Scientific report on the stay of Dimitri MARKUSHEVICH in Italy from 24/06/2008 to 05/08/2008 in the framework of the research project "Lagrangian fibrations" in collaboration with Ugo BRUZZO

During the stay, Ugo BRUZZO organized two conferences on the themes directly related to the research project of our collaboration ; I took part in both of them and gave talks there :

1) Workshop on Moduli spaces, enumerative problems and integrable systems, Genova, 24/06/2008-29/06/2008, talk "Lagrangian family of Prym surfaces with polarization of type (1,2)".

2) Workshop on geometric methods in Mathematical Physics, Trieste, 30/06/2008-05/07/2008, talk "Integrable systems associated to K3-Fano flags".

In Trieste, I also gave a colloquium talk entitled "Abel-Jacobi map for hypersurfaces and noncommutative Calabi-Yau's".

During my stay, we started a collaboration with Ugo BRUZZO and Alessandro TANZINI on the **framed bundles over rational ruled surfaces**. The main objective of this work is to construct and to study the moduli space of framed torsion-free sheaves on the Hirzebruch surface $\mathbb{F}_p$. Such a moduli space, denoted $M^D_{\mathbb{F}_p}(r;k,n)$, parametrizes the isomorphism classes of framed torsion-free sheaves $(\mathcal{E},\varphi)$ on $\mathbb{F}_p$ with $\operatorname{rk}\mathcal{E}=r$, $c_1(\mathcal{E})=kE$, $c_2(\mathcal{E})=n$. The framing is an isomorphism $\varphi:\mathcal{E}_{|D}\to\mathcal{O}_D^{\oplus r}$, E denotes the exceptional curve of $\mathbb{F}_p$ with self-intersection $E^2=-p$, and D the movable cross-section of the projection $\mathbb{F}_p\to\mathbb{P}^1$.

We endow $M^D_{\mathbb{F}_p}(r;k,n)$ with a natural structure of a smooth quasi-projective variety. To this end, we show that any $(\mathcal{E},\varphi)\in M^D_{\mathbb{F}_p}(r;k,n)$ is (H,δ)-stable in the sense of Huybrechts-Lehn for some polynomial $\delta(t)=\delta_1 t+\delta_0$ of degree 1 with positive leading coefficient and for all ample divisors $H\simeq aD+bF$ with $a/b\gg 0$, so that $M^D_{\mathbb{F}_p}(r;k,n)$ is identified with an open subscheme of a quasi-projective moduli space $\mathcal{M}^s_{H,\delta}(\mathbb{F}_p;D,P)$ introduced in Theorem 0.1 of loc. cit. The notion of framing we adopt here naturally arises in the ADHM construction of instantons and is different from that of Huybrechts-Lehn : in their paper, a framing of $\mathcal{E}$ is any morphism of $\mathcal{O}$-modules $\varphi:\mathcal{E}\to\mathcal{D}$ for some fixed coherent $\mathcal{O}$-module $\mathcal{D}$. In our paper, $\mathcal{D}$ is the torsion sheaf supported on the curve D, which is trivial and has rank r as an $\mathcal{O}_D$-module, and moreover, we require that φ induces an isomorphism $\mathcal{E}_{|D}\simeq\mathcal{D}$. Thus moduli spaces of stable framed sheaves in the sense of our definition are open subschemes of the moduli spaces defined by Huybrechts-Lehn.

The next parts of the work we plan : description of the open piece of the moduli space $M^D_{\mathbb{F}_p}(r;k,n)$ representing the framed sheaves that are locally free on the exceptional curve in terms of the elementary transforms of some "sample" sheaves ; computation of the cohomology and the Poincaré polynomial of the moduli space via the count of the fixed points of a $\mathbb{C}^*$-action and Bott's formula. A preprint on this subject is being written.

We also discussed and outlined the work on two more themes, which will be continued after we finish writing the preprint on the instanton calculus over $\mathbb{F}_p$.

