

Modulation of Camassa–Holm equation and reciprocal transformations

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Abstract

We derive the modulation equations or Whitham equations for the Camassa–Holm (CH) equation. We show that the modulation equations are hyperbolic and admit bi-Hamiltonian structure. Furthermore they are connected by a reciprocal transformation to the modulation equations of the first negative flow of the Korteweg de Vries (KdV) equation. The reciprocal transformation is generated by the Casimir of the second Poisson bracket of the KdV averaged flow. We show that the geometry of the bi-Hamiltonian structure of the KdV and CH modulation equations is quite different: indeed the KdV averaged bi-Hamiltonian structure can always be related to a semisimple Frobenius manifold while the CH one cannot.

1 Introduction

In 1993 R. Camassa and D. Holm [7] proposed a new equation

$$u_t + 3uu_x = (u_{xxt} + 2u_x u_{xx} + uu_{xxx}) - 2\nu u_x, \quad (1)$$

with ν a constant parameter, deriving it as the governing equation for waves in shallow water when surface tension is present. This involves an asymptotic expansion in small amplitude of the incompressible Euler equation for unidirectional motion under the influence of gravity that extends one order beyond the Korteweg-de Vries (KdV) equation. (1) is also an element in a class of equations introduced by A.Fokas and B.Fuchssteiner [20] through the method of recursion operators in 1981.

Equation (1) is strongly nonlinear, admits a bi-Hamiltonian structure [20], a Lax pair [7] and it is formally integrable through the inverse scattering method [9]. The bi-Hamiltonian structure of the CH equation can be described as follows

$$m_t = -P_1 \frac{\delta H_2}{\delta m}, \quad m = u - u_{xx}, \quad P_1 = \partial_x - \partial_x^3,$$

with

$$H_2 = \frac{1}{2} \int (u^3 + uu_x^2 + 2\nu u^2) dx, \quad (2)$$

or

$$m_t = -P_2 \frac{\delta H_1}{\delta m}, \quad P_2 = \partial_x m + m \partial_x + 2\nu \partial_x,$$

and

$$H_1 = \frac{1}{2} \int (u^2 + u_x^2) dx. \quad (3)$$

The bi-Hamiltonian structure implies that the CH equation has an infinite number of conserved quantities. The functionals H_k , $k \in \mathbb{Z}$, defined by

$$P_2 \frac{\delta H_k}{\delta m} = P_1 \frac{\delta H_{k+1}}{\delta m}, \quad k \in \mathbb{N}, \quad (4)$$

are conserved quantities in involution with respect to the Poisson bracket determined either by P_1 or P_2 . The Hamiltonian $H_0 = \int m dx$ is the Casimir of the first Poisson tensor P_1 . For $k > 2$ the Hamiltonian densities of H_k are not local functions of u and its spatial derivatives.

In the case $\nu = 0$, R. Camassa and D. Holm [7] proved the existence of solutions that are continuous but only piece-wise analytic (peakons). The CH equation possesses soliton solutions, periodic finite-gap solutions [8, 10, 4], real finite-gap solutions [22] and, for $\nu = 0$, multi-peakons [6]. In particular, the algebro-geometric solutions of (CH) are described as Hamiltonian flows on nonlinear subvarieties (strata) of generalized Jacobians. This implies that the associated finite dimensional integrable systems may be described in the framework of integrable systems with deficiency [36, 1] whose algebraic-geometrical structure has much in common with the

celebrated algebraically completely integrable systems introduced and thoroughly studied by M. Adler and P. van Moerbeke [3].

In this work we derive the Whitham modulation equations for the CH flows. Whitham modulation equations for a nonlinear evolution system describe slow modulations of parameters over a family of periodic travelling wave solutions (or families of multi-phase solutions which are so far known to exist only for integrable systems). Contrary to the Korteweg de Vries case [26], it is an open problem to show that the Cauchy problem for CH with slowly varying initial data is described by the Whitham equations. Both for KdV and CH equations, this approximation is physically meaningful when the ratio between the water depth and the wavelength is very small [15].

The Whitham equations are a system of hydrodynamic type equations [12] and in the Riemann invariant coordinates take the form

$$u_t^i = v^i(\mathbf{u})u_x^i, \quad i = 1, \dots, N, \quad \mathbf{u} = (u_1, \dots, u_N).$$

The original evolution system is usually Lagrangian or Hamiltonian and this property is usually inherited by the equations of slow modulations. To average the original equations in the Lagrangian form, Whitham [37] introduced the pseudo-phase method and then he constructed the corresponding Hamiltonian structure. For local Hamiltonian structures, B.A. Dubrovin and S.P. Novikov [12] introduced a procedure for averaging local Poisson brackets and obtained the corresponding modulation equations. A third method to derive the Whitham modulation equations is the nonlinear analog of the WKB method [11]. It can be proven that the three methods lead to the same equations for the case in which the original equation has a local Hamiltonian structure and local Hamiltonian densities (see [12] and references therein).

The Camassa-Holm equation does not possess a local Hamiltonian structure: indeed in the variable u the Hamiltonian operator is strongly nonlocal and the Hamiltonian densities of H_k are non-local for $k > 2$. In the variable m the Hamiltonian densities of H_k are nonlocal for $k > 0$. Therefore the CH equation does not fit into the method of averaging local Hamiltonian structure [12] nor even in the Maltsev-Novikov method of averaging weakly nonlocal Hamiltonian structures [28] (an Hamiltonian structure is weakly nonlocal if it is polynomial in ∂_x and its higher derivatives and linear in ∂_x^{-1}) or in the Maltsev method [29] of averaging weakly non-local symplectic form (inverse of the Hamiltonian operator). The latter method applies to Camassa-Holm only when averaging one-phase solutions [30].

The CH equation can be written as a local Lagrangian system and we use the Whitham method (modulation equations in Lagrangian form) to derive the modu-

lation equations for the one-phase periodic solution. Then following Hayes [25] and Whitham [37] the equations can be written in Hamiltonian form with a local Poisson bracket of Dubrovin-Novikov type

$$u_t^i = v^i(\mathbf{u})u_x^i = A^{ij} \frac{\partial h}{\partial u^j}$$

where

$$A^{ij} = g^{ii} \delta^{ij} \frac{d}{dx} - g^{ii} \Gamma_{ik}^j u_x^k \quad (5)$$

is the Hamiltonian operator and h the Hamiltonian density. As pointed out by Dubrovin and Novikov, A^{ij} defines a Hamiltonian operator if and only if $g^{ii} = g^{ii}(\mathbf{u})$ is a flat non degenerate metric and Γ_{ik}^j are the Christoffel symbols of the corresponding Levi-Civita connection. We also find a second local compatible Hamiltonian structure which is obtained from the flat metric $g^{ii}(\mathbf{u})(u^i + \nu)$ where ν is the constant in the CH equation. Therefore the nonlocal bi-Hamiltonian structure of the original CH equation averages to a local bi-Hamiltonian structure of Dubrovin-Novikov type.

A reciprocal transformation is a closed form which changes the independent variables of the equation and maps conservation laws into conservation laws, but it does not preserve the Poisson structure as shown by E.V. Ferapontov and M.V. Pavlov [18],[17]. The Camassa Holm equation can be transformed by a reciprocal transformation into the first negative flow of the KdV hierarchy [21] (also known as AKNS equation [2]). An elegant treatment of the relations among positive and negative flows of the CH and KdV hierarchies can be found in [27].

Let g_{ii}^{KdV} and g_{ii}^{KdV}/β^i be the flat compatible metrics associated to the bi-Hamiltonian structure of the KdV modulation equations [35][13] with respect to the usual Riemann invariants $\beta^1, \beta^2, \beta^3$ as defined in [37]. Then the reciprocal transformation is generated by the Casimir \mathcal{K}_0 of the Hamiltonian operator associated to the metric g_{ii}^{KdV}/β^i . According to the results in [18],[17] the reciprocal transformation maps the two KdV flat metrics to the CH metrics

$$\frac{g_{ii}^{KdV}}{\mathcal{K}_0^2}, \quad \frac{g_{ii}^{KdV}}{\mathcal{K}_0^2 \beta^i}$$

which are not flat. The relation between the CH Riemann invariants and the KdV Riemann invariants is $\beta^i = 1/(u^i + \nu)$. The corresponding CH modulation equations are Hamiltonian with respect to two non-local operators of Mokhov-Ferapontov and Ferapontov type [16],[17] which are of the form (5) plus a nonlocal tail. However

from the Lagrangian averaging, we independently prove the existence of one local Hamiltonian structure. We show that the two metrics

$$\frac{g_{ii}^{KdV}}{\mathcal{K}_0^2(\beta^i)^2}, \quad \frac{g_{ii}^{KdV}}{\mathcal{K}_0^2(\beta^i)^3}$$

are flat and define a flat pencil, that is, the CH modulation equations are bi-Hamiltonian with respect to two local Hamiltonian operator of the form (5). We remark that the two flat KdV metrics g_{ii}^{KdV} and g_{ii}^{KdV}/β^i are related to a semisimple Frobenius manifold [14]. More in general, B. Dubrovin [14] proves that, under certain assumptions, a flat pencil of contravariant metrics on a manifold induces a Frobenius structure on it. One of the assumptions is the requirement that one of the two flat metrics is of Egorov type (namely its rotation coefficients are symmetric). Since none of the two CH flat metrics have the Egorov property, there is no Frobenius structure associated to this system. Therefore, from the geometric point of view, the KdV modulation equations and the CH modulation equation belong to two different classes.

