0) = u_0.$$

Applying the transformation (1.43) to the vector-function $u(x,t)$ we obtain a vector-function $\tilde u(x,t;\epsilon)$. Finally the substitution

$$q^i(\tilde u(x,t;\epsilon)) =: w^i(x,t;\epsilon), \quad i = 1, \dots \qquad (1.47)$$

yields n functions $w^1(x,t;\epsilon)$, $\dots$, $w^n(x,t;\epsilon)$. Here the functions $q^i(u)$ are defined as in (1.32).

Corollary 1.9 *The functions (1.47) satisfies (1.27). Conversely, any* monotone *at* $x = x_0$, $t = t_0$ *solution to (1.27) can be obtained by this procedure.*

By definition, the solution $w(x,t;\epsilon)$ is called monotone if all the x-derivatives

$$\partial_x u^1(w(x,t;\epsilon)), \dots, \partial_x u^n(w(x,t;\epsilon))$$

do not vanish for $x = x_0$, $t = t_0$, $\epsilon = 0$.

Finally, we can combine the Quasitriviality Theorem with the main result of the recent paper [36] in order to describe the complete set of invariants of bihamiltonian structures of the above form with the *given* leading term $\{w^i(x), w^j(y)\}_a^{[0]}$.

Introduce the following combinations of the coefficients of $\epsilon\,\delta''(x-y)$ and $\epsilon^2\delta'''(x-y)$ of the bihamiltonian structure

$$P_a^{ij}(u) = \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l}A_{1,2;a}^{kl}(w),\ \ Q_a^{ij}(u) = \frac{\partial u^i}{\partial w^k}\frac{\partial u^j}{\partial w^l}A_{2,3;a}^{kl}(w),\ \ i,j = 1,\dots,n, a = 1,2. \tag{1.48}$$

Define the functions

$$c_i(u) = \frac{1}{3(f^i(u))^2}\left(Q_2^{ii} - u^i Q_1^{ii} + \sum_{k\neq i}\frac{(P_2^{ki} - u^i P_1^{ki})^2}{f^k(u)(u^k - u^i)}\right), \quad i = 1,\dots,n. \tag{1.49}$$

The functions $c^i(u)$ are called *central invariants* of the bihamiltonian structure (1.28). The main result of [36] on the classification of infinitesimal deformations of bihamiltonian structures of hydrodynamic type can be reformulated as follows

Corollary 1.10 *i) Each function $c_i(u)$ defined in (1.49) depends only on u^i. ii) Two semisimple bihamiltonian structures (1.28) with the same leading terms $\{\ ,\ \}_a^{[0]}$, $a = 1,2$ are equivalent* iff *they have the same set of central invariants $c_i(u^i), i = 1,\dots,n$. In particular, a polynomial in the derivatives reducing transformation exists* iff *all the central invariants vanish:*

$$c_1 = c_2 = \dots = c_n = 0.$$

The papers is organized as follows. In Section 2 we recall some basic notions of the theory of Poisson structures for PDEs and prove Lemma 1.2, 1.5 and Lemma 1.6. In Section 3 and Section 4 we give the proofs of the Quasitriviality Theorem and the Corollaries 1.8, 1.9. In Section 5 we reformulate the main result of [36] on the classification of infinitesimal deformations of a semisimple bihamiltonian structure of hydrodynamic type and prove Corollary 1.10. In the final section we give some examples of bihamiltonian structures of the class studied in this paper and formulate some open problems. In the Appendix we briefly present, following [17], the theory of semisimple bihamiltonian structures of hydrodynamic type.

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2 Some basic notions about Poisson structures for PDEs

Like in finite dimensional Poisson geometry, an infinite dimensional Poisson structure of the form (1.30) or (1.28) can be represented by a local bivector on the formal loop space of the manifold M. Recall that in our considerations the manifold M will always be a n-dimensional ball. In general, let $w^1, \dots, w^n$ be a local coordinate system of a chart of the manifold M. A *local translation invariant* k-vector [16] is a formal infinite sum of the form

$$\alpha = \sum \frac{1}{k!} \partial_{x_1}^{s_1} \dots \partial_{x_k}^{s_k} A^{i_1 \dots i_k} \frac{\partial}{\partial w^{i_1, s_1}(x_1)} \wedge \dots \wedge \frac{\partial}{\partial w^{i_k, s_k}(x_k)}. \tag{2.1}$$

Here the coefficients A's have the expressions

$$A^{i_1 \dots i_k} = \sum_{p_2, \dots, p_k \geq 0} B^{i_1 \dots i_k}_{p_2 \dots p_k}(w(x_1); w_x(x_1), \dots) \delta^{(p_2)}(x_1 - x_2) \dots \delta^{(p_k)}(x_1 - x_k). \tag{2.2}$$

with only a finite number of nonzero terms in the summation, and $B^{i_1 \dots i_k}_{p_2 \dots p_k}(w; w_x, \dots)$ is a smooth function on a domain in the jet space $J^N(M)$ for certain integer N that depends on the indices $i_1, \dots, i_k$ and $p_2, \dots, p_k$. The delta-function and its derivatives are defined formally by

$$\int f(w(y); w_y(y), w_{yy}(y), \dots)\, \delta^{(k)}(x - y)\, dy = \partial_x^k f(w(x); w_x(x), w_{xx}(x), \dots). \tag{2.3}$$

In this formula the operator of total derivative ∂_x is defined by

$$\partial_x f(w; w_x, w_{xx}, \dots) = \sum w^{i, s+1} \frac{\partial f}{\partial w^{i, s}}. \tag{2.4}$$

Note the useful identity

$$\begin{aligned} & f(w(y); w_y(y), w_{yy}(y), \dots)\, \delta^{(k)}(x - y) \\ & = \sum_{m=0}^{k} \binom{k}{m} \partial_x^m f(w(x); w_x(x), w_{xx}(x), \dots)\, \delta^{(k-m)}(x - y). \end{aligned} \tag{2.5}$$

The distributions

$$A^{i_1 \dots i_k} = A^{i_1 \dots i_k}(x_1, \dots, x_k; w(x_1), \dots, w(x_k), \dots) \tag{2.6}$$

are antisymmetric with respect to the simultaneous permutations $i_p, x_p \leftrightarrow i_q, x_q$. They are called *the components* of the local k-vector α. Note that in the definition of local k-vectors given in [16] it is required that the functions $B^{i_1 \dots i_k}_{p_2 \dots p_k}$ are differential polynomials. Here we drop this requirement for the convenience of our use of these notations during our proof of the theorem. The space of all such local k-vectors is still denoted by Λ^k_{loc} as it is done in [16]. For $k = 0$ by definition Λ^0_{loc} is the space of local functionals of the form

$$I = \int f(w, w_x, \dots, w^{(m)})\, dx \tag{2.7}$$

The Schouten-Nijenhuis bracket is defined on the space of local multi-vectors

$$[\ ,\ ]:\ \Lambda^k_{loc} \times \Lambda^l_{loc} \to \Lambda^{k+l-1}_{loc}, \quad k, l \geq 0. \tag{2.8}$$

It generalizes the usual commutator of two local vector fields and possesses the following properties

$$[\alpha, \beta] = (-1)^{kl}[\beta, \alpha], \tag{2.9}$$

$$(-1)^{km}[[\alpha, \beta], \gamma] + (-1)^{kl}[[\beta, \gamma], \alpha] + (-1)^{lm}[[\gamma, \alpha], \beta] = 0 \tag{2.10}$$

for any $\alpha \in \Lambda^k_{loc}$, $\beta \in \Lambda^l_{loc}$, $\gamma \in \Lambda^m_{loc}$. For the definition of the Schouten-Nijenhuis bracket see [16] and references therein. Here we write down the formulae, used below, for the bracket of a local bivector with a local functional and with a local vector field. Let a local vector field ξ and a local bivector ϖ have the representation

$$\xi = \sum_{i=1}^{n} \sum_{s \geq 0} \partial_x^s \xi^i(w(x); w_x(x), \dots, w^{(m_i)}) \frac{\partial}{\partial w^{i,s}(x)}, \tag{2.11}$$

$$\varpi = \frac{1}{2} \sum \partial_x^s \partial_y^t \varpi^{ij} \frac{\partial}{\partial w^{i,s}(x)} \wedge \frac{\partial}{\partial w^{j,t}(y)}. \tag{2.12}$$

Here we assume that

$$\varpi^{ij} = \sum_{k \geq 0} A^{ij}_k(w(x), w_x(x), \dots, w^{(m_k)}(x))\, \delta^{(k)}(x - y). \tag{2.13}$$

Then the components of $[\varpi, I]$ and of $[\varpi, \xi]$ are given respectively by

$$[\varpi, I]^i = \sum_{j,k} A^{ij}_k \partial_x^k \frac{\delta I}{\delta w^j(x)}, \tag{2.14}$$

$$[\varpi, \xi]^{ij} = \sum_{k,t} \left(\partial_x^t \xi^k(w(x); \dots) \frac{\partial \varpi^{ij}}{\partial w^{k,t}(x)} - \frac{\partial \xi^i(w(x); \dots)}{\partial w^{k,t}(x)} \partial_x^t \varpi^{kj} \right.$$
$$\left. - \frac{\partial \xi^j(w(y); \dots)}{\partial w^{k,t}(y)} \partial_y^t \varpi^{ik} \right). \tag{2.15}$$

In the last formula it is understood that

$$\partial_y^t \varpi^{ij} = \sum_{k \geq 0} (-1)^t A^{ij}_k(w(x), w_x(x), \dots, w^{(m_k)}(x))\, \delta^{(k+t)}(x - y).$$

and the identity (2.5) has been used in order to represent the resulting bivector in the normalized form (2.13).

Let us denote by ϖ_1, ϖ_2 the two bivectors that correspond to the bihamiltonian structure (1.30), the components $\varpi_a^{ij}, a = 1, 2$ are given by the right hand side of (1.30). The bihamiltonian property is equivalent to the following identity that is valid for an arbitrary parameter λ:

$$[\varpi_2 - \lambda\,\varpi_1, \varpi_2 - \lambda\,\varpi_1] = 0. \tag{2.16}$$

Denote by ∂_1, ∂_2 the differentials associated with ϖ_1, ϖ_2. By definition

$$\partial_a : \ \Lambda^k_{loc} \to \Lambda^{k+1}_{loc}, \quad \partial_a \alpha = [\varpi_a, \alpha], \quad \forall \alpha \in \Lambda^k_{loc}, \quad a = 1, 2. \tag{2.17}$$

The bihamiltonian property (2.16) can be recast in the form

$$\partial_1^2 = \partial_2^2 = \partial_1\partial_2 + \partial_2\partial_1 = 0. \tag{2.18}$$

The important fact that we need to use below is vanishing of the first and second Poisson cohomologies

$$H^k(\mathcal{L}(M), \varpi_a) = \mathrm{Ker}\,\partial_a|_{\Lambda^k_{loc}} \,/\, \mathrm{Im}\,\partial_a|_{\Lambda^{k-1}_{loc}}, \quad a = 1, 2, \ k = 1, 2. \tag{2.19}$$

This fact is proved in [25, 10, 16]. It readily implies, along with the results of [13] the reducibility of any Poisson bracket of the form (1.10) - (1.13) to the constant form (1.17).

Let us now give

Proof of Lemma 1.2. For the Poisson bracket written in the form (1.17) the Hamiltonian system reads

$$\tilde{w}^i_t = \{H, \tilde{w}^i(x)\} = -\eta^{ij}\partial_x \frac{\delta H}{\delta \tilde{w}^j(x)}.$$

This gives a system of conservation laws form with

$$\psi^i = \eta^{ij} \frac{\delta H}{\delta \tilde{w}^j(x)}.$$

The equations (1.21) is nothing but the spelling of the classical Helmholtz criterion [27] for functions ψ_i to be representable in the form of variational derivatives. □

We pass now to the theory of canonical coordinates.

Proof of Lemma 1.5. From the result of [17, 41] it follows the existence of a system of local coordinates $\hat{u}^1, \ldots, \hat{u}^n$ such that both of the metrics become diagonal and $g_1^{ij} = \delta_{ij} h^i(\hat{u})$, $g_2^{ij} = \delta_{ij}\lambda_i(\hat{u}^i) h^i(\hat{u})$. Since by our assumption $\det(a_1 g_1^{kl}(w) + a_2 g_2^{kl}(w))$ does not vanish identically for $w \in M$ unless $a_1 = a_2 = 0$, we can choose $u^1 = \lambda_1(\hat{u}^1), \ldots, u^n = \lambda_n(\hat{u}^n)$ as a system of local coordinates which are just the canonical coordinates. □

We now proceed to

Proof of Lemma 1.6. In the canonical coordinates a bihamiltonian system

$$u_t^i = \{H_1, u^i(x)\}_1^{[0]} = \{H_2, u^i(x)\}_2^{[0]},$$

$$H_a = \int h_a^{[0]}(u)\, dx, \quad a = 1,\, 2$$

has the expression

$$u_t^i = -\sum_{j=1}^n V_j^i(u) u_x^j, \quad i = 1, \dots, n \tag{2.20}$$

where

$$V_j^i(u) = f^i(u)\mathcal{A}_{ij} h_1^{[0]}(u) = u^i f^i(u)\mathcal{A}_{ij} h_2^{[0]}(u), \quad \text{for } i \neq j. \tag{2.21}$$

Here the linear differential operators $\mathcal{A}_{ij}$ are defined by

$$\mathcal{A}_{ij} = \frac{\partial^2}{\partial u^i \partial u^j} + \frac{1}{2}\frac{\partial\,(\log f^i(u))}{\partial u^j}\frac{\partial}{\partial u^i} + \frac{1}{2}\frac{\partial\,(\log f^j(u))}{\partial u^i}\frac{\partial}{\partial u^j}. \tag{2.22}$$

Symmetry with respect to to the indices i, j implies

$$(u^i - u^j)\mathcal{A}_{ij} h_2^{[0]}(u) = 0, \quad i \neq j. \tag{2.23}$$

Thus $V_j^i(u) = 0$ when $i \neq j$. This proves the first part of the lemma.

To prove the converse statement we use the following result of [45]: the diagonal system

$$u_t^i + V^i(u) u_x^i = 0, \quad i = 1, \dots, n$$

is Hamiltonian with respect to to the Poisson bracket associated with the diagonal metric of zero curvature

$$ds^2 = \sum_{i=1}^n g_{ii}(u)(du^i)^2$$

iff the following equations holds true

$$\partial_k V^i(u) = \big(V^k(u) - V^i(u)\big)\ \partial_k \log\sqrt{g_{ii}(u)}, \quad i \neq k. \tag{2.24}$$

By assumption these equations hold true for the first metric

$$g_{ii}(u) = 1/f_i(u).$$

For the pencil $\{\ ,\ \}_2^{[0]} - \lambda\,\{\ ,\ \}_1^{[0]}$ one has to replace

$$g_{ii}(u) = \frac{1}{(u^i - \lambda) f^i(u)}.$$

Such a replacement does not change the equations (2.24). The Lemma is proved. □

3 Proof of the Quasitriviality Theorem

In this and the forthcoming sections we assume that the bihamiltonian structure (ϖ_1, ϖ_2) defined by (1.30) is semisimple and we work in canonical coordinates $u^1, \ldots, u^n$. We first formulate and prove some intermediate results crucial for the proof of the Quasitriviality Theorem.

Theorem 3.1 *Assume that a vector fields X has components of the form*

$$X^i = \sum X^i_j(u)\, u^j_x, \quad i = 1, \ldots, n \tag{3.1}$$

and satisfies

$$\partial_1 \partial_2 X = 0. \tag{3.2}$$

Then there exist two local functionals I, J of the form

$$I = \int G(u(x))dx, \quad J = \int \tilde{G}(u(x))dx \tag{3.3}$$

such that X has the representation $X = \partial_1 I - \partial_2 J$.