The second paper we plan to work on with Ugo BRUZZO is concerned with the search of **new Lagrangian fibrations** on the varieties $S^{[d]}$, Hilbert powers of K3 surfaces. It is natural to expect, by analogy with K3 surfaces, that a rational Lagrangian fibration on $S^{[d]}$ exists if and only if V has a divisor D with Beauville–Bogomolov square 0. The necessity to go over to *rational* LFs is due to the fact that birational ISV's have the same Beauville–Bogomolov form, and a birational transformation (flop) can destroy the regularity of a LF making it a rational map. This question was addressed in my previous paper, were I proved [M2] that $S^{[d]}$ admits a rational LF if the Beauville–Bogomolov form on the cohomology $H^2(S^{[d]})=H^2(S)\overset{\perp}{\oplus}\langle 2d-2\rangle$ has an isotropic vector $f+ke$, where e is a basis of $\langle 2d-2\rangle = \mathbb{Z}e$, $\mathrm{Pic}(S) = \mathbb{Z}f$ and $k \in \mathbb{Z}$. The construction uses a twisted Fourier–Mukai transform which induces a birational isomorphism μ between $S^{[d]}$ and a certain moduli space of twisted sheaves on another K3 surface M, obtained from S as its (twisted) Fourier–Mukai partner.

Using more sophisticated twisted Fourier-Mukai transforms, we plan to extend this result to the general case when the Beauville–Bogomolov form represents zero on $H^2(S^{[d]})$, that is, the case when there exists an isotropic vector $v \in H^2(S^{[d]})$ of the form $kf + le$ with $k, l \in \mathbb{Z}$ mutually prime, $f \in H^2(S)$ a primitive class. As we have verified, the orbits of such classes $kf + le$ under the orthogonal group action exhaust all the integer isotropic vectors in $H^2(S^{[d]})$. Thus our result will imply that for each isotropic vector $v \in H^2(S^{[d]})$, the hyperplane section $v^\perp$ of the period domain has a dense open subset of points representing periods of regular LFs given by the linear system of the divisor class v. Stated in this form, the result will establish a complete analogy with the description of periods of K3 surfaces having an elliptic pencil with class v.

The third joint work we have outlined during our stay will be effectuated in collaboration with Ugo BRUZZO and Atanas ILIEV from Sophia (Bulgaria). This is the work on **another kind of Lagrangian fibration** (or integrable system) related to irreducible symplectic varieties (ISV). Up to now, all the known Lagrangian fibrations on ISVs are obtained by finding a birational isomorphism between the given ISV M and some torsor T under the compactified relative Jacobian J of a complete linear system $|C|$ of genus-g curves on a K3 surface, or else between M and the fiber of the Albanese map $T \to \mathrm{Alb}(T) = A \times \hat{A}$, where T is some torsor under the compactified relative Jacobian J of a linear system of curves on an abelian surface A (see [M2], [Gu], [De] and references therein).

On the other hand, there are examples of LFs over a noncompact base. These are the Donagi–Markman integrable system [DoM] of the intermediate Jacobians of the hyperplane sections of a smooth cubic 4-fold in $\mathbb{P}^5$, and the integrable system of intermediate Jacobians of the complete family of Fano varieties with fixed hyperplane section [B], [IM1], [IM2], [M3]. The case considered in [M3] is that of the relative intermediate Jacobian of the moduli space of the K3–Fano flags $S \subset X$, where S is a fixed K3 surface and X a Fano 3-fold containing S as an anticanonical divisor.

The fibers of these LFs are in general not Jacobians of curves, and it is a very interesting question whether their total space can be compactified into a smooth symplectic variety on which the LF remains regular. If yes, this will give first examples of LFs on ISVs whose fibers are principally polarized, but are not Jacobians of curves.

Atanas ILIEV suggested an original (at the moment conjectural) construction which would relate the Donagi-Markman Lagrangian fibration for a special cubic 4-fold Y to the sporadic 10-dimensional ISV of O'Grady on the K3 surface S naturally associated to Y. One of the outcomes od this construction would be a compactification of the Donagi-Markman fibration, which seems to be unaffordable in terms of just intermediate Jacobians of the hyperplane sections.

One more theme which was suggested in my research proposal, was the continuation of the work, started in [MT] and aimed at new constructions of generalized ISVs with certain mild singularities (with a hope that one may occasionally get nonsingular ones, too). We decided not to develop this theme by the forces of only two of us, but to suggest it as a theme for a co-advised PhD thesis, as soon as Ugo BRUZZO or I find an appropriate PhD candidate. Narrowly understood, this is the study of the **relative compactified Prym varieties** of τ-invariant parts of linear systems of curves on K3 surfaces with an anti-symplectic involution. In a larger sense, these are components of the fixed loci of finite groups of *symplectic* automorphisms of holomorphically symplectic varieties (the compactified Pryms are particular cases of these). One may also consider fibrations in generalized Prym–Tyurin varieties as those studied in the recent work of Izadi–Lange and try to find out, which of them admit a symplectic structure making them LFs.

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