All the results presented here for the one-phase CH modulation equations may be generalized to the multi-phase case in a straightforward way. However, in the present paper we have decided to concentrate only on the one-phase case to better clarify similarities and differences with the KdV case, and we will present the discussion of the multi-phase case in a future publication.

2 Whitham modulation equations

In this section we use Lagrangian formalism to average the Camassa-Holm equation in the genus one case and refer to [12],[31] for a general exposition of the method we use. Introducing the potential

$$\phi : \quad \phi_x = u,$$

equation (1) takes the form

$$\phi_{xt} - \phi_{xxxt} + 3\phi_x\phi_{xx} - 2\phi_{xx}\phi_{xxx} - \phi_x\phi_{xxxx} + 2\nu\phi_{xx} = 0,$$

and a Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\phi_x\phi_t + \frac{1}{2}\phi_{xxx}\phi_t - \frac{1}{2}\phi_x^3 - \nu\phi_x^2 + \frac{1}{4}\phi_x^2\phi_{xxx}. \quad (6)$$

We consider 2π -periodic solutions of the form

$$u = u(\theta), \quad \theta = kx - \omega t.$$

Following Whitham [37], we introduce the pseudo-phase

$$\phi = \psi + \Phi(\theta), \quad \psi = \beta x - \gamma t, \quad \theta = kx - \omega t,$$

where $\Phi(\theta)$ is a 2π periodic function of θ with zero average. The averaged Lagrangian over the one-dimensional real torus is

$$\begin{aligned} \bar{\mathcal{L}} = \frac{1}{2\pi} \int d\theta & \left[-\frac{1}{2}(\beta + k\Phi_\theta)(\gamma - \omega\Phi_\theta) + \frac{1}{2}k^3\Phi_{\theta\theta\theta}(\gamma - \omega\Phi_\theta) - \right. \\ & \left. -\frac{1}{2}(\beta + k\Phi_\theta)^3 - \nu(\beta + k\Phi_\theta)^2 + \frac{1}{4}(\beta + k\Phi_\theta)^2k^3\Phi_{\theta\theta\theta} \right] \end{aligned}$$

Now we suppose that the constants β , k , γ and ω are slowly varying function of time, that is $\beta = \beta(X, T)$, $\gamma = \gamma(X, T)$, $k = k(X, T)$ and $\omega = \omega(X, T)$ with X and T “slow variables”. Then

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}(k, \omega, \beta, \gamma; X, T).$$

The equations of slow modulation of the parameters k, ω, β, γ are the extremals of the functional [37]

$$\int \bar{\mathcal{L}}(k, \omega, \beta, \gamma) dX dT,$$

and take the form

$$\begin{aligned} \partial_X \bar{\mathcal{L}}_k + \partial_T \bar{\mathcal{L}}_\omega &= 0, & k_T + \omega_X &= 0, \\ \partial_X \bar{\mathcal{L}}_\gamma + \partial_T \bar{\mathcal{L}}_\beta &= 0, & \beta_T + \gamma_X &= 0. \end{aligned} \tag{7}$$

As in the KdV case, substituting the fourth equation into the third in (7), the X -derivative disappears and we get a constraint. Thus the number of equations reduces from four to three.

We get to the same conclusion transforming the modulation equations in Hamiltonian form. Following Hayes [25] and Whitham [37], let us introduce the Hamiltonian density

$$\mathcal{H} = \mathcal{H}(k, \omega, \beta, \gamma) = \omega \bar{\mathcal{L}}_\omega + \gamma \bar{\mathcal{L}}_\gamma - \bar{\mathcal{L}}.$$

Then the modulation equations are Hamiltonian with respect to the canonical Poisson bracket

$$\{k(X), \bar{\mathcal{L}}_\omega(Y)\} = \delta'(X - Y), \quad \{\beta(X), \bar{\mathcal{L}}_\gamma(Y)\} = \delta'(X - Y)$$

Since

$$\bar{\mathcal{L}}_\gamma(Y) = \frac{1}{2}\beta$$

the Lagrangian $\bar{\mathcal{L}}$ degenerates and the number of fields reduces from four to three. This is connected with the Dirac reduction. Therefore the Whitham equations can be written in Hamiltonian form with a local Dubrovin-Novikov Poisson bracket [30]

$$\{k(X), \bar{\mathcal{L}}_\omega(Y)\} = \delta'(X - Y), \quad \{\beta(X), \beta(Y)\} = 2\delta'(X - Y)$$

and with Hamiltonian $\mathcal{H} = \mathcal{H}(k, J, \beta)$, $J = \bar{\mathcal{L}}_\omega$. In these variables the equations of motion are Hamiltonian

$$k_T = \partial_X \mathcal{H}_J, \quad J_T = \partial_X \mathcal{H}_k, \quad \beta_T = \partial_X \mathcal{H}_\beta. \quad (8)$$

3 Modulation equations for CH in Riemann invariant form

The modulation equations (8) can also be written in Riemann invariant form. For the purpose we introduce the spectral curve associated to the periodic travelling wave solution $u = u(kx - \omega t)$. When we plug $u(\theta)$, $\theta = kx - \omega t$, into the CH equation (1), we get, after integration,

$$k^2(c - u)u_\theta^2 + (2\nu - c)u^2 + u^3 + 2Bu - 2A = 0, \quad (9)$$

where A and B are constants of integration and $c = \omega/k$. The CH one-phase solution $u = u(kx - \omega t)$ is obtained by inverting the third kind differential

$$\int_{u_0}^u \frac{(\eta - c)d\eta}{\sqrt{(\eta - c)(\eta^3 + (2\nu - c)\eta^2 + 2B\eta - 2A)}} = kx - \omega t.$$

The inversion of the above integral is discussed in [4] where the solution $u(x, t)$ is expressed in terms of convenient generalized theta-functions in two variables, which are constrained to the generalized theta-divisor. Integration of (9) over θ yields the amplitude dependent dispersion relation for the nonlinear dispersive wave

$$k \oint \frac{(\eta - c)d\eta}{\sqrt{(\eta - c)(\eta^3 + (2\nu - c)\eta^2 + 2B\eta - 2A)}} = 2\pi, \quad (10)$$

where the integration is taken on a close path between e^2 and e^1 where $c > e^1 > e^2 > e^3$ are the roots of the polynomial

$$\eta^3 - (c - 2\nu)\eta^2 + 2B\eta - 2A,$$

with the constraint

$$2\nu = c - e^1 - e^2 - e^3.$$

Here and below, we indicate vectors with upper indices.

From now on, we will use small letters x, t for the ‘slow variables’ X, T introduced in the previous section, since we deal only with modulation equations and no ambiguity may occur.

Theorem 3.1 *The one-phase CH modulation equations (8) take the Riemann invariant form*

$$\partial_t u^i + C^i(\mathbf{u})\partial_x u^i = 0, \quad i = 1, \dots, 3, \quad (11)$$

where the Riemann invariants $\mathbf{u} = (u^1, u^2, u^3)$ are

$$u^1 = \frac{1}{2}(e^2 + e^3), \quad u^2 = \frac{1}{2}(e^1 + e^3), \quad u^3 = \frac{1}{2}(e^1 + e^2),$$

and the speeds $C^i(\mathbf{u})$ take the form

$$C^i(\mathbf{u}) = \frac{\partial_{u^i} \omega(\mathbf{u})}{\partial_{u^i} k(\mathbf{u})}. \quad (12)$$

The wave number and frequency are given by the Abelian integrals

$$k = 2\pi \left(\oint_a \frac{(\lambda + \nu)d\lambda}{\sqrt{R(\lambda)}} \right)^{-1}, \quad R(\lambda) = (\lambda + \nu)(\lambda - u^1)(\lambda - u^2)(\lambda - u^3), \quad (13)$$

$$\omega = (2\nu + u^1 + u^2 + u^3)k,$$

and the integration is taken a close cycle a passing between u^2 and u^1 (see figure 1).

The existence of the Riemann invariants and equations (12) is proven directly starting from (8) and using variational identities of Abelian integrals. To write the velocities in an explicit form, we introduce the integrals

$$I_k = \oint_a \frac{\lambda^k}{\sqrt{R(\lambda)}} d\lambda, \quad k \geq 0, \quad (14)$$

Figure 1: The homology basis.

where $R(\lambda)$ is defined in (13).