Proof In this proof summations over repeated Greek indices are assumed. Let us redenote the components of the two bivectors that correspond to the bihamiltonian structure (1.30) in the form

$$\begin{aligned}
\varpi_1^{ij} &= g^{ij}\delta'(x-y) + \Gamma^{ij}_\alpha u^\alpha_x \delta(x-y),\\
\varpi_2^{ij} &= \tilde{g}^{ij}\delta'(x-y) + \tilde{\Gamma}^{ij}_\alpha u^\alpha_x \delta(x-y),
\end{aligned}$$

The Levi-Civita connections of these two metrics $g^{ij}, \tilde{g}^{ij}$ are denoted by ∇ and $\tilde{\nabla}$ respectively. Denote $\nabla_i, \tilde{\nabla}_i$ the covariant derivatives of these two connections along $\frac{\partial}{\partial u^i}$. We also introduce the notations $\nabla^i = g^{i\alpha}\nabla_\alpha, \tilde{\nabla}^i = \tilde{g}^{i\alpha}\tilde{\nabla}_\alpha$.

The condition (3.2) implies existence of a vector field Y with components of the form

$$Y^i = \sum Y^i_j(u) u^j_x, \quad i = 1, \ldots, n \tag{3.4}$$

such that $\partial_1 X = \partial_2 Y$. Denote by

$$Z^{ij} = \sum_{p \geq 0} Z^{ij}_p(u(x), u_x(x), \ldots)\, \delta^{(p)}(x-y) := (\partial_1 X - \partial_2 Y)^{ij} \tag{3.5}$$

the components of the bivector $\partial_1 X - \partial_2 Y$, and by $Z^{ij}_{p,(k,m)}$ the derivatives $\frac{\partial Z^{ij}_p}{\partial u^{k,m}}$. Then we have

$$Z^{ij}_2 = (X^{ij} - X^{ji}) - (Y^{ij} - Y^{ji}) = 0, \tag{3.6}$$

$$\begin{aligned}
Z^{ij}_{0,(k,2)} &= \left(\nabla_k X^{ij} - \nabla^i X^j_k + \Gamma^j_{k\alpha}\left(X^{\alpha i} - X^{i\alpha}\right)\right)\\
&\quad - \left(\tilde{\nabla}_k Y^{ij} - \tilde{\nabla}^i Y^j_k + \tilde{\Gamma}^j_{k\alpha}\left(Y^{\alpha i} - Y^{i\alpha}\right)\right) = 0
\end{aligned} \tag{3.7}$$

where $X^{ij} = g^{i\alpha} X^j_\alpha, Y^{ij} = \tilde{g}^{i\alpha} Y^j_\alpha$. From the above two equations we obtain

$$\tilde{\nabla}^k Y^{ij} - \tilde{\nabla}^i Y^{kj} = \tilde{g}^{k\alpha} \left(\nabla_\alpha X^{ij} - \nabla^i X^j_\alpha + T^j_{\alpha\beta}(X^{i\beta} - X^{\beta i}) \right). \tag{3.8}$$

Here the components of the $(1,2)$-tensor T are defined by $T^j_{\alpha\beta} = \tilde{\Gamma}^j_{\alpha\beta} - \Gamma^j_{\alpha\beta}$. Since the l.h.s of the above equation is antisymmetric with respect to k, i, we have

$$\begin{aligned} &\tilde{g}^{k\alpha} \left(\nabla_\alpha X^{ij} - \nabla^i X^j_\alpha + T^j_{\alpha\beta}(X^{i\beta} - X^{\beta i}) \right) + \\ &\quad \tilde{g}^{i\alpha} \left(\nabla_\alpha X^{kj} - \nabla^k X^j_\alpha + T^j_{\alpha\beta}(X^{k\beta} - X^{\beta k}) \right) = 0. \end{aligned} \tag{3.9}$$

The following trivial identity

$$\begin{aligned} &\left(\tilde{\nabla}^k Y^{ij} - \tilde{\nabla}^i Y^{kj}\right) + \left(\tilde{\nabla}^i Y^{jk} - \tilde{\nabla}^j Y^{ik}\right) + \left(\tilde{\nabla}^j Y^{ki} - \tilde{\nabla}^k Y^{ji}\right) \\ &\quad = \tilde{\nabla}^k \left(Y^{ij} - Y^{ji}\right) + \tilde{\nabla}^i \left(Y^{jk} - Y^{kj}\right) + \tilde{\nabla}^j \left(Y^{ki} - Y^{ik}\right), \end{aligned}$$

implies that

$$\begin{aligned} &\tilde{\nabla}^k \left(X^{ij} - X^{ji}\right) + \tilde{\nabla}^i \left(X^{jk} - X^{kj}\right) + \tilde{\nabla}^j \left(X^{ki} - X^{ik}\right) \\ = {}&\tilde{g}^{k\alpha} \left(\nabla_\alpha X^{ij} - \nabla^i X^j_\alpha + T^j_{\alpha\beta}(X^{i\beta} - X^{\beta i}) \right) + \\ &\tilde{g}^{i\alpha} \left(\nabla_\alpha X^{jk} - \nabla^j X^k_\alpha + T^k_{\alpha\beta}(X^{j\beta} - X^{\beta j}) \right) + \\ &\tilde{g}^{j\alpha} \left(\nabla_\alpha X^{ki} - \nabla^k X^i_\alpha + T^i_{\alpha\beta}(X^{k\beta} - X^{\beta k}) \right). \end{aligned} \tag{3.10}$$

By using the formula

$$\tilde{\nabla}_k A^{ij} = \nabla_k A^{ij} + T^i_{k\alpha} A^{\alpha j} + T^j_{k\alpha} A^{i\alpha}$$

we can simplify the equations (3.10) to the form

$$\begin{aligned} &\tilde{g}^{k\alpha} \left(\nabla_\alpha X^{ji} - \nabla^i X^j_\alpha + T^i_{\alpha\beta}(X^{j\beta} - X^{\beta j}) \right) + \\ &\tilde{g}^{i\alpha} \left(\nabla_\alpha X^{kj} - \nabla^j X^k_\alpha + T^j_{\alpha\beta}(X^{k\beta} - X^{\beta k}) \right) + \\ &\tilde{g}^{j\alpha} \left(\nabla_\alpha X^{ik} - \nabla^k X^i_\alpha + T^k_{\alpha\beta}(X^{i\beta} - X^{\beta i}) \right) = 0. \end{aligned} \tag{3.11}$$

Let us employ the equations (3.9) and (3.11) to prove the existence two local functionals I, J of the form (3.3) such that $X = \partial_1 I - \partial_2 J$. Equivalently, we need to find functions $G, \tilde{G}$ that satisfy the conditions

$$X^i_j = g_{j\alpha} \nabla^i \nabla^\alpha G - \tilde{g}_{j\alpha} \tilde{\nabla}^i \tilde{\nabla}^\alpha \tilde{G} \tag{3.12}$$

To this end, we first define two symmetric $(2,0)$-tensors $A, \tilde{A}$ such that

$$X^i_j = g_{j\alpha} A^{\alpha i} - \tilde{g}_{j\alpha} \tilde{A}^{\alpha i} \tag{3.13}$$

In the canonical coordinates the off-diagonal components of $A, \tilde{A}$ are uniquely determined by the above relations and have the explicit forms

$$A^{ij} = \frac{g^i X^j_i - g^j X^i_j}{u^i - u^j}, \quad \tilde{A}^{ij} = \frac{u^j g^i X^j_i - u^i g^j X^i_j}{u^i - u^j} \quad \text{for } i \neq j. \tag{3.14}$$

Here we use the fact that in the canonical coordinates the two metrics have components of the form $g^{ij} = \delta_{ij} f^i$, $\tilde{g}^{ij} = \delta_{ij} g^i = \delta_{ij} u^i f^i$. For an arbitrary choice of the diagonal components A^{ii} the above relation uniquely determines the diagonal components $\tilde{A}^{ii}$ by

$$\tilde{A}^{ii} = u^i \left(f^i X^i_i - A^{ii} \right). \tag{3.15}$$

We will specify the choice of $A^{ii}, i = 1, \ldots, n$ in a moment. Let us now express the equations (3.9) and (3.11) in terms of the components of the tensors $A, \tilde{A}$. By substituting the the expression (3.13) of X^i_j into (3.9) and (3.11) and by using the fact that $g_{\alpha\beta}, \tilde{g}_{\alpha\beta}, T^i_{\alpha\beta}$ are all diagonal with respect to α, β in the canonical coordinates, we arrive at

$$\left(u^k - u^i\right)\left(\nabla^k A^{ij} - \nabla^i A^{kj}\right) + \left(\frac{1}{u^k} - \frac{1}{u^i}\right)\left(\tilde{\nabla}^k \tilde{A}^{ij} - \tilde{\nabla}^i \tilde{A}^{kj}\right) = 0, \tag{3.16}$$

$$\begin{aligned} & u^k \left(\nabla^k A^{ij} - \nabla^i A^{kj}\right) - \left(\frac{1}{u^j} \tilde{\nabla}^k \tilde{A}^{ij} - \frac{1}{u^i} \tilde{\nabla}^i \tilde{A}^{kj}\right) \\ & \quad + u^i \left(\nabla^i A^{jk} - \nabla^j A^{ik}\right) - \left(\frac{1}{u^k} \tilde{\nabla}^i \tilde{A}^{jk} - \frac{1}{u^j} \tilde{\nabla}^j \tilde{A}^{ik}\right) \\ & \quad + u^j \left(\nabla^j A^{ki} - \nabla^k A^{ji}\right) - \left(\frac{1}{u^i} \tilde{\nabla}^j \tilde{A}^{ki} - \frac{1}{u^k} \tilde{\nabla}^k \tilde{A}^{ji}\right) = 0. \end{aligned} \tag{3.17}$$

Rewrite (3.17) into the form

$$\begin{aligned} & u^k \left(\nabla^k A^{ij} - \nabla^i A^{kj}\right) + \frac{1}{u^k} \left(\tilde{\nabla}^k \tilde{A}^{ij} - \tilde{\nabla}^i \tilde{A}^{kj}\right) \\ & \quad + \text{terms obtained by cyclic permutations of } (i, j, k) = 0. \end{aligned} \tag{3.18}$$

By using (3.16), we can replace u^k by u^i in the first two terms of (3.18). Then after the cancellation of some terms we arrive at the simplification of (3.17)

$$\left(u^i - u^j\right)\left(\nabla^j A^{ki} - \nabla^k A^{ij}\right) + \left(\frac{1}{u^i} - \frac{1}{u^j}\right)\left(\tilde{\nabla}^j \tilde{A}^{ki} - \tilde{\nabla}^k \tilde{A}^{ij}\right) = 0. \tag{3.19}$$

Changing the indices $(i, j, k) \mapsto (j, k, i)$ we obtain

$$\left(u^j - u^k\right)\left(\nabla^k A^{ij} - \nabla^i A^{kj}\right) + \left(\frac{1}{u^j} - \frac{1}{u^k}\right)\left(\tilde{\nabla}^k \tilde{A}^{ij} - \tilde{\nabla}^i \tilde{A}^{kj}\right) = 0. \tag{3.20}$$

From the equations (3.16), (3.20) it readily follows that

$$\nabla^k A^{ij} = \nabla^i A^{kj}, \ \tilde{\nabla}^k \tilde{A}^{ij} = \tilde{\nabla}^i \tilde{A}^{kj} \quad \text{for } i \neq j, \ k \neq j. \tag{3.21}$$

Now let us proceed to choosing the diagonal components A^{ii} in such a way to ensure that the components of the tensor $\nabla^k A^{ij}$ are totaslly symmetric in i, j, k. This amounts to require that A^{ii} should satisfy

$$\nabla^k A^{ii} = \nabla^i A^{ki}, \ i, k = 1, \ldots, n, \ k \neq i. \tag{3.22}$$

The existence of solutions A^{ii} is guaranteed by the compatibility of the above systems due to the equalities

$$\nabla^j\left(\nabla^i A^{ki}\right) = \nabla^k\left(\nabla^i A^{ji}\right), \quad \text{for distinct } i, j, k. \tag{3.23}$$

Fix a solution $A^{ii}, i = 1, \dots, n$ of the system (3.22). From the validity of the equations (3.16) and (3.21) we know that the tensor $\tilde{A}$ with components $\tilde{A}^{ij}$ determined by (3.14) and (3.15) also have the property of symmetry of $\tilde{\nabla}^k \tilde{A}^{ij}$ in i, j, k. Thus we can find functions $G(u), \tilde{G}(u)$ such that

$$A^{ij} = \nabla^i \nabla^j G, \;\; \tilde{A}^{ij} = \tilde{\nabla}^i \tilde{\nabla}^j \tilde{G}.$$

The lemma is proved. □

The above theorem implies that the linear in ϵ terms of the bihamiltonian structure (1.28) can be eliminated by a Miura-type transformation.

Denote by $\mathcal{A}$ the space of functions that can be represented as a finite sum of rational functions of the form

$$\frac{P^i_{j_1,\dots,j_m}(u; u_x, \dots)}{u_x^{j_1} \dots u_x^{j_m}}, \quad m \geq 0 \tag{3.24}$$

(no denominator for $m = 0$). Here $P^i_{j_1,\dots,j_m}$ are quasihomogeneous differential polynomials. Define a gradation on the ring $\mathcal{A}$ by

$$\deg u^{i,m} = m, \quad i = 1, \dots, n, \;\; m \geq 0. \tag{3.25}$$

We call elements of $\mathcal{A}$ *almost differential polynomials*. Below we will also encounter functions that belong to the ring

$$\tilde{\mathcal{A}} = \mathcal{A}[\log u_x^1, \dots, \log u_x^n].$$

It is also a graded ring with the definition of degree (1.8) and

$$\deg(\log u_x^i) = 0, \quad i = 1, \dots, n. \tag{3.26}$$

Theorem 3.2 *Let $X \in \Lambda^1_{loc}$ be a local vector field with components*

$$X^i(u, u_x, \dots, u^{(N)}), \;\; i = 1, \dots, n, \;\; N \geq 1, \tag{3.27}$$

where X^i are homogeneous almost differential polynomials of degree $d \geq 3$. If X satisfies the condition

$$\partial_1 \partial_2 X = 0 \tag{3.28}$$

then there exist local functionals

$$I = \int f(u, u_x, \dots, u^{([\frac{N}{2}])}) dx, \quad J = \int g(u, u_x, \dots, u^{([\frac{N}{2}])}) dx, \tag{3.29}$$

with densities that are homogeneous almost differential polynomials of degree $d-1$ such that $X = \partial_1 I - \partial_2 J$.

We will prove the Quasitriviality Theorem by induction on the highest order of the x-derivatives of u^k on which the components X^i of the vector field X depend. The following lemmas spell out some important properties of the vector field X that are implied by the condition (3.28).

In what follows, for a function $A = A(u, u_x, \dots)$ we will use the subscript (k, m) to indicate the derivative of A with respect to $u^{k,m}$, i.e., $A_{(k,m)} = \frac{\partial A}{\partial u^{k,m}}$.

