Bihamiltonian structures with symmetries, Lie pencils, and integrable systems

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The aim of this project is to develop a technique in the theory of bihamiltonian structures [Mag78,GZ93] coming from the following "physical" motivation. Consider the n-dimensional free rigid body system on $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$. This is a hamiltonian system with respect to the canonical Lie-Poisson structure on $\mathfrak{g}^*$. After identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the "trace form" the corresponding hamiltonian function becomes $H(M) = (1/2)\,\mathrm{Tr}(M \cdot L^{-1}M), M \in \mathfrak{g}$. Here L is an operator on $\mathfrak{g}$ defined by $L : M \mapsto DM + MD$, D being the "inertia" matrix of the rigid body, a diagonal matrix with positive simple spectrum [MP96].

There are several approaches to studying the complete integrability of this system. One of them called "the argument translation method", which goes back to Manakov [Man76] and was fully developed by Mishchenko-Fomenko [MF78], uses integrals of the form $\mathrm{Tr}((M + \lambda D^2)^k)$. In fact these integrals are related to the Poisson pencil defined on $\mathfrak{sl}(n, \mathbb{R}) \supset \mathfrak{so}(n, \mathbb{R})$ that isgenerated by two Poisson structures: the Lie-Poisson structure $\vartheta_{\mathfrak{sl}(n,\mathbb{R})}$ and the constant one obtained by "freezing" $\vartheta_{\mathfrak{sl}(n,\mathbb{R})}$ at the point $D^2 \in \mathfrak{sl}(n, \mathbb{R})$.

Another approach, first appeared in [Bol91] and then in [MP96], exploits another Poisson pencil. It is defined on $\mathfrak{so}(n, \mathbb{R})$ itself and is generated by two Lie-Poisson structures, one related to the standard commutator $[,]$ on $\mathfrak{g}$, another to the modified commutator $[,]_{D^2}$, $[X,Y]_{D^2} := XD^2Y - YD^2X$. The corresponding integrals are of the form $\mathrm{Tr}(((I + \lambda D^2)^{-1/2}M(I + \lambda D^2)^{-1/2})^k)$. Although looking differently from that defined above they in fact define the same invariant tori.

Now assume the matrix D has multiplicities in its spectrum. Then in general neither of the above series of integrals is sufficient for the integrability of the system. However, the nonsimplicity of the spectrum of D is equivalent to existence of inner symmetries of the body and in order to prove the complete integrability of the system one needs to add to the above mentioned integrals the Noether integrals induced by these symmetries.

The aim of this project is to develop a similar approach in the more general setting of the Poisson pencils with an additional property. This property (which holds for both the examples above, as well as for many other natural examples) supposes the constancy of the eigenvalues of the corresponding recursion operator (we don't assume that the generic Poisson bracket in the pencil is nondegenerate, i.e. we mean some generalization of the recursion operator in the "usual" sense). In such an approach one should consider on the same level the "standard" functions in involution coming from the Casimir functions of the generic elements of the Poisson pencil (the Manakov and Bolsinov integrals of the examples above) and, on the other hand, the integrals coming from "inner" symmetries of the pencil.

I hope to discuss the related techniques during my visit with the members of the research groups of Prof. B. Dubrovin (SISSA, Trieste) and Prof. F. Magri (Bicocca University, Milan) in both of which the study of bihamiltonian structures is one of the main research directions.

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