Let $\sigma_1(\lambda)$ be the normalized third kind differential with first order pole at $(\infty, \pm\infty)$ with residue ± 1 , respectively, and let σ_2 be the normalized second kind differential with second order pole at infinity, namely

$$\sigma_1(\lambda) = \frac{P_1(\lambda)d\lambda}{\sqrt{R(\lambda)}}, \quad P_1(\lambda) = \lambda + \gamma_1, \quad (15)$$

$$\sigma_2(\lambda) = \frac{P_2(\lambda)d\lambda}{\sqrt{R(\lambda)}}, \quad P_2(\lambda) = \lambda^2 - \frac{1}{2}(u^1 + u^2 + u^3 - \nu)\lambda + \gamma_2 \quad (16)$$

where the constants $\gamma_1 = -\frac{I_1}{I_0}$ and $\gamma_2 = -\frac{I_2}{I_0} + \frac{1}{2}(u^1 + u^2 + u^3 - \nu)\frac{I_1}{I_0}$ are uniquely determined by

$$\oint_a \sigma_i(\lambda) = 0, \quad i = 1, 2. \quad (17)$$

Let $\phi(\lambda)$ be the normalized holomorphic differential,

$$\phi(\lambda) = \frac{d\lambda}{I_0\sqrt{R(\lambda)}}, \quad \oint_a \phi(\lambda) = 1,$$

and let $\Omega_\nu(\lambda)$ be the second kind Abelian differential with second order pole at $\lambda = -\nu$ with asymptotic behavior for $\lambda \rightarrow -\nu$

$$\Omega_\nu(\lambda) \rightarrow \frac{dt}{t^2}, \quad t^2 = \lambda + \nu,$$

and normalized by the condition

$$\oint_a \Omega_\nu(\lambda) = 0. \quad (18)$$

The differential $\Omega_\nu(\lambda)$ is explicitly given by the expression

$$\begin{aligned}\Omega_\nu(\lambda) &= \frac{P_\nu(\lambda)d\lambda}{\sqrt{R(\lambda)}\sqrt{(-\nu-u^1)(-\nu-u^2)(-\nu-u^3)}}, \\ P_\nu(\lambda) &= \frac{-(\nu+u^1)(\nu+u^2)(\nu+u^3)}{2(\lambda+\nu)} + P_2(-\nu),\end{aligned}\tag{19}$$

where $P_2(\lambda)$ has been defined in (16).

Using the above differentials we may write the speeds in (12) explicitly.

Theorem 3.2 *The speeds $C^i(\mathbf{u})$, $i = 1, 2, 3$ defined in (12) take the form*

$$C^i(\mathbf{u}) = u^1 + u^2 + u^3 + 2\nu - \frac{P_1(-\nu)}{P_\nu(u^i)} \prod_{j \neq i, j=1}^3 (u^i - u^j), \quad i = 1, 2, 3,\tag{20}$$

where the rational function $P_1(\lambda)$ and $P_\nu(\lambda)$ are as in (15) and (19) respectively. The equations (11) are hyperbolic and satisfy

$$C^1(\mathbf{u}) < C^2(\mathbf{u}) < C^3(\mathbf{u}), \quad \text{for } -\nu < u^1 < u^2 < u^3.$$

In the limit when two Riemann invariants coalesce, the modulation equations reduce to the dispersionless CH equation

$$\partial_t u + (3u + 2\nu)\partial_x u = 0.$$

To prove (20), we observe that the wave number k defined in (13) takes the form

$$k = \frac{\phi(\lambda)}{\sigma_1(\lambda)} \Big|_{\lambda=-\nu} = \frac{\oint_b \Omega_\nu(\lambda)}{\sigma_1(-\nu)},\tag{21}$$

where

$$\sigma_1(-\nu) := \frac{\sigma_1(\lambda)}{dt} \Big|_{\lambda=-\nu}, \quad t^2 = \lambda + \nu,$$

and ϕ , σ_1 and Ω_ν have been defined in (15) and (19). The following variational formulas hold [23]

$$\frac{\partial}{\partial u^i} \oint_b \Omega_\nu(\lambda) = \frac{1}{2} \phi(u^i) \Omega_\nu(u^i), \quad \frac{\partial}{\partial u^i} \sigma_1(-\nu) = \frac{1}{2} \sigma_1(u^i) \Omega_\nu(u^i),\tag{22}$$

where

$$\phi(u^i) := \frac{\phi(\lambda)}{dt} \Big|_{\lambda=u^i}, \quad \Omega_\nu(u^i) := \frac{\Omega_\nu(\lambda)}{dt} \Big|_{\lambda=u^i}, \quad t^2 = \lambda - u^i. \quad (23)$$

Inserting (21) and (22) into (12), we finally obtain the explicit expressions (20).

We now prove hyperbolicity. Since $-\nu < u^1 < u^2 < u^3$, $\lim_{\lambda \rightarrow -\nu^+} P_\nu(\lambda) = -\infty$, and, by monotonicity, there is only one point $\lambda^* > -\nu$ for which

$$\frac{(\nu + u^1)(\nu + u^2)(\nu + u^3)}{2(\lambda^* + \nu)} = P_2(-\nu);$$

moreover, because of (18), the point λ^* satisfies the inequality $u^1 < \lambda^* < u^2$. Therefore

$$P_\nu(u^1) < 0, \quad P_\nu(u^3) > P_\nu(u^2) > 0.$$

In the same way, using (17) we conclude that $P_1(-\nu) < 0$. Using the above inequalities, it is straightforward to verify that

$$C^1(\mathbf{u}) < C^2(\mathbf{u}) < C^3(\mathbf{u}), \quad -\nu < u^1 < u^2 < u^3.$$

We end the proof of the theorem studying the behavior of the speeds $C^i(\mathbf{u})$, $i = 1, 2, 3$, when two of the Riemann invariants coalesce. In the limiting case $u^2 \rightarrow u^3$, we have

$$P_1(\lambda)|_{u^2 \rightarrow u^3} = \lambda - u^3, \quad P_\nu(\lambda)|_{u^2 \rightarrow u^3} = \frac{1}{2(\lambda + \nu)}(\lambda - u^3)(u^1 + \nu)(u^3 + \nu),$$

so that

$$\lim_{u^2 \rightarrow u^3} \frac{P_1(-\nu)}{P_\nu(u^1)}(u^1 - u^2)(u^1 - u^3) = 2(u^3 - u^1)$$

and

$$\lim_{u^2 \rightarrow u^3} \frac{P_1(-\nu)}{P_\nu(u^2)}(u^2 - u^1)(u^2 - u^3) = \lim_{u^2 \rightarrow u^3} \frac{P_1(-\nu)}{P_\nu(u^3)}(u^3 - u^2)(u^3 - u^1).$$

Combining the above relations, we have

$$C^2(u^1, u^3, u^3) = C^3(u^1, u^3, u^3), \quad C^1(u^1, u^3, u^3) = 3u^1 + 2\nu,$$

which gives the dispersionless CH equation $\partial_t u^1 + (3u^1 + 2\nu)\partial_x u^1 = 0$. In the same way, it can be proved that, in the limit $u^2 = u^1$, the speeds $C^1(u^1, u^2, u^3) = C^2(u^1, u^2, u^3)$ and $C^3(u^1, u^1, u^3) = 3u^3 + 2\nu$.

3.1 Hamiltonian structure and integration

In this section, we investigate the bi-Hamiltonian structure of the one-phase Whitham equations

$$\partial_t u^i + C^i(\mathbf{u}) \partial_x u^i = 0, \quad i = 1, \dots, 3,$$

with $C^i(\mathbf{u})$ as in (12) or (20). In section 2 we have proven that the above equations are Hamiltonian with respect to a canonical Poisson bracket. In the following we show that the CH modulation equations are bi-Hamiltonian with respect to local Poisson brackets of Dubrovin-Novikov type.

Proposition 3.3 *The speeds $C^i(\mathbf{u})$, $i = 1, 2, 3$, satisfy the following relations*

$$\frac{\partial_{u^j} C^i}{C^j - C^i} = \partial_{u^j} \log \sqrt{g_{ii}}, \quad (24)$$

where $g_{ii} = g_{ii}(\mathbf{u})$ is the covariant metric defined by the relation

$$g_{ii}(\mathbf{u}) = 4(u^i + \nu) \frac{\operatorname{Res}_{\lambda=u^i} \left\{ \frac{(\Omega_\nu(\lambda))^2}{d\lambda} \right\}}{\operatorname{Res}_{\lambda=-\nu} \left\{ \frac{(\sigma_1(\lambda))^2}{d\lambda} \right\}}, \quad (25)$$

with $\Omega_\nu(\lambda)$ the second kind differential defined in (19) and $\sigma_1(\lambda)$ the third kind differential defined in (15). The metric $g_{ii}(\mathbf{u})$ is flat. The metric $g_{ii}(\mathbf{u})$ is defined up to multiplication by an arbitrary function $f_i(u^i)$. The metrics

$$\frac{g_{ii}(\mathbf{u})}{2(u^i + \nu)}, \quad \frac{g_{ii}(\mathbf{u})}{4(u^i + \nu)^2}, \quad \frac{g_{ii}(\mathbf{u})}{8(u^i + \nu)^3}, \quad (26)$$

are respectively flat, of constant curvature $R_{ij}^{ij} = 1$ and conformally flat with curvature tensor $R_{ij}^{ij} = C^i(\mathbf{u}) + C^j(\mathbf{u}) + 2\nu$. The pencil of metrics

$$g_{ii}(\mathbf{u}) + \lambda \frac{g_{ii}(\mathbf{u})}{(u^i + \nu)} \quad (27)$$

is flat for any real λ . None of the metrics defined in (25) and in (26) is of Egorov type.