Lemma 3.3 *For any two vector fields X, Y with components of the form*

$$X^i = X^i(u, u_x, \dots, u^{(N)}), \quad Y^i = Y^i(u, u_x, \dots, u^{(N)}), \quad N \geq 1, \tag{3.30}$$

vanishing of $Z^{ij}_{0,(k,2N+1)}$ and $Z^{ij}_{0,(k,2N)}$, where Z^{ij} are defined as in (3.5), implies that the components of the vector fields X, Y must take the form

$$\begin{aligned} X^i &= \sum_{j=1}^n \left(u^j G^i_j(u, \dots, u^{(N-1)}) + F^i_j(u, \dots, u^{(N-1)})\right) u^{j,N} + Q^i(u, \dots, u^{(N-1)}), \\ Y^i &= \sum_{j=1}^n G^i_j(u, \dots, u^{(N-1)})\, u^{j,N} + R^i(u, \dots, u^{(N-1)}). \end{aligned} \tag{3.31}$$

Moreover, when $N \geq 2$ the functions F^i_j, G^i_j must satisfy the following equations:

$$F^j_{i,(k,N-1)} - F^j_{k,(i,N-1)} + (u^i - u^k) G^j_{k,(i,N-1)} = 0. \tag{3.32}$$

Proof By definition, we have

$$Z^{ij}_{0,(k,2N+1)} = (-1)^{N+1} \left(f^i\, X^j_{(i,N)(k,N)} - g^i\, Y^j_{(i,N)(k,N)}\right).$$

So vanishing of $Z^{ij}_{0,(k,2N+1)}$ implies that the functions X^i, Y^i can be represented as

$$\begin{aligned} X^i &= \sum_{j=1}^n (u^j \tilde{G}^i_j(u, \dots, u^{(N-1)}; u^{j,N}) + \tilde{F}^i_j(u, \dots, u^{(N-1)}) u^{j,N}) + \tilde{Q}^i(u, \dots, u^{(N-1)}), \\ Y^i &= \sum_{j=1}^n \tilde{G}^i_j(u, \dots, u^{(N-1)}; u^{j,N}). \end{aligned}$$

When $N \geq 2$ the equations (3.31) of X^i, Y^i follow from vanishing of

$$Z^{ij}_{0,(i,2N)} = (-1)^{N+1} \left(N + \frac{1}{2}\right) f^i u^i_x G^j_{i,(i,N)(i,N)}.$$

In the case of $N = 1$, the formulae (3.31) follow from vanishing of $Z^{ij}_{0,(i,2N)}$ and of Z^{ij}_2. Finally, for $N \geq 2$ the equation (3.32) is derived from vanishing of

$$Z^{ij}_{0,(k,2N)} = (-1)^{N+1} f^i \left(F^j_{i,(k,N-1)} - F^j_{k,(i,N-1)} + (u^i - u^k) G^j_{k,(i,N-1)}\right).$$

The lemma is proved. $\square$

Lemma 3.4 *(a) Assume that the vector fields X, Y have components of the form (3.31) and $N \geq 2$. Then for any $m = 1, 2, \ldots, \left[\frac{N}{2}\right]$ the following identity holds true:*

$$\sum_{l\geq 0}(-1)^{m-l}\binom{N-l}{m-l} Z^{ij}_{l,(k,2N+1-m-l)} = (-1)^{N+1} f^i\, F^j_{i,(k,N-m)}. \tag{3.33}$$

Here and below we denote Z^{ij} as in (3.5) of Lemma 3.1.
(b) Assume that the vector field X, Y have components of the following form

$$\begin{aligned} X^i &= \sum_{j=1}^n \left(u^j G^i_j(u,\ldots,u^{(N-m-1)}; u^{j,N-m}) + F^i_j(u,\ldots,u^{(N-m-1)})\right) u^{j,N} \\ &\qquad + Q^i(u,\ldots,u^{(N-1)}), \\ Y^i &= \sum_{j=1}^n G^i_j(u,\ldots,u^{(N-m-1)}; u^{j,N-m})\, u^{j,N} + R^i(u,\ldots,u^{(N-1)}) \end{aligned} \tag{3.34}$$

with $N \geq 3$ and $1 \leq m \leq \left[\frac{N-1}{2}\right]$. Then we have

$$\begin{aligned} &\sum_{l\geq 0}(-1)^{m-l}\binom{N-l}{m-l} Z^{ij}_{l,(k,2N-m-l)} \\ &= (-1)^{N+1}\Big[f^i\left(F^j_{i,(k,N-m-1)} - F^j_{k,(i,N-m-1)}\right) + f^i(u^i-u^k)G^j_{k,(i,N-m-1)} \\ &\quad + (N-m+\frac{1}{2}) f^i u^i_x G^j_{k,(i,N-m)}\delta^{ik} + (A^{ik} - u^k B^{ik}) G^j_{k,(k,N-m)}\Big] \end{aligned} \tag{3.35}$$

Here A^{ik}, B^{ik} are defined by (1.35).

Proof By using the formula (2.15) the components Z^{ij} of the bivector $\partial_1 X - \partial_2 Y$ have the explicit expressions

$$\begin{aligned} Z^{ij} &= \sum_{s\geq 0}(-1)^{s+1}\left(f^i\partial_x^{s+1}(X^j_{(i,s)}\delta) + \frac{\partial_x f^i}{2}\partial_x^s(X^j_{(i,s)}\delta) + \sum_{k=1}^n A^{ik}\partial_x^s(X^j_{(k,s)}\delta)\right. \\ &\quad \left. - g^i\partial_x^{s+1}(Y^j_{(i,s)}\delta) - \frac{\partial_x g^i}{2}\partial_x^s(Y^j_{(i,s)}\delta) - \sum_{k=1}^n B^{ik}\partial_x^s(Y^j_{(k,s)}\delta)\right) + \cdots. \end{aligned}$$

Here $\delta = \delta(x-y)$. It is easy to see that when $m \leq \left[\frac{N}{2}\right]$, the first two terms in the formula (2.15) don't appear in the identity (3.33). So we denote them by periods in the above formula and will omit them in the calculations below. Then Z^{ij}_p reads

$$\begin{aligned} Z^{ij}_p &= \sum_{s\geq 0}(-1)^{s+1}\left(f^i\binom{s+1}{p}\partial_x^{s+1-p}X^j_{(i,s)} + \frac{\partial_x f^i}{2}\binom{s}{p}\partial_x^{s-p}X^j_{(i,s)}\right. \\ &\quad + \sum_{k=1}^n A^{ik}\binom{s}{p}\partial_x^{s-p}X^j_{(k,s)} - g^i\binom{s+1}{p}\partial_x^{s+1-p}Y^j_{(i,s)} \\ &\quad \left. - \frac{\partial_x g^i}{2}\binom{s}{p}\partial_x^{s-p}Y^j_{(i,s)} - \sum_{k=1}^n B^{ik}\binom{s}{p}\partial_x^{s-p}Y^j_{(k,s)}\right) \end{aligned}$$

Denote by $l.h.s.$ the left hand side of the identity (3.33). We obtain

$$
\begin{aligned}
&l.h.s. = \\
&\sum_{p,s\geq 0}(-1)^{m-p+s+1}\binom{N-p}{m-p}\left(f^i\binom{s+1}{p}\sum_{t\geq 0}\binom{s+1-p}{t}\partial_x^t X^j_{(i,s)(k,2N-m-s+t)}\right.\\
&\quad+\frac{\partial_x f^i}{2}\binom{s}{p}\sum_{t\geq 0}\binom{s-p}{t}\partial_x^t X^j_{(i,s)(k,2N+1-m-s+t)}\\
&\quad+\sum_{l=1}^{n}A^{il}\binom{s}{p}\sum_{t\geq 0}\binom{s-p}{t}\partial_x^t X^j_{(l,s)(k,2N+1-m-s+t)}\\
&\quad-g^i\binom{s+1}{p}\sum_{t\geq 0}\binom{s+1-p}{t}\partial_x^t Y^j_{(i,s)(k,2N-m-s+t)}\\
&\quad-\frac{\partial_x g^i}{2}\binom{s}{p}\sum_{t\geq 0}\binom{s-p}{t}\partial_x^t Y^j_{(i,s)(k,2N+1-m-s+t)}\\
&\quad\left.-\sum_{l=1}^{n}B^{il}\binom{s}{p}\sum_{t\geq 0}\binom{s-p}{t}\partial_x^t Y^j_{(l,s)(k,2N+1-m-s+t)}\right).
\end{aligned}
$$

Here we used the commutation relations

$$\frac{\partial}{\partial u^{i,q}}\partial_x^m=\sum_{t\geq 0}\binom{m}{t}\partial_x^t\frac{\partial}{\partial u^{i,q-m+t}}$$

By using the identity

$$\sum_{p\geq 0}(-1)^p\binom{N-p}{m-p}\binom{s}{p}\binom{s-p}{t}=\binom{s}{t}\binom{N-s+t}{m}$$

and by changing the order of summation, we can rewrite $l.h.s.$ as follows

$$
\begin{aligned}
&l.h.s. = \\
&\sum_{s,t\geq 0}(-1)^{m+s+1}\left(\binom{s+1}{t}\binom{N-s+t-1}{m}f^i\partial_x^t X^j_{(i,s)(k,2N-m-s+t)}\right.\\
&\quad+\binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x f^i}{2}\partial_x^t X^j_{(i,s)(k,2N+1-m-s+t)}\\
&\quad+\binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^{n}A^{il}\partial_x^t X^j_{(l,s)(k,2N+1-m-s+t)}\\
&\quad-\binom{s+1}{t}\binom{N-s+t-1}{m}g^i\partial_x^t Y^j_{(i,s)(k,2N-m-s+t)}\\
&\quad-\binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x g^i}{2}\partial_x^t Y^j_{(i,s)(k,2N+1-m-s+t)}\\
&\quad\left.-\binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^{n}B^{il}\partial_x^t Y^j_{(l,s)(k,2N+1-m-s+t)}\right)
\end{aligned}
\tag{3.36}
$$

Now we substitute the expression (3.31) of X, Y into the right hand side of the above formula. By using the properties of binomial coefficients, it is easy to see that all terms in the above summation vanish but the terms with $s = N, t = 0$, so the above formula can be simplified to

$$\begin{aligned} l.h.s. &= (-1)^{m+N+1}\left((-1)^m f^i X^j_{(i,N)(k,N-m)} - (-1)^m g^i Y^j_{(i,N)(k,N-m)}\right) \\ &= (-1)^{N+1} f^i F^j_{i,(k,N-m)} \end{aligned}$$

So part (a) of the lemma is proved.

Similarly to the derivation of (3.36), we can prove the following formula:

$$\begin{aligned} &\sum_{l\geq 0}(-1)^{m-l}\binom{N-l}{m-l} Z^{ij}_{l,(k,2N-m-l)} = \\ &\sum_{s,t\geq 0}(-1)^{m+s+1}\left(\binom{s+1}{t}\binom{N-s+t-1}{m} f^i\partial_x^t X^j_{(i,s)(k,2N-1-m-s+t)}\right. \\ &\qquad +\binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x f^i}{2}\partial_x^t X^j_{(i,s)(k,2N-m-s+t)} \\ &\qquad +\binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^n A^{il}\partial_x^t X^j_{(l,s)(k,2N-m-s+t)} \\ &\qquad -\binom{s+1}{t}\binom{N-s+t-1}{m} g^i\partial_x^t Y^j_{(i,s)(k,2N-1-m-s+t)} \\ &\qquad -\binom{s}{t}\binom{N-s+t}{m}\frac{\partial_x g^i}{2}\partial_x^t Y^j_{(i,s)(k,2N-m-s+t)} \\ &\qquad \left. -\binom{s}{t}\binom{N-s+t}{m}\sum_{l=1}^n B^{il}\partial_x^t Y^j_{(l,s)(k,2N-m-s+t)}\right). \end{aligned} \tag{3.37}$$

The summands of the r.h.s. vanish except for the terms with $(s,t) = (N,0), (N-m-1,0), (N-m,1), (N-m,0)$. Then the $l.h.s$ of identity (3.35) reads

$$\begin{aligned} l.h.s. = &(-1)^{N+1}\left(f^i X^j_{(i,N)(k,N-m-1)} - g^i Y^j_{(i,N)(k,N-m-1)}\right) \\ &+(-1)^N\left(f^i X^j_{(k,N)(i,N-m-1)} - g^i Y^j_{(k,N)(i,N-m-1)}\right) \\ &+(-1)^{N+1}(N-m+1)\left(f^i\partial_x X^j_{(k,N)(i,N-m)} - g^i\partial_x Y^j_{(k,N)(i,N-m)}\right) \\ &+(-1)^{N+1}\left(\frac{\partial_x f^i}{2}X^j_{(k,N)(i,N-m)} - \frac{\partial_x g^i}{2}Y^j_{(k,N)(i,N-m)}\right. \\ &\qquad \left.+\sum_{l=1}^n\left(A^{il}X^j_{(k,N)(l,N-m)} - B^{il}Y^j_{(k,N)(l,N-m)}\right)\right) \end{aligned} \tag{3.38}$$

Note that our X, Y have properties

$$X^j_{(k,N)(i,N-m)} = u^k G^j_{k,(i,N-m)}\delta^{ki},\ \ Y^j_{(k,N)(i,N-m)} = G^j_{k,(i,N-m)}\delta^{ki}$$

Then the identity (3.35) follows from (3.38) immediately. Part (b) of the lemma is proved. □

Lemma 3.5 *Let X, Y be two local vector fields that have components of the form (3.30) and satisfy the relation*

$$\partial_1 X = \partial_2 Y. \tag{3.39}$$

Then the following statements hold true:
i) When $N = 2M + 1$ the components of these vector fields have the expressions

$$X^i = \sum_{j=1}^{n} X^i_j(u, u_x, \dots, u^{(M)}) u^{j,2M+1} + Q_i(u, u_x, \dots, u^{(2M)}), \tag{3.40}$$

$$Y^i = \sum_{j=1}^{n} Y^i_j(u, u_x, \dots, u^{(M)}) u^{j,2M+1} + R_i(u, u_x, \dots, u^{(2M)}). \tag{3.41}$$

ii) When $N = 2M$ there exist local functionals $I_a, a = 1, 2, 3$ such that the components of the vector fields X, Y have, after the modification

$$X \mapsto X - (\partial_1 I_1 - \partial_2 I_2), \quad Y \mapsto Y - (\partial_1 I_2 - \partial_2 I_3) \tag{3.42}$$

(if necessary), the expressions

$$X^i = \sum_{j \neq i} X^i_j(u, u_x, \dots, u^{(M-1)}) u^{j,2M} + Q_i(u, u_x, \dots, u^{(2M-1)}), \tag{3.43}$$

$$Y^i = \sum_{j \neq i} Y^i_j(u, u_x, \dots, u^{(M-1)}) u^{j,2M} + R_i(u, u_x, \dots, u^{(2M-1)}). \tag{3.44}$$

In the case when the components X^i of the vector field X are homogeneous almost differential polynomials of degree $d \geq 3$, we can choose the densities of the local functionals I_a such that they are homogeneous almost differential polynomials of degree $d - 1$.

Proof For the case when $N = 1$ the result of the lemma follows from Lemma 3.3, so we assume that $N \geq 2$. It follows from Lemma 3.3 that the components of X, Y must take the form (3.31). The result of part (a) of Lemma 3.4 then shows that F^i_j are independent of $u^{(N-m)}$ for $m = 1, \dots, \left[\frac{N}{2}\right]$, so the identities in (3.32) read

$$(u^i - u^k) G^j_{k,(i,N-1)} = 0,$$

thus G^j_k are independent of $u^{i,N-1}$ when $i \neq k$. When $N \geq 3$ we use the identity (3.35) of Lemma 3.4 with $m = 1$ to obtain, by putting $i = k$,

$$G^j_{i,(i,N-1)} = 0.$$

So G^j_k are in fact independent of $u^{(N-1)}$. Now the identity (3.35) shows that

$$G^j_{k,(i,N-2)} = 0, \quad i \neq k.$$

By using repeatedly the identity (3.35) we know that G^i_j is independent of $u^{(N-m)}$ for $m = 1, \dots, \left[\frac{N-1}{2}\right]$, and $G^j_{i,(k,M)} = 0$ for $N = 2M \geq 2$ and $i \neq k$.

For the case of $N = 2M+1, M \geq 1$ the above argument shows that the components of the vector fields X, Y have the form (3.40), (3.41) where X^i_j, Y^i_j are given by the expressions

$$\begin{aligned} X^i_j &= u^j G^i_j(u, u_x, \dots, u^{(M)}) + F^i_j(u, u_x, \dots, u^{(M)}), \\ Y^i_j &= G^i_j(u, u_x, \dots, u^{(M)}. \end{aligned} \tag{3.45}$$

For the case of $N = 2M, M \geq 1$ the above argument shows that X^i, Y^i have the form

$$\begin{aligned} X^i = &\sum_{j=1}^n \left(u^j G^i_j(u, \dots, u^{(M-1)}, u^{j,M}) + F^i_j(u, \dots, u^{(M-1)})\right) u^{j,N} \\ &\quad + Q^i(u, \dots, u^{(N-1)}), \end{aligned} \tag{3.46}$$

$$Y^i = \sum_{j=1}^n G^i_j(u, \dots, u^{(M-1)}, u^{j,M})\, u^{j,N} + R^i(u, \dots, u^{(N-1)}) \tag{3.47}$$

From vanishing of the coefficients of $\delta^{(2M+1)}(x-y)$ in the expression of $(\partial_1 X - \partial_2 Y)^{ij}$ it follows that

$$f^i(u) F^j_i + f^j(u) F^i_j = 0. \tag{3.48}$$

In particular, we have

$$F^i_i = 0, \quad i = 1, \dots, n. \tag{3.49}$$

Define the functionals

$$I_k = (-1)^M \int \sum_{i=1}^n \partial^{-2}_{u^{i,M}} \frac{(u^i)^{3-k}\, G^i_i}{(M+\frac{1}{2}) f^i u^i_x} dx, \quad k = 1, 2, 3 \tag{3.50}$$

Then the components of the vector fields $\tilde{X} = X - (\partial_1 I_1 - \partial_2 I_2)$, $\tilde{Y} = Y - (\partial_1 I_2 - \partial_2 I_3)$ have the form of (3.46), (3.47) with $G^i_i = 0$. Since the vector fields $\tilde{X}, \tilde{Y}$ still satisfy the relation $\partial_1 \tilde{X} = \partial_2 \tilde{Y}$, we can assume without loss of generality that the components of X, Y have the form (3.46), (3.47) with vanishing G^i_i. By using the equations (3.48) we check that the identity (3.35) is still valid for $i = k$ when $m = M$ and $N \geq 2$. This leads to the fact that the functions G^i_j for $i \neq j$ do not depend on $u^{j,M}$. Thus we proved that the components of X, Y have the form (3.43), (3.44) after the modification (3.42) if necessary.

