REPORT

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Critical behaviour of the focusing nonlinear Schrödinger equation and nonlocal perturbations

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1. PURPOSE OF THE VISIT

The Nonlocal nonlinear Schrödinger (NNLSE) is of interest in many area of modern Physics, from nonlinear optics to Bose-Einstein condensation.[1, 2, 4, 5] In particular, we focus on the models which are relevant in the so-called nonlinear geometric optics, which involves the study of the NNLSE in the low dispersion limit. The main question arising in this context is how nonlocal potentials affect the behaviour of solutions. One of the most important features of the NNLSE so far observed, is that the nonlocality tends to stabilize the well known wave collapse phenomenon occurring in $2+1-$dimensions. It is a general believing that nonlocality contributes to smooth out singular behaviours of the solutions. In a recent paper, Dubrovin, Grava and Klein have showed that, at the critical point of the dispersionless $1+1$D NLSE, solutions to the low dispersion NLSE exhibit a universal behaviour: at the leading order, the analytic form of the solution does not depend on the initial condition. In their approach, the integrability of the $1+1$D NLSE is crucial. In $2+1-$dimensions the NNLSE, as well as the standard NLSE, is not integrable. Nevertheless, integrable reductions of the $2+1$D NNLS equation might exist. Study of nonlocal effects at the critical point of the dispersionless limit is of particular interest and could suggest a classification tool of different types of nonlocality. Moreover, the analysis of robustness of the universality property under different kind on nonlocal perturbations (integrable and non integrable) is a completely open problem. A first step in this direction, is the study of the extendibility of the Dubrovin-Grava-Klein approach to nonlocal models, which is the main purpose of the present collaboration.

2. DESCRIPTION OF THE WORK

Consider the $1+1-$dimensional reduction of the NNLS equation

$$\begin{aligned} &u_t + (uv)_x = 0 \\ &v_t + vv_x - \eta_x + \epsilon^2 \left(\frac{1}{2}\frac{u_x^2}{u^2} - \frac{u_x x}{u} \right)_x = 0. \end{aligned} \tag{1}$$

where the nonlocal potential η is of the form

$$\eta = \int_{-\infty}^{\infty} R\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}, \sigma\right) u(y) d\left(\frac{y}{\epsilon}\right). \tag{2}$$

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The distribution $R(x, y, \sigma; \epsilon)$ depends on a parameter σ which is the typical range on nonlocality, and ϵ is a small parameter associated with the dispersion. The kernel $R(x, y, \sigma; \epsilon)$ can be implicitly specified via a spatial constraint on the potential η of the general form

$$L(\eta, \eta_x, \eta_{xx}, \dots) = u.$$

As an axample, consider an isotropic medium modelled by a Gaussian nonlocal response of the form

$$\eta = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}\sigma} e^{-\frac{(x-y)^2}{\epsilon^2\sigma^2}} u(y) d\left(\frac{y}{\epsilon}\right).$$

We immediately see that if the range of nonlocality σ is fixed there are no nonlocal contributions in the dispersionless limit

$$\lim_{\epsilon\to 0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}\sigma} e^{-\frac{(x-y)^2}{\epsilon^2\sigma^2}} u(y) d\left(\frac{y}{\epsilon}\right) = \delta(x-y).$$

Only very high nonlocalities of typical range $\sigma = \alpha/\epsilon$ survive in the dispersionless limit. Thus, for finite range nonlocalities we can adopt a kernel expansion

$$R(x, y, \sigma) = \delta(x-y) + \epsilon^2\sigma^2 R_2(x, y) + \dots \quad (3)$$

The isotropy of the medium suppresses odd powers in ϵ. Functions $R_i(x, y)$'s correspond to a set of nonlocal perturbations.

For the standard NLS equation, Dubrovin, Grava and Klein showed that, at the critical point of the dispersionless limit, solutions exhibit a universal behaviour, that is, their analytical form near the breaking point is independent on the initial condition and it is given by the *tritronqueé* solution to the Painlevé I. In their approach, the integrability of the $1+1-$dimensional NLS equation, plays a crucial role. It is based on the property that any hydrodynamic first integral of the dispersionless NLS equation can be extended to an integral of the dispersionfull NLS.

We have analyzed in detail one of the most studied nonlocal models in nonlinear optics, which is a particular case of (3). The nonlocal potential η obeys the following equation

$$\eta = u + \epsilon^2\sigma^2\eta_{xx}. \quad (4)$$

In the low dispersion limit, equation (4) implies that

$$\eta = u + \epsilon^2\sigma^2 u_{xx} + \epsilon^4\sigma^4 u_{xxxx} + \dots \quad (5)$$

Then, the nonlocal model (4) induces a tail of dispersive corrections of the form (5) to the standard dispersionless NLS equation

$$\begin{aligned} u_t + (uv)_x &= 0 \\ v_t + vv_x - u_x &= 0. \end{aligned} \quad (6)$$

Using the approach, described in [8] we proved that the hydrodynamic first integral of the dispersionless NLS equation cannot be uniformly extended to obtain a first integral of the perturbed equation. Then, the model (4) is not integrable. Nevertheless, this specific example suggests that *in the low dispersion regime, any physical nonlocal model of the form* (2) *is equivalent to a dispersive perturbation of the form*

$$\eta = u + \epsilon^2 (A_1(u,v)u_{xx} + A_2(u,v)v_{xx} + B_1 u_x^2 + B_2 u_x v_x + B_3 v_x^2) + \dots \tag{7}$$

In other words, nonlocality acts as a dispersive effect which may be useful to explain the regularization mechanism of singular solutions. This conjecture addresses the problem of classification of all integrable models in the class (7). A preliminary investigation showed that there are no nontrivial integrable examples in the class

$$\eta = u + \epsilon^2 A(u,v)u_{xx} + \dots$$

Even thought the classification procedure is rather clear, it is computationally very intensive and it is still under investigation. A comparison with the similar analysis of all dispersive corrections to the dKV equation suggests the solution to be rather nontrivial.[6,7] A dependence on a limited number of arbitrary functions of one variable is also expected.

3. CONCLUSIONS AND PERSPECTIVES

The analysis of one of the most studied nonlocal models in nonlinear optics leads to the identification of nonlocal effects with low dispersive corrections. Nevertheless, this model and its simplest generalizations are not integrable and suggest to look for integrable cases in the most general class of dispersive corrections. The present collaboration with Proffs. Boris Dubrovin and Tamara Grava aims to carry on and provide a full classification of physical integrable nonlocal models and a consequent analysis of solutions at the critical point of the dispersionless limit as in [7, 8]. The visit in Trieste has been important to set up the collaboration and to formulate correctly the problem. I'm confident that the present work will lead to a high quality publication in a suitable mathematical physics journal which will be also of sure interest for the nonlinear optics community.

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