We denote the diagonal metric in covariant form by g_{ii} and its inverse by g^{ii} . To prove the proposition we evaluate the non-zero elements of the curvature tensor R_{il}^{ij}

$$R_{il}^{ij} = \frac{1}{\sqrt{g_{ii}g_{jj}}} \{ \partial_l r_{ji} - r_{jl} r_{li} \}, \quad i \neq j \neq l, \quad (28)$$

$$R_{ij}^{ij} = \frac{1}{\sqrt{g_{ii}g_{jj}}} \left\{ \partial_i r_{ij} + \partial_j r_{ji} + \sum_{p \neq i, j} r_{pj} r_{pi} \right\}. \quad (29)$$

Here r_{ij} are the rotation coefficients defined by

$$r_{ij} = \frac{\partial_{u^i} \sqrt{g_{jj}}}{\sqrt{g_{ii}}}, \quad i \neq j. \quad (30)$$

A metric is of Egorov type if $r_{ij} = r_{ji}$, $\forall i \neq j$.

To evaluate (28-30) the following additional variational formulas are needed

$$\begin{aligned} \frac{\partial}{\partial u^i} \gamma_1 &= -\frac{1}{2} + \sigma_1(u^i) \sigma_2(u^i), \quad i = 1, 2, 3, \\ \frac{\partial}{\partial u^i} \gamma_2 &= \frac{1}{4}(u^1 + u^2 + u^3 - \nu) - \frac{1}{2}u^i + (\sigma_2(u^i))^2, \quad i = 1, 2, 3, \end{aligned} \quad (31)$$

where σ_1 and σ_2 have been defined in (15) and (16) respectively and

$$\sigma_k(u^i) := \left. \frac{\sigma_k(\lambda)}{dt} \right|_{\lambda=u^i}, \quad t^2 = \lambda - u^i, \quad i = 1, 2, 3, \quad k = 1, 2.$$

By direct calculation we obtain

$$r_{ij} = \frac{1}{2} \Omega_{u^i}(u^j) - \frac{1}{2} \frac{\Omega_\nu(u^j) \sigma_1(u^i)}{\sigma_1(-\nu)}$$

where $\Omega_{u^i}(\lambda)$ is a normalized second kind differential with second order pole at $\lambda = u^i$. The explicit form of $\Omega_{u^i}(\lambda)$ can be obtained from (19) by replacing $-\nu$ by u^i . The quantities $\Omega_\nu(u^j)$ and $\Omega_{u^i}(u^j)$ are defined as in (23). From the rotation coefficient formula (30) it is clear that $r_{ij} \neq r_{ji}$, for all the four metrics and therefore none of them is an Egorov metric. Using the variational formulas (31) we obtain

that $R_{il}^{ij} = 0$ for all the four metrics and

$$\begin{aligned} g_{ii}(\mathbf{u}) &\rightarrow R_{ij}^{ij} = 0, \\ \frac{g_{ii}(\mathbf{u})}{2(u^i + \nu)} &\rightarrow R_{ij}^{ij} = 0, \\ \frac{g_{ii}(\mathbf{u})}{4(u^i + \nu)^2} &\rightarrow R_{ij}^{ij} = 1, \\ \frac{g_{ii}(\mathbf{u})}{8(u^i + \nu)^3} &\rightarrow R_{ij}^{ij} = 2\nu + C^i(\mathbf{u}) + C^j(\mathbf{u}). \end{aligned}$$

The flatness of the pencil of metrics (27) can be obtained from the results in [13]. An elegant proof of the flatness of the metrics, valid for any genus can be obtained in a more convenient set of coordinates $u^i \rightarrow \frac{1}{u^i + \nu}$ that makes the spectral curve odd with a branch point at infinity. We use these coordinates in the next section to compare the Whitham equations for CH and for KdV.

Remark 3.4 *The non-existence of a flat Egorov metric is related to the non-existence of conservation laws of the form $a_t = b_x$ and $b_t = c_x$ [32]. Furthermore the non-existence of a flat Egorov metric implies that the CH modulation equations cannot be associated to a Frobenius manifold. We recall that, under certain assumptions, a flat pencil of contravariant metrics on a manifold induces a Frobenius structure on it [14]. One of the assumptions is the requirement that one of the two flat metrics is an Egorov metric. Therefore the geometric structure of the CH modulation equations is substantially different from that of the KdV modulation equations.*

To any flat diagonal Riemannian contravariant metric g^{ii} , Dubrovin and Novikov [12] associate a local homogenous Poisson bracket of hydrodynamic type

$$\{F, G\} = \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta w^j}$$

defined by the Hamiltonian operator

$$A^{ij} = g^{ii} \delta^{ij} \frac{d}{dx} - g^{ii} \Gamma_{ik}^j u_x^k. \quad (32)$$

Indeed in [12], they prove that A^{ij} defines a Poisson tensor if and only if g^{ii} is a flat non-degenerate metric and Γ_{ik}^j are the Christoffel symbols of the Levi-Civita connection compatible with the metric (the metric is not necessary diagonal in their

formulation). If the metric g^{ii} is not flat, the Poisson tensor needs to be modified adding a non-local tail [17],[16] of the form

$$A^{ij} = g^{ii}\delta^{ij}\frac{d}{dx} - g^{ii}\Gamma_{ik}^j u_x^k + cu_x^i \left(\frac{d}{dx}\right)^{-1} u_x^j, \quad (33)$$

for metrics of constant curvature c , and of the form

$$A^{ij} = g^{ii}\delta^{ij}\frac{d}{dx} - g^{ii}\Gamma_{ik}^j u_x^k + \eta^i u_x^i \left(\frac{d}{dx}\right)^{-1} u_x^j + u_x^i \left(\frac{d}{dx}\right)^{-1} \eta^j u_x^j, \quad (34)$$

for conformally flat metrics. In the above relation the affinors η^j satisfy the equations

$$R_{ij}^{ij} = \eta^i + \eta^j,$$

where R_{ij}^{ij} is the curvature tensor.

In the following let $g_{ii}(\mathbf{u})$ be as in (25). Let A_1^{ij} and A_2^{ij} be the Hamiltonian operators of the form (32) that correspond to the flat metric $g_{ii}(\mathbf{u})$ and to $\frac{g_{ii}(\mathbf{u})}{2(u^i + \nu)}$, respectively. The linear combination $A_1^{ij} + \lambda A_2^{ij}$ is an Hamiltonian operator for any λ because $g_{ii}(\mathbf{u}) + \lambda g_{ii}(\mathbf{u})/u^i$ is a flat pencil of metrics for any λ .

Remark 3.5 *The Hamiltonian structure obtained in (8) coincides with Hamiltonian operator A_1 . Indeed the coordinates $k = k(u^1, u^2, u^3)$, $\beta = \beta(u^1, u^2, u^3)$ and $J = J(u^1, u^2, u^3)$ defined in (8) are the densities of the Casimirs for the Hamiltonian operator A_1^{ij} , namely*

$$A_1^{ij} \frac{\delta k}{\delta u^j} = A_1^{ij} \frac{\delta \beta}{\delta u^j} = A_1^{ij} \frac{\delta J}{\delta u^j} = 0,$$

and therefore they are the flat coordinates for the metric $g_{ii}(\mathbf{u})$ [12].

Defining $h_0 := \oint md\theta = \beta$, we obtain the Hamiltonian densities $h_k = h_k(\mathbf{u})$, $k \geq 0$, by the recursion relation

$$A_2^{ij} \frac{\partial h_k}{\partial u^j} = A_1^{ij} \frac{\partial h_{k+1}}{\partial u^j}, \quad k \geq 0.$$

The CH modulation equations can be written in the bi-Hamiltonian form

$$\partial_{t_k} u^i = -A_1^{ij} \frac{\delta h_{k+1}}{\delta u^j} = -A_2^{ij} \frac{\delta h_k}{\delta u^j} = 0, \quad i = 1, 2, 3.$$

Theorem 3.6 For $k = 1$ and $t_1 = t$, the above equations coincide with the modulation equations obtained in (11) and take the form

$$u_t^i = A_1^{ij} \frac{\delta h_2}{\delta u^j} = A_2^{ij} \frac{\delta h_1}{\delta u^j}$$

where the Hamiltonian densities are the averaged Hamiltonians

$$h_2 = \frac{1}{2} \oint (u^3 + uu_x^2 + 2\nu u^2) d\theta, \quad h_1 = \frac{1}{2} \oint (u^2 + u_x^2) d\theta.$$

Moreover, the generating function for the Hamiltonian densities h_k , $k \geq 0$, is given by the coefficients of the expansion as $\lambda \rightarrow \infty$ of the differential

$$\frac{\Omega_\nu(\lambda)}{\left\{ \text{Res}_{\lambda=-\nu} \frac{(\sigma_1(\lambda))^2}{d\lambda} \right\}^{\frac{1}{2}}} \rightarrow (\xi_0 + \xi_1 \frac{1}{\lambda} + \xi_2 \frac{1}{\lambda^2} + \dots) \frac{d\lambda}{\lambda^3}.$$

In particular, the first Hamiltonians (modulo Casimirs) are

$$h_0 = 2\xi_0, \quad h_1 = 2\xi_1 + 2\nu\xi_0, \quad h_2 = \frac{8}{3}\xi_2 + 6\nu\xi_1. \quad (35)$$

Remark 3.7 For $\nu = 0$ the Hamiltonian densities are $h_k = \frac{2^{k+1}}{2k-1} \xi_k$, $k \geq 0$.