When the components of the vector field X are homogeneous almost differential polynomials of degree $d \geq 3$, the equations (3.49) and the expression (3.46) imply that the fuctions G^i_i are also homogeneous almost differential polynomials of degree $d - 2M$. So, when $M \geq 2$ we can choose the densities of the functionals I_a defined in (3.50) to be homogeneous almost differential polynomials of degree $d-1$. In the case

$M=1$, since the functions $G^i_i=G^i_i(u;u^i_x)$ are homogeneous of degree $d-2\ge 1$ (recall our assumption $d\ge 3$), the function $\frac{G^i_i(u;u^i_x)}{u^i_x}$ is in fact a polynomial in u^i_x. Thus in this case we can still choose the densities of the functionals I_a defined in (3.50) to be homogeneous almost differential polynomials of degree $d-1$. The lemma is proved. □

Lemma 3.6 *Let the vector fields X,Y have components of the form (3.30) with $N\ge 2$ and satisfy the relation (3.39). If the functions $X^i, i=1,\dots,n$ do not depend on $u^{(N)}$, then we can modify the vector field Y by*

$$Y\mapsto Y-\partial_2 J \tag{3.51}$$

for certain local functional J such that the components of this modified vector field Y depend at most on $u,\dots,u^{(N-1)}$ and the relation (3.39) still holds true.

Proof We first assume that $N=2M+1$. From the assumption of the lemma and the result of Lemma 3.5 we know that the components of the vector fields X,Y have the form (3.40), (3.41) with $X^i_j=0$. To prove the lemma we need to find a local functional J with density $h(u,u_x,\dots,u^{(M)})$ which satisfies the conditions

$$\frac{\partial^2 h}{\partial u^{i,M}\partial u^{j,M}}=(-1)^M\frac{1}{g^i(u)}Y^i_j,\quad i,j=1,\dots,n. \tag{3.52}$$

Denote by A_{ij} the r.h.s. of the above formulae. Then from vanishing of the coefficients of $\delta^{(2M+2)}(x-y)$ in the expression of the components of the bivector $\partial_1 X-\partial_2 Y$ it follows that the functions A_{ij} are symmetric with respect to the indices i,j. From the equation (3.35) with $m=M$ we also know that

$$\frac{\partial Y^j_i}{\partial u^{k,M}}=\frac{\partial Y^j_k}{\partial u^{i,M}}. \tag{3.53}$$

So the functions $\frac{\partial A_{ij}}{\partial u^{k,M}}$ are symmetric with respect to the indices i,j,k which implies the existence of a function $h(u,u_x,\dots,u^{(M)})$ satisfying the requirement (3.52).

Next let us assume that $N=2M$. As we did in the proof of Lemma 3.5 we can show that the components of X,Y have the form (3.46), (3.47) with

$$F^i_j=-u^j G^i_j. \tag{3.54}$$

Since the functions F^i_j do not depend on $u^{(M)}$ we deduce that the functions X^i,Y^i must have the expressions (3.43) with $X^i_j=0$. From (3.49) and the independence of X^i on $u^{(N)}$ it also follows that $G^i_i=0$ for $i=1,\dots,n$. Now by using the vanishing of the coefficients of $\delta^{(2M+1)}$ of the components of $\partial_1 X-\partial_2 Y$ and that of the l.h.s. of (3.35) with $m=M$ we obtain

$$\hat{Y}_{ij}+\hat{Y}_{ji}=0, \tag{3.55}$$

$$\hat{Y}_{jk,(i,M-1)}-\hat{Y}_{ik,(j,M-1)}-\hat{Y}_{ji,(k,M-1)}=0. \tag{3.56}$$

Here $\hat{Y}_{ij} = \frac{1}{g^i} Y^i_j$. The above two equations ensure the existence of a 1-form $\alpha = \sum_{i=1}^n h_i(u, \dots, u^{(M-1)}) du^{i,M-1}$ such that

$$d\alpha = \frac{1}{2} \sum_{i,j} \hat{Y}_{ij}\, du^{i,M-1} \wedge du^{j,M-1}. \tag{3.57}$$

Now the functional J defined by

$$J = \int \sum_{i=1}^n h_i(u(x), \dots, u^{(M-1)}(x))\, u^{i,M}(x) dx \tag{3.58}$$

meets the requirement of the lemma and we finished the proof. □

Lemma 3.7 *Let X be a local vector field with components*

$$X^i = X^i(u; u_x, \dots, u^{(N)}), \quad i = 1, \dots, n,\ N \geq 4.$$

that are homogeneous almost differential polynomials of degree $d \geq 3$. If X also has the following properties:
(a) When $N = 2M+2$, the components of X have the form

$$X^i = \sum_{j \neq i} X^i_j(u, \dots, u^{(M)}) u^{j,2M+2} + Q^i(u, \dots, u^{(2M+1)}) \tag{3.59}$$

and satisfy the conditions

$$(u^k - u^j)\left(\hat{X}_{ij,(k,M)} - \hat{X}_{ik,(j,M)}\right) + (u^k - u^i)\hat{X}_{jk,(i,M)} + (u^j - u^i)\hat{X}_{kj,(i,M)} = 0 \tag{3.60}$$

for

$$\hat{X}_{ij} = \frac{1}{f^i(u)} X^i_j(u, \dots, u^{(M)}). \tag{3.61}$$

(b) When $N = 2M+1$, X has components of the form

$$X^i = \sum_{j=1}^n X^i_j(u, \dots, u^{(M)}) u^{j,2M+1} + Q^i(u, \dots, u^{(2M)}) \tag{3.62}$$

and satisfies the conditions

$$X^i_{j,(k,M)} - X^i_{k,(j,M)} = 0 \tag{3.63}$$

$$(u^k - u^j)\hat{X}_{ij,(k,M)} + (u^i - u^k)\hat{X}_{jk,(i,M)} + (u^j - u^i)\hat{X}_{ki,(j,M)} = 0. \tag{3.64}$$

Then there exist two local functionals I_1, I_2 with densities that are homogeneous almost differential polynomials of degree $d-1$ such that the components of the vector field $X - (\partial_1 I_1 - \partial_2 I_2)$ depend at most on $u, u_x, \dots, u^{(N-1)}$.

Proof We first prove the lemma for the case when $d \geq 3, N \geq 5$. Assume $N = 2M+2$ and the vector field X satisfies the conditions (3.59), (3.60). We want to find two local functionals I_1, I_2 with densities of the form

$$h_a = \sum_{j=1}^{n} h_{a;j}(u, u_x, \ldots, u^{(M)}) u^{j,M+1}, \quad a = 1, 2 \tag{3.65}$$

such that they meet the requirement of the lemma. For this we need to find the functions $h_{a;j}, a = 1, 2, j = 1, \ldots, n$ satisfying the following equations:

$$(-1)^{M+1} X^i_j = f^i \left(\frac{\partial h_{1;i}}{\partial u^{j,M}} - \frac{\partial h_{1;j}}{\partial u^{i,M}} \right) - g^i \left(\frac{\partial h_{2;i}}{\partial u^{j,M}} - \frac{\partial h_{2;j}}{\partial u^{i,M}} \right). \tag{3.66}$$

Denote

$$P_{ij} = (-1)^M \frac{g^j X^i_j + g^i X^j_i}{f^i g^j - f^j g^i}, \quad Q_{ij} = (-1)^M \frac{f^j X^i_j + f^i X^j_i}{f^i g^j - f^j g^i} \tag{3.67}$$

Then it follows from (3.60) that the two-forms

$$\varpi_1 = \frac{1}{2} \sum_{i,j} P_{ij} du^{i,M} \wedge du^{j,M}, \quad \varpi_2 = \frac{1}{2} \sum_{i,j} Q_{ij} du^{i,M} \wedge du^{j,M} \tag{3.68}$$

are closed. So there exist one-forms

$$\alpha_a = \sum_j h_{a;j}(u, u_x, \ldots, u^{(M)})\, du^{j,M}, \ a = 1, 2 \tag{3.69}$$

such that $d\alpha_a = \varpi_a$ and the functions $h_{a;j}$ are homogeneous almost differential polynomials of degree $d - M - 2$. Now it's easy to see that the functions $h_{a;j}$ satisfy (3.66). So we proved the lemma for $N = 2M + 2 > 4$.

Next we assume that $N = 2M + 1 > 4$ and the vector field X satisfy the condition (3.62)-(3.64). Let the two local functionals I_1, I_2 that we are looking for have densities of the form

$$h_a(u, u_x, \ldots, u^{(M)}), \quad a = 1, 2. \tag{3.70}$$

Since the i-th component of the vector field $\partial_1 I_1 - \partial_2 I_2$ depends at most on $u, \ldots, u^{(2M+1)}$ and, moreover, it depends linearly on $u^{k,2M+1}$, we only need to find functions h_1, h_2 such that

$$X^i_j = \frac{\partial(\partial_1 I_1 - \partial_2 I_2)^i}{\partial u^{j,2M+1}} = (-1)^M (f^i\, h_{1,(i,M)(j,M)} - g^i\, h_{2,(i,M)(j,M)}). \tag{3.71}$$

To this end, let us define $P_{ii}, Q_{ii},\ i = 1, \ldots, n$ by solving the following systems

$$\begin{aligned} \frac{\partial P_{ii}}{\partial u^{j,M}} &= \frac{(-1)^{M+1}}{u^i - u^j} \left(u^j \frac{X^i_{j,(i,M)}}{f^i} - u^i \frac{X^j_{i,(i,M)}}{f^j} \right), \quad j \neq i, \\ \frac{\partial Q_{ii}}{\partial u^{j,M}} &= \frac{(-1)^{M+1}}{u^i - u^j} \left(\frac{X^i_{j,(i,M)}}{f^i} - \frac{X^j_{i,(i,M)}}{f^j} \right), \ j \neq i. \end{aligned} \tag{3.72}$$

The conditions (3.63), (3.64) implies the compatibility of the above systems, i.e.,

$$\frac{\partial}{\partial u^{k,M}}\left(\frac{\partial P_{ii}}{\partial u^{j,M}}\right)=\frac{\partial}{\partial u^{j,M}}\left(\frac{\partial P_{ii}}{\partial u^{k,M}}\right),\quad j,k\neq i. \tag{3.73}$$

So we have a set of functions $P_{ii}=P_{ii}(u,u_x,\dots,u^{(M)}), Q_{ii}=Q_{ii}(u,u_x,\dots,u^{(M)})$ satisfying the conditions (3.72). The ambiguity in the definition of these functions is the following shifts:

$$\begin{aligned} &P_{ii}\mapsto P_{ii}+W_{ii}(u,u_x,\dots,u^{(M-1)},u^{i,M}),\\ &Q_{ii}\mapsto Q_{ii}+R_{ii}(u,u_x,\dots,u^{(M-1)},u^{i,M}). \end{aligned} \tag{3.74}$$

Here W_{ii}, R_{ii} are arbitrary functions to be specified later. We also define functions P_{ij}, Q_{ij} with $i\neq j$ by the following formulae:

$$P_{ij}=\frac{(-1)^{M+1}}{u^i-u^j}\left(u^j\frac{X^i_j}{f^i}-u^i\frac{X^j_i}{f^j}\right),\ Q_{ij}=\frac{(-1)^{M+1}}{u^i-u^j}\left(\frac{X^i_j}{f^i}-\frac{X^j_i}{f^j}\right). \tag{3.75}$$

By using the conditions (3.63), (3.64) we easily verify that

$$\frac{\partial P_{ij}}{\partial u^{k,M}},\quad \frac{\partial Q_{ij}}{\partial u^{k,M}}$$

are symmetric with respect to their indices i,j,k. So there exist functions $h_a(u,u_x,\dots,u^{(M)}),\ a=1,2$ such that

$$\frac{\partial^2 h_1}{\partial u^{i,M}\partial u^{j,M}}=P_{ij},\quad \frac{\partial^2 h_2}{\partial u^{i,M}\partial u^{j,M}}=Q_{ij}. \tag{3.76}$$

Now it's easy to verify that, when $i\neq j$, the functions h_1,h_2 satisfy the conditions given in (3.71). When $i=j$ we have

$$X^i_i-(-1)^M(f^i\,h_{1,(i,M)(i,M)}-g^i\,h_{2,(i,M)(i,M)})=X^i_i-(-1)^M(f^i\,P_{ii}-g^i\,Q_{ii}). \tag{3.77}$$

It follows from the defintion of P_{ii}, Q_{ii} and the condtions given in (3.63) that the r.h.s. of the above formulae does not depend on $u^{k,M}$ for any $k\neq i$, so we can make it to be zero by adjusting the functions P_{ii}, Q_{ii} as in (3.74). By the above construction, the functions h_1,h_2 can be chosen to be homogeneous almost differential polynomials of degree $d-1$. So the lemma is proved for the case mentioned above.

Now let us consider the case $d\geq 3, N=4$. Proceeding in the same way as for the case of $N=2M+2, M\geq 2$ we can find the 1-forms (3.69) such that the 2-forms ϖ_1,ϖ_2 that are defined as in (3.68) can be represented by $\varpi_a=d\alpha_a,\ a=1,2$. The pecularity of this particular case of $M=1$ lies in the fact that the functions $h_{a,j}$ that we constructed above are in general no longer rational functions of the jet coordinates $u^{i,k}, k\geq 1$, they can be chosen to have the form

$$h_{a,j}=\sum_{k=1}^n W_{a,j;k}(u;u_x)\log u^k_x+U_{a,j}(u;u_x),\quad a=1,2,\ j=1,\dots,n. \tag{3.78}$$

Here $W_{a,j;k}, U_{a,j} \in \mathcal{A}$ and are homogeneous of degree $d-1$. Since

$$\frac{\partial h_{a,j}}{\partial u_x^i} - \frac{\partial h_{a,i}}{\partial u_x^j} \in \mathcal{A} \tag{3.79}$$

we must have

$$\frac{\partial W_{a,j;k}}{\partial u_x^i} = \frac{\partial W_{a,i;k}}{\partial u_x^j}, \quad i,j = 1,\dots,n. \tag{3.80}$$

This implies the existence of functions $A_{a,k}(u;u_x) \in \tilde{A}$ of degree $d-2$ such that

$$W_{a,j;k} = \frac{\partial A_{a,k}(u;u_x)}{\partial u_x^j}, \quad a = 1,2;\ j,k = 1,\dots,n. \tag{3.81}$$

Since $W_{a,j;k}$ are almost differential polynomials, the functions $A_{a,k}$ can also be chosen as homogeneous almost differential polynomials of degree $d-2 \geq 1$ up to the addition of terms of the form

$$\sum_{l=1}^{n} B_{a,k,l}(u) \log u_x^l. \tag{3.82}$$

However, such functions have degree zero, so they are not allowed to appear in the expression of $A_{a,k}$. Now the needed functionals I, J that satisfy the requirements of the lemma can be chosen to have densities $\tilde{h}_1, \tilde{h}_2$ of the form

$$\tilde{h}_a = \sum_{k=1}^{n} \left(U_{a,k}(u,u_x) - A_{a,k}(u,u_x)\frac{1}{u_x^k} \right) u_{xx}^k, \quad a = 1,2. \tag{3.83}$$

The lemma is proved. $\square$

Proof of Theorem 3.2. Assume that the components X^i of the vector field X have the form

$$X^i = X^i(u,\dots,u^{(N)}), \quad N \geq 4,\ i = 1,\dots,n. \tag{3.84}$$

The relation (3.28) implies the existence of a local vector field Y such that $\partial_1 X = \partial_2 Y$. By using Lemma 3.6 we can choose the vector field Y such that its components depend at most on the coordinates $u,\dots,u^{(N)}$. Then it follows from the results of Lemma 3.5 that the components of X, Y have the expressions (3.40),(3.41) when N is odd and the one of (3.43),(3.44) when N is even (after a modification of (3.42) which does not affect our result). We now proceed to employ the result of Lemma 3.7 in order to find two local functionals I, J with densities that are almost differential polynomials of degree $d-1$ such that $X - (\partial_1 I - \partial_2 J)$ depends at most on $u,\dots,u^{(N-1)}$. To this end we need to verify that X^i satisfy the equations (3.60) when N is even and that of (3.63), (3.64) when N is odd.

