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Orthogonal Polynomials with discontinuous weight

We consider orthogonal polynomials

$$\int_0^\infty w(x)P_n(x)P_m(x) = h_n\delta_{nm}$$

where $w(x)$ is a deformed Laguerre weight with one jump

$$w(x) = x^\alpha e^{-x}(1 - \frac{\beta}{2} + \beta\theta(x - x_0)), \quad 0 < \beta < 2,$$

with θ the Heaviside step function and $0 < x_0 < \infty$. Here β is the height of the jump. The polynomials $P_n(x)$ are monic

$$P_n(x) = x^n + \text{lower terms.}$$

From the orthogonality condition there follows the recurrence relations

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), n = 0, 1, ..., \ \beta_0 P_{-1}(z) := 0,$$

We study the dependence of the recurrence coefficient α_n as a function of x_0 and show that it satisfies a Painlevè type equation. A further step in the analysis is related to the study of the large n asymptotics of α_n.

We also consider a similar problem for the Hermite polynomials orthogonal over $(-\infty, +\infty)$ with deformed weight with two jumps

$$w(x) = e^{-x^2}(1 - \frac{\beta}{2} + \beta\theta(x - x_0) + \delta\theta(x - x_1)) \quad 0 < \beta < 2$$

where $\delta > 0$ and $x_1 > x_0$. We derive the differential equations of the recursion coefficients as a function of x_0 and x_1. Such problem has been considered earlier for the case of one simple jump in [CP], and [IK]. In particular in [CP] and [IK] the large n limit of the Hankel determinant

$$D_n(\beta) = \det\left(\int_{-\infty}^{\infty} x^{j+k} w(x)dx\right)_{j,k=0}^{n-1}$$

with Hermite weight with one jump $w(x) = e^{-x^2}(1 - \frac{\beta}{2} + \beta\theta(x - x_0))$,has been derived. We planned to derive a similar formula for the Hermite weight with more than one discontinuity. The two problems above originate in the works of [B], [BC], [BCW] and [W] in the study of Toeplitz and Hankel determinants with discontinuous symbols.

Bibliography

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