In the following we write the CH-modulation equation in Hamiltonian form with respect to a nonlocal Hamiltonian operator of Mokhov-Ferapontov and Ferapontov type. Let A_3^{ij} be the Hamiltonian operator of the form (33) associated to the metric $\frac{g_{ii}(\mathbf{u})}{4(\nu + u^i)^2}$ of constant curvature $c = 1$, and let A_4^{ij} be the Hamiltonian operator of the form (34) associated to the conformally flat metric $\frac{g_{ii}(\mathbf{u})}{8(\nu + u^i)^3}$ with affinors $\eta_i = C^i + \nu$, $i = 1, 2, 3$.

The CH modulation equations (11) can also be written in a non-local Hamiltonian form

$$\partial_t u^i = -A_3^{ij} \frac{\delta h_0}{\delta u^j} = -A_4^{ij} \frac{\delta h_{-1}}{\delta u^j} = 0, \quad i = 1, 2, 3,$$

where

$$h_{-1} = 1 - \frac{\nu}{\left\{ \text{Res}_{\lambda=-\nu} \frac{(\sigma_1(\lambda))^2}{d\lambda} \right\}^{\frac{1}{2}}}.$$

We remark that the Hamiltonian operators A_3 and A_4 can be obtained from the recursion $A_3 = \mathcal{R}^2 A_1$ and $A_4 = \mathcal{R}^3 A_1$ where $\mathcal{R} = A_2 A_1^{-1}$ is the recursion operator and A_1^{-1} denotes the inverse of A_1 .

In the limit when two Riemann invariants coalesce, the Hamiltonian operators A_1^{ij} and A_2^{ij} reduce to the “dispersionless limit” of the Poisson operators of the CH equation

$$\mathcal{P}_1 = \frac{d}{dx}, \quad \mathcal{P}_2 = 2(u + \nu) \frac{d}{dx} + u_x,$$

respectively. In the same limit, the Hamiltonian operators A_3^{ij} and A_4^{ij} reduce to $\mathcal{P}_2 \mathcal{P}_1^{-1} \mathcal{P}_2$ and $\mathcal{P}_2 \mathcal{P}_1^{-1} \mathcal{P}_2 \mathcal{P}_1^{-1} \mathcal{P}_2$, respectively.

3.2 Integration of the Whitham equations

In this subsection we show how to integrate the Whitham equations (11). All hydrodynamic systems satisfying (24) are integrable, via the generalized hodograph transform introduced by Tsarev [35]. Indeed such systems (not necessarily with local Hamiltonian) possess an infinite number of commuting flows

$$\partial_t u^i = w^i(\mathbf{u}) \partial_x u^i,$$

where the w^i are solutions of the linear overdetermined system

$$\frac{\partial_{u^j} w^i}{w^i - w^j} = \frac{\partial_{u^j} C^i}{C^i - C^j}, \quad i \neq j, \quad (36)$$

where $C^i = C^i(u^1, u^2, u^3)$, $i = 1, 2, 3$, are the speeds in (12). Then the solution $\mathbf{u}(x, t) = (u^1(x, t), u^2(x, t), u^3(x, t))$ of the so-called hodograph transform

$$x = -C^i(\mathbf{u})t + w^i(\mathbf{u}) \quad i = 1, 2, 3, \quad (37)$$

satisfies the system (11). Conversely, any solution $(u^1(x, t), u^2(x, t), u^3(x, t))$ of (11) can be obtained in this way. For monotone decreasing initial data $x = f(u)|_{t=0}$, the general solution of the system (36) can be obtained following the work of Fei-Ran Tian [34] and the algebraic-geometric integration of Krichever [24]. For simplicity, we restrict ourselves to the case $\nu = 0$.

Proposition 3.8 *For $\nu = 0$, and monotone increasing initial data $x = f(u)|_{t=0}$, the solution of the system (36) is*

$$w^i(\mathbf{u}) = q(\mathbf{u}) + (C^i(\mathbf{u}) - u^1 - u^2 - u^3) \frac{\partial q(\mathbf{u})}{\partial u^i}, \quad (38)$$

where the function $q = q(\mathbf{u})$ solves the linear over-determined system of Euler-Poisson-Darboux type

$$\begin{aligned}\partial_{u^i} q(\mathbf{u}) - \partial_{u^j} q(\mathbf{u}) &= 2(u^i - u^j) \partial_{u^i} \partial_{u^j} q(\mathbf{u}), \\ q(u, u, u) &= f(u).\end{aligned}\tag{39}$$

The proof of the above statement follows from [33]. Equation (39) can be integrated and the explicit expression of the function $q(u^1, u^2, u^3)$ is

$$q(u^1, u^2, u^3) = \frac{1}{2\sqrt{2\pi}} \int_{-1}^1 \int_{-1}^1 \frac{f\left(\frac{1+\mu}{2} \frac{1+\eta}{2} u^1 + \frac{1+\mu}{2} \frac{1-\eta}{2} u^2 + \frac{1-\mu}{2} u^3\right)}{\sqrt{(1-\mu)(1-\eta^2)}} d\mu d\eta.$$

4 CH modulation equations versus KdV and reciprocal transformations

In this section we compare CH and KdV modulation equations. We start recalling the reciprocal transformation which links the CH equation to the first negative KdV flow. We show that the CH modulation equations (11) are transformed to the modulation equations of the first negative KdV flow by the averaged reciprocal transformation. Finally we compare the averaged Hamiltonian operators of the two systems.

4.1 A reciprocal transformation between Camassa Holm equation and the first negative flow of KdV

In this subsection we summarize the relation between the Camassa Holm equation and the first negative flow of KdV hierarchy [21]. The (associated) Camassa-Holm equation is transformed into the first negative KdV flow by a reciprocal transformation. In the following, to distinguish between CH and KdV, we use (x, t) for Camassa-Holm variables and (y, τ_-) for the KdV variables.

The change of dependent variable $\rho^2 = m + \nu$ transforms the Camassa-Holm equation

$$m_t = -2mu_x - um_x - 2\nu u_x, \quad m = u - u_{xx}.$$

into the associated Camassa-Holm equation

$$\begin{cases} \rho_t = -\left(u\rho\right)_x, \\ \rho^2 = u - u_{xx} + \nu, \end{cases}\tag{40}$$

which, via the reciprocal transformation introduced by Fuchssteiner [21],

$$\begin{cases} dy = \rho dx - u \rho dt, \\ d\tau_- = dt, \end{cases} \quad (41)$$

is finally transformed into

$$\begin{cases} u = \rho^2 - \nu - \rho_{y\tau_-} + \frac{\rho_{\tau_-}\rho_y}{\rho}, \\ \left(\frac{1}{\rho}\right)_{\tau_-} = 2\rho\rho_y - \left(\rho(\log \rho)_{y\tau_-}\right)_y. \end{cases} \quad (42)$$

The transformation (41) is a reciprocal transformation because the one-form $\rho dx - u \rho dt$ is closed with respect to the CH flow.

The equation (42) is equivalent to the first negative flow of the KdV hierarchy

$$(\partial_y^2 + 2U + U_y \partial_y^{-1}) U_{\tau_-} = 0, \quad (43)$$

under the condition $U_{\tau_-} = -2\rho_y$. Equation (43) may be re-expressed as

$$\begin{cases} U = \frac{\rho_y^2 - 2\rho\rho_{yy} - 1}{2\rho^2}, \\ \left(\frac{\rho_y^2 - 2\rho\rho_{yy} - 1}{4\rho^2}\right)_{\tau_-} = -\rho_y. \end{cases} \quad (44)$$

Finally, we observe that $\int \rho(x, t) dx$ is a Casimir of the second Hamiltonian operator of the Camassa–Holm equation, $P_2 = m\partial_x + \partial_x m + 2\nu\partial_x$.

The 2π -periodic solutions of the first negative KdV flow (43), $U(\Theta)$, $\Theta = Ky - \Omega\tau$, satisfy

$$d\Theta = \frac{KdU}{\sqrt{-U^3 + \alpha U^2 - \beta U + \gamma}}. \quad (45)$$

We may express such solutions also in the form $\rho(\Theta)$ or $u(\Theta)$ and we easily get

$$\begin{cases} u_{\Theta}^2 = \frac{1}{C^2 K^2} (u - c)(u^3 - (c - 2\nu)u^2 + 2Bu - 2A), \\ \rho(\Theta) = -\frac{C}{u(\Theta) - c}, \\ U(\Theta) = -\frac{2}{u(\Theta) - c}, \end{cases} \quad (46)$$

where $C = \Omega/K$. A, B, c are constants which satisfy

$$Bc + \nu c^2 - A = C^2, \quad \alpha = -\frac{c^2 + 2B + 4\nu c}{C^2}. \quad (47)$$

The periodic solutions $u(\theta), \rho(\theta)$, $\theta = kx - \omega t$ of the (associated) Camassa–Holm equation, satisfy

$$d\theta = \frac{k(u-c)du}{\sqrt{(u-c)(u^3 - (c-2\nu)u^2 + 2Bu - 2A)}}.$$

We observe that the reciprocal transformation sends 2π -periodic solutions in Θ into 2π -periodic solution in θ (and vice versa). Indeed, let T be the period of $u(\theta)$, then

$$T = \frac{k}{K} \int_0^{2\pi} \frac{c-u}{C} d\Theta = k \oint \frac{(c-u)du}{\sqrt{(u-c)(u^3 - (c-2\nu)u^2 + 2Bu - 2A)}} = 2\pi.$$

It is then natural to expect that the average of the reciprocal transformation connects the CH modulation equations to the modulation equations of the first negative KdV flow.