Let N be an even integer $2M+2$. Then by using vanishing of the l.h.s. of (3.35) with $m = M$ we obtain

$$\hat{X}_{jk,(i,M)} - \hat{X}_{ik,(j,M)} - \hat{X}_{ji,(k,M)} = u^i u^j \left(\hat{Y}_{jk,(i,M)} - \hat{Y}_{ik,(j,M)} - \hat{Y}_{ji,(k,M)} \right). \tag{3.85}$$

Here $\hat{Y}_{ij}$ are defined as in (3.56) and $\hat{X}_{ij}$ are defined by (3.61). We also have the following equations

$$\hat{X}_i^j + \hat{X}_j^i = u^i u^j \left(\hat{Y}_{ji} + \hat{Y}_{ij}\right) \tag{3.86}$$

due to vanishing of the coefficients of $\delta^{(2M+1)}$ in the components of $\partial_1 X - \partial_2 Y$. Denote by $L_{i,j,k}$, $R_{i,j,k}$ the expression of the l.h.s. and r.h.s. of (3.85) multiplied by u^k. Then we have

$$\begin{aligned} &R_{i,j,k} - R_{j,k,i} - R_{k,i,j} - R_{i,k,j} = u^i u^j u^k \left(\hat{Y}_{jk,(i,M)} + \hat{Y}_{kj,(i,M)}\right) \\ &= u^i \left(\hat{X}_{jk,(i,M)} + \hat{X}_{kj,(i,M)}\right). \end{aligned}$$

Here in the last equality we used the equations in (3.86). By equating the last expression with $L_{i,j,k} - L_{j,k,i} - L_{k,i,j} - L_{i,k,j}$ we arrive at the proof that the vector field X satisfies the conditions given in (3.60).

Now let us assume that $N = 2M + 1$. Then from (3.35) with $m = M$ we know that

$$f^i(X^j_{i,(k,M)} - X^j_{k,(i,M)}) = g^i(Y^j_{i,(k,M)} - Y^j_{k,(i,M)}), \tag{3.87}$$

This equation together with the one that is obtained from it by exchange the indices i and k implies that

$$X^j_{i,(k,M)} = X^j_{k,(i,M)}, \quad Y^j_{i,(k,M)} = Y^j_{k,(i,M)}. \tag{3.88}$$

So the components X^i, Y^i satisfy the conditions of the form (3.63). We are left to prove that they also satisfy the conditions of the form (3.64). For this, let us consider the coefficients of $\delta^{(2M+2)}(x-y)$ in the expression of the components of the bivector $\partial_1 X - \partial_2 Y$. Vanishing of these coefficients leads to the equations

$$\hat{X}_{ji} - \hat{X}_{ij} = u^i u^j \left(\hat{Y}_{ji} - \hat{Y}_{ij}\right). \tag{3.89}$$

By taking derivative with respect to $u^{k,M}$ and multiplying by u^k on both sides of the above equation we obtain

$$u^k \left(\hat{X}_{ji,(k,M)} - \hat{X}_{ij,(k,M)}\right) = u^k u^i u^j \left(\hat{Y}_{ji,(k,M)} - \hat{Y}_{ij,(k,M)}\right). \tag{3.90}$$

Denote the r.h.s. of the last equation by $W_{i,j,k}$. Then the condition (3.64) follows from $W_{i,j,k} + W_{j,k,i} + W_{k,i,j} = 0$.

Above we showed that the vector field X satisfies the requirements of Lemma 3.7. So we can find local functionals I, J with densities that are almost differential polynomials of degree $d-1$ such that $X - (\partial_1 I - \partial_2 J)$ depends at most on $u, \dots, u^{(N-1)}$. Repeating this procedure by the subtraction of terms of the form $\partial_1 I - \partial_2 J$, we reduce the proof of the Theorem 3.2 to the case when the components X^i of the vector field X have the form (3.84) with $N = 1, 2, 3$.

Note that in the case when $N = 3$ or $N = 2$, the components of the vector field X and the accompanying one Y also have the forms (3.40),(3.41) and (3.43),(3.44) with

$M=1$, and the above equations (3.85), (3.86) and (3.87), (3.89) still hold true. Thus when $N=3$ the vector field X fulfils the requirements of Lemma 3.7, and we can find local functionals I_1, I_2 such that $X-(\partial_1 I_1-\partial_2 I_2)$ depend at most on u, u_x, u_{xx}. The difference of this special case from the general one lies in the fact that now it is not obvious that we can choose the densities $h_a(u,u_x), a=1,2$ to be almost differential polynomials. What can easily be seen from our construction is that they can be chosen to have the form

$$h_a(u,u_x)=\sum_{i\neq j} V_{a;i,j}(u,u_x)\log u_x^i \log u_x^j+\sum_{i=1}^n V_{a;i}(u,u_x)\log u_x^i+U_a(u,u_x). \tag{3.91}$$

Here the functions $V_{a;i,j}, V_{a;i}, U_a$ are homogeneous almost differential polynomials of degree $d-1$. The vector field $\tilde{X}=X-(\partial_1 I_1-\partial_2 I_2)$ still has the property $\partial_1\partial_2\tilde{X}=0$, so the same argument as above shows that the components of $\tilde{X}$ have the form (3.59) with $M=0$ and satisfy the equations (3.60). By using the construction of Lemma 3.7 we can find local functionals I_3, I_4 such that the vector field $\tilde{X}-(\partial_1 I_3-\partial_2 I_4)$ depends at most on u, u_x. A careful analysis of this construction shows that the densities of these local functionals can also be chosen to have the form of (3.91). Let us denote the densities $I=I_1+I_3,\ J=I_2+I_4$ also by h_1, h_2 which have the expression (3.91). A simple calculation shows that

$$\frac{\partial(\partial_1 I-\partial_2 J)^i}{\partial u^j_{xxx}}=g^i\frac{\partial^2 h_2}{\partial u^i_x\partial u^j_x}-f^i\frac{\partial^2 h_1}{\partial u^i_x\partial u^j_x}, \tag{3.92}$$

$$\frac{\partial(\partial_1 I-\partial_2 J)^i}{\partial u^i_{xx}}=-\frac{3}{2}f^i(u)u^i_x\frac{\partial^2 h_2}{\partial u^i_x\partial u^i_x}. \tag{3.93}$$

The r.h.s. of the above two identities equal respectively to $\frac{\partial X^i}{\partial u^j_{xxx}}$ and $\frac{\partial X^i}{\partial u^j_{xx}}$. We deduce that the functions

$$\frac{\partial^2 h_a}{\partial u^i_x\partial u^j_x},\quad a=1,2;\ i,j=1,\dots,n \tag{3.94}$$

are homogeneous almost differential polynomials of degree $d-3$. This fact yields the restriction on the coefficients $V_{a;i,j}, V_{a;i}$ in the expression of the densities (3.91) that they depend on u_x at most linearly. Since h_1, h_2 have degree $d-1\geq 2$ (recall that we assume $d\geq 3$), it follows that the functions $V_{a;i,j}, V_{a;i}$ must vanish, and as a result the densities h_1, h_2 of the local functionals I, J are homogeneous almost differential polynomials of degree $d-1$. Now let us prove that we have in fact

$$X=\partial_1 I-\partial_2 J. \tag{3.95}$$

This is due to the fact that the vector field $\bar{X}=X-(\partial_1 I-\partial_2 J)$ still satisfies the property $\partial_1\partial_2\bar{X}=0$. So, by using Lemma 3.6 we can find a vector field $\bar{Y}$ that depends at most on u, u_x such that $\partial_1\bar{X}=\partial_2\bar{Y}$. Then by using Lemma 3.3 we know that the components of $\bar{X}$ depend at most linearly on u_x, since they are homogeneous almost differential polynomials of degree $d\geq 3$ we must have $\bar{X}=0$. Thus we proved the theorem. □

Theorem 3.8 *Let the vector field X satisfying the condition (3.28) have components of the form*

$$X^i = \sum_{j=1}^n X^i_j(u)u^j_{xx} + \sum_{k,l} Q^i_{kl}(u)u^k_x u^l_x. \tag{3.96}$$

Then each function $\hat{X}_{ii} = (f^i(u))^{-1}X^i_i(u)$ depends only on u^i and there exist local functionals $\tilde{I}_1, \tilde{I}_2$ with densities that are homogeneous differential polynomials of degree 1 such that

$$X = \partial_1(I_1 + \tilde{I}_1) - \partial_2(I_2 + \tilde{I}_2) \tag{3.97}$$

where

$$I_a = -\frac{2}{3}\sum_{i=1}^n \int (u^i)^{2-a}\hat{X}_{ii}(u^i)u^i_x \log u^i_x dx, \quad a = 1, 2. \tag{3.98}$$

Note that the densities h_1, h_2 of the functionals I_1, I_2 can be chosen as

$$h_a = \sum_{i=1}^n V_{a,i}(u^i)\frac{u^i_{xx}}{u^i_x}\, dx, \quad a = 1, 2 \tag{3.99}$$

which are homogeneous almost differential polynomials of degree $d - 1 = 1$. Here the functions $V_{a,i}$ are defined by $V_{a,i}(u^i)' = 2(u^i)^{2-a}\hat{X}_{ii}(u^i)$.

Proof Since X^i are differential polynomials the condition (3.28) implies existence of a vector field Y with components Y^i of the same form (3.96) of X^i such that $\partial_1 X = \partial_2 Y$. By using vanishing of the l.h.s. of (3.35) with $k = i \neq j$ we deduce that

$$\frac{\partial \hat{X}_{ii}}{\partial u^j} - u^i u^j \frac{\partial \hat{Y}_{ii}}{\partial u^j} = 0, \quad i \neq j = 1, \ldots, n. \tag{3.100}$$

Since (3.86) also holds true in this case, we obtain

$$\hat{Y}_{ii} = \frac{1}{(u^i)^2}\hat{X}_{ii} \tag{3.101}$$

which yields, together with (3.100), the first result of the theorem

$$\frac{\partial \hat{X}_{ii}}{\partial u^j} = 0, \quad i \neq j = 1, \ldots, n. \tag{3.102}$$

From the definition (3.40) it is easy to see that the components of the vector field $\tilde{X} = X - (d_1 I_1 - d_2 I_2)$ are still homogeneous differential polynomials of the form (3.49) with $X^i_i(u) = 0,\ i = 1, \ldots, n$. Then by using the same construction as we give in the proof of Lemma 3.7 for the local functionals with the densities (3.65) we can find functionals $\tilde{I}_1, \tilde{I}_2$ with homegeneous differential polynomial densities of degree 1 such that the vector field $\bar{X} = \tilde{X} - (\partial_1\tilde{I}_1 - \partial_2\tilde{I}_2)$ depends at most on u, u_x. The equation

$\partial_1\partial_2\bar{X}=0$ then implies that $\bar{X}$ depends at most linearly on u_x. Since the components of $\bar{X}$ are homogeneous differential polynomials of degree 2, we arrive at the equalities

$$X=\partial_1(I_1+\tilde{I}_1)-\partial_2(I_2+\tilde{I}_2). \tag{3.103}$$

The Theorem is proved. □

Proof of the Quasitriviality Theorem Due to the triviality of the Poisson cohomology $H^2(\mathcal{L}(M),\varpi_1)$ the bihamiltonian structure (1.28) can always be assumed, if necessary by performing a usual Miura-type transformation, to have the following form:

$$\{u^i(x),u^j(y)\}_1=\{u^i(x),u^j(y)\}^{[0]}, \tag{3.104}$$
$$\{u^i(x),u^j(y)\}_2=\{u^i(x),u^j(y)\}^{[0]}+\sum_{k\ge1}\epsilon^k Q_k^{ij}. \tag{3.105}$$

Here Q_k^{ij} are the components of the bivectors Q_k and have the expressions

$$Q_k^{ij}=\sum_{l=0}^{k+1}Q_{k,l}^{ij}(u;u_x,\dots,u^{(k+1-l)})\,\delta^{(l)}(x-y).$$

We also denote by Q_0 the bivector corresponding to the undeformed second Poisson structure. The coefficients $Q_{k,l}^{ij}$ are homogeneous differential polynomials of degree $k+1-l$. The compatibility of the above two Poisson brackets implies the existence of vector fields $X_k,\ k\ge1$ such that

$$Q_k=\partial_1X_k,\quad \partial_1\partial_2X_1=0,\quad k\ge0 \tag{3.106}$$

and the components of X_k are homogeneous differential polynomials of degree k. By using Theorem 3.2, we know the existence of two local functionals

$$I_1=\int h_{1,1}(u(x))dx,\quad J_1=\int h_{2,1}(u(x))dx \tag{3.107}$$

such that $X_1=\partial_1I_1-\partial_2J_1$. By performing the Miura-type transformation

$$u^i\mapsto\exp(\epsilon\,\partial_1J_1)\,u^i \tag{3.108}$$

the first Poisson structure remains the same while the second Poisson structure (3.105) is transformed to

$$\exp(-\epsilon\,\mathrm{ad}_{\partial_1J_1})(Q_0+\sum_{k\ge1}\epsilon^k\partial_1X_k)=Q_0+\sum_{k\ge2}\epsilon^k\partial_1\tilde{X}_k \tag{3.109}$$

where the vector fields $\tilde{X}_k$ have the expressions

$$\tilde{X}_k=\sum_{l=0}^{k}(-1)^l\frac{(\mathrm{ad}_{\partial_1J_1})^l}{l!}X_{k-l},\quad k\ge2. \tag{3.110}$$

The components of these vector fields are homogeneous differential polynomials of degree k. So we can use Theorem 3.2 again to find two local functionals I_2, J_2 with densities $h_{a,2}(u, u_x, u_{xx})$ that are homogeneous almost differential polynomials of degree 1 such that $X_2 = \partial_1 I_2 - \partial_2 J_2$. Then the Miura-type transformation

$$u \mapsto \exp(\epsilon^2\, \partial_1 J_2)\, u^i \tag{3.111}$$

leaves the form of the first Poisson structure unchanged while transforms the second one to the form

$$\exp(-\epsilon^2\, \mathrm{ad}_{\partial_1 J_2})(Q_0 + \sum_{k\geq 2} \epsilon^k \partial_1 \tilde{X}_k) = Q_0 + \sum_{k\geq 3} \epsilon^k \partial_1 \bar{X}_k \tag{3.112}$$

Here the vector fields $\bar{X}_k$ have the expressions

$$\bar{X}_k = \sum_{l=0}^{[k/2]} (-1)^l \frac{(\mathrm{ad}_{\partial_1 J_2})^l}{l!} \tilde{X}_{k-2l}, \quad k \geq 3. \tag{3.113}$$

and their components are homogeneous almost differential polynomials of degree k which depend only on finitely many of the jet coordinates. Here $\tilde{X}_1$ is assumed to be zero. By repeatedly using the above procedure, we see that the deformation part of (1.28) can be absorbed by a series of quasi-Miura transformations. So it is quasitrivial, and the Theorem is proved. $\square$

From the proof of the Quasitriviality Theorem we see that the reducing transformation

$$u^i \mapsto u^i + \sum_{k\geq 1} \epsilon^k F_k(u, \dots, u^{(m_k)}) \tag{3.114}$$

of the bihamiltonian structure (1.28) has the properties that F_k are homogeneous almost differential polynomials of degree k and that $m_k \leq k$ when k is odd and $m_k \leq \frac{3}{2}k$ when k is even.