4.2 The modulation equations of the negative KdV flow

In this subsection, we compute the modulation equations of the KdV negative flow in the Riemann invariant coordinates used in the literature, namely the branch points $\beta^1, \beta^2, \beta^3$, of the odd elliptic curve [37]

$$w^2 = (\eta - \beta^1)(\eta - \beta^2)(\eta - \beta^3), \quad (48)$$

where

$$\beta^1 + \beta^2 + \beta^3 = \alpha,$$

with α defined in (45). It turns out from (47) that

$$\beta^i = \frac{1}{u^i + \nu}, \quad i = 1, 2, 3, \quad (49)$$

where u^i are the CH Riemann invariants defined in theorem 3.1. On the Riemann surface (48) we define the second kind normalized differential

$$dp(\eta) = \frac{\eta + \alpha_1}{\sqrt{(\eta - \beta^1)(\eta - \beta^2)(\eta - \beta^3)}} d\eta, \quad (50)$$

with second order pole at infinity and asymptotic behavior

$$dp(\eta) \rightarrow d\eta(\eta^{-\frac{1}{2}} + O(\eta^{-\frac{3}{2}})), \quad \eta \rightarrow \infty.$$

The constant α_1 is uniquely determined by the normalization condition

$$\oint_a dp(\eta) = 0.$$

In the KdV literature $dp(\eta)$ is known as quasi-momentum. Now we can compute the velocities of the modulation equations of the negative KdV flow

Proposition 4.1 *The one phase Whitham equations of the first negative KdV flow are*

$$\partial_{\tau_-} \beta^i + v_i(\beta) \partial_y \beta^i = 0, \quad (51)$$

where

$$v_i(\beta) := \frac{\partial_i \Omega(\beta)}{\partial_i K(\beta)} = \frac{2}{\sqrt{-\beta^1 \beta^2 \beta^3}} \left(1 - \frac{\prod_{j \neq i} (\beta^i - \beta^j)}{\beta^i (\beta^i + \alpha_1)} \right), \quad (52)$$

with α_1 as in (50). In the above relations Ω and K are the frequency and wave-number of the one-phase KdV negative flow.

The proof of the proposition is as follows. From (45) and (46), we immediately get $K = 2\pi \mathcal{J}_0^{-1}$ and $\Omega = CK = 4\pi \mathcal{J}_0^{-1} (-\beta^1 \beta^2 \beta^3)^{-1/2}$, where

$$\mathcal{J}_0 = \oint \frac{d\lambda}{\sqrt{(\lambda - \beta^1)(\lambda - \beta^2)(\lambda - \beta^3)}}.$$

Then the expressions for the velocities (52) are computed from the definition using the following variational formula

$$\frac{\partial \mathcal{J}_0}{\partial \beta^i} = \frac{1}{2} \mathcal{J}_0 \frac{\beta^i + \alpha_1}{\prod_{j \neq i} (\beta^i - \beta^j)}, \quad i = 1, 2, 3. \quad (53)$$

Remark 4.2 *In the limit $\beta^2 = \beta^3$, the equations (51) converge to $\beta_{\tau_-}^1 = -2(\beta^1)^{-\frac{3}{2}} \beta_y^1$, which is the dispersionless limit of the first negative KdV flow.*

Next, we write the negative KdV equations in Hamiltonian form. In the β s' coordinates, the flat metric associated to the first local KdV-Whitham Hamiltonian operator is [13]

$$g_{ii}^{KdV}(\beta) = \text{Res}_{\eta=\beta^i} \left\{ \frac{(dp(\eta))^2}{d\eta} \right\}, \quad i = 1, 2, 3, \quad (54)$$

where the differential dp has been defined in (50).

Remark 4.3 *The flat metrics $g_{ii}^{KdV}(\boldsymbol{\beta})$ and $g_{ii}^{KdV}(\boldsymbol{\beta})/\beta^i$ can be related to a Frobenius manifold defined on the moduli space of elliptic curves $w^2 = (\eta - \beta^1)(\eta - \beta^2)(\eta - \beta^3)$ [14].*

Let J_1 and J_2 be the local KdV Hamiltonian operators of the form (32) associated to the flat metrics $\frac{1}{8}g_{ii}^{KdV}(\boldsymbol{\beta})$ and $\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{4\beta^i}$. Let J_3 be the nonlocal Hamiltonian operator of the form (33) associated to the metric $\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{2(\beta^i)^2}$ of constant curvature $R_{ij}^{ij} = -\frac{1}{2}$. Let J_4 be the nonlocal Hamiltonian operator of the form (34) associated to the conformally flat metric $\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{(\beta^i)^3}$ with Riemannian curvature $R_{ij}^{ij} = -\frac{1}{8}(w_+^i + w_+^j)$, where

$$w_+^i = \left(\beta^1 + \beta^2 + \beta^3 + \frac{2 \prod_{j \neq i} (\beta^i - \beta^j)}{\beta^i + \alpha_1} \right),$$

with α_1 defined in (50). The w_+^i , $i = 1, \dots, 3$, are the velocities (originally obtained by Whitham [37]) of the usual positive KdV modulated flow

$$\frac{\partial \beta^i}{\partial \tau_+} = w_+^i(\boldsymbol{\beta}) \frac{\partial \beta^i}{\partial y}.$$

Remark 4.4 *The Hamiltonian operator J_1 corresponds to the average over a one-dimensional torus of the Gardner-Zakharov KdV Hamiltonian structure $\mathcal{P}_1 = 8\partial_y$ while the Hamiltonian operator J_2 corresponds to the average over a one-dimensional torus of the Lenard-Magri local Hamiltonian structure $\mathcal{P}_2 = 2\partial_{yyy} + 2u\partial_y + 2\partial_y u$. We use this unusual normalization in order to be consistent with the normalization of the CH equation. Defining the recursive operator $\mathcal{R} = \mathcal{P}_2(\mathcal{P}_1)^{-1}$, we obtain the family of non-local Hamiltonian operators $\mathcal{P}_{k+1} = \mathcal{R}\mathcal{P}_k$, $k \geq 1$. The averaged KdV non-local Hamiltonian operators J_3 and J_4 corresponds to the average over a one-dimensional torus of the non-local Hamiltonian operators \mathcal{P}_3 and \mathcal{P}_4 , respectively.*

The modulation equations of the negative KdV flow is generated by \mathcal{K}_0 , which is the average over the one dimensional torus of the Casimir generating the KdV negative flow, namely

$$\mathcal{K}_0 = \oint \frac{d\Theta}{\rho(\Theta)}. \quad (55)$$

In the β s' coordinates by direct calculation it can be verified that

$$\mathcal{K}_0 = \oint \frac{d\Theta}{\rho(\Theta)} = - \left. \frac{(\Lambda_0(\eta))}{2d\xi} \right|_{\xi=0}, \quad \eta = \frac{1}{\xi^2},$$

where

$$\Lambda_0(\eta) = \sqrt{-\beta^1\beta^2\beta^3} \frac{\frac{1}{\eta} + \alpha_0}{\sqrt{(\eta - \beta^1)(\eta - \beta^2)(\eta - \beta^3)}} d\eta. \quad (56)$$

$\Lambda_0(\eta)$ is a normalized third kind differential with first order poles at the points $O^\pm = (0, \pm\sqrt{-\beta^1\beta^2\beta^3})$ with residue ± 1 , respectively. The constant α_0 in (56) is uniquely determined by the condition $\oint_a \Lambda_0(\eta) = 0$. From the Riemann bilinear relation, it is finally immediate to verify that

$$\mathcal{K}_0 = - \left. \frac{(\Lambda_0(\eta))}{2d\xi} \right|_{\xi=0} = \frac{1}{2}p(0), \quad \eta = \frac{1}{\xi^2}, \quad (57)$$

where $p(\eta)$ is the Abelian integral of the quasi-momentum $dp(\eta)$ defined in (50). The expression (57) for the Casimir can be found [14]. Finally the following result holds.

Proposition 4.5 *The first negative KdV averaged flow can be written in the Hamiltonian form*

$$\beta_{\tau_-}^i = J_1^{ij} \frac{\delta \mathcal{K}_0}{\delta \beta^j} = J_2^{ij} \frac{\delta \mathcal{K}_{-1}}{\delta \beta^j} = J_3^{ij} \frac{\delta \mathcal{K}_{-2}}{\delta \beta^j} = J_4^{ij} \frac{\delta \mathcal{K}_{-3}}{\delta \beta^j}, \quad (58)$$

where \mathcal{K}_0 is the Casimir of J_2 defined in (55) and the Hamiltonian densities are determined by the recursion scheme

$$J_1^{ij} \frac{\delta \mathcal{K}_{-s}}{\delta \beta^j} = J_2^{ij} \frac{\delta \mathcal{K}_{-s-1}}{\delta \beta^j}, \quad s \geq 0.$$

The Hamiltonian \mathcal{K}_{-s} , $s \geq 1$, are generated by the expansion for $\eta \rightarrow 0$ of the quasi-momentum $dp(\eta)$. Indeed,

$$dp(\eta) \rightarrow d\eta(\mathcal{K}_{-1} + \eta\mathcal{K}_{-2} + \frac{3}{4}\eta^2\mathcal{K}_{-3} + \dots), \quad \eta \rightarrow 0. \quad (59)$$

4.3 CH versus KdV modulation equations

In the previous section we computed the modulation equations of the first negative KdV flow. In this subsection, we compute the average of the reciprocal transformation defined in section 4.1 and we show that the negative KdV modulation equations are transformed to the CH modulation equations. Finally, both KdV-Whitham and CH-Whitham systems are Hamiltonian systems, so we end the section investigating how the reciprocal transformation acts on the Hamiltonian structures of the two systems.

To compare KdV and CH, we first need to reduce the even spectral curve of CH to the odd spectral curve of KdV. The natural change of coordinates $(\lambda, y) \rightarrow (\eta, w)$

$$\eta = \frac{1}{\lambda + \nu}, \quad w^2 = \frac{y^2}{(\lambda + \nu)^4 \prod_{i=1}^3 (\nu + u^i)},$$

maps the even spectral curve $y^2 = (\lambda + \nu)(\lambda - u^1)(\lambda - u^2)(\lambda - u^3)$ to the odd KdV spectral curve

$$w^2 = (\eta - \beta^i)(\eta - \beta^2)(\eta - \beta^3), \quad \beta^i = \frac{1}{\nu + u^i}.$$

The differentials $\Omega_\nu(\lambda)$ and $\sigma_1(\lambda)$ defined in (19) and (15) transform to

$$\Omega_\nu(\lambda) \rightarrow \frac{1}{2} dp(\eta), \quad (60)$$

$$\sigma_1(\lambda) \rightarrow \Lambda_0(\eta), \quad (61)$$

with $dp(\eta)$ as in (50) and $\Lambda_0(\eta)$ as in (56). It follows from (60) and (61) that the change of coordinates $\beta^i = 1/(\nu + u^i)$ transforms the speeds $C^i(\mathbf{u})$ defined in (20) to

$$\tilde{C}^i(\boldsymbol{\beta}) = \frac{1}{\beta^1} + \frac{1}{\beta^2} + \frac{1}{\beta^3} - \nu + 2 \frac{\alpha_0 \prod_{j \neq i, j=1}^3 (\beta^i - \beta^j)}{\beta^i (\beta^i + \alpha_1)}. \quad (62)$$

Now we show that the averaged reciprocal transformation maps the modulation equations of the KdV negative flow to the CH modulation equations. Indeed averaging over a period the inverse of (41)

$$dx = 1/\rho dy + u d\tau_-, \quad dt = d\tau_-,$$

we get the averaged reciprocal transformation

$$dx = dy \oint \frac{d\Theta}{\rho(\Theta)} + d\tau_- \oint u(\Theta) d\Theta, \quad dt = d\tau_-. \quad (63)$$

Proposition 4.6 *The averaged reciprocal transformation (63) takes the form*

$$dx = \mathcal{K}_0 dy + \mathcal{N} d\tau_-, \quad dt = d\tau_-, \quad (64)$$

where \mathcal{K}_0 is the Casimir defined in (55) and

$$\mathcal{N} = \frac{1}{2}(\nabla\mathcal{K}_0)^2 - \nu, \quad (65)$$

where $(\nabla\mathcal{K}_0)^2 = 8 \sum_i (g_{ii}^{KdV})^{-1} (\partial_{\beta^i} \mathcal{K}_0)^2$. Finally (64) is a reciprocal transformation for the KdV-Whitham negative flow.

To prove the proposition we observed that the one form (64) is closed by (65). The proof of (65) follows from the identity $\mathcal{K}_0 = \frac{p(0)}{2}$, and the variational formula $\partial_{\beta^i} p(0) = \frac{1}{2} dp(\beta^i) \Lambda_0(\beta^i)$.

Next we show that the modulation equations of the first negative KdV flow are mapped by the reciprocal transformation (64) to the CH modulation equations.

Proposition 4.7 *The reciprocal transformation $dx = \mathcal{K}_0 dy + \mathcal{N} d\tau_-$, where \mathcal{K}_0 and \mathcal{N} are as in (55) and (65) respectively, transforms the modulation equations (51) of the first negative KdV flow*

$$\partial_{\tau_-} \beta^i + v_i(\boldsymbol{\beta}) \partial_y \beta^i = 0, \quad (66)$$

where $v_i(\boldsymbol{\beta})$, $i = 1, \dots, 3$ are as in (52) to the CH modulation equations

$$\partial_t \beta^i + \tilde{C}^i(\boldsymbol{\beta}) \partial_x \beta^i = 0, \quad (67)$$

where the CH velocities are defined in (62).

Viceversa, the inverse reciprocal transformation $dy = \frac{1}{\mathcal{K}_0} dx - \frac{\mathcal{N}}{\mathcal{K}_0} dt$ transforms (67) into (66).

Indeed plugging (64) into (66) we get $\partial_t \beta^i + V_i(\boldsymbol{\beta}) \partial_x \beta^i = 0$, where

$$V_i(\boldsymbol{\beta}) = v_i(\boldsymbol{\beta}) \mathcal{K}_0 + \mathcal{N} = \tilde{C}^i(\boldsymbol{\beta}).$$

To compare the CH and KdV Hamiltonian structures, we express the CH-Whitham Hamiltonian operators introduced in subsection 3.1 in the β s' coordinates.

In these coordinates, using (57), (59) and (60), the CH flat metric $g_{ii}(\mathbf{u})$ defined in (25) takes the form

$$\tilde{g}_{ii}(\boldsymbol{\beta}) = \frac{1}{(\beta^i)^3} \frac{\text{Res}_{\eta=\beta^i} \left\{ \frac{(dp(\eta))^2}{d\eta} \right\}}{p(0)^2/4} = \frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{(\beta^i)^3 \mathcal{K}_0^2}, \quad i = 1, 2, 3. \quad (68)$$

The other metrics defined in (26) transform, respectively, to

$$\frac{1}{2} \tilde{g}_{ii}(\boldsymbol{\beta}) \beta^i = \frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{2(\beta^i)^2 \mathcal{K}_0^2}, \quad \frac{1}{4} \tilde{g}_{ii}(\boldsymbol{\beta}) (\beta^i)^2 = \frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{4\beta^i \mathcal{K}_0^2}, \quad \frac{1}{8} \tilde{g}_{ii}(\boldsymbol{\beta}) (\beta^i)^3 = \frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{8\mathcal{K}_0^2}, \quad (69)$$

and are, respectively, flat, of constant curvature and conformally flat.

Let \tilde{A}_1 and \tilde{A}_2 be the local Hamiltonian operators associated to the CH flat metrics $\tilde{g}_{ii}(\boldsymbol{\beta})$ and $\frac{1}{2} \tilde{g}_{ii}(\boldsymbol{\beta}) \beta^i$, respectively. Moreover, let \tilde{A}_3 and \tilde{A}_4 be the non-local Hamiltonian operators associated to the CH constant curvature and conformally flat metrics, $\frac{1}{4} \tilde{g}_{ii}(\boldsymbol{\beta}) (\beta^i)^2$ and $\frac{1}{8} \tilde{g}_{ii}(\boldsymbol{\beta}) (\beta^i)^3$, respectively. Then, in the β s' coordinates the CH modulation equations in Hamiltonian form are

$$\beta_t^i = \tilde{A}_1^{ij} \frac{\delta \tilde{h}_2}{\delta \beta^j} = \tilde{A}_2^{ij} \frac{\delta \tilde{h}_1}{\delta \beta^j} = \tilde{A}_3^{ij} \frac{\delta \tilde{h}_0}{\delta \beta^j} = \tilde{A}_4^{ij} \frac{\delta \tilde{h}_{-1}}{\delta \beta^j} \quad (70)$$

where the Hamiltonians \tilde{h}_j , $i = -1, \dots, 2$, are the averaged conservation laws introduced in subsection 3.1 expressed in the β s' coordinates. Indeed, following (59), the positive CH Hamiltonians are obtained from the coefficients of the expansion for $\eta \rightarrow 0$ of the differential $\frac{dp(\eta)}{2\mathcal{K}_0}$ and take the form

$$\tilde{h}_{-1} = 1 - \frac{\nu}{\mathcal{K}_0}, \quad \tilde{h}_0 = \frac{\mathcal{K}_{-1}}{\mathcal{K}_0} - \nu, \quad \tilde{h}_j = \frac{\mathcal{K}_{-j-1}}{\mathcal{K}_0} - \nu \frac{\mathcal{K}_{-j}}{\mathcal{K}_0}, \quad j \geq 1. \quad (71)$$

The following theorem by Ferapontov and Pavlov describes the action of a reciprocal transformation for an Hamiltonian hydrodynamic equation and we use it to clarify the relation between the Hamiltonian structures of the averaged KdV and CH Hamiltonian structures.