4 Reducing bihamiltonian PDEs

In this section we study properties of the bihamiltonian systems (1.27).

Lemma 4.1 *Let I, J be two local functionals*

$$I = \int p(u, u_x, \dots, u^{(N)})\, dx, \quad J = \int q(u, u_x, \dots, u^{(N)})\, dx \tag{4.1}$$

that satisfy the relation

$$\partial_1 I = \partial_2 J. \tag{4.2}$$

Then up to additions of total x-derivatives, the densities p, q do not depend on the jet coordinates $u_x, \dots, u^{(N)}$.

Proof Denote by X the vector field $\partial_1 I - \partial_2 J$ and by X^i its components. Then from vanishing of

$$\frac{\partial X^i}{\partial u^{j,2N+1}} = (-1)^N \left(f^i \frac{\partial^2 p}{\partial u^{i,N} \partial u^{j,N}} - g^i \frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N}} \right), \tag{4.3}$$

we see that the functions p, q satisfy

$$(u^i - u^j) \frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N}} = 0, \quad \frac{\partial^2 p}{\partial u^{i,N} \partial u^{j,N}} - u^i \frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N}} = 0. \tag{4.4}$$

So the functions p, q can be represented by some functions p_i, r_i, s as

$$\begin{aligned} p &= \sum_{i=1}^n p_i(u, \dots, u^{(N-1)}, u^{i,N}), \\ q &= \sum_{i=1}^n \left[u^i p_i(u, \dots, u^{(N-1)}, u^{i,N}) + r_i(u, \dots, u^{(N-1)}) u^{i,N} + s(u, \dots, u^{(N-1)}) \right] \end{aligned} \tag{4.5}$$

By substituting these expressions of p, q into the equations

$$\frac{\partial X^i}{\partial u^{i,2N}} = 0 \tag{4.6}$$

we deduce that

$$f^i u^i \frac{\partial^2 p_i}{\partial u^{i,N} \partial u^{i,N}} = 0 \tag{4.7}$$

Thus we can rewrite the functions p, q into the form

$$\begin{aligned} p &= \sum_{i=1}^n a_i(u, \dots, u^{(N-1)}) u^{i,N} + c(u, \dots, u^{(N-1)}), \\ q &= \sum_{i=1}^n b_i(u, \dots, u^{(N-1)}) u^{i,N} + d(u, \dots, u^{(N-1)}), \end{aligned} \tag{4.8}$$

From the identity $\frac{\partial X^i}{\partial u^{j,2N}} = 0$ for $i \neq j$ we have

$$\left(\frac{\partial^2 p}{\partial u^{i,N} \partial u^{j,N-1}} - \frac{\partial^2 p}{\partial u^{j,N} \partial u^{i,N-1}} \right) = u^i \left(\frac{\partial^2 q}{\partial u^{i,N} \partial u^{j,N-1}} - \frac{\partial^2 q}{\partial u^{j,N} \partial u^{i,N-1}} \right) \tag{4.9}$$

These equations imply that the functions a_i, b_i satisfy

$$\frac{\partial a_i}{\partial u^{j,N-1}} = \frac{\partial a_j}{\partial u^{i,N-1}}, \quad \frac{\partial b_i}{\partial u^{j,N-1}} = \frac{\partial b_j}{\partial u^{i,N-1}}. \tag{4.10}$$

So there exist functions $A(u, \dots, u^{(N-1)}), B(u, \dots, u^{(N-1)})$ such that

$$a_i = \frac{\partial A}{\partial u^{i,N-1}}, \quad b_i = \frac{\partial B}{\partial u^{i,N-1}}, \quad i = 1, \dots, n. \tag{4.11}$$

Now we can replace the densities p, q of the Hamiltonians I, J respectively with

$$\tilde{p} = p - \partial_x A, \quad \tilde{q} = q - \partial_x B \tag{4.12}$$

Then the new densities become independent of the jet variables $u^{i,N}, i = 1, \dots, N$. Repeating the above procedure successively, we arrive at the result of the lemma. $\square$

Proof of Corollary 1.8. Let us assume that after the quasi-Miura transformation the Hamiltonians of the systems (1.27) have the expansion in ϵ:

$$H_a = \sum_{k\geq 0} \epsilon^k H_a^{[k]} = \sum_{k\geq 0} \epsilon^k \int \tilde{h}_a^{[k]}(u, u_x, \dots, u^{(m_k)})dx, \quad a = 1, 2. \tag{4.13}$$

Here m_k are some positive integers which may also depend on the index a, and the functions $h_a^{[k]}$ have degrees k. Due to the bihamiltonian property

$$\partial_1 H_1 = \partial_2 H_2 \tag{4.14}$$

we know in particular that

$$\partial_1 H_1^{[1]} = \partial_2 H_2^{[1]}. \tag{4.15}$$

Then the result of the above lemma implies that $H_1^{[1]} = H_2^{[1]} = 0$. Similarly, we prove that all other Hamiltonians $H_a^{[k]},\ k \geq 2$ are trivial. The Theorem is proved. $\square$

Corollary 4.2 *Any two bihamiltonian flows of the form (1.27) that correspond to the same bihamiltonian structure (1.28) mutually commute.*

Proof Denote by X, Y the vector fields corresponding to the given bihamiltonian systems, then their commutator $[X, Y]$ is also a bihamiltonian vector field of degree greater than 1. From Lemma 4.1 it follows that under the quasi-Miura transformation reducing the bihamiltonian structure (1.28) to (1.30) this vector field must vanishes. Thus we proved the corollary. $\square$

5 Central invariants of bihamiltonian structures

One of the important applications of the property of quasitriviality is the classification of deformations of a given bihamiltonian structure of hydrodynamic type. The problem of classification of quasitrivial infinitesimal deformations was solved in [36]. It was also conjectured that all deformations of the form (1.28) have reducing transformations. The Quasitriviality Theorem proves this conjecture. In this section we reformulate the main result of [36] in order to describe the complete list of invariants of a bihamiltonian structure with a given leading order $\{\ ,\ \}_{1,2}^{[0]}$ modulo Miura-type transformations (1.39). Recall that these transformations must depend polynomially on the derivatives in every order in ϵ.

Let us rewrite the bihamiltonian structure (1.28) in terms of the canonical coordinates $u^i = u^i(w),\ i = 1, \dots, n$

$$\{u^i(x), u^j(y)\}_a = \{u^i(x), u^j(y)\}_a^{[0]}$$
$$+ \sum_{m\geq 1}\sum_{l=0}^{m+1} \epsilon^m A^{ij}_{m,l;a}(u; u_x, \dots, u^{(m+1-l)})\delta^{(l)}(x-y),\ a = 1, 2, . \tag{5.1}$$

Then the functions P^{ij}_a, Q^{ij}_a defined in (1.48) have the expressions

$$P^{ij}_a(u) = A^{ij}_{1,2;a}(u),\ Q^{ij}_a(u) = A^{ij}_{2,3;a}(u),\ i, j = 1, \dots, n, a = 1, 2. \tag{5.2}$$

Proof of Corollary 1.10. Let us first assume that the bihamiltonian structure (5.1) has the following special form

$$(\varpi_1, \varpi_2 + \epsilon^2\gamma + \mathcal{O}(\epsilon^3)). \tag{5.3}$$

Here (ϖ_1, ϖ_2) denote the bihamiltonian structure given by the leading terms of (5.1), and γ is a bivector which can be represented as $\gamma = \partial_1 X$ through a vector field with components that are homogenous differential polynomials of degree 2. Due to Theorem 3.8, the vector field X can be represented up to a Miura-type transformation in the form

$$X = \partial_1 I - \partial_2 J \tag{5.4}$$

where the functionals I, J are defined by

$$I = \int \sum_{i=1}^n u^i \hat{c}_i(u^i) u^i_x \log u^i_x dx, \quad J = \int \sum_{i=1}^n \hat{c}_i(u^i) u^i_x \log u^i_x dx \tag{5.5}$$

with

$$\hat{c}_i(u^i) = \frac{1}{3(f^i(u))^2}(\gamma)^{ii}_3, \quad i = 1, \dots, n. \tag{5.6}$$

Here $(\gamma)^{ii}_3$ denote the coefficients of $\delta'''(x-y)$ in the components $(\gamma)^{ii}$ of the bivector γ. The main result of [36] together with the Quasitriviality Theorem shows that any two bihamiltonian structures of the form (5.3) are equivalent *iff* they correspond to the same set of functions $\hat{c}_i, i = 1, \dots, n$. In the case that the two bihamiltonian structures of the present theorem have the form (5.3), it is easy to see that $\hat{c}_i(u^i) = c_i(u)$, and the result of the theorem follows.

Now return to the general form (5.1) of the a bihamiltonian structure. We redenote it as

$$(\varpi_1 + \epsilon\alpha_1 + \epsilon^2\beta_1, \varpi_2 + \epsilon\alpha_2 + \epsilon^2\beta_2) + \mathcal{O}(\epsilon^3). \tag{5.7}$$

By using the result of Theorem 3.1 we can eliminate the linear in ϵ terms by a Miura-type transformation

$$u^i \mapsto \exp(-\epsilon X)u^i, \quad i = 1, \dots, n \tag{5.8}$$

given by a local vector field X with components of the form

$$X^i = \sum_{j=1}^n X^i_j(u) u^j_x \tag{5.9}$$

This implies that $P_1 = \partial_1 X,\ P_2 = \partial_2 X$ and this in turn yields

$$P_1^{ik} = -f^k(u)X_k^i(u) + f^i(u)X_i^k(u), \quad P_2^{ik} = -g^k(u)X_k^i(u) + g^i(u)X_i^k(u) \tag{5.10}$$

where the functions P_1^{ij}, P_2^{ij} are defined by (5.2). Solving the above system we obtain

$$X_k^i(u) = \frac{P_2^{ki} - u^i P_1^{ki}}{f^k(u)(u^k - u^i)}, \quad k \neq i. \tag{5.11}$$

After the Miura-type transformation (5.8), the bihamiltonian structure (5.7) becomes

$$(\varpi_1 + \epsilon^2(\beta_1 - \frac{1}{2}[X, \alpha_1]), \varpi_2 + \epsilon^2(\beta_2 - \frac{1}{2}[X, \alpha_2])) + \mathcal{O}(\epsilon^3). \tag{5.12}$$

Then there exists a local vector field Y such that

$$\partial_1 Y = \beta_1 - \frac{1}{2}[X, \alpha_1]. \tag{5.13}$$

So the Miura-type transformation

$$u^i \mapsto \exp(-\epsilon Y)u^i, \quad i = 1, \ldots, n \tag{5.14}$$

reduces the bihamiltonian structure (5.12) to the form of (5.8)

$$(\varpi_1, \varpi_2 + \epsilon^2(\beta_2 - \frac{1}{2}[X, \alpha_2] - \partial_2 Y)) + \mathcal{O}(\epsilon^3). \tag{5.15}$$

and we need to compute the coefficients χ_3^{ij} of $\delta'''(x-y)$ in the components of the bivector $\chi = \beta_2 - \frac{1}{2}[X, \alpha_2] - \partial_2 Y$. By using the notations introduced in (5.2) we have

$$\begin{aligned}
&[X, \alpha_1]_3^{ii} = \sum_k X_k^i (P_1^{ik} - P_1^{ki}) = -2 \sum_{k \neq i} X_k^i P_1^{ki}, \\
&[X, \alpha_2]_3^{ii} = \sum_k X_k^i (P_2^{ik} - P_2^{ki}) = -2 \sum_{k \neq i} X_k^i P_2^{ki}, \\
&(\partial_2 Y)_3^{ii} = u^i (\partial_1 Y)_3^{ii} = u^i(\beta_1 - \frac{1}{2}[X, \alpha_1])_3^{ii} = u^i Q_1^{ii} - \frac{1}{2} u^i [X, \alpha_1]_3^{ii}
\end{aligned} \tag{5.16}$$

Here as above for any bivector η we denote by η_3^{ij} the coefficient of $\delta'''(x-y)$ in the components η^{ij}. These formulae together with the expressions (5.11) for $X_k^i, k \neq i$ show that the functions $\hat{c}_i(u^i)$ that are defined by (5.6) with γ replaced by χ coincide with the functions $c_i(u)$ introduced in (1.49). Thus we proved the theorem. □

From this theorem it also easily follows the following corollary:

Corollary 5.1 *Any deformation (1.28) of the bihamiltonian structure (1.30) is equivalent, under a appropriate Miura-type transformation, to a deformation in which only even powers of ϵ appear.*

This result can also be seen from the construction of the functionals I, J in the proof of Theorem 3.2 and the argument given in the proof of the Quasitriviality Theorem.

Theorem 5.2 *If we choose another representative*

$$\{\ ,\ \}\tilde{}_1 = c\{\ ,\ \}_2 + d\{\ ,\ \}_1, \quad \{\ ,\ \}\tilde{}_2 = a\{\ ,\ \}_2 + b\{\ ,\ \}_1, \quad ad - bc \neq 0 \tag{5.17}$$

of the bihamiltonian structure (5.1), then the functions $c_i(u)$ that we define in (1.49) are changed to

$$\tilde{c}_i(\tilde{u}^i) = \frac{cu^i + d}{ad - bc} c_i(u^i), \quad i = 1, \dots, n. \tag{5.18}$$

where

$$\tilde{u}^i = \frac{au^i + b}{cu^i + d}, \quad i = 1, \dots, n \tag{5.19}$$

are the canonical coordinates of the bihamiltonian structure (5.1) with respect to the new representative (5.17).

Proof The result of the theorem is obtained by a straightforward calculation with the help of the formula (5.19) and the tensor rule abided by P_a^{ij}, Q_a^{ij} under the change of coordinates $u^i \mapsto \tilde{u}^i(u)$. □

From the above theorem we see that, for the bihamiltonian structure (5.1), the following 1/2-forms

$$\Omega_i = c_i(u^i)(du^i)^{\frac{1}{2}} \tag{5.20}$$

are invariant, up to a permutation, under the change of representative (5.17) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$.

6 Examples and concluding remarks.

Let us give some examples of bihamiltonian structures, their central invariants and reducing transformations.

Example 1. The Bihamiltonian structure of the KdV hierarchy [23, 39, 48] has the form

$$\begin{aligned} &\{w(x), w(y)\}_1 = \delta'(x-y), \\ &\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2} w'(x)\delta(x-y) + 3\, c\, \epsilon^2\, \delta'''(x-y), \end{aligned} \tag{6.1}$$

The canonical coordinate is $u = w$, and the constant c is the central invariant. Up to terms of the order ϵ^6 the reducing transformation [16] is given[4]

$$w = v + c\,\epsilon^2 \partial_x^2 \left(\log v_x\right) + \frac{c^2 \epsilon^4}{10} \partial_x^2 \left(5 \frac{v^{(4)}}{v_x^2} - 21 \frac{v_{xx} v_{xxx}}{v_x^3} + 16 \frac{v_{xx}^3}{v_x^4}\right) + O(\epsilon^6). \tag{6.2}$$

[4]The (inverse to) the reducing transformation for the KdV equation was constructed in [4]. However, the action of this transformation onto the Poisson pencil was not studied.

The Poisson pencils (6.1) with different values of c are inequivalent with respect to Miura-type transformations.