Theorem 4.8 [18] *Let $\beta_\tau^i = J^{ij} \frac{\partial h}{\partial \beta^j}$ be an Hamiltonian system associated to the local operator $J^{ij} = g^{ii} \delta_i^j \frac{d}{dy} - g^{ii} \Gamma_{ik}^j \beta_y^k$. Then, under the action of the reciprocal*

transformation $dx = Ady + Bd\tau$, $dt = d\tau$, where $d(Ady + Bd\tau) = 0$, the transformed system is Hamiltonian $\beta_i^i = \tilde{J}^{ij} \frac{\partial \tilde{h}}{\partial \beta^j}$, with nonlocal operator

$$\tilde{J}^{ij} = \tilde{g}^{ii} \delta^{ij} \frac{d}{dx} - \tilde{g}^{ii} \tilde{\Gamma}_{ik}^j \beta_x^k + \tilde{w}^i \beta_x^i \left(\frac{d}{dx} \right)^{-1} \beta_x^j + \beta_x^i \left(\frac{d}{dx} \right)^{-1} \tilde{w}^j \beta_x^j,$$

and hamiltonian density $\tilde{h} = h/A$. The transformed metric is $\tilde{g}_{ii} = g_{ii}/A^2$, $\tilde{\Gamma}$ is the Levi-Civita connection and

$$\tilde{w}^i = \nabla^i \nabla_i A \cdot A - \frac{1}{2} (\nabla A)^2 = g^{ii} \left(\partial_i^2 A - \sum_j \Gamma_{ii}^j \partial_j A \right) A - \frac{1}{2} \sum_j g^{jj} (\partial_j A)^2. \quad (72)$$

Moreover, the transformed metric is conformally flat with curvature tensor

$$\tilde{R}_{ij}^{ij} = \tilde{w}^i + \tilde{w}^j.$$

By the above theorem and by inspection of (58) and (70), where the metrics have been defined in (68), (69) and (54), we conclude that the local KdV-Whitham Poisson operators, J_1 and J_2 , are mapped to the nonlocal CH-Whitham Poisson operators \tilde{A}_4 and \tilde{A}_3 , respectively, by the reciprocal transformation $dx = \mathcal{K}_0 dy + \mathcal{N} d\tau$.

Remark 4.9 *The application of theorem 4.8 deserves a special attention in the case $\nu \neq 0$ for the computation of the Hamiltonian densities \tilde{h}_k . Indeed neither the KdV-Hamiltonian operators J_i nor the averaged negative KdV Hamiltonian densities \mathcal{K}_{-s} , $s = 0, 1, 2, 3$, depend on the parameter ν while the CH averaged Hamiltonian densities \tilde{h}_s , $s = -1, 0, 1, 2$, in (71) do. To solve this apparent contradiction, we recall that the Hamiltonian densities are defined modulo Casimirs.*

In particular, the Casimir $-\nu \mathcal{K}_0$ of the second KdV-Whitham Hamiltonian operator is transformed to the CH Hamiltonian density $-\nu$ which generates the constant flow term associated to \tilde{A}_3 . Similarly, the term $-\nu$ is mapped by the reciprocal transformation to the CH Hamiltonian density $-\nu/\mathcal{K}_0$ which generates the constant flow term associated to \tilde{A}_4 .

Applying theorem 4.8 to the local CH-Whitham Poisson operators, we prove that the local Hamiltonian CH operators \tilde{A}_1 and \tilde{A}_2 , are mapped to the nonlocal

KdV–Whitham Poisson operators J_4 and J_3 , respectively, by the reciprocal transformation $dy = 1/\mathcal{K}_0 dx - \mathcal{N}/\mathcal{K}_0 dt$. By the same theorem 4.8, the corresponding KdV Hamiltonian densities are related to the CH ones by

$$\tilde{h}_2 \mathcal{K}_0 = \mathcal{K}_{-3}, \quad \tilde{h}_1 \mathcal{K}_0 = \mathcal{K}_{-2},$$

where the above identities again hold modulo Casimirs of the corresponding KdV Hamiltonian operators. We summarize the results of this section in the following.

Theorem 4.10 *The local KdV Hamiltonian operator J_1 is mapped by $dx = \mathcal{K}_0 dy + \mathcal{N} d\tau$ to the nonlocal CH Poisson operator \tilde{A}_4 associated to the covariant conformally flat metric $\frac{1}{8} \tilde{g}_{ii}(\boldsymbol{\beta})(\beta^i)^3$. The Hamiltonian density of the negative KdV flow $\mathcal{K}_0 - \nu$ is mapped to the corresponding CH Hamiltonian density $\tilde{h}_{-1} = 1 - \frac{\nu}{\mathcal{K}_0}$.*

The local KdV Hamiltonian operator J_2 is mapped by $dx = \mathcal{K}_0 dy + \mathcal{N} d\tau$ to the nonlocal Poisson CH operator \tilde{A}_3 associated to the constant curvature metric $\frac{1}{4} \tilde{g}_{ii}(\boldsymbol{\beta})(\beta^i)^2$. The Hamiltonian density of the negative KdV flow $\mathcal{K}_{-1} - \nu \mathcal{K}_0$ is mapped to the corresponding CH Hamiltonian density $\tilde{h}_0 = -\nu + \frac{\mathcal{K}_{-1}}{\mathcal{K}_0}$.

The local CH Hamiltonian operator \tilde{A}_1 is mapped by $dy = 1/\mathcal{K}_0 dx - \mathcal{N}/\mathcal{K}_0 dt$ into the nonlocal KdV Poisson operator J_4 associated to the covariant conformally flat metric $\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{(\beta^i)^3}$. The CH Hamiltonian density $\tilde{h}_2 = \frac{\mathcal{K}_{-3}}{\mathcal{K}_0} - \nu \frac{\mathcal{K}_{-2}}{\mathcal{K}_0}$ is mapped to the corresponding KdV Hamiltonian density \mathcal{K}_{-3} modulo a Casimir.

The local CH Hamiltonian operator \tilde{A}_2 is mapped by $dy = 1/\mathcal{K}_0 dx - \mathcal{N}/\mathcal{K}_0 dt$ into the nonlocal KdV Poisson operator J_3 associated to the covariant constant curvature metric $\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{2(\beta^i)^2}$. The CH Hamiltonian density $\tilde{h}_1 = \frac{\mathcal{K}_{-2}}{\mathcal{K}_0} - \nu \frac{\mathcal{K}_{-1}}{\mathcal{K}_0}$ is mapped to the corresponding KdV Hamiltonian density \mathcal{K}_{-2} modulo a Casimir.

We conclude this section by illustrating with a table all the reciprocal transformations between the various Hamiltonian operators.

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KdV Poisson tensor	\mathcal{P}_1	\mathcal{P}_2	$\mathcal{R}^2\mathcal{P}_1$	$\mathcal{R}^3\mathcal{P}_1$
averagedKdV Poisson tensor	J_1	J_2	J_3	J_4
KdV metric	$\frac{1}{8}g_{ii}^{KdV}(\boldsymbol{\beta})$	$\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{4\beta^i}$	$\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{2(\beta^i)^2}$	$\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{(\beta^i)^3}$
averaged KdV Hamiltonian	$\mathcal{K}_0 - \nu$	$\mathcal{K}_{-1} - \nu\mathcal{K}_0$	$\mathcal{K}_{-2} - \nu\mathcal{K}_{-1}$	$\mathcal{K}_{-3} - \nu\mathcal{K}_{-2}$
Reciprocal transformation	$dx = \mathcal{K}_0 dy + \mathcal{N} d\tau$ ↓		↑ $dy = \frac{1}{\mathcal{K}_0} dx - \frac{\mathcal{N}}{\mathcal{K}_0} dt$	
averagedCH Poisson tensor	$\tilde{\mathcal{A}}_4$	$\tilde{\mathcal{A}}_3$	$\tilde{\mathcal{A}}_2$	$\tilde{\mathcal{A}}_1$
CH metric	$\frac{1}{8\mathcal{K}_0^2}g_{ii}^{KdV}(\boldsymbol{\beta})$	$\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{4\mathcal{K}_0^2\beta^i}$	$\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{2\mathcal{K}_0^2(\beta^i)^2}$	$\frac{g_{ii}^{KdV}(\boldsymbol{\beta})}{\mathcal{K}_0^2(\beta^i)^3}$
averagedCH Hamiltonian	$1 - \frac{\nu}{\mathcal{K}_0}$	$\frac{\mathcal{K}_{-1}}{\mathcal{K}_0} - \nu$	$\frac{\mathcal{K}_{-2}}{\mathcal{K}_0} - \nu\frac{\mathcal{K}_{-1}}{\mathcal{K}_0}$	$\frac{\mathcal{K}_{-3}}{\mathcal{K}_0} - \nu\frac{\mathcal{K}_{-2}}{\mathcal{K}_0}$

Table 1: In the above table \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{R} have been defined in remark 4.4. The averaged KdV Poisson tensors J_i , $i = 1, \dots, 4$ and corresponding metrics have been defined at the beginning of section 4.3. The quantities \mathcal{K}_{-s} are defined in (55) and (59). The averaged KdV Hamiltonian densities for the operators J_i , $i = 1, 2, 3, 4$ are defined modulo Casimirs. The CH Hamiltonian operators $\tilde{\mathcal{A}}_s$ are defined below formula (69) and the corresponding Hamiltonians are defined in (71).

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