Example 2. The Bihamiltonian structure related to the Camassa-Holm hierarchy [7, 8, 20, 21, 22] has the expression

$$\begin{aligned} \{w(x), w(y)\}_1 &= \delta'(x-y) - \frac{\epsilon^2}{8}\delta'''(x-y), \\ \{w(x), w(y)\}_2 &= w(x)\delta'(x-y) + \frac{1}{2}\, w'(x)\delta(x-y). \end{aligned} \tag{6.3}$$

The canonical coordinate u also coincides with w, and the central invariant $c(u) = \frac{1}{24}u$. As it was shown in [37] the reducing transformation is, up to ϵ^4, given by

$$\begin{aligned} w = v &+ \epsilon^2\, \partial_x\left(\frac{v\, v_{xx}}{24\, v_x} - \frac{v_x}{48}\right) \\ &+\epsilon^4\partial_x\left(\frac{7\, v_{xx}^2}{2880\, v_x} + \frac{v\, v_{xx}^3}{180\, v_x^3} - \frac{v^2\, v_{xx}^4}{90\, v_x^5} - \frac{v_{xxx}}{512} - \frac{59\, v\, v_{xx}\, v_{xxx}}{5760\, v_x^2}\right. \\ &\left. + \frac{37\, v^2\, v_{xx}^2\, v_{xxx}}{1920\, v_x^4} - \frac{7\, v^2\, v_{xxx}^2}{1920\, v_x^3} + \frac{5\, v\, v^{(4)}}{1152\, v_x} - \frac{31\, v^2\, v_{xx}\, v^{(4)}}{5760\, v_x^3} + \frac{v^2\, v^{(5)}}{1152\, v_x^2}\right). \end{aligned} \tag{6.4}$$

Example 3. The Bihamiltonian structure related to the multi-component KdV-CH (Camassa-Holm) hierarchy. Define

$$\mathcal{D}_i = w^i\delta'(x-y) + \frac{w_x^i}{2}\delta(x-y) + a_i\frac{\epsilon^2}{8}\delta'''(x-y), \quad i = 0, 1, \dots, n \tag{6.5}$$

Here $w^0 = 1$ and $a_0, \dots, a_n$ are given constants with at least one nonzero. Define also the numbers

$$f_m^{ij} = \begin{cases} -1 & i, j \le m \\ +1 & i, j \ge m+1 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

Then we have the following $n+1$ compatible Hamiltonian structures

$$\{w^i(x), w^j(y)\}_m = (-1)^m f_m^{ij}\, \mathcal{D}_{i+j-m-1}, \quad 1 \le i, j \le n, \ m = 0, 1, \dots, n. \tag{6.7}$$

When $i < 0$ or $i > n$ we assume $\mathcal{D}_i = 0$. These Hamiltonian structures were introduced[5] in the study of the hierarchies of integrable systems (called the coupled KdV hierarchies) associated with the compatibility conditions of the linear systems of the form [1, 2, 3, 40]

$$\left(\frac{1}{2}\,(\epsilon\,\partial_x)^2 + A(w;\lambda)\right)\psi = 0, \tag{6.8}$$

$$\psi_t = \frac{1}{2}B\psi_x - \frac{1}{4}B_x\psi. \tag{6.9}$$

[5] To our best knowledge, connections of these bihamiltonian structures with the Camassa - Holm equation and its multicomponent generalizations was never considered in the literature.

Here $A(w;\lambda)$ has the expression

$$A(w;\lambda)=\frac{\sum_{i=0}^{n}(-1)^i w^i\lambda^{n-i}}{\sum_{i=0}^{n}(-1)^i a_i\lambda^{n-i}} \tag{6.10}$$

and B is certain polynomial or Laurent polynomial in λ with coefficients that are differential polynomials of $w^1,\dots,w^n$ which can be chosen according to the equation

$$A_t = AB_x+\frac{A_x}{2}B+\frac{\epsilon^2}{8}B_{xxx}. \tag{6.11}$$

As it was shown by Ferapontov in [18], if a system of hydrodynamic type with n depend variables possesses $n+1$ compatible Hamiltonian structures of hydrodynamic type, then this $n+1$-Hamiltonian structure must be equivalent to the one obtained from the leading terms of (6.7).

From (6.7) we readily have the following bihamiltonian structures

$$\mathcal{B}_{k,l}=(\{\ ,\ \}_k,\{\ ,\ \}_l),\quad k\neq l. \tag{6.12}$$

Denote by $\lambda_1(w),\dots,\lambda_n(w)$ the roots of the polynomial $P(\lambda)=\lambda^n+w^1\lambda^{n-1}+\dots+w^n$. Then the canonical coordinates for the bihamiltonian structure $\mathcal{B}_{k,l}$ are given by $u^i=(\lambda_i)^{k-l},\ i=1,\dots,n$, and the central invariants $c_1,\dots,c_n$ have the expressions

$$c_i(u^i)=\frac{\sum_{i=0}^{n}(-1)^i a_i(\lambda_i)^{n-i}}{24\,(l-k)(\lambda_i)^{n-1-l}},\quad i=1,\dots,n. \tag{6.13}$$

In particular, for the one-component case $n=1$, choosing $a_0=0, a_1=1$ we get the bihamiltonian structure $\mathcal{B}_{1,0}$ which coincides with (6.1) for the KdV hierarchy. The choice $a_0=-1, a_1=0$ yields the bihamiltonian structure (6.3) of the Camassa-Holm hierarchy. In general, we call the hierarchy generated by the bihamiltonian structure $\mathcal{B}_{k,l}$ the *multi-component KdV-CH hierarchy.*

For the case when $n=2$, the above defined bihamiltonian structure $\mathcal{B}_{2,1}$ yields, with different choices of the constants a_0,a_1,a_2 and up to certain Miura-type transformation and rescaling of the Poisson structures, the following four bihamiltonian structures that appeared in the literature. They have the same leading terms

$$\begin{aligned}&\{\varphi(x),\varphi(y)\}_1^{[0]}=0,\ \{\rho(x),\varphi(y)\}_1^{[0]}=\delta'(x-y),\ \{\rho(x),\rho(y)\}_1^{[0]}=0,\\ &\{\varphi(x),\varphi(y)\}_2^{[0]}=2\,\delta'(x-y),\ \{\rho(x),\varphi(y)\}_2^{[0]}=\varphi(x)\,\delta'(x-y),\\ &\{\rho(x),\rho(y)\}_2^{[0]}=2\rho(x)\,\delta'(x-y)+\rho'(x)\,\delta(x-y).\end{aligned} \tag{6.14}$$

A bihamiltonian structure related to the nonlinear Schrödinger hierarchy is given by the above brackets with the only difference [5, 36]

$$\{\rho(x),\varphi(y)\}_2=\{\rho(x),\varphi(y)\}_2^{[0]}+\epsilon\delta''(x-y). \tag{6.15}$$

After the Miura-type transformation

$$w^1 = 2\varphi, \quad w^2 = \varphi^2 - 4\rho + 2\epsilon\varphi_x \tag{6.16}$$

it is transformed to the bihamiltonian structure $8\,\mathcal{B}_{2,1}$ with the choice of constants $a_0 = a_1 = 0, a_2 = -8$. Here $8\,\mathcal{B}_{2,1}$ denotes the bihamiltonian structure obtained from $\mathcal{B}_{2,1}$ by the mutiplication of a overall factor 8.

In [36] a generalization of the Camassa-Holm hierarchy is introduced which is called the 2-component Camassa-Holm hierarchy. It is reduced to the usual Camassa-Holm hierarchy under a natural constraint on its two dependent variables. The related bihamiltonian structure is defined by the brackets (6.14) except

$$\{\rho(x), \varphi(y)\}_1 = \{\rho(x), \varphi(y)\}_1^{[0]} + \epsilon\delta''(x-y). \tag{6.17}$$

After the Miura-type transformation

$$w^1 = 2\varphi + 2\epsilon\varphi_x, \quad w^2 = \varphi^2 - 4\rho \tag{6.18}$$

it is converted, up to the approximation to ϵ^2, to the bihamiltonian structure $8\,\mathcal{B}_{2,1}$ with the choice of constants $a_0 = -8, a_1 = 0, a_2 = 0$.

In [29] the bihamiltonian structure for the so called classical Boussinesq hierarchy is given. It is defined by the brackets (6.14) except for

$$\{\rho(x), \rho(y)\}_2 = \{\rho(x), \rho(y)\}_2^{[0]} + \frac{1}{2}\epsilon^2\delta'''(x-y). \tag{6.19}$$

After the Miura-type transformation

$$w^1 = 2\varphi, \quad w^2 = \varphi^2 - 4\rho \tag{6.20}$$

it is transformed to the bihamiltonian structure $8\,\mathcal{B}_{2,1}$ with the choice of constants $a_0 = 0, a_1 = 0, a_2 = -8$.

Note that, for the bihamiltonian structure related to the nonlinear Schrödinger hierarchy, by moving the perturbation term from the second Poisson bracket to the first one we obtain the bihamiltonian structure of the 2-component Camassa-Holm hierarchy. Doing precisely in the same way we obtain from the above bihamiltonian structure of the classical Boussinesq hierarchy the one that is defined by (6.14) except for the bracket

$$\{\rho(x), \rho(y)\}_1 = \{\rho(x), \rho(y)\}_1^{[0]} - \frac{1}{2}\epsilon^2\delta'''(x-y). \tag{6.21}$$

After the change of dependent variables

$$w^1 = 2\varphi, \quad w^2 = \varphi^2 - 4\rho \tag{6.22}$$

it is transformed to the bihamiltonian structure $8\,\mathcal{B}_{2,1}$ with the choice of constants $a_0 = 0, a_1 = 8, a_2 = 0$. This bihamiltonian structure is related to the Ito type equations [28, 33].

The bihamiltonian structures related to the nonlinear Schrödinger hierarchy and the classical Boussinesq hierarchy are equivalent. Indeed, their central invariants are given by $c_1 = c_2 = \frac{1}{24}$. The central invariants for the bihamiltonian structure related to the 2-component Camassa-Holm hierarchy are given by $c_1 = \frac{(u^1)^2}{24}$, $c_2 = \frac{(u^2)^2}{24}$, and those for the bihamiltonian structure defined by (6.14) and (6.21) have the form $c_1 = \frac{u^1}{24}$, $c_2 = \frac{u^2}{24}$.

We omit here the presentation of the reducing transformations of the above bihamiltonian structures due to their cumbersome expressions.

Example 4. The equations of motion of one-dimensional isentropic gas with the equation of state $p = \frac{\kappa}{\kappa+1}\rho^{\kappa+1}$ read

$$u_t + \left(\frac{u^2}{2} + \rho^\kappa\right)_x = 0, \quad \rho_t + (\rho\, u)_x = 0. \tag{6.23}$$

Here κ is an arbitrary parameter, $\kappa \neq 0,\, -1$. For a gas with m degrees of freedom one has

$$\kappa = \frac{2}{m}$$

(see, e.g., [9]). This is a weakly simmetrizable system with

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives the first Poisson structure of the equations with the Poisson brackets

$$\{u(x), \rho(y)\}_1^{[0]} = \delta'(x-y), \tag{6.24}$$

other brackets vanish. The second hamiltonian structure

$$\begin{aligned}
\{u(x), u(y)\}_2^{[0]} &= 2\rho^{\kappa-1}(x)\,\delta'(x-y) + \rho_x^{\kappa-1}\,\delta(x-y),\\
\{u(x), \rho(y)\}_2^{[0]} &= u(x)\,\delta'(x-y) + \frac{1}{\kappa}\,u'(x)\,\delta(x-y),\\
\{\rho(x), \rho(y)\}_2^{[0]} &= \frac{1}{\kappa}\left(2\,\rho(x)\,\delta'(x-y) + \rho'(x)\,\delta(x-y)\right).
\end{aligned} \tag{6.25}$$

was found in [42, 43].

As it was shown in [16] the isentropic gas equations have the following deformation which preserves the bihamiltonian property (up to corrections[6] of order ϵ^6):

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \partial_x \Bigg\{ \frac{u^2}{2} + \rho^\kappa + \epsilon^2 \left[\frac{\kappa(\kappa-2)}{8}\rho^{\kappa-3}\,\rho_x^2 + \frac{\kappa^2}{12}\,\rho^{\kappa-2}\rho_{xx}\right] \\
&\quad + \epsilon^4(\kappa-2)(\kappa-3)\,\big[a_1\,\rho^{-4}\,u_x^2\,\rho_x^2 + a_2\,\rho^{\kappa-6}\,\rho_x^4 + a_3\,\rho^{-3}\,u_{xx}\,u_x\,\rho_x \\
&\qquad + a_4\,\rho^{-2}\,u_{xx}^2 + a_5\,\rho^{-3}\,u_x^2\,\rho_{xx} + a_6\,\rho^{\kappa-5}\,\rho_x^2\,\rho_{xx} + a_7\,\rho^{\kappa-4}\,\rho_{xx}^2 + a_8\,\rho^{-2}\,u_x\,u_{xxx} \\
&\qquad\quad + a_9\,\rho^{\kappa-4}\,\rho_x\,\rho_{xxx} + a_{10}\rho^{-\kappa-2}u_x^4\big] + \epsilon^4\frac{\kappa\,(\kappa^2-4)}{360}\rho^{\kappa-3}\,\rho_{xxxx}\Bigg\} = \mathcal{O}(\epsilon^6),
\end{aligned}$$

[6] In principle one can continue the expansions till an arbitrary order in ϵ. However, the computations become very involved.

$$
\frac{\partial\rho}{\partial t}+\partial_x\left\{\rho\,u+\epsilon^2\,\left(\frac{(2-\kappa)(\kappa-3)}{12\,\kappa\,\rho}\,u_x\,\rho_x+\frac{1}{6}\,u_{xx}\right)\right.
$$

$$
\begin{aligned}
&+\epsilon^4(\kappa-2)(\kappa-3)\left[b_1\,\rho^{-4}\,u_x\,\rho_x^3+b_2\,\rho^{-3}\,\rho_x^2\,u_{xx}+b_3\,\rho^{-3}\,u_x\,\rho_x\,\rho_{xx}\right.\\
&\quad+b_4\,\rho^{-2}\,u_{xx}\,\rho_{xx}+b_5\,\rho^{-2}\,u_{xxx}\,\rho_x+b_6\,\rho^{-2}\,u_x\,\rho_{xxx}+b_7\,\rho^{-1}\,u_{xxxx}\\
&\quad\left.\left.+b_8\,\rho^{-\kappa-1}u_x^2u_{xx}+b_9\,\rho^{-\kappa-2}u_x^3\rho_x\right]\right\}=\mathcal{O}(\epsilon^6)
\end{aligned}
\tag{6.26}
$$

The coefficients are given by

$$
\begin{aligned}
a_1&=\frac{18+75\,\kappa-15\,\kappa^2+20\,\kappa^3+2\kappa^4}{2880\,\kappa^3},\quad a_2=\frac{6+113\,\kappa+409\,\kappa^2-185\,\kappa^3+17\,\kappa^4}{5760\,\kappa^2}\\
a_3&=-\frac{18+11\,\kappa+3\,\kappa^2}{720\,\kappa^2},\quad a_4=\frac{7}{720\,\kappa},\quad a_5=\frac{-6+3\,\kappa-\kappa^2}{480\,\kappa^2}\\
a_6&=\frac{-6-39\,\kappa-10\,\kappa^2+5\,\kappa^3}{480\,\kappa},\quad a_7=\frac{14+5\,\kappa+5\,\kappa^2}{1440},\quad a_8=\frac{1}{120\,\kappa}\\
a_9&=\frac{2+5\,\kappa}{240},\quad a_{10}=-\frac{(\kappa+2)(\kappa+3)(\kappa^2-1)}{5760\,\kappa^4}.
\end{aligned}
\tag{6.27}
$$

$$
\begin{aligned}
b_1&=\frac{42+83\,\kappa-53\,\kappa^2+8\,\kappa^3}{1440\,\kappa^3},\quad b_2=-\frac{6+35\,\kappa-24\,\kappa^2+5\,\kappa^3}{720\,\kappa^3}\\
b_3&=-\frac{12+40\,\kappa-13\,\kappa^2+5\,\kappa^3}{720\,\kappa^3},\ b_4=\frac{6-4\,\kappa+\kappa^2}{180\,\kappa^2},\ b_5=\frac{6+\kappa+\kappa^2}{720\,\kappa^2}\\
b_6&=\frac{6+\kappa+\kappa^2}{720\,\kappa^2},\ b_7=-\frac{1}{360\,\kappa},\quad b_8=-\frac{(\kappa+2)(\kappa+3)}{720\,\kappa^4},\\
b_9&=\frac{(\kappa+1)(\kappa+2)(\kappa+3)}{1440\,\kappa^4}.
\end{aligned}
\tag{6.28}
$$

The corresponding bihamiltonian structure (at the approximation up to ϵ^4) is given in Section 4.2.3 of [16], the central invariants are $c_1=c_2=\frac{1}{24}$. The above system can be represented as

$$
\begin{aligned}
\frac{\partial u}{\partial t}&=\{H_1,u(x)\}_1=\frac{\kappa}{\kappa+1}\{H_2,u(x)\}_2,\\
\frac{\partial\rho}{\partial t}&=\{H_1,\rho(x)\}_1=\frac{\kappa}{\kappa+1}\{H_2,\rho(x)\}_2.
\end{aligned}
\tag{6.29}
$$

Here the densities h_1,h_2 of the Hamiltonians H_1,H_2 have the expressions

$$
\begin{aligned}
h_1&=\frac{1}{2}\rho\,u^2+\frac{\rho^{\kappa+1}}{\kappa+1}+\Delta h_1\\
h_2&=\rho\,u+\Delta h_2
\end{aligned}
$$

where

$$\Delta h_1 = u\,\Delta h_2 - \frac{\epsilon^2}{24\kappa}\,\left((\kappa^2-3\,\kappa+6)\,u_x^2+\kappa(2\,\kappa^2-5\,\kappa+6)\,\rho^{\kappa-2}\,\rho_x^2\right)$$
$$+\epsilon^4\,\frac{(\kappa-2)\,(\kappa-3)}{240\,\kappa^3}\left(-\frac{1}{3}\,\kappa\,\left(\kappa^2-4\,\kappa+6\right)\,\rho^{-1}\,u_x\,u_{xxx}\right.$$
$$+\frac{1}{3}\,\kappa\,\left(2\,\kappa^2-13\,\kappa+12\right)\,\rho^{-2}\,u_x\,u_{xx}\,\rho_x+\frac{1}{72}\,\frac{(3\,\kappa+5)\,(\kappa+3)\,(\kappa+2)}{\kappa}\,\rho^{-\kappa-1}\,u_x^4$$
$$-\frac{1}{12}\,(2\,\kappa-3)\,(\kappa+3)\,(\kappa+2)\,\rho^{-3}\,u_x^2\,\rho_x^2+\frac{\kappa^2\,(\kappa-1)\,(3\,\kappa^2-8\,\kappa+12)}{2(\kappa-3)}\,\rho^{\kappa-3}\,\rho_{xx}^2$$
$$\left.-\frac{1}{72}\,\kappa\,(\kappa-1)\left(16\,\kappa^4-100\,\kappa^3+229\,\kappa^2-211\,\kappa+6\right)\,\rho^{\kappa-5}\,\rho_x^4\right), \tag{6.30}$$

$$\Delta h_2 = -\epsilon^2\,\frac{(\kappa-2)\,(\kappa-3)}{12\,\kappa}\rho^{-1}\,u_x\,\rho_x$$
$$+\epsilon^4\,\frac{(\kappa-2)\,(\kappa-3)}{720\,\kappa^3}\left[-2\,\kappa\,\left(\kappa^2-8\,\kappa+6\right)\,\rho^{-2}\,u_x\,\rho_{xxx}\right.$$
$$+\kappa\,\left(7\,\kappa^2-61\,\kappa+42\right)\,\rho^{-3}\,u_x\,\rho_x\,\rho_{xx}+(-5\,\kappa^3+\frac{79}{2}\,\kappa^2-\frac{55}{2}\,\kappa+3)\,\rho^{-4}\,u_x\,\rho_x^3$$
$$\left.+\frac{1}{6\kappa}\,(\kappa+3)\,(\kappa+2)\,(\kappa+1)\,\rho^{-k-2}\,u_x^3\,\rho_x\right]. \tag{6.31}$$

To write down the reducing transformation of the perturbed system of the one dimensional isentropic gas and its bihamiltonian structure, we introduce the operators $\mathcal{T}_1, \mathcal{T}_2$

$$\mathcal{T}_1\,u^{(m)} = u^{(m+1)},\quad \mathcal{T}_1\,\rho^{(m)} = \rho^{(m+1)},$$
$$\mathcal{T}_2\,u^{(m)} = \partial_x^m\left(\kappa\rho^{\kappa-2}\,\rho_x\right),\quad \mathcal{T}_2\,\rho^{(m)} = u^{(m+1)},\ m\geq 0. \tag{6.32}$$

We will use Greek subscripts for the result of acting of the operators on the functions ρ and u, i.e.

$$\rho_{\alpha_1\alpha_2\ldots} := \mathcal{T}_{\alpha_1}\mathcal{T}_{\alpha_2}\ldots\rho,\quad u_{\alpha_1\alpha_2\ldots} := \mathcal{T}_{\alpha_1}\mathcal{T}_{\alpha_2}\ldots u.$$

Define the functions

$$\mathcal{F}_1 = \frac{1}{24}\log\left(\kappa\rho^{\kappa-2}\rho_x^2-u_x^2\right)-\frac{1}{24}\frac{(\kappa-2)(\kappa-3)}{\kappa}\log\rho, \tag{6.33}$$

$$
\begin{aligned}
\mathcal{F}_2 &= \frac{1}{1152}\,\rho_{\alpha_1\alpha_2\alpha_3\alpha_4} M^{\alpha_1\alpha_2}\,M^{\alpha_3\alpha_4} - \frac{1}{360}\,\rho_{\alpha_1\alpha_2\alpha_3}\rho_{\alpha_4\alpha_5\alpha_6} M^{\alpha_1\alpha_4}\,M^{\alpha_2\alpha_5}\,M^{\alpha_3\alpha_6}\\
&-\frac{1}{1152}\,\rho_{\alpha_1\alpha_2}\rho_{\alpha_3\alpha_4\alpha_5\alpha_6} M^{\alpha_1\alpha_3}\,M^{\alpha_2\alpha_4}\,M^{\alpha_5\alpha_6}\\
&+\frac{1}{360}\,\rho_{\alpha_1\alpha_2}\rho_{\alpha_3\alpha_4\alpha_5}\rho_{\alpha_6\alpha_7\alpha_8} M^{\alpha_1\alpha_3}\,M^{\alpha_2\alpha_6}\,M^{\alpha_4\alpha_7}\,M^{\alpha_5\alpha_8}\\
&+(\kappa-2)(\kappa-3)\,D^{-2}\left[-\frac{1}{240}\kappa\,\rho^{2k-5}\,\rho_{xxx}\,\rho_x^3 + \frac{11}{2880}\kappa\,\rho^{2\kappa-5}\,\rho_{xx}^2\,\rho_x^2\right.\\
&+\left(-\frac{7}{5760}\kappa^2+\frac{19}{5760}\kappa+\frac{7}{960}\right)\,\rho^{2\kappa-6}\,\rho_{xx}\,\rho_x^4 + \frac{11}{2880}\,\rho^{\kappa-3}\,\rho_x^2\,u_{xx}^2\\
&-\frac{1}{5760\kappa}(\kappa^4-9\kappa^3+\kappa^2+53\kappa+6)\,\rho^{2\kappa-7}\,\rho_x^6 + \frac{1}{240}\,\rho^{\kappa-3}\,\rho_x^2\,u_{xxx}\,u_x\\
&+\frac{1}{240}\,\rho^{\kappa-3}\,\rho_x\,\rho_{xxx}\,u_x^2 - \frac{11}{720}\,\rho^{\kappa-3}\,\rho_x\,u_x\,u_{xx}\,\rho_{xx} + \frac{11}{2880}\,\rho^{\kappa-3}\,u_x^2\,\rho_{xx}^2\\
&-\frac{1}{1440}(11\kappa-21)\,\rho^{\kappa-4}\,\rho_x^3\,u_{xx}\,u_x + \frac{1}{2880\kappa}(22\kappa^2-47\kappa-42)\,\rho^{\kappa-4}\,\rho_x^2\,u_x^2\,\rho_{xx}\\
&+\frac{1}{5760\kappa^2}(12\kappa^4-45\kappa^3+15\kappa^2+101\kappa+6)\,\rho^{\kappa-5}\,u_x^2\,\rho_x^4 - \frac{1}{240\kappa}\,\rho^{-1}\,u_{xxx}\,u_x^3\\
&+\frac{11}{2880\kappa}\,\rho^{-1}\,u_x^2\,u_{xx}^2 + \frac{1}{1440\kappa}\,\rho^{-2}\,u_{xx}\,u_x^3\,\rho_x + \frac{1}{5760\kappa^2}(7\kappa^2-13\kappa+42)\,\rho^{-2}\,\rho_{xx}\,u_x^4\\
&\left.-\frac{1}{5760\kappa^3}(8\kappa^3-31\kappa^2+43\kappa-6)\,\rho^{-3}\,u_x^4\,\rho_x^2 - \frac{1}{5760\kappa^4}(\kappa+3)(\kappa+2)\,\rho^{-\kappa-1}\,u_x^6\right]
\end{aligned}
$$

Here the matrix $M(M^{\alpha\beta})$ and differential polynomial D read

$$
M = D^{-1}\begin{pmatrix} -\kappa\rho^{\kappa-2}\,\rho_x & u_x\\ u_x & -\rho_x\end{pmatrix},\quad D = u_x^2 - \kappa\,\rho^{\kappa-2}\rho_x^2, \tag{6.34}
$$

Then the reducing transformation is given by the formula

$$
u \mapsto u + \mathcal{T}_1\mathcal{T}_2\left(\epsilon^2\mathcal{F}_1+\epsilon^4\mathcal{F}_2\right),\quad \rho\mapsto\rho+\mathcal{T}_1\mathcal{T}_1\left(\epsilon^2\mathcal{F}_1+\epsilon^4\mathcal{F}_2\right). \tag{6.35}
$$

We leave as an exercise for the reader to check that the denominator $D\neq 0$ on the monotone solutions.

In conclusion let us formulate some open problems.

Problem 1. Study convergence of the reducing transformations for the case of analytic in ϵ perturbations (1.7) (e.g., for the case of polynomial dependence on ϵ).

Problem 2. Are there more wide classes of perturbations of systems of hyperbolic PDEs admitting reducing transformation? The natural candidate to be considered is the perturbations of the so-called *semi-hamiltonian* systems in the Tsarev's sense [45], i.e., hyperbolic systems written in the diagonal form and possessing a complete family of commuting flows.

Problem 3. According to our results, classes of equivalence of semisimple bihamiltonian structures depend at most on $n(n+1)$ arbitrary functions of one variable. Prove existence of such bihamiltonian structures for an arbitrary choice of these functional parameters.

Appendix: Bihamiltonian structures of hydrodynamic type

In this Appendix we will describe in more details, following [17], the defining equations for semisimple bihamiltonian structures of hydrodynamic type as well as their Lax pair representation.

We will work in the canonical coordinates u^1, ..., u^n (see Lemma 1.5 above). Introduce the classical *Lamé coefficients*

$$H_i(u) := f_i^{-1/2}(u), \quad i = 1, \dots, n$$

and the *rotation coefficients*

$$\gamma_{ij}(u) := H_i^{-1} \partial_i H_j, \quad i \neq j. \tag{A.1}$$

Here, as usual

$$\partial_i = \frac{\partial}{\partial u^i},$$

no summation over repeated indices will be assumed within this section. The classical *Lamé equations*

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i,\ j,\ k \text{ distinct} \tag{A.2}$$

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i,j} \gamma_{ki}\gamma_{kj} = 0, \quad i \neq j \tag{A.3}$$

describe diagonal metrics of curvature zero[7]. Adding the equations

$$u^i \partial_i \gamma_{ij} + u^j \partial_j \gamma_{ji} + \sum_{k \neq i,j} u^k \gamma_{ki}\gamma_{kj} + \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) = 0, \quad i \neq j \tag{A.4}$$

one obtains the defining relations for semisimple Poisson pencils of hydrodynamic type. The solutions to the system (A.2) - (A.4) are parametrized by $n(n-1)$ arbitrary functions of one variable. Indeed, one can freely choose the functions

$$\gamma_{ij}(u_0^1, \dots, u^j, \dots, u_0^n)$$

near a given point

$$u_0 = (u_0^1, \dots, u_0^n), \quad u_0^i \neq u_0^j, \quad u_0^i \neq 0. \tag{A.5}$$

[7] Integrability of the system (A.2) - (A.3) was discovered by V.Zakharov[47].

The equations (A.2) - (A.4) can be represented as the compatibility conditions of the linear system

$$\begin{aligned}
&\partial_i \psi_j = \gamma_{ji} \psi_i, \quad i \neq j \\
&\partial_i \psi_i + \sum_{k \neq i} \gamma_{ki} \frac{u^k - \lambda}{u^i - \lambda} \psi_k + \frac{1}{2\,(u^i - \lambda)} \psi_i = 0.
\end{aligned} \tag{A.6}$$

("Lax pair" with the spectral parameter λ for (A.2) - (A.4)). The solutions to the linear system (A.6) are closely related with the common first integrals of the bihamiltonian systems of hydrodynamic type, i.e. with the Casimirs of the Poisson pencil

$$\begin{aligned}
&\{\ , I\}_2^{[0]} - \lambda \{\ , I\}_1^{[0]} = 0, \quad I = \int P(u)\, dx \\
&\partial_i P(u) = \psi_i H_i, \quad i = 1, \ldots, n.
\end{aligned} \tag{A.7}$$

As we already know (see Lemma 1.6 above) the bihamiltonian systems are all diagonal in the canonical coordinates

$$u_t^i + V^i(u) u_x^i = 0, \quad i = 1, \ldots, n. \tag{A.8}$$

The characteristic velocities $V^i(u)$ are determined from the following linear system

$$\partial_k \chi_i = \gamma_{ki} \chi_k, \quad i \neq k \tag{A.9}$$

$$\chi_i = H_i V^i, \quad i = 1, \ldots, n. \tag{A.10}$$

For the given rotation coefficients $\gamma_{ij}(u)$ satisfying (A.2) - (A.4) the solutions to (A.9) can be reconstructed with ambiguity of n arbitrary functions of one variable. In particular, the Lame coefficients $\chi_i = H_i(u)$ give a solution to (A.9). They correspond to the spatial translations $V^i(u) \equiv 1$, $i = 1, \ldots, n$.

Finally, to reconstruct the flat pencil of metrics starting from given solution to (A.2) - (A.4) near a given point (A.5) one has to choose a solution $\chi_1(u), \ldots, \chi_n(u)$ such that

$$\chi_i(u_0) \neq 0, \quad i = 1, \ldots, n.$$

Then we put

$$g_1^{ij}(u) = \chi_i^{-2}(u) \delta_{ij}, \quad g_2^{ij}(u) = u^i \chi_i^{-2}(u) \delta_{ij}. \tag{A.11}$$

The flat coordinates of the metrics correspond to particular solutions of the system (A.6). Namely, to find flat coordinates for the first metric one has to choose a fundamental system of solutions

$$\psi_i^\alpha(u), \quad \alpha = 1, \ldots, n, \quad \det\left(\psi_i^\alpha(u_0)\right) \neq 0$$

to the following linear overdetermined system

$$\begin{aligned}
&\partial_i \psi_j = \gamma_{ji} \psi_i, \quad i \neq j \\
&\partial_i \psi_i + \sum_{k \neq i} \gamma_{ki} \psi_k = \frac{1}{2} \psi_i
\end{aligned} \tag{A.12}$$

obtained from (A.6) at $\lambda = \infty$. Then the flat coordinates v^α are defined by quadratures

$$dv^\alpha = \sum_{i=1}^n \chi_i \psi_i^\alpha du^i, \quad \alpha = 1, \dots, n. \tag{A.13}$$

Flat coordinates for the second metric are constructed in a similar way by using a fundamental system of solutions to (A.6) at $\lambda = 0$.

We deduce that semisimple bihamiltonian structures of hydrodynamic type with n dependent variables are parametrized by n^2 arbitrary functions of one variable. For $n \leq 2$ the equations (A.2) - (A.4) are linear. So an explicit parametrization of the Poisson pencils is available [41]. The equations become nonlinear starting from $n \geq 3$. All known so far nontrivial solutions are obtained within the theory of Frobenius manifolds. In this case the rotations coefficients are symmetric

$$\gamma_{ji} = \gamma_{ij}$$

(the so-called *Egoroff metrics*) and the equations (A.2) - (A.4) are reduced to isomonodromy deformations [12]. We will study more general case in subsequent